Pub. Mat. UAB
Vol. 26 Nº 2 Juny 1982

REALIZABILITY OF COHOMOLOGY ALGEBRAS: A SURVEY

J. AGUADÉ
(Universitat Autònoma de Barcelona)

Rebut el 1 de desembre del 1981

0. Introduction

Let R be a commutative ring and let A be a graded commutative (in the graded sense) R-algebra. Is there a space X such that $H^*(X;R) \cong A$ as graded R-algebras?

This is the realizability problem for cohomology algebras. It asks for a characterisation of those graded algebras that appear as the cohomology algebra of some space. It is an extremely difficult problem, even for very simple algebras. It was stated by Steenrod in 1960 ([46]) but it has been considered, at least, since Hopf.

In these notes I want to give a survey of the methods that have been used in solving special cases of the above problem and, moreover, to provide the hypothetical reader with a quite complete information about the results which are currently known and the most interesting open problems.

Since the emphasis is put in pointing out the main ideas in the field, there is almost no proof in these notes. The interested reader can found them in the references listed at the end of the paper.

This paper has been written while I had a grant from the Fundación Juan March and I was in the Forschungsinstitut für Mathematik of the Eidgenössische Technische Hochschule in Zürich. I whish to sincerely thank both institutions.

1. The Steenrod algebra

Let us consider $H^*(X;Z_p)$. It is a graded commutative Z_p-algebra but it is actually much more than that: it is an algebra over the algebra of stable Z_p-cohomology operations.

A cohomology operation of type (n,m,R,R') is a natural transformation

$$H^n(\ ;R) \longrightarrow H^n(\ ;R').$$

A stable operation of degree n is a family of natural transformations

$$H^i(\ ;R) \longrightarrow H^{i+n}(\ ;R'),\ i=0,1,2,\ldots$$

which commute with the suspension isomorphism. If we take $R=R'=Z_p$, the stable cohomology operations form a graded Z_p-algebra called the mod p Steenrod algebra, A_p. It is clear that, for any space X, the algebra A_p acts on $H^*(X;Z_p)$, i.e. the mod p cohomology is not only a graded Z_p-algebra but also a module over the Steenrod algebra. For that reason, it is of the greatest importance to know the algebraic structure of A_p. The study of this structure was done in the work of Steenrod, Adem, Cartan and Serre. The standard reference is [48].

If p=2, the Steenrod algebra is generated by some operations

$$Sq^i : H^n(\ ;Z_2) \longrightarrow H^{n+i}(\ ;Z_2),\ n,i=0,1,2,\ldots$$

with the following properties:

1) $Sq^o = id.$;
2) $Sq^n x = x^2$ if $\deg x = n$;
3) $Sq^n x = 0$ if $\deg x > n$;
4) $Sq^n xy = \sum_{i+j=n} Sq^i x\, Sq^j y$;
5) $Sq^1 = \beta$ (the Bockstein homomorphism).

And these operations verify the "Adem relations":

$$0 < a < 2b, \qquad Sq^a\, Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j}\, Sq^j.$$

Moreover, the operations Sq^i, $i \geq 0$, and the Adem relations are a complete set of generators and relations for A_2. It follows easily from the Adem relations that the operations Sq^i with $i=2^j$, $j \geq 0$, generate A_2.

If $p \neq 2$ the results are similar. The algebra A_p is generated by some stable operations

$$P^i : H^n(\;;Z_p) \longrightarrow H^{n+2i(p-1)}(\;;Z_p), \; n=0,1,2,\ldots$$

$i \geq 0$, together with the Bockstein homomorphism β. It holds:

1) $P^0 = id.$;
2) $P^n x = x^p$ if $\deg x = 2n$;
3) $P^n x = 0$ if $\deg x < 2n$;
4) $P^n xy = \sum_{i+j=n} P^i x \; P^j y$;

and the generators verify the following relations:

$$a < pb, \qquad P^a\, P^b = \sum_{t=0}^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} P^{a+b-t}\, P^t;$$

$$a \leq b, \qquad P^a\, \beta\, P^b = \sum_{t=0}^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)}{a-pt} \beta\, P^{a+b-t}\, P^t$$

$$\sum_{t=0}^{[(a-1)/p]} (-1)^{a+t-1} \binom{(p-1)(b-t)-1}{a-pt-1} P^{a+b-t}\, \beta\, P^t.$$

From these relations it is easy to see that the operations P^i with $i=p^j$, $j \geq 0$, together with β generate A_p.

