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Realizability of localized groups and spaces

J. Aguadé

The theory of localization of nilpotent groups and spaces (see [4] for a reference) associates to each nilpotent group (space) G, a family $\{G_p\}$ of nilpotent groups (spaces), G_p p-local. In this paper we study the problem of deciding if given a family $\{G(p)\}$ of groups (spaces) there is a group (space) G such that $\{G(p)\}$ coincides with the family of localizations of G. We obtain necessary and sufficient conditions for an affirmative answer (see § 3 for a precise definition).

In the last section of this paper we apply the preceding results to the problem of fibering a space by a subspace. We show that under certain conditions it is a "local" problem in the sense that a space E can be fibered by a subspace F if and only if the localizations E_p can be fibered by F_p for all p.

All spaces are assumed to be of the homotopy type of CW complexes.

1. Realizability of localized groups

In this section we consider the following problem: Let $\{G(p)\}$ be a family of nilpotent groups of class $\leq c$, G(p) p-local, and let $G(p) \longrightarrow G(o)$ be o-localization (i.e. all groups G(p) have isomorphic rationalizations). We want to obtain necessary and sufficient conditions in order to insure the existence of a group G with p-localizations isomorphic to G(p). More precisely, we say that a nilpotent group G of class $\leq c$ solves the problem if:

a) There are isomorphisms $G_p \xrightarrow{\cong} G(p)$ and $G_o \xrightarrow{\cong} G(o)$;

b) the following diagram is commutative:

$$\begin{array}{ccc} G_p & \xrightarrow{\cong} & G(p) \\ \downarrow & & \downarrow \\ G_o & \xrightarrow{\cong} & G(o) \end{array}$$

Notice that the homomorphisms $G(p) \longrightarrow G(o)$ are data of the problem. This is important because it is known that there are non-isomorphic groups with isomorphic localizations (see [4], p.33), whereas, at least if the group G is finitely generated, G is completely determined by the homomorphism $G_p \longrightarrow G_o$. Note also that the problem does not always have a solution. A counterexample can be constructed by taking $G(p) = \mathbb{Z}_{(p)}$, $G(o) = \mathbb{Q}$ and $G(p) \longrightarrow G(o)$ multiplication by p. We will see later that there is no group G solving the problem in this case. Clearly, if we omit the condition b), we can take $G = \mathbb{Z}$.

Theorem 1.1 With the above notations let us consider the following conditions:

i) the problem has a solution;

ii) there exists $\rho: G(o) \longrightarrow (\Pi G(p))_o$ such that if h_p is the rationalization of the canonical projection $\Pi G(p) \longrightarrow G(p)$, then the following diagram is commutative:

$$\begin{array}{ccc} G(o) & \longrightarrow & G(p) \\ & {\scriptstyle h_p\rho}\searrow \quad \swarrow & \\ & G(p)_o & \end{array}$$

iii) let us denote $H_p = \mathrm{Im}(G(p) \longrightarrow G(o))$, $H = \cap H_p$. Given $x \in G(o)$ there exists n such that $x^n \in H$.

Then we have: i $\Leftrightarrow$ ii $\Rightarrow$ iii and if the groups $G(p)$ are torsion free abelian groups then all three conditions are equivalent.

Proof: i $\Rightarrow$ ii. Let G be a group solving the problem . We can define ρ as the composition $G(o) \xleftarrow{\cong} G_o \longrightarrow (\Pi G(p))_o$ where the second map is the rationalization of the composition $G \longrightarrow \Pi G \xrightarrow{\cong} \Pi G(p)$.

i $\Rightarrow$ iii. It suffices to prove iii for G_p and G_o instead of $G(p)$ and $G(o)$. Given $x \in G_o$, there exist n such that $x^n = ry$, $y \in G$, $r : G \longrightarrow G_o$ the rationalization. Let us consider the p-localizations of y, $x_p \in G_p$. Then x_p rationalizes to x and $x^n \in H$.

ii $\Rightarrow$ i. If there exists ρ, we define G as the pullback

$$\begin{array}{ccc} G & \longrightarrow & \Pi G(p) \\ \downarrow & & \downarrow r \\ G(o) & \xrightarrow{\rho} & (\Pi G(p))_o \end{array}$$

G is a nilpotent group of class $\leqslant c$. Composing the top homomorphism with the canonical projections $\Pi G(p) \longrightarrow G(p)$ we obtain homomorphisms g_p: $G \longrightarrow G(p)$. We will show that g_p is a p-localization i.e. g_p is a p-isomorphism. From the hypothesis on ρ we obtain the commutativity of the diagram:

$$\begin{array}{ccc} G & \xrightarrow{g_p} & G(p) \\ & \searrow \quad \swarrow & \\ & G(o) & \end{array} \qquad (1)$$

We have

$$G = \{((x_q),y) \mid x_q \in G(q),\ y \in G(o) \ \text{ and } \ r((x_q)) = \rho y\}.$$

Let us assume $g_p((x_q),y) = 1$, i.e. $x_p = 1$. Then the above diagram yields $y = 1$ and so $r((x_q)) = \rho y = 1$. Since r is a 0-isomorphism, there exists n such that $(x_q^n) = 1$. But x_q belongs to the q-local group $G(q)$, hence we can assume $(n,p) = 1$ and so we have proved that g_p is a p-monomorphism.

