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THE DIVISOR GROUP OF A FIR

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Introduction

There have been many attempts to define determinants on non-commutative rings (cf.e.g.[13] and the references quoted there), of which perhaps the most successful is the definition of Dieudonné [10], leading for any skew field K and any $n \geqslant 1$ (except when n=2 and $K=F_2$) to an isomorphism

$$(1) \qquad GL_n(K)^{ab} \cong K^{*ab}.$$

Suppose now that K is obtained from a ring R by inverting certain matrices over R, forming a set Σ. The way in which the elements of K are obtained from R and Σ was described in Ch.7 of [3], and we may ask whether $GL_n(K)^{ab}$ can be described directly in terms of R and Σ. Since (1) is an isomorphism for all n, we can limit ourselves to a single value of n, or we may simply take the limit $GL(K)=\varinjlim GL_n(K)$. Our aim here is to describe the Whitehead group $K_1(K) = GL(K)^{ab}$ in terms of Σ; this can be done under fairly general conditions, though for more precise results we need to take R to be a fir and K its universal field of fractions. In parti cular, by taking R to be a free associative algebra we obtain an explicit expression for determinants over a free field (Th.5.2).

To state the results, let $f : R \longrightarrow K$ be a homomorphism of any rings and suppose that every element of K can be obtained from the entries of the formal inverses of the matrices from a set Σ, which is multiplicative (as defined below, cf. also [3], p.249), then it is not hard to show that f induces an epimorphism of abelian groups

$$(2) \qquad f^* : \Sigma^{ab} \longrightarrow K_1(K),$$

where Σ^{ab} is the universal abelian group of Σ (Th.2.2 and Cor.). In general there is no reason for f^* to be injective, but when K is the universal field of fractions of a Sylvester domain R and Σ the set of all full matrices over R, then (2) is an isomorphism. This is proved (for the slightly larger class of pseudo-Sylvester domains) in Th.3.1 by constructing an inverse mapping to f^*. For a somewhat different treatment of the same problem see [12] and also [6].

For Sylvester domains it is difficult to say more because little is known about factorization in such rings. But when we have a fir R, or more generally a fully atomic semifir (i.e. one in which every full matrix can be expressed as a product of atoms) then a more precise statement is possible. In R define a *prime* as a class of stably associated atoms and the *divisor group* $D(R)$ as the free abelian group on all the primes, and let U be the universal field of fractions of R, then we prove in Th.4.4 that

$$K_1(U) \cong U^{*ab} \cong D(R) \times [GL(R)/GL(R) \cap GL(U)'].$$

In particular, when $R = k \langle X \rangle$ is a free algebra, this becomes

$$U^{*ab} \cong D(R) \times k^*$$

(cf. Th.5.2.). These results have also been obtained by G.Révész [12] by a different method.

Our second main result is concerned with localization of firs. Let R be a fully atomic semifir and Σ a multiplicative set of matrices such that R_Σ is again a semifir, then R_Σ is also fully atomic and the divisor group of R_Σ is isomorphic to $D_\Sigma(R)$, the subgroup of $D(R)$ generated by the primes which survive in R_Σ (Th.6.3).

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1. Notation and general background

Let R be any ring; we write ${}^mR^n$ for the set of all $m \times n$ matrices over R and also put ${}^mR$ for ${}^mR^1$ and R^n for ${}^1R^n$. The *characteristic* of an $m \times n$ matrix is defined as $n - m$. If a matrix is expressed in block form $\binom{P}{Q}$, we often write this as $(P,Q)^T$ to save space;here T indicates that the blocks P, Q are to be written as a column, but are not themselves transposed. Similarly for more than two blocks.

The *diagonal sum* of two matrices A, B is defined as $A \dotplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. The set of all invertible $n \times n$ matrices over R is denoted by $GL_n(R)$ and we embed $GL_n(R)$ in $GL_{n+1}(R)$ by the rule $A \longmapsto A \dotplus 1$. Further we put $GL(R) = \varinjlim GL_n(R)$. As usual, a matrix is said to be *elementary* if it differs from the unit matrix in at most one off-diagonal entry; the group generated by all elementary $n \times n$ matrices is written $E_n(R)$ and as before we put $E(R) = \varinjlim E_n(R)$.

For any $A \in {}^mR^n$, the least integer r such that $A = PQ$, where $P \in {}^mR^r$, $Q \in {}^rR^n$ is called the *inner rank* and a matrix is said to be *full* if it is square, say $n \times n$ and of inner rank n. In general, if A is full, it need not be the case that $A \dotplus 1$ is full; if $A \dotplus I$ is full for any unit matrix I, A is called *stably full* and the set of all stably full $n \times n$ matrices over R is written $F_n(R)$; we embed $F_n(R)$ in $F_{n+1}(R)$ as for $GL(R)$ and write $F(R) = \varinjlim F_n(R)$. Sometimes we shall need a generalization of the inner rank. The *stable rank* of a matrix A is defined as $\lim \{rk(A \dotplus I_n) - n\}$, where rk denotes the inner rank. This limit always exists and is an integer or $-\infty$, but if I_n is full for all n, the inner rank is actually non-negative (cf. [8]). We note that an $n \times n$ matrix is stably full precisely when it has stable rank n.

Two matrices A, B are said to be *associated* if $A = PBQ$ for some invertible matrices P, Q. If P, Q can be taken in $E_n(R)$, we call A and B *E-associated*. If $A \dotplus I$ is (E-) associated to $B \dotplus I$ (where the unit matrices need not be of the same size), then we say that A and B are *stably* (E-) associated. Two stably associated matrices are not necessarily of the same size, but they have the same characteristic, provided that R is a ring with invariant basis number (i.e. all invertible matrices are square).

By a *field* we understand a not necessarily commutative division ring; sometimes we use the prefix 'skew' for emphasis. If G is any group, its derived group is denoted by G' and we write $G^{ab} = G/G'$ for the abelianization of G. For a field K we write K* for the group of its non-zero elements and by abuse of notation we simply write K^{ab} for K^{*ab}. If R' is any ring, an *R-field* is a field K with a homomorphism $R \longrightarrow K$; if K is generated, as a field, by the image of R, we speak of an *epic* R-field.