The properties 2 and 4 of the Steenrod operations relate the algebra structure of $H^*(X;Z_p)$ with the A_p-module structure. As a consequence, we obtain restrictions on the algebras that can be isomorphic to $H^*(X;Z_p)$ for some X. For instance, if we have an algebra A with $x^p \neq 0$ for some $x \in A$ and $A \cong H^*(X;Z_p)$, then A

should admit a non-trivial action of A_p. To say it more precisely we can introduce the following definition: Let A be a graded commutative Z_p-algebra. We say that A is an *unstable A_p-algebra* if it is a graded A_p-module and the properties 1 to 4 of the Steenrod operations hold on A.

It is now clear that only the algebras that admit a structure of unstable A_p-algebras (we say sometimes: *they admit Steenrod operations*) can be the cohomology algebra of some space. Not every algebra admit such a structure, for instance, consider the polynomial algebra $Z_2[x]$ with $\deg x = n$. If this algebra admits Steenrod operations, then $Sq^n x = x^2 \neq 0$ and since Sq^{2^j}, $j \geqslant 0$, generate A_2, we have $Sq^{2^j} x \neq 0$ for some $j \geqslant 0$. This implies $n = 2^j$, $j \geqslant 0$.

Theorem *Let $H^*(X;Z_p) \cong Z_p[x]/x^{p+1}$ with $\deg x = h$. Then*

a) if $p=2$, then $h = 2^j$, $j \geqslant 0$;

b) if $p \neq 2$, then $h = 2p^j r$, $j \geqslant 0$, r divides $p-1$.

This is proved using the fact that, since $x^p \neq 0$, the Steenrod algebra does not operate trivially and hence there is some generator of A_p which operates non-trivially. This method can always be used in studing the realizability of some algebra but, even for the case of a polynomial algebra on two generators, it is very difficult to see if a given algebra admits a structure of unstable A_p-algebra. There are some results that are very useful in proving that a given algebra does not admit Steenrod operations:

Theorem ([54]) *Let A be a polynomial algebra over Z_2 which has a generator in dimension n. If A admits Steenrod operations then A has a generator in dimension n-k for every $k \geqslant 0$ such that $\binom{n-k-1}{k} \equiv 1$ (2).*

Theorem ([16]) *Let A be a polynomial algebra over Z_p (p odd) and suppose that A admits Steenrod operations. If 2m is the degree of a generator of A, then A has a generator in some degree 2n for which $n \equiv 1-p$ (m), or* *else* $m \equiv 0\ (p)$.

Another result of this kind is given in [38]. In the case of a polynomial algebra over Z_p with only two generators, a (non complete) set of necessary conditions for admiting Steenrod operations can be found in [37] (notice that this paper contains a mistake in the case p=3). The converse problem: to show algebraically that some algebra does admit Steenrod operations, is also very hard. The reader can look at [47] where this is done for a polynomial algebra over Z_p with generators in degrees 4 and $2(p+1)$.

If p=2 and there are no two generators in the same degree, the classification of the polynomial algebras that admit Steenrod operations was achieved in [49]. The result is as follows: Let us call a monotone increassing sequence of positive integers $I = (a_1,\ldots,a_r)$ *allowable* if the polynomial algebra over Z_2 with generators in degrees $a_1,\ldots,a_r$ admits Steenrod operations. We write $2^s I = (2^s a_1,\ldots,2^s a_r)$.

<u>Theorem</u> *([49]) I is allowable if and only if it is a disjoint union* $I = \cup 2^{s_i} I_i$ *where the sequences* I_i *belong to the following list:*

a) $T_1 = (1)$;

b) $A_h = (2,3,\ldots,h+1)$, $h \geq 2$, $h+1$ *not a power of 2;*

c) $G_m = A_{2^{m+1}-2} - (2^t+1;\ 0 \leq t \leq m)$, $m \geq 2$;

d) $E_n = (2^{n+1}-2^s;\ 0 \leq s \leq n)$, $n \geq 3$.

We get in this way a large set of polynomial algebras which are unstable A_2-algebras. However, not all these algebras are realizable as cohomology algebras. To admit a structure of unstable A_p-algebra is in some sense the first obstruction to realizability, but there should be higher obstructions. For example, observe that, using the Steenrod algebra we cannot settle the realizability problem for $A = Z_2[x]/x^3$ because A admits Steenrod operations for deg $x = 2^i$, $i \geq 0$, and only the cases deg $x = 1,2,4,8$ are known to be realizable. It can be proved that no other case is, indeed, realizable, but one needs more powerful methods, as we will see in the forthcoming section.