Let $x_p \in G(p)$. We have to see that there exists m such that $(m,p)=1$ and $x_p^m = g_p a$ for some $a \in G$. Let $y = rx_p \in G(o)$, $z = \rho\, y \in (\Pi G(p))_o$. Then, $h_p z = y$. Since r: $\Pi G(p) \longrightarrow (\Pi G(p))_o$ is a 0-isomorphism, there exists n such that $z^n = r((\bar{x}_q))$. Since $\bar{x}_q \in G(q)$ and this group is q-local, if $q \neq p$ we can take $\bar{x}_q = x_q'^{\,p^k}$ with $h = p^k m$ and $(p,m) = 1$. On the other hand $\bar{x}_p$ goes to $y^n = rx_p^n$. Since $G(p) \longrightarrow G(o)$ is a q-isomorphism, we have $\bar{x}_p^{p^t} = x^{p^{k+t}m}$ and we take $x_p' = x_p^m$. Let us consider $(x_q') \in \Pi G(p)$. We have:

$$z^{p^{k+t}m} = r((\bar{x}_q)^{p^t}) = r((x_q')^{p^{k+t}}) = (r((x_q')))^{p^{k+t}} \in (\Pi G(p))_o$$

Since $(\Pi G(p))_o$ is o-local, we obtain $r((x_q')) = z^m$ and $g_p((x_q'),y^m) = x_p' = x_p^m$ with $(m,p) = 1$. This proves that g_p is a p-epimorphism.

Let us see now that the group G solves the problem. Since we have proven that g_p: $G \longrightarrow G(p)$ is a p-localization, we have an isomorphism $G_p \xrightarrow{\cong} G(p)$. Moreover, since the diagram (1) is commutative, the homomorphism $G \longrightarrow G(o)$ is a 0-isomorphism and we have an isomorphism $G_o \xrightarrow{\cong} G(o)$. We only have to see that the diagram

$$\begin{array}{ccc} G_p & \xrightarrow{\cong} & G(p) \\ \downarrow & & \downarrow \\ G_o & \xrightarrow{\cong} & G(o) \end{array}$$

is commutative, but this follows from the fact that it is obtained from

$$\begin{array}{ccc} G & \xrightarrow{g_p} & G(p) \\ \downarrow & & \downarrow \\ G_o & \longrightarrow & G(o) \end{array}$$

by localization. This ends the proof of ii $\Rightarrow$ i. Let us assume now that the groups $G(p)$ are torsion free abelian groups and let us show that iii $\Rightarrow$ ii. Given $x \in G(o)$, let n be such that $nx \in H$. Then for each p there is a uniquely determined $x_p \in G(p)$ such that $nx = rx_p$. We take $z = (x_p) \in \Pi G(p)$ and we define $\rho x = z' \in (\Pi G(p))_o$ where z' is such that $nz' = rz$. It is then clear that z' does not depend on the n we have chosen. In this way we obtain an homomorphism $\rho : G(o) \longrightarrow (\Pi G(p))_o$.

This ends the proof of the theorem. □

Now we can see that if we take $G(p) = \mathbb{Z}_{(p)}$, $G(o) = \mathbb{Q}$ and $G(p) \longrightarrow G(o)$ multiplication by p, then there is no group G solving the problem because condition iii in the above theorem is not satisfied.

Theorem 3.1 in [3] proves that for a given ρ the solution is uniquely determined.

We will study now under what conditions a family $\{G \longrightarrow H_p\}_p$ of homomorphisms, where G and H are nilpotent groups, comes from a homomorphism $G \longrightarrow H$. A necessary condition is that the family $\{G \longrightarrow H_p\}$ should be rationaly coherent i.e. for all primes p,q the diagram

$$\begin{array}{ccc} G & \longrightarrow & H_p \\ \downarrow & & \downarrow \\ H_q & \longrightarrow & H_o \end{array}$$

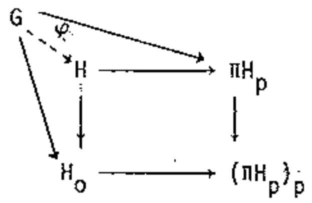
should be commutative. If H is finitely generated this condition is also sufficient ([4], p.26). In general we have:

Proposition 1.2 A rationaly coherent family of homomorphism $\{G \longrightarrow H_p\}_p$ comes from a homomorphim $G \longrightarrow H$ if and only if the induced diagram

$$\begin{array}{ccc} H_o & \xrightarrow{\rho} & (\Pi H_p)_o \\ \uparrow & & \uparrow \\ G_o & \xrightarrow{\rho} & (\Pi G_p)_o \end{array}$$

is commutative.