An m x n matrix over a field K, of rank r, is said to have *left nullity* m-r and *right nullity* n-r. These nullities are only defined over a field, but if A is a matrix over R and K is any R-field, we can consider the nullities of A over K; they are simply the nullities of the image of A in K.

We shall not repeat the definitions of fir and semifir (cf. [3], Ch. 1 or [4], Ch. 4). We merely recall that every semifir R has a universal field of fractions U, obtained by formally inverting all full matrices over R (cf. [3], p. 282 f.). More generally, if R is any ring and Σ a set of square matrices over R, then a map $f : R \longrightarrow S$ is called a *Σ-inverting epimorphism* if f is a homomorphism mapping each matrix of Σ to an invertible matrix over S and S consists of the entries of inverses of the matrices A^f, $A\in\Sigma$. It is easily seen that this is in fact an epimorphism in the category of rings. In what follows, Σ will generally be *multiplicative*, i.e. $1\in\Sigma$ and if A, $B\in\Sigma$, then $\begin{pmatrix} A & 0 \\ C & B \end{pmatrix}\in\Sigma$ for all C of appropriate size. For any ring R and set Σ of full matrices over R, the *localization* R_Σ of R by Σ is defined as the ring obtained from R by formally inverting all the matrices in Σ. Then in any Σ-inverting epimorphism $f : R \longrightarrow S$, S is clearly a homomorphic image of R_Σ. Suppose now that Σ is multiplicative;then each element of S may be obtained as the last component u_n of the solution of a matrix equation

$$(1) \qquad A^f u = 0, \quad A = (A_o, A_1, \dots, A_n) \in {}^nR^{n+1},$$

where $(A_1,\dots,A_n)\in\Sigma$ and $u = (1,u_1,\dots,u_n)^T$. If $p = u_n$, we say that (1) is an S-admissible system for p and call $(A_o,A_1,\dots,A_{n-1})$ the *numerator*, $(A_1,\dots,A_n)$ the *denominator* of p (cf. [5], § 4). It is often convenient to put $(A_1,\dots,A_{n-1}) = A_*$ and $A_n = A_\infty$), then the numerator will be (A_o,A_*) and the denominator (A_*, A_∞).

2. The calculation of K_1 for a localization

It is a well known fact that for any ring R, $E(R) = GL(R)'$ (cf. [1], V.1.5, p.223), so that $GL(R)^{ab} = GL(R)/E(R)$; by definition this is the Whitehead group $K_1(R)$. If we have a skew field K, then for any non-singular matrix A over

K there exists $\alpha \in K^*$ such that

$$(1) \qquad A \equiv \alpha \pmod{E(K)},$$

and here α is determined mod $K^{*\prime}$. The residue class α^{ab} of α in K^{ab} is called the *Dieudonné determinant* of A, written det A. We note that by (1),

$$(2) \qquad K_1(K) \cong K^{ab},$$

for any skew field K.

Suppose now that R, S are any rings, Σ is a multiplicative set of matrices over R and $f:R \longrightarrow S$ is Σ-inverting epimorphism. Our object is to express $K_1(S)$ in terms of R and Σ. In order to do this we need to express matrices over S as solutions of systems of equations over R (as was done in (1) of § 1 for elements).

Proposition 2.1 *Let* R, S *be any rings,* Σ *a multiplicative set of matrices over* R, *and let* $f:R \longrightarrow S$ *be a* Σ*-inverting epimorphism. Given any* $P \in {}^mS^n$, *there exists an integer* $r \geq 0$ *and matrices*

$$(3) \qquad A = (A_0, A_*, A_\infty) \in {}^{r+m}R^{n+r+m}, \quad u = (u_0, u_*, u_\infty)^T \in {}^{n+r+m}S^n,$$

where the numbers of columns of A_0, A_*, A_∞ *and likewise the numbers of rows of* u_0, u_*, u_∞ *are* n, r, m, *respectively, such that*

$$(4) \qquad A^f u = 0, \qquad (A_*, A_\infty) \in \Sigma,$$

and

$$(5) \qquad u_0 = I, \; u_\infty = P.$$

Moreover, u *is the unique element of* ${}^{n+r+m}S^n$ *satisfying* (4), (5) *for the given matrix* A.

We shall call A an S-*admissible* system for P.

Proof. The uniqueness of u follows from (4), (5), since any matrix in Σ is invertible over S.

To prove the main assertion we note that if it holds for two matrices $P', P'' \in {}^mS^n$, then it also holds for $P = P' + P''$. Indeed, if P', P'' are determined by systems $A'^f u' = 0$, $A''^f u'' = 0$, analogous to A above, then (as in the case of elements, cf. [3], p.250), P is given by the system

$$\begin{pmatrix} A'_0 & A'_* & A'_\infty & 0 & 0 \\ A''_0 & 0 & -A''_\infty & A''_* & A''_\infty \end{pmatrix}^f (I, u'_*, P', u''_*, P)^T = 0.$$

Hence it suffices to prove the result for matrices with only one non-zero entry, since the general case may be obtained by adding such matrices. By row and column transformations we can reduce everything to the case $P = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$, $p \in S$. Let $A^f u = 0$ be an S-admissible system for p, then

$$\begin{pmatrix} A_0 & 0 & A_* & A_\infty & 0 \\ 0 & 0 & 0 & 0 & I_{m-1} \end{pmatrix}^f \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \\ u_* & 0 \\ p & 0 \\ 0 & 0 \end{pmatrix} = 0$$

is a system for P, as required.

The equation (4) may again be written in a form of Cramer's rule ([3], p.251 and [5], § 4):

$$(6) \qquad (A_*, A_\infty)^f \begin{pmatrix} I_r & u_* \\ 0 & P \end{pmatrix} = (A_*, -A_0)^f.$$

We shall again call (A_*, A_∞) the denominator of the system A, but define the *numerator* as $(A_*, -A_0)$. This is right associated to the numerator as defined in [5] (and recalled in § 1), so the change has no effect on the considerations of [5] except notationally. From (6) we see that P is stably associated over S to its numerator, in particular it has same characteristic, and it is invertible if the numerator is invertible over S, i.e. if the system can be chosen so as to have a numerator in Σ. Moreover, when S is a field, the left and right nullities of P over S agree with those of its numerator.