2. Secondary cohomology operations

Roughly speaking, a secondary cohomology operation is a cohomology operation defined only for some cohomology classes and which takes values modulo some indeterminacy. Their existence is important for the realizability problem because they provide restrictions on the action of the Steenrod algebra on the cohomology of a space, showing that some unstable A_p-algebras cannot arise as cohomology algebras.

Let us give more precise definitions. Let $\sum_{k=1}^{m} a_k b_k = 0$ be an homogeneous relation of degree n+1 in A_p. A *stable secondary operation* ϕ associated to this relation is a natural transformation

$$\phi : N^i(b, \ ) \longrightarrow H^{i+n}(\ ;Z_p)/I^{i+n}(a, \ ) \ , \ i \geq 0,$$

where

$$N^i(b,X) = \{ x \in H^i(X;Z_p) \ : \ b_k(x)=0, \ k=1,\dots,m \}$$

$$I^i(a,X) = \sum_{k=1}^{m} a_k \ H^{i+n-\deg a_k}(X;Z_p)$$

and such that ϕ verifies the following properties:

1) ϕ commutes with suspensions (i.e. it is stable);

2) (Peterson-Stein formula) Let (X,Y) be a couple of spaces and let $i:Y \longrightarrow X$, $j:X \longrightarrow (X,Y)$ be the inclusions. Let $v \in H^i(X,Y)$ and assume $j^*v \in N^i(b,X)$. Then the exact sequence of (X,Y) shows that there are elements $w_k \in H^{i+\deg b_k-1}(Y)$ such that $\delta w_k = b_k(v)$. We require that

$$i^* \phi(j^*v) = \sum_{k=1}^{m} a_k(w_k).$$

The key results are the following:

a) Given any relation $\Sigma a_k b_k = 0$, there is at least one stable secondary operation associated to it;

b) two stable secondary operations associated to the same relation differ by a primary operation.

Of course, we are using here the word relation in a quite intuitive sense. To be more precise, we can define a relation in A_p as a pair (d,z) where d is a map $d:C \longrightarrow A_p$ and

1) C is a finitely generated free graded A_p-module,

2) d is an A_p-module homomorphism,

3) z is an homogeneous element of ker d.

Then, the relation in the intuitive sense associated with the pair (d,z) is $\Sigma a_k b_k = 0$, where a_k are the coordenates of z in a base of C and b_k are the images by d of the elements of this base.

The construction of a secondary operation associated to a given relation is done by the method of the universal example: one constructs a space E and defines the secondary operation on E. Moreover, the space E is universal in the sense that for every space X and every cohomology class $x \in H^*(X)$ on which the secondary operation has to be defined, there is a map $f:X \longrightarrow E$ such that $x = f^*y$ and the secondary operation is defined on y. We will illustrate this method in the very simple case of the relation $Sq^1 Sq^1 = 0$.

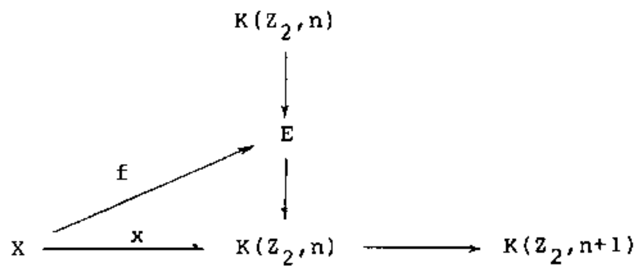
Let us consider $K(Z_2,n)$ and let us take a map $K(Z_2,n) \longrightarrow K(Z_2,n+1)$ corresponding to $Sq^1\iota \in H^{n+1}(Z_2,n;Z_2)$, where $\iota \in H^n(Z_2,n;Z_2)$ is the fundamental class. This map induces a fibration of fibre $K(Z_2,n)$ over $K(Z_2,n)$:

$$\begin{array}{ccc} K(Z_2,n) & & \\ \downarrow & & \\ E & & \\ \downarrow & & \\ K(Z_2,n) & \xrightarrow{Sq^1} & K(Z_2,n+1) \end{array}$$