Proof: The "only if" part is trivial. If The above diagram commutes we have:



and we get φ because the square is a pullback ([3]) □

2. Realizability of localized spaces

Let $\{B(p)\}$ be a family of nilpotent connected spaces, $B(p)$ p-local, and let $B(p) \longrightarrow B(o)$ be rationalizations (i.e. all spaces $B(p)$ have homotopy equivalent rationalizations). We ask for the existence of a nilpotent space B and homotopy equivalences $B_p \xrightarrow{\sim} B(p)$, $B_o \xrightarrow{\sim} B(o)$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} B_p & \xrightarrow{\sim} & B(p) \\ \downarrow & & \downarrow \\ B_o & \xrightarrow{\sim} & B(o) \end{array}$$

If such a space B exists we say that B solves the problem. First of all, a necessary condition for the existence of a solution is that $\Pi B(p)$ must be a nilpotent space. It is not difficult to see that this is equivalent to say that there exist integers c_n, $n \geq 1$ such that $\pi_1 B(p)$ is a nilpotent group of class $\leq c_1$ and $\pi_i B(p)$ is a nilpotent $\pi_1 B(p)$-module of class $\leq c_n$, for all p. From now on we assume $\Pi B(p)$ nilpotent.

We have seen in the last section that the realizability problem for groups does not always have a solution. The same holds for spaces because if $G(p) \longrightarrow G(o)$ is a counterexample for groups, we can consider $K(G(p),1) \longrightarrow K(G(o),1)$.

<u>Theorem 2.1</u> There exists a nilpotent space B solving the problem if and only if there is a map $\rho : B(o) \longrightarrow (\Pi B(p))_o$ such that if h_p is the rationalization of the map $\Pi B(p) \longrightarrow B(p)$, then the following diagram commutes up to homotopy:

$$\begin{array}{ccc} B(o) & \longrightarrow & B(p) \\ & {\scriptstyle h_p\rho}\searrow & \swarrow \\ & B(p)_o & \end{array}$$

Proof: If B is given we take ρ to be the rationalization of the composition $B \longrightarrow \Pi B_p \xrightarrow{\sim} \Pi B(p)$. Conversely, let us assume that there exists a map ρ satisfiying the hypothesis of the theorem. For each $i \geq 1$, we define the group G^i as the pullback

$$\begin{array}{ccc} G^i & \longrightarrow & \prod_p \pi_i B(p) \\ \downarrow & & \downarrow r_* \\ \pi_i B(o) & \xrightarrow{\rho_*} & (\prod_p \pi_i B(p))_o \end{array}$$

Then G^i is a nilpotent group (abelian if $i > 1$) whose localized groups coincide with the $\pi_i B(p)$. By [3], the diagram is bicartesian and we have exact sequences:

$$G^i \rightarrowtail \pi_i B(o) \oplus \Pi \pi_i B(p) \xrightarrow{<\rho_*, -r_*>} (\Pi \pi_i B(p))_o$$
$$\qquad\qquad (2)$$
$$G^1 \rightarrowtail \pi_1 B(o) \times \Pi \pi_1 B(p) \rightrightarrows (\Pi \pi_1 B(p))_o$$

We define the space B as the (weak) pullback

$$\begin{array}{ccc} B & \longrightarrow & \Pi B(p) \\ \downarrow & & \downarrow r \\ B(o) & \xrightarrow{\rho} & (\Pi B(p))_o \end{array} \qquad (3)$$

If we apply ([3], 3.4) to the diagram (1) we see that every $z \in (\Pi \pi_1 B(p))_o$ can be expressed as $z = r_* x . \rho_* y$ and this implies, by [4], II. 7.11, that B is connected. Since $\Pi B(p)$ is nilpotent, [4], II.7.6 implies that B is also nilpotent.

The homotopy Mayer-Vietoris exact sequence of the (weak) pullback (3) yields ([2]):

$$\ldots \longrightarrow \pi_i B \longrightarrow \pi_i B(o) \oplus \Pi \pi_i B(p) \xrightarrow{<\rho_*, -r_*>} (\Pi \pi_i B(p))_o \longrightarrow \ldots \quad (4)$$

Since (1) is a pullback we have a canonical homomorphism $\pi_i B \longrightarrow G^i$ and it follows from (2) and (4) that it is an isomorphism. Then $B \longrightarrow B(p)$ is a p-localization because $\pi_i B \longrightarrow \pi_i B(p)$ is also a p-localization. The rest of the proof is formally analogous to that of 1.1. □

Theorem 3.3 in [3] proves that for a given map ρ the solution is uniquely determined up to homotopy.