Let R, S be any rings, Σ a multiplicative set of full matrices over R and $f:R \longrightarrow S$ a Σ-inverting epimorphism. Since Σ is multiplicative, its matrices are even stably full and we may embed Σ in $F(R)$ by the rule $A \longmapsto A \dot{+} I$; this allows us to regard Σ as a submonoid of $F(R)$, with the multiplication AB. Now consider the universal abelian group Σ^{ab} of Σ; this is defined as an abelian group Σ^{ab} with a homomorphism $\Sigma \longrightarrow \Sigma^{ab}$ which is universal for all homomorphisms of Σ into abelian groups. To describe Σ^{ab} explicitly, let us denote by $[A]$ or $[A]_\Sigma$ when confusion is possible, the class of $A \in \Sigma$ under stable E-association over R_Σ. Since $E(R_\Sigma) = GL(R_\Sigma)'$, we may regard $[A]$ as the residue class of A (mod $GL(R_\Sigma)'$). We define a binary operation on the set G of all these calsses by putting

$$[A] + [B] = [A \dotplus B].$$

This is well-defined, since replacing A or B by a stable E-associate replaces $A \dotplus B$ by a stable E-associate. It is clear that the multiplication is associative, with neutral [1], and it is commutative by Whitehead's lemma ([1],p.226). Moreover, the mapping $A \longmapsto [A]$ is a homomorphism, because AB is stably E-associated to $A \dotplus B$. Thus G is essentially Σ made commutative, and so Σ^{ab} is the universal group of the monoid G. The elements of Σ^{ab} are of the form $[A] - [B]$, where $A, B \in \Sigma$, with $[A] - [B] = [A'] - [B']$ if and only if $[A \dotplus B' \dotplus C] = [A' \dotplus B \dotplus C]$ for some $C \in \Sigma$.

Now the matrices of Σ are all inverted over S, so we have a map from Σ to $GL(S)$ induced by $f:R \longrightarrow S$. Let us write $[A]_S$ for the class of A (mod $E(S)$), just as $[A]_\Sigma$ is the class of A (mod $E(R_\Sigma)$). Since $[A]_S + [B]_S = [A \dotplus B]_S$ in $K_1(S)$, this map f gives rise to a homomorphism

$$(7) \qquad f^*: \Sigma^{ab} \longrightarrow K_1(S),$$

obtained by mapping $[A]_\Sigma$ to $[A]_S$. We claim that f^* is surjective. For let $P \in GL(S)$ and take an S-admissible system $Au = 0$ for P (as in Prop.2.1), then (6) holds; hence on passing to $K_1(S)$ we find

$$[(A_*,A_\infty)]_S + [P]_S = [(A_*,-A_0)]_S.$$

This shows $[P]_S$ to be the image of $[(A_*,-A_0)]_\Sigma - [(A_*,A_\infty)]_\Sigma$ and so f_* is surjective. Thus we have

Theorem 2.2. *Let* R, S *be any rings,* Σ *a multiplicative set of full matrices over* R *and* $f:R \longrightarrow A$ *a* Σ*-inverting epimorphism. Denote by* Σ^{ab} *the universal abelian group for* Σ, *then there is a natural epimorphism*

$$f^*: \Sigma^{ab} \longrightarrow K_1(S), \tag{7}$$

where $A, B \in \Sigma$ *have the same image under* f^* *if and only if there exists* $C \in \Sigma$ *such that* $A \dotplus C$ *is stably* E*-associated to* $B \dotplus C$ *over* S.

In case $S = K$ is a field, we have the isomorphism (2) by the Dieudonné determinant, hence we obtain the

Corollary. *Let* R *be a ring,* Σ *a multiplicative set of full matrices over* R *and* K *an epic* R*-field such that the natural map* $R \longrightarrow K$ *is* Σ*-inverting, then there is an epimorphism* $\Sigma^{ab} \longrightarrow K^{ab}$.

3. The case of pseudo-Sylvester domains

In general there is no reason for the map f^* in Th.2.2 to be injective, because f need not be so (and even the injectivity of f will not guarantee that of f^*), but we now turn to a case where f^* is an isomorphism. We saw that the universal field of fractions of a semifir R may be described as the localization R_F, where $F = F(R)$ is the set of all full matrices over R (of course over a semifir every full matrix is stably full). The rings R such that R_F is a field, - necessarily the universal field of fractions of R - have been studied under the name *Sylvester domain* by Dicks and Sontag [9]. Thus Sylvester domains form a class including semifirs; an example of a Sylvester domain not a semifir is given by the free Z-algebra on a non-empty set $X:Z < X >$. Still more generally, we may define a *pseudo-Sylvester domain* as a ring R with a universal field of fractions U obtained by inverting all stably full matrices, cf.[8]. This seems to be the widest class to which the method used here is applicable.

Let R be a pseudo-Sylvester domain and $U = R_F$ its universal field of fractions; we claim that the induced map

$$(1) \qquad f^*: F(R)^{ab} \longrightarrow K_1(U) = GL(U)^{ab}$$

is an isomorphism. We shall prove this (following a suggestion of Bergman) by constructing an inverse for f^*. Thus let $P \in GL_n(U)$ and take a U-admissible system $Au = 0$ for P, as in Prop.2.1. In detail we have

$$(A_0, A_*, A_\infty)(I_n, u_*, P)^T = 0.$$

Since P is invertible over U, so is its numerator $(A_*, -A_0)$, hence the latter is stably full over R. We define a map δ_0: $GL(U) \longrightarrow F(R)^{ab}$ by the rule

$$P^{\delta_0} = [(A_*, -A_0)]_F - [(A_*, A_\infty)]_F,$$

where $F = F(R)$. To prove that δ_0 is well-defined, we take another system for P, say $Bv = 0$, then we have to show that in $F(R)^{ab}$,

$$[(A_*,-A_0)]_F - [(A_*,A_\infty)]_F = [(B_*,-B_0)]_F - [(B_*,B_\infty)]_F \text{ , i.e.}$$

$$(2)\ [(A_*,-A_0)]_F + [(B_*,B_\infty)]_F = [(B_*,-B_0)]_F + [(A_*,A_\infty)]_F.$$