Let us consider the Serre exact sequence of this fibration:

$$H^{n+1}(E;Z_2) \xrightarrow{i^*} H^{n+1}(Z_2,n;Z_2) \xrightarrow{\tau} H^{n+2}(Z_2,n;Z_2)$$

Since $Sq^1\,\iota = Sq^1\,Sq^1\,\iota = 0$, there is an element $z \in H^{n+1}(E)$ such that $i^*z = Sq^1\iota$. We define ϕ in the following way: if $x \in H^n(X;Z_2)$ and $Sq^1x=0$, we have a lifting



We define

$$\phi\ x = \{\ f^*z\ :\ z \in (i^*)^{-1}(Sq^1\iota)\}$$

and we have a secondary operation associated to the relation $Sq^1\ Sq^1=0$. (Of course, ϕ is nothing but the secondary Bockstein). The method of this example can be easily generalized to any relation.

The next step should be to determine how many secondary operations exist and what are the relations that they verify. Since secondary operations arise from relations in the Steenrod algebra, homological algebra and the homology of the Steenrod algebra will be the main topics in the study and classification of secondary operations. We can give an intuitive idea of how homological algebra enters on this context.

We can consider Z_p as an A_p-module (concentrated in degree zero) in the obvious way. Let C_o be an A_p-module free on one generator of degree zero. If we have a resolution

$$C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_o \longrightarrow Z_p$$

of Z_p by finitely generated free A_p-modules and we consider elements in $\operatorname{Im} d_2$, we get relations on A_p and, associated with these relations, we have secondary operations. If the above resolution is "minimal" (this means intuitively that at each stage there are as few generators as possible) and we consider the secondary operations ϕ_i associated with the elements $z_i = d_2\, c_{2,i}$ where $\{c_{2,i}\}$ is an A_p-base of C_2, then it holds that any other secondary operation is a linear combination of the operations ϕ_i. Moreover, if $c_3 \epsilon C_3$, we have $d_3\, c_3 = \Sigma\, a_i\, c_{2,i}$, $a_i \epsilon A_p$ and applying d_2 we have $\Sigma\, a_i\, z_i = 0$. Since we know that secondary operations associated with the same relation differ by a primary operation, we have a relation of the form $\Sigma\, a_i\, \phi_i = [b]$, $a_i, b \epsilon A_p$. Hence, if c_3 runs over an A_p-base of C_3, we obtain the relations between the operations ϕ_i.

Then, the basic secondary operations are in 1-1 correspondence with a Z_p-base of $\operatorname{Tor}_2^A(Z_p, Z_p)$ and the relations between these operations are in 1-1 correspondence with a Z_p-base of $\operatorname{Tor}_3^A(Z_p, Z_p)$. Hence, in order to understand the secondary cohomology operations we have to compute the homology of the Steenrod algebra. This homology is known only up to some low dimension. For example, for p=2 we have:

<u>Theorem</u> *a)* $H^0(A_2; Z_2)$ *has a base formed by the unit element;*

b) $H^1(A_2; Z_2)$ *has a base formed by elements* h_i, $i=0,1,2,\ldots$ *with* $\deg h_i = 2^i$;

c) $H^2(A_2; Z_2)$ *has as a base the products* $h_j h_i$, $j \geqslant i \geqslant 0$, $j \neq i+1$, *and* $h_{i+1} h_i = 0$;

d) in $H^3(A_2; Z_2)$ *holds* $h_k^2 h_o \neq 0$ *if* $k \geqslant 3$.

Let us denote by ϕ_{ij} the secondary operations corresponding to $h_j h_i \epsilon H^2(A_2; Z_2)$ (or to their duals in $H_2(A_2; Z_2)$) and assume $k \geq 3$. Since $h_k^2 h_o \neq 0$ in $H^3(A_2;Z_2)$, there is a non-trivial relation between the operations ϕ_{ij}, $j \leq k$, $k \geq 3$, and it can be seen that this relation has the form

$$\sum_{\substack{0 \leq i \leq j \leq k \\ j \neq i+1}} a_{ij} \phi_{ij} = [\lambda\, Sq^{2^{k+1}}] \qquad (\star)$$

which holds modulo the total indeterminacy of the left-hand side, when the operations ϕ_{ij} are defined (i.e. on classes x such that $Sq^{2^r}x=0$, $0 \leq r \leq k$). It remains only to compute the value of the coefficient λ. This is done by applying the formula $(\star)$ to a test space that in this case is the complex projective space of infinitely many dimensions. In this way one proves $\lambda \neq 0$.