There is also an analogous of proposition 1.2.:

<u>Proposition 2.2</u> A rationaly coherent family of maps $\{X \longrightarrow Y_p\}_p$ comes from a map $X \longrightarrow Y$ if and only if the induced diagram

$$\begin{array}{ccc} Y_o & \xrightarrow{\rho} & (\Pi Y_p)_o \\ \uparrow & & \uparrow \\ X_o & \xrightarrow{\rho} & (\Pi X_p)_o \end{array}$$

commutes up to homotopy. □

3. The problem of fibering a space by a subspace

Let (E,F) be a couple of nilpotent spaces, i.e. F is a subspace of E. We say that (E,F) is a <u>fiber couple</u> if there exists a nilpotent space B and a map $E \longrightarrow B$ such that $F \longrightarrow E \longrightarrow B$ is homotopically equivalent to a fibration. In other words, there is a homotopy commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & \nearrow & \\ \bar{F} & \longrightarrow & \bar{E} & & \end{array}$$

where $\bar{F} \longrightarrow \bar{E} \longrightarrow B$ is a fibration and the vertical arrows are homotopy equivalences. By [1] p.60, the fibration $\bar{F} \longrightarrow \bar{E} \longrightarrow B$ turns out to be nilpotent.

To characterize fiber couples is one of the problems listed in [5].

It is not difficult to prove the following result:

<u>Lemma 3.1</u> (E,F) is a fiber couple if and only if there exists a nilpotent space B and a map $p: E \longrightarrow B$ such that i) $p_{|F} \sim *$; ii) $p_*: \pi_i(E,F) \longrightarrow \pi_i B$ is an isomorphism for all i. □

Our goal is to relate the fact that (E,F) is a fiber couple to the fact that (E_p,F_p) are fiber couples for all primes p. The equivalence of bouth assertions will be obtained only under certain hypothesis.

We say that (E,F) is a <u>nice couple</u> if $F_o = *$ or $F \longrightarrow E$ is a rational homotopy equivalence. Recall that a space X is called quasifinite if the homotopy groups $\pi_n X$ are finitely generated for all $n \geq 1$ and $H_n X = 0$ for n sufficiently large.

<u>Theorem 3.2</u> Let F be a quasifinite space and let (E,F) be a nice couple. (E,F) is a fiber couple if and only if (E_p, F_p) is a fiber couple for all primes p.

Proof: Since localization preserves fibrations, only the part "if" of the theorem needs a proof. Let us assume we have nilpotent fibrations $F_p \longrightarrow E_p \longrightarrow B(p)$ for all p. The exact homotopy sequence of these fibrations yields that B(p) is a p-local space. Since (E,F) is a nice couple we have homotopy equivalences $B(p)_o \sim B(q)_o$. In order to construct a space B whose localizations coincide with the B(p), we have to see that $\Pi B(p)$ is nilpotent but since we have fibrations $F_p \longrightarrow E_p \longrightarrow B(p)$, the nilpotency class of the homotopy groups of B(p) is bounded because the same holds for E_p and F_p. Let us consider the diagram:

$$\begin{array}{ccccc} F_o & \longrightarrow & E_o & \longrightarrow & B_o \\ \bar{\bar{\rho}} \downarrow & & \bar{\rho} \downarrow & & \rho \vdots \\ (\Pi F_p)_o & \longrightarrow & (\Pi E_p)_o & \longrightarrow & (\Pi B(p))_o \end{array}$$

and the existence of the dotted map ρ follows from the fact that the couple (E,F) is a nice one. Moreover the hypothesis of theorem 2.1 are fullfilled and we obtain a space B such that $B_p \sim B(p)$.

We have to construct a map $E \longrightarrow B$. Since we have compatible maps $E_p \longrightarrow B_p$ we can apply proposition 2.2 and we get a map $E \longrightarrow B$. It remains only to show that F E B is homotopy equivalent to a fibration.

Since F is quasifinite, the composition $F \to E \to B$ is homotopically trivial ([4], p.89) and since $\pi_i(E_p, F_p) \to \pi_i(B_p)$ is an isomorphism for all p, all i, then $\pi_i(E,F) \to \pi_i B$ is also an isomorphism. Hence (E,F) is a fiber couple. □

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Universitat Autònoma de Barcelona

and

Forschungsinstitut für Mathematik, ETH-Zürich.