Now consider the relation

$$(3)\qquad \begin{pmatrix} A_0 & A_* & 0 & A_\infty \\ B_0 & 0 & B_* & B_\infty \end{pmatrix}^f (I_n,u_*,v_*,P)^T = 0.$$

We shall need to know the rank of the left-hand matrix over U; this is the stable rank over R and may well be less than the inner rank, but if we can form its diagonal sum with a sufficiently large unit matrix, the two ranks will be equal. This can be done by modifying A_* or B_* as follows. Since $Au = 0$ is a U-admissible system for P, so is

$$\begin{pmatrix} A_0 & A_* & 0 & A_\infty \\ 0 & 0 & I & 0 \end{pmatrix}^f (I,u_*,0,P)^T = 0.$$

Moreover, if we modify A_* in this way, the values of $[(A_*,-A_0)]_F$ and $[(A_*,-A_0)]_F$ remain unchanged. We may thus assume A, B modified in such a way that the left-hand matrix in (3) is stabilized, i.e. its stable rank is just its (inner) rank. Let the number of columns in A_*, B_* be r, s respectively, then the left-hand matrix in (3) is square of order $r + s + 2n$; by (3) it has right mullity at least n over U, hence its inner rank over R (or also the stable rank) is at most $r + s + n$. Thus we can write it in the form

$$(4) \qquad \begin{pmatrix} A_0 & A_* & 0 & A_\infty \\ B_0 & 0 & B_* & B_\infty \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix} (D_0, D', D'', D_\infty),$$

where $P \in {}^{r+n}R^{r+s+n}$, $Q \in {}^{s+n}R^{r+s+n}$ and D_0, D', D'', D_∞ have n, r, s, n columns respectively. From (4) we obtain the following factorizations:

$$(5) \qquad \begin{pmatrix} A_* & -A_0 & 0 & 0 \\ 0 & -B_0 & B_* & B_\infty \end{pmatrix} = \begin{pmatrix} P & 0 \\ Q & B_\infty \end{pmatrix} \begin{pmatrix} D' & -D_0 & D'' & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix},$$

$$(6) \qquad \begin{pmatrix} A_* & -A_0 & 0 & -A_\infty \\ 0 & -B_0 & B_* & 0 \end{pmatrix} = \begin{pmatrix} P & 0 \\ Q & B_\infty \end{pmatrix} \begin{pmatrix} D' & -D_0 & D'' & -D_\infty \\ 0 & 0 & 0 & I_n \end{pmatrix}.$$

If we apply $[\,]_F$ to both sides and bear in mind the evident relation

$$\left[\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\right]_F = \left[\begin{pmatrix} A & 0 \\ D & B \end{pmatrix}\right]_F = [A]_F + [B]_F,$$

we find that the right-hand sides of (5), (6) are equal, while the left hand side of (5) gives just the left hand side of (4). The left-hand side of (6) will similarly give the right-hand side of (4) if we can interchange the second and fourth column blocks and change the sign of the latter. Now any two columns, x and y say, can be interchanged with the sign of one of them changed, by elementary column operations:

$$(x,y) \longrightarrow (x,x+y) \longrightarrow (-y,x+y) \longrightarrow (-y,x).$$

Hence we can in (6) exchange the columns of $(-A_0,-B_0)^T$ against those of $(-A_\infty,0)^T$ one by one and change the signs of the latter. In this way we obtain the right-hand side of (4); this then shows that (4) holds and it proves that δ_0 is well-defined. Since $F(R)^{ab}$ is abelian, we can factor δ_0 via $GL(U)^{ab}$ and so obtain $\delta : GL(U)^{ab} \longrightarrow F(R)^{ab}$, defined by

$$[P]_U^{\delta} = P^{\delta o}.$$

From the definition it is clear that

$$[P]_U^{\delta f^*} = \{[(A_*,-A_0)]_F - [(A_*,A_\infty)]_F\}^{f^*} = [P]_U,$$

using an admissible system A for P. Next, if $P \in F(R)$, then by taking the admissible system

$$(-P,I)\begin{pmatrix} I \\ P \end{pmatrix} = 0,$$

we see that $[P]_F^{f^*\delta} = [P]_U^{\delta} = [P]_F - [I]_F = [P]_F$. Thus f^*, δ are mutually inverse, and this proves incidentally that δ is a homomorphism. Hence we have proved

Theorem 3.1. *Let* R *be a pseudo-Sylvester domain,* $F = F(R)$ *the set of all stably full matrices over* R *and* $U = R_F$ *its universal field of fractions, then*

$$K_1(U) \cong U^{ab} \cong F(R)^{ab}.$$

In particular this provides a means of calculating determinants of matrices over pseudo-Sylvester domains:

Corollary. *Let* R *be a pseudo-Sylvester domain and* U *its universal field of fractions, then for any stably full matrix* A *over* R *we have*

$$\det A = [A]_F^{f^*},$$

where f^* *is the map* (1) *induced by* $f:R \longrightarrow U$ *and det is taken over* U.

For over U, A is stably E-associated to $\alpha \in U$, such that $\alpha^{ab} = \det A$. Hence $[A]_F^{f^*} = \alpha^{ab} = \det A$.

For Sylvester domains Th.3.1 has also been obtained by G.Révész [12], by another method, based on the above Corollary (for the case of firs there is yet another proof in [6]).

4. The divisor group of a fully atomic semifir

In order to investigate the structure of U^{ab} more fully we need to assume the existence of complete factorizations in our ring R. We recall that a square matrix A is called an *atom* if it is a non-unit and cannot be written as a product of two (square) non-unit matrices; it is clear from this that an atom is necessarily full. A ring is said to be *fully atomic* if every full matrix can be written as a product of a finite number of atoms, or is a unit. In particular, every element not zero or a unit then has a complete factorization into atoms.

Let R be a semifir and U its universal field of fractions. By a Z-*value* on R we shall understand a homomorphism $v:GL(U) \longrightarrow Z$ such that $v(A) \geqslant 0$ for all $A \in F(R)$.

To give an example, let us assume that R is a fully atomic semifir and recall from [3],p.201 the unique factorization property: Every full matrix over R is either a unit or has a factorization into atoms which is unique up to stable association and the order of the factors. Now let P be an atom and for any $A \in F(R)$ define $v(A) = r$ if in any complete factorization of A the number of atomic factors stably associated to P is just r. By unique factorization this is well-defined and we obtain a Z-value on R by putting

$$v([A]_F - [B]_F) = v(A) - v(B).$$

This is called the *simple* Z-value associated with the atom P.