The formula $(\star)$ is of trascendental importance for the realizability problem. For instance, it solves the Hopf invariant one problem which is equivalent to the realizability problem for $Z_2[x]/x^3$.

<u>Theorem</u> *If $Sq^1 = Sq^2 = Sq^4 = Sq^8 = 0$ in $H^*(X;Z_2)$, then $Sq^i = 0$, $i > 0$.*

The proof is as follows: since the elements Sq^{2^i} generate the Steenrod algebra, it is enough to prove that $Sq^{2^i}=0$, $i \geq 0$. Let $x \in H^*(X;Z_2)$. Since $Sq^1x = Sq^2x = Sq^4x = Sq^8x = 0$, the operations ϕ_{ij}, $j \leq 3$ are defined on x and have zero indeterminacy. Then, by $(\star)$, $Sq^{16}x = \Sigma\, a_{ij}\, \phi_{ij}x$ and, since $\deg a_{ij} < 16$, we have $a_{ij}=0$ on $H^*(X;Z_2)$ and so $Sq^{16}x=0$. Inductively we prove that $Sq^{2^i}=0$ for all $i \geq 0$.

Notice that the fact that $Sq^1 = Sq^2 = Sq^4 = Sq^8 = 0$ implies $Sq^i=0$ for all $i > 0$ is not a consequence of the structure of the Steenrod algebra but an additional restriction verified by realizable algebras, due to the existence of secondary operations.

The theory of secondary operations and the proof of $(\star)$ are the goals of a celebrated paper by Adams ([2]). For another proof of the Hopf invariant one problem, based on a deeper knowledge of the cohomology of the Steenrod algebra, see [57]. There are also similar results for odd primes. Liulevicius ([33]), Shimada ([41]) and Yamanoshita ([61]) proved that the operations P^j, $j=p^i$, $i \geq 1$, can be factorized by secondary cohomology operations. For example, if $i=1$, we have a formula

$$\lambda\ P^p = \beta\phi - 2\ P^{p-2}\chi$$

where $\lambda \neq 0$ and ϕ and χ are secondary operations defined on classes x such that $\beta x = P^1 x = 0$. As an immediate consequence of this factorisation, we obtain:

<u>Theorem</u> *If* $\beta = P^1 = 0$ *in* $H^*(X;Z_p)$, $p > 2$, *then* $P^i = 0$, $i > 0$.

As an example, we apply the above results to the algebra $Z_p[x]$:

<u>Theorem</u> *Suppose* $H^*(X;Z_p) \cong Z_p[x]$, *dim* $x = h$. *Then:*

a) if $p=2$, *then* $h=1,2,4$;

b) if $p \neq 2$, *then* $h=2r$ *and* r *divides* $p-1$.

This result is a trivial corollary of the above theorems, except for the case $p=2$, $h=8$, which requires some extre work on secondary operations (see [22]).

If $p=2$, the spaces RP^∞, CP^∞ and HP^∞ are examples of spaces realizing $Z_p[x]$ with dim $x = 1,2,4$ respectively. If $p > 2$, we have to wait until chapter 5 to see examples of spaces which have $Z_p[x]$ as cohomology algebra.

This same method was applyed to algebras with several generators but the difficulties increase very fast and even in the case of two generators only partial results were obtained ([1], [37], [20]).

3. K-theory

The existence of a space whose cohomology realizes a given algebra A implies that A has to be compatible with the action of all primary and high-order operations but, actually, the existence of X produces not only ordinary cohomology groups but also cohomology groups $h^*(X)$ for any generalized cohomology theory h^* and these groups (or rings) have to be compatible with the operations of the cohomology theory h^*. In this way we can obtain a new method to find necessary conditions for realizability. This method was first considered for the case of complex K-theory and produced many important advances in the realizability problem.

Let us briefly review the most important features of complex K-theory (see [12]). It is a functor which associates to any finite CW complex X a commutative ring K(X). Moreover, there is a filtration in K(X) defined by

$$K(X)_n = \mathrm{Ker}\ (i_{n-1}! : K(X) \longrightarrow K(X^{n-1}))$$

where X^{n-1} denotes the (n-1)-skeleton of X. Thus, $K(X)_o=K(X)$ and $K(X)_n=0$ if $\dim X<n$. Moreover, we have $K(X)_{2q}=K(X)_{2q-1}$. With this filtration K(X) is a filtered ring. On this ring act the called Adams operations ψ^i, $i\geqslant 0$, which verify the following properties:

i) ψ^k is a ring homomorphism;

ii) $\psi^{kh} = \psi^k \psi^h$;

iii) $\psi^k x - k^n x \in K_{2n+1}$ if $x \in K_{2n}$;

iv) if p is prime then $\psi^p x \equiv x^p$ (p).