Proposition 4.1. *Let* R *be a fully atomic semifir and let* v *be any* Z-*value on* R. *Then* (i) $v(P)=0$ *for* $P \in GL(R)$, *and* (ii) $v(A)=v(A')$ *whenever* A, A' *are stably associated.*

Proof. (i) Let $P \in GL(R)$, then $v(P) \geqslant 0$, $v(P^{-1}) \geqslant 0$, but $v(P) + v(P^{-1}) = v(I) = 0$, hence $v(P) = 0$. (ii) Let A, A' be stably associated, say

$$(A \dotplus I)U = V(A' \dotplus I), \qquad U,V \in GL(R);$$

since $v(U) = v(V) = 0$, we have $v(A) = v(A')$ as claimed.

Let us define a *prime* of R as a class of stably associated atomic matrices. With each prime p_i there is associated a simple Z-value v_i. More generally, pick an integer

$n_i \geqslant 0$ for each prime p_i, then $w = \Sigma n_i v_i$ is a Z-value, for it is defined on each full matrix A: $w(A) = \Sigma n_i v_i(A)$, where the sum on the right is finite because $v_i(A) = 0$ for almost all i. We observe that every Z-value arises in this way; for if w is a Z-value on R, let P_i be an atom in the class p_i and put $n_i = w(P_i)$, then w and $\Sigma n_i v_i$ have the same value on each atom and hence on all of $F(R)^{ab}$, so $w = \Sigma n_i v_i$. This proves

Theorem 4.2. *Let* R *be a fully atomic semifir and let* (v_i) *be the simple* Z-*values corresponding to the primes of* R. *For any family* (n_i) *of non-negative integers,* $\Sigma n_i v_i$ *is a* Z-*value, and conversely, every* Z-*value on* R *is of this form.*

We remark that with every full matrix A there is associated a Z-value w_A which is simple if and only if A is an atom, viz. $w_A = \Sigma n_i v_i$, where the v_i are all the simple Z-values and $n_i = v_i(A)$.

We can also use Z-values to characterize fully atomic semifirs:

Proposition 4.3. *Let* R *be a semifir, then* R *is fully atomic if and only if there is a* Z-*value* w *on* R *such that* $w(A) = 0$ *precisely when* A *is a unit.*

Proof. If R is a fully atomic semifir and v_i are the simple Z-values corresponding to the different primes of R, then $w = \Sigma v_i$ has the desired property. Conversely, when w exists, take any factorization $A \in F(R)$ and factorize it into non-units in any way:

(2) $$A = P_1 \dots P_r.$$

Since $w(P_i) \geqslant 1$ by hypothesis, we have $w(A) = \Sigma w(P_i) \geqslant r$, and this provides a bound on the number of factors in (2). By taking a factorization with maximal r we obtain a complete factorization of A. This completes the proof.

Now take a fully atomic semifir R and let p_i $(i \in I)$ be the family of all primes. For each p_i we have a homomorphism: v_i: $F(R)^{ab} \longrightarrow Z$, and combining all these maps, we have a homomorphism

$$F(R)^{ab} \longrightarrow Z^I.$$

But each full matrix maps to 0 in almost all factors of Z^I, hence the image lies in the weak direct power $Z^{(I)}$. Let us write $D = D(R)$ for the free abelian group on the p_i (written additively). then we have a homomorphism $\lambda : F(R)^{ab} \longrightarrow D$ and hence, by Th.3.1,

(3) $$\lambda^*: K_1(U) \longrightarrow D(R).$$

From its construction the map λ is surjective, hence so is (3). We claim that its kernel is $GL(R)/(GL(R) \cap E(U))$. For any $A \in GL(R)$ satisfies $v_i(A) = 0$ for all i, hence $A \in \ker \lambda^*$. Conversely, if $([A] - [B])^{\lambda^*} = 0$, then $A^\lambda = B^\lambda$, hence A, B have the same atomic factors, up to order and stable association. Let $A = P_1 \dots P_r$ be a complete factorization and let B be the product (in some order) of $Q_1, \dots, Q_r$, where Q_i is stably associated to P_i. Replacing A, B by $A \dotplus I$, $B \dotplus I$ for suitably large I, we may assume Q_i to be associated to P_i, say $P_i = U_iQ_iV_i$, where $U_i, V_i \in GL(R)$. Then except for the order of the factors we can write $A = Q_1 \ldots Q_r U_1 \ldots U_r V_1 \ldots V_r = BF$, where $F \in GL(R)$. Hence $A \equiv BF$ (mod $GL(U)'$) and so $[A] - [B] = [F] \in GL(R).GL(U)'$. It follows that $\ker \lambda^* = GL(R).GL(U)'/GL(U)' \cong GL(R)/GL(R) \cap GL(U)'$. Here we may replace $GL(U)'$ by $E(U)$; moreover, since D is free abelian, λ^* is split by D over its kernel and we obtain

Theorem 4.4. *Let* R *be a fully atomic semifir with universal field of fractions* U *and divisor group* $D(R)$, *then*

$$(4) \qquad K_1(U) \cong U^{ab} \cong D \times [GL(R)/(GL(R) \cap E(U))].$$

The divisor group D inherits a partial ordering from R, by writing $\pi > 0$ whenever π is positive on R. However, the ordering on D is not enough to define R within U, as is shown by the fact that the determinant of a matrix over R is usually a proper fraction (i.e. has no representative in R).

It is also of interest to compare Z-values with valuations (cf. [11]). Clearly a Z-value v will be a valuation if and only if

$$(5) \qquad v(p - 1) \geqslant \min\{v(p), 0\}, \text{ for any } p \in U.$$

Let $A = (A_0, A_*, A_\infty)$ be an admissible matrix for p, then an admissible matrix for $p-1$ is $(A_0+A_\infty, A_*, A_\infty)$, so the condition (5) becomes, after a slight rearrangement,

$$(6) \qquad v(A_0 + A_\infty, A_*) \geqslant \min\{v(A_0, A_*), \; v(A_\infty, A_*)\}.$$

We recall that when two matrices differ in only one column, say the first: $A = (A_1, \ldots, A_n)$, $B = (B_1, A_2, \ldots, A_n)$, then the matrix obtained by adding the first columns and leaving the other columns unchanged is called the *determinantal sum* and is written

$$A \triangledown B = (A_1 + B_1, A_2, \ldots, A_n).$$

With this notation we see that v is a valuation if and only if

$$(7) \qquad v(A \triangledown B) \geqslant \min\{v(A), v(B)\},$$

whenever the determinantal sum is defined (cf. [11]). In general this condition need not hold, e.g. in $k\langle x,y\rangle$ consider the simple Z-value v associated with x. We have $v(xy) = v(yx) = 1$, but $v(xy - yx) = 0$. Nevertheless there is a valuation on the universal field of fractions U associated with x; to obtain it we write U as a skew function field $K(x;\alpha)$, where K is the universal field of fractions of $k\langle y_i \mid i \in Z\rangle$ and α is the shift automorphism $y_i \longmapsto y_{i+1}$ (thus y_i is realized as $x^{-i}yx^i$). On $K(x;\alpha)$ the order in x is the required valuation. In terms of Z-values this valuation is obtained as the sum of certain simple Z-values, but this is not a very efficient way of constructing this valuation.

5. The case of free algebras

To illustrate Th.4.4 we shall consider the case of free algebras, where it is possible to compute the second factor on the right of (4) of §4. We first prove a lemma.

Lemma 5.1. *Let* k *be a commutative field and* $U = k \langle\!\langle X \rangle\!\rangle$ *the universal field of fractions of the free* k*-algebra* $k\langle X\rangle$*, then* $E(U) \cap GL_1(k) = 1$.

Proof. Let $A = \alpha \dot{+} I \in E(U)$, where $\alpha \in k$; we have to show that $\alpha = 1$. Write A as a product of elementary matrices over U and let P be the diagonal sum of all the denominators of the entries occurring in these matrices. Our plan will be to find a k-field K such that we can specialize X to values in K so that P remains invertible and A maps to I. For each n not divisible by χ, the characteristic of k, we adjoin a primitive nth root of 1, ω_n say, to k and define

$$K(n) = k(x,y \mid yx = \omega_n xy). \tag{1}$$

It is easily seen that $K(n)$ is then a skew field, in fact a division algebra of index n. Let K be an ultraproduct of the $K(n)$ with a non-principal ultrafilter, and denote by x', y', ω' the elements of K whose components are all x, y, ω_n respectively, then $y'x' = \omega'x'y'$ and $\omega'^n \neq 1$ for all n. It follows that K is infinite-dimensional over its centre. We now apply the specialization lemma from [4], p.141. Clearly the centre of K, C say, is infinite and $k\langle\!\langle X\rangle\!\rangle$ is embedded in $K_C\langle\!\langle X\rangle\!\rangle$ so we can specialize X to values in K so that P remains invertible. It follows that for all but finitely many n not divisible by χ we have a specialization from X to $K(n)$ making P invertible. In each of these fields $K(n)$ the reduced norm maps each matrix in $E(U)$ to I, hence $\alpha^n = 1$ for all but finitely many n not divisible by χ. This still leaves infinitely many values of n and so is impossible unless $\alpha = 1$.

For the free algebra $k\langle X\rangle = R$, every invertible matrix is a product of elementary and diagonal matrices, i.e. $GL(R) = E(R).k^*$ (by Prop.2.7.2 of [3],p.95), hence $GL(R)\cap E(U) = E(R).k^*\cap E(U) = E(R)\ (k^*\cap E(U)) = E(R)$, by the lemma. Therefore $GL(R)/GL(R)\cap E(U) = E(R).k^*/E(R) \cong k^*/k^*\cap E(R) \cong k^*$, and so we find

Theorem 5.2. *Let* $R = k\langle X\rangle$ *be the free* k-*algebra on a set* X *and* $U = k\langle\!\langle X\rangle\!\rangle$ *its field of fractions and* $D(R)$ *its divisor group, then*

$$U^{ab} \cong D(R) \times k^*$$

This solves Exercise 7.6.10 of [3]. In many cases it is true that $E(U)\cap GL(R) = E(R)$, as in the case of $k\langle X\rangle$, but by no means always. For a study of the general case we refer to Révész [12].

At the other extreme, let K be a skew field in which every non-zero element is a commutator (cf.[6]), let C be its centre and consider the free K-ring $K_C\langle X\rangle$ and its universal field of fractions $U = K_C\langle\!\langle X\rangle\!\rangle$. The ring R has a weak algorithm (cf. [3], p.78), hence $GL(R) = GL_1(R).E(R)$, and so $GL(R)/GL(R)\cap E(U) = GL(R).E(U)/E(U) = GL_1(R).E(U)/E(U)$. Now $GL_1(R) = K^*\cap E(U)$, hence the second factor on the right of (4) in § 4 is trivial and so

$$K_C\langle\!\langle X\rangle\!\rangle^{ab} \cong D,$$

where D is free abelian of countable rank (or of rank $|X|$ if this is larger).

6. Localization

Let R be a semifir and Σ any set of square matrices over R; it is natural to ask under what conditions the localization R_Σ is again a semifir. This has been answered in [7], where it is shown that R_Σ is a semifir if and only if Σ is *factor complete*, i.e. whenever $AB \in \Sigma$, then there exists a matrix C over R_Σ such that (B,C) is invertible over R_Σ. We shall show that when R is fully atomic, then so is R_Σ and our aim will be to study the relation between the divisor groups of R and R_Σ in that case.

An atom A in R and also the associated simple Z-value is called Σ-*irrelevant* if A becomes a unit in R_Σ and Σ-*relevant* otherwise.

Theorem 6.1. *Let* R *be a fully atomic semifir and let* Σ *be a factor complete set of matrices over* R, *then* R_Σ *is again a fully atomic semifir, and every atom over* R *either becomes a unit or remains an atom over* R_Σ.