K-theory and ordinary cohomology are related by the Atiyah-Hirzebruch spectral sequence, whose E_2^* term is $E_2^* \cong H^*(X;Z)$ and which converges to K(X) with the filtration described above. Moreover, this spectral sequence collapses if $H^*(X)$ has no torsion and so in this case we have an isomorphism of graded rings $H^*(X;Z) \cong Gr(K(X))$ and then $H^{2n}(X;Z) \cong K(X)_{2n}/K(X)_{2n+1}$ as abelian groups. The relation between the multiplicative structures in $H^*(X)$ and $K(X)$ is not so straightforward. However, there is a ring isomorphism

$$K(X) \otimes \mathbf{Q} \cong \bigoplus_{n \text{ even}} H^n(X;\mathbf{Q}).$$

Assume that we are studing the realizability of some algebra A. If there is a space X such that $A \cong H^*(X;Z)$ and we are able to compute $K(X)$ then we can obtain restrictions on A by considering that the structure of $K(X)$ has to be compatible with the action of the operations ψ^i. Let us consider a simple example: Take $A = Z[x]/x^3$. Using the methods of chapter 2 we know that A is realizable if and only if $\dim x = 2,4,8$. We will see that this result can also be obtained by using K-theory. Assume $H^*(X;Z) \cong Z[x]/x^3$ with $\dim x = 2n$. Then we can compute $K(X)$ and we get that $K(X) \cong Z[y]/y^3$, where y has exact filtration degree 2n, i.e. $y \in K(X)_{2n}$, $y \notin K(X)_{2n+1}$. Then the operations ψ^2 and ψ^3 will act in the following way:

$$\psi^2 y = 2^n y + a\, y^2 ;$$
$$\psi^3 y = 3^n y + b\, y^2 ;$$

where a is some odd integer (because $\psi^2 y \equiv y^2$ (2)). Since $\psi^6 = \psi^2 \psi^3 = \psi^3 \psi^2$, we can compute $\psi^6 y$ in two different ways and comparing the coefficients in both expressions for $\psi^6 y$ we obtain the equation

$$2^n(2^n-1)b = 3^n(3^n-1)a.$$

Since a is odd, 2^n must divide 3^n-1 and it is just a matter of elementary number theory to see that this can happen only if n= 1,2,4. Hence, we have a new proof of the Hopf invariant one theorem (due to Adams and Atiyah ([4])).

We have seen that a direct application of primary K-theory operations allows to give a very short proof of the Hopf invariant one theorem, a result whose first proof required the machinery of secondary cohomology operations and the cohomology of the Steenrod algebra considered in the preceeding chapter. This fact gives us an idea of the usefulness of K-theory in the problem of realizability of cohomology algebras. The method that we have used in the simple case $Z[x]/x^3$ can be applied to more complex algebras. The steps are the following:

i) begin with $H^*(X;Z)$;

ii) compute $K(X)$ using the Atiyah - Hirzebruch spectral sequence (trivial if there is no torsion);

iii) compute the multiplicative structure of $K(X)$;

iv) consider the action of theoperations ψ^i on $K(X)$;

v) apply $\psi^r\psi^s = \psi^s\psi^r$.

Hubbuck developed this program in a very general setting in the paper "*Generalized cohomology operations and H-spaces of low rank*" ([28]). Hubbuck's *generalized operations* are the translation to cohomology of the operations ψ^i. These operations show some relations in $H^*(X;Z)$ that cannot be directely obtained from the Steenrod algebra or the secondary operations. He proves:

<u>Theorem</u> ([28]) *Let Y be a CW complex with finite skeletons and with $H_*(Y;Z)$ free of 2-torsion. Suppose that $H^{n+2(q-s)}(Y;Z_2) = 0$ for $1 \leq s \leq 2 + \nu_2(q)$, if q is even, or $H^{n+2(q-1)}(Y;Z_2) = 0$, if q is odd; then $Sq^{2q}\colon H^n(Y;Z_2) \longrightarrow H^{n+2q}(Y;Z_2)$ vanishes.*