Proof. We begin by proving the last part. Let A be an atom over R and suppose that over R_Σ we have $A = B_1B_2$, where the B_i are non-units. Then by Cramer's rule, $U_i(B_i \dotplus I)V_i = C_i$ $(i = 1,2)$, where C_i is a matrix over R and $U_i, V_i \in GL(R_\Sigma)$ Hence

$$A \dotplus I = U_1^{-1}C_1V_1^{-1}U_2^{-1}C_2V_2^{-1}. \tag{1}$$

Let v be the simple Z-value defined by A, take complete factorizations of C_1, C_2 over R and let w_1, w_2 be the Z-values corresponding to C_1, C_2 but counting only Σ-relevant atoms. Them by (1), $v = w_1 + w_2$. But $w_i(C_i) \geq 1$ and so

$$2 \leq w_1(C_1) + w_2(C_2) = v(A) = 1$$

a contradiction, and this shows that A is an atom or a unit over R_Σ.

Now let P be any full matrix over R_Σ and write

(2) $$U(P \dot{+} I)V = A,$$

where $A \in F(R)$, U, $V \in GL(R_\Sigma)$. We can write A as a product of r atoms say, over R; each will be either an atom or a unit over R_Σ, hence P can be written as a product of at most r atoms over R_Σ and this shows R_Σ to be fully atomic.

The fact that R_Σ is fully atomic may also be proved as follows: Denote by w the sum of all Σ-relevant simple Z-values on R, then w is a Z-value on R_Σ and $w(A) = 0$ for $A \in F(R_\Sigma)$ only if A is invertible (by Cramer's rule), hence the criterion of Prop.4.3 is satisfied.

By Prop.6.1 we can define the divisor groups of both R and R_Σ; to describe the mapping between them we need

Proposition 6.2. *Let* R *be a fully atomic semifir and* Σ *a factor complete set of matrices over* R, *so that* R_Σ *is again fully atomic. Then* (i) *any two atoms over* R *that are not stably associated over* R *are not stably associated over* R_Σ, *unless both become units,* (ii) *every matrix* P

over R_Σ *is stably associated to the image of a matrix* P' *over* R, *and if* P *is an atom, then so is* P'.

Proof. (i) Let A, A' be atoms over R, not stably associated, and suppose that A is Σ-relevant. Let v be the simple Z-value corresponding to A, then v is a Z-value on R_Σ and $v(A) = 1$, $v(A') = 0$, hence A, A' cannot be stably associated over R_Σ. (ii) Let P be a matrix over R_Σ, then we again have an equation (2), hence P is stably associated to $A \in F(R)$. Now suppose that P is an atom and denote by w the sum of all Σ-relevant simple Z-values on R, then w is a Z-value on R_Σ. Since P is an atom, we have $1 = w(P) = w(A)$; this means that in a complete factorization of A over R there is only one factor, P' say, which is Σ-relevant, and clearly P is stably associated over R_Σ to P'. This completes the proof.

Let A be an atom over R and denote by $[A]_R$ the corresponding prime of R; if A is Σ-relevant, it remains an atom over R_Σ and so defines a prime $[A]_\Sigma$ there. It is clear that stably associated atoms over R remain stably associated over R_Σ, hence the correspondence $[A]_R \longmapsto [A]_\Sigma$ defines a homomorphism

$$\lambda : D(R) \longrightarrow D(R_\Sigma).$$

Let $D_\Sigma(R)$ be the subgroup of $D(R)$ generated by the Σ-relevant primes; we claim that $D_\Sigma(R) \cong D(R_\Sigma)$. For the restriction of λ to $D_\Sigma(R)$ is injective by Prop.6.2 (i) and surjective by (ii). Thus we have proved

Theorem 6.3. *Let* R *be a fully atomic semifir,* Σ *a factor complete set of matrices and denote by* $D_\Sigma(R)$ *the subgroup of* $D(R)$ *generated by the* Σ*-relevant primes of* R. *Then the embedding* $R \longrightarrow R_\Sigma$ *induces an isomorphism*

$$D_\Sigma(R) \cong D(R_\Sigma).$$

Moreover, if $\lambda : D(R) \longrightarrow D(R_\Sigma)$ *is the induced homomorphism, then*

$$D(R) = D_\Sigma(R) \times \ker \lambda;$$

here $\ker \lambda$ *is the subgroup of* $D(R)$ *generated by the* Σ*-irrelevant primes.*

We conclude by discussing an example, suggested by A.H. Schofield. Consider the free algebra $k\langle X\rangle$; we first examine the form of atoms stably associated to the generators.

Proposition 6.4. *Let* $x \in X$, *then over* $k\langle X\rangle$, *any* $n \times n$ *matrix stably associated to* x *is associated to* $x \dotplus I_{n-1}$. *In particular, any element stably associated to* x *has the form* λx $(\lambda \in k^*)$.

Proof. Let A' be an $n \times n$ matrix stably associated to x, then (by Prop.2.2, [5]), there is a comaximal relation

$$xb' = bA', \tag{3}$$

where $b, b' \in R^n$. By the weak algorithm in $R = k\langle X\rangle$ we can reduce b to $e_1 = (1,0,\dots,0)$; then (3) becomes $xb'_j = a'_{1j}$, hence $A' = (x \dotplus I)A''$ and here A'' must be a unit, by unique factorization. This proves the first part; now the rest is clear since any associate of x has the form

λx, $\lambda \in k^*$.

We now assume X to be infinite and partition it into two parts X', X'' of which X'' is again infinite. Let $\Sigma = \Sigma(X')$ be the set of all full matrices over $k<K>$ which are totally coprime to X', i.e. which have no factor stably associated to an element of X'. We claim that Σ is factor complete. Let $C \in \Sigma$, and suppose that $C = AB$, where $A \in {}^nR^N$, $B \in {}^NR^n$ $(n \leqslant N)$. Given that C is totally coprime to X', we have to find $D \in {}^{N-n}R^N$ such that $(A,D)^T$ is full and totally coprime to X'. We shall take the entries of D to be distinct elements of X'' not occurring in A or B; this is possible because X'' is infinite. Since C is full, A has rank n, so we can choose n columns of A forming a full matrix, say the first n, then $(A,D_o)^T$ will be full if we choose $D_o = (0,I)$. This can always be done by specializing the choice of D made earlier, so it follows that $(A,D)^T$ is full. It remains to show that $(A,D)^T$ is totally coprime to X'. Suppose that

$$(4) \qquad \begin{pmatrix} A \\ D \end{pmatrix} = PTQ, \; T = x \dotplus I_{N-1}, \quad x \in X'.$$

We partition P in accordance with the left-hand side of (4), i.e. we put $P = (P_1,P_2)^T$, so that $A = P_1TQ$, $D = P_2TQ$. Write $P_2 = (p_2,P_2')$, where p_2 is the first column, Further, write $X_o = X\setminus\{x\}$, $x_o' = X'\cap X_o$, $S_o = k<X_o>_{\Sigma(X_o')}$, then S is a localization of $S_o \underset{k}{*} k[x]$, again a fir, and over the latter ring we again have a factorization (4). Consider the homomorphism $f \longmapsto \bar{f}$ obtained by putting $x = 0$. This does not affect D, so $D = \overline{D} = (0,\overline{P}'_2)\overline{Q}$. But this means that $D \in {}^{N-n}R^N$ has inner rank at most $N-n-1$, which is clearly false. Hence no equation (4) can exist and $(A,D)^T$ is totally coprime to X'. This shows $\Sigma(X')$ to be factor complete and it proves

Theorem 6.5. *Let* $k\langle X\rangle$ *be the free algebra on an infinite set* X, *let* X' *be a subset of* X *with an infinite complement in* X, *and denote be* $\Sigma = \Sigma(X')$ *the set of all full matrices over* $k\langle X\rangle$ *totally coprime to* X', *then the localization* $k\langle X\rangle_\Sigma$ *is a fir.*

For when Σ is factor complete, the localization is a semifir by [7]; it is hereditary by [2], and hence a fir.

We now partition X into X', X'', where both X' and X'' are infinite. Our aim will be to prove that in this case $k\langle X\rangle_{\Sigma(X')}$ is simple. Let us write $R = k\langle X\rangle$, $S = R_\Sigma$ and take $c \in S$, $c \notin k$. Choose $x \in X'$ such that x does not occur in c, then we claim that $cx - xc$ is a unit in S. Once this is proved, it will follow that c cannot lie in any two-sided ideal $\neq 0$ of S, and since c was any element of S not in k, it follows that S is simple.

Let X_0 be the subset of X involved in c and let Σ_0 be the set of matrices in Σ with entries involving only X_0, and put $S_0 = R_{\Sigma_0}$, then S is a localization of S_0, and the latter is a fir.

Consider $cx - xc$ in S_0; if this is not a unit or an atom, then

(5) $$cx - xc = ab, \quad a,b \in S_0, \quad a,b \text{ non-units.}$$

Let us write $a = a(x)$, $b = b(x)$ to indicate the dependence on x; we note that $f(x) \longmapsto f(0)$ is a homomorphism from S_0 (to the corresponding algebra with x replaced by 0, again a fir), hence by (5), $a(0)b(0) = 0$, so $a(0) = 0$ or $b(0) = 0$, say the former. If t is a commuting indeterminate, then by (5),

$$(6) \qquad a(tx)b(tx) = t(cx - xc) = ta(x)b(x).$$

Clearly a,b are polynomials in t, and $a(0) = 0$, so by (6) a,b are homogeneous of degrees 1, 0 respectively in t, in particular, $b(x) = b(0)$ is independent of x. So we have

$$(7) \qquad cx - xc = a(x).b.$$

By hypothesis b is a non-unit in S_0, say it has a factor stably associated to $x_1 \in X'$, and $x_1 \neq x$ by what has been proved. Then on the right of (7) all terms have x to the left of the right-most factor x_1 and likewise in xc, whereas in cx, x occurs on the right. Thus the terms in cx must cancel, i.e. $cx = 0$, which is not the case. Hence b is a unit, and this shows that $cx - xc$ is an atom. Suppose that it is stably associated to an element of X'. Now S_0 may be written as $T \underset{k}{*} k[x]$, where T is a localization of the free algebra in the elements $\neq x$. Let U be the field of fractions of T and form $U \underset{k}{*} k(x)$; this is a localization of S_0 in which all the elements of X' occurring are invertible, hence $cx - xc$ must be a unit in $U * k(x)$, but that is clearly not so, hence $cx - xc$ is totally coprime to X', and it is therefore a unit in S. This then shows S to be simple.

Next we show that S is an Ore domain, and hence principal. Take $p,q \in S$, $p,q \neq 0$ and without loss of generality p,q have no common left factor (apart from units). Take $x_1, x_2 \in X'$ not occurring in p,q and form $c = px_1 - qx_2$. Let $X_0 = X \setminus \{x_1, x_2\}$, $R_0 = k\langle X_0 \rangle$ $S_0 = R_{0\Sigma(X_0')}$, where $X_0' = X' \cap X_0$, then c is an atom in $S_0 \underset{k}{*} k\langle x_1, x_2 \rangle$, for if not, consider an equation

$$c = ab.$$

We have $b = b_1x_1 + b_2x_2$, hence $p = ab_1$, $q = -ab_2$, hence a is a unit, by the choice of p,q. This then shows c to be an atom. If c is stably associated to $x \in X'$, then in $V_0 * k(x_1) * k(x_2)$, where V_0 is the universal field of fractions for S_0, c will be a unit. But it is clearly not a unit, hence it must be totally coprime to X' and so is a unit in S. Now we have $px_1c^{-1} - qx_2c^{-1} = 1$, hence $p(x_1c^{-1}p - 1) = q(x_2c^{-1}p)$ is a common right multiple.

Thus $R_{\Sigma(X')}$ is a simple PID whenever X' is an infinite subset of X with infinite complement. Suppose now that X_0 is a finite subset of X; let X' be any subset of X containing X_0 and having an infinite complement in X, then it is clear that $R_{\Sigma(X_0)}$ is a localization of $R_{\Sigma(X')}$ and the latter has been shown to be a simple PID. Any localization is again a simple PID, so we have

Theorem 6.6. *Let* $R = k\langle X \rangle$ *be the free algebra on an infinite set* X, *and let* X_0 *be a subset of* X *with an infinite complement. Denote by* $\Sigma = \Sigma(X_0)$ *the set of all full matrices totally coprime to* X_0, *then* R_Σ *is a simple principal ideal domain.*

In particular, taking X_0 to consist of a single element, we obtain a simple PID with a single atom, but not a local ring.

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