<u>Theorem</u> ([27]) *Let $p > 2$ and let X be a CW complex with finite skeletons and with $H_*(X;Z)$ free of p-torsion, such that $H^{n+2(q-s)(p-1)}(X;Z_p) = 0$ for $1 \leq s \leq 1 + \nu_p(q)$; then $P^q\colon H^n(X;Z_p) \longrightarrow H^{n+2q(p-1)}(X;Z_p)$ vanishes.*

One interesting feature of the use of K-theory in the realizability problem is that it works over the integers (or over $Z_{(p)}$) and so it is possible to obtain properties of $H^*(X;Z)$ which are not visible from the point of view of the Steenrod algebra. For instance, we have the following result:

Theorem ([29]) *Assume $H^*(X;Z)$ finitely generated in each dimension and free of p-torsion. Let $x \in H^n(X;Z_{(p)})$, $n>0$ and assume $x^p = py$ for some y. Then, $y^p = pz$ for some z.*

Using the method described above, Hubbuck solved the realizability problem for truncated polynomial algebras over Z_2 of height ≥ 3 and not more than 5 generators. A *truncated polynomial algebra of height k* is an algebra with generators $x_1,\ldots,x_m$ and relations $x_1^{r_1}\ldots x_m^{r_m} = 0$ if $r_1 + \ldots + r_m \geq k$. Hubbuck's result is as follows:

Theorem ([28]) *Let $H^*(X;Z_2)$ be a truncated polynomial algebra of height ≥ 3 on not more than five generators all with even dimensions. Then the set of dimensions of the generators is a union of sets taken from (2), (4), (8), (4,6), (4,8), (4,6,8), (4,8,12), (4,6,8,10), (4,8,12,16), (4,6,8,10,12), (4,8,12,16,20).*

The same method was applied by Wilkerson to $Z_p[x]$ in [58] and to $Z_p[x,y]$ in [59]. For other applications of K-theory to realizability of cohomology algebras see [42], [34], [10], [9], [25], [26].

4. Rational homotopy

In the last chapters we have mainly considered cohomology with coefficients in one of the rings Z, Z_p or $Z_{(p)}$. These are the most important cases but it is also worthwhile to devote a chapter to descrive what happens if we take as ring of coefficients the field Q of rational numbers. In this case the shape of the problem changes completely. This is a general phenomenon in algebraic topology: the homotopy theory over the field Q has its proper features which differ very much from the modulo p case and which convert rational homotopy theory in an independent field inside algebraic topology. The fundational work in rational homotopy theory is the paper [39] by Quillen. In this paper he shows that the category of rational homotopy types is equivalent to some algebraic category. More precisely, denote by (T) the category of 1-connected pointed spaces and base point preserving maps and by $Ho_Q(T)$ the localisation of (T) with respect to rational homotopy equivalences. Denote by (DGC) the category of 2-reduced differential graded cocommutative coassociative coalgebras over Q and by Ho(DGC) the localisation of (DGC) with respect to homomorphisms which induce isomorphisms in homology. Then Quillen proves that there is an equivalence of categories

$$Ho_Q(T) \xrightarrow{\lambda} Ho(DGC)$$

and an isomorphism of functors $H \cong H \cdot \lambda$ where H denotes rational homology. As a corollary he obtains the following theorem:

Theorem ([39]) *Let A be a graded commutative Q-algebra with A^q of finite dimension for each q and $A^1=0$, $A^0=Q$. Then, there is a space X such that $H^*(X;Q) \cong A$.*

Thus, over the rationals any algebra is realizable. This result was conjectured by Hopf (see [39]). In [53] Thom raised the problem of constructing commutative cochains over Q for simply-connected spaces, a problem closely related to the algebraisation of rational homotopy. In 1967, Paul Olum in a lecture at the Cornell Topology Conference (see [43]) proved the above theorem, but there was a mistake in solving the *commutative cochains problem*. About the work of P. Baum, L. Smith says the following ([43]): *The proof was based on an erroneous construction of commutative cochains over Q. Unfortunately, none of us was then aware that Whitney had already constructed the requesite functorial commutative cochains in his work 'Geometric Integration Theory'. The influence of this lecture of Baum should not be underestimated.*

Actually, the existence of commutative cochains over fields of characteristic zero is what makes possible the algebraic description of rational homotopy types. The problem is the following: we know that the cohomology algebra of a space is commutative, but at the cochain level we have only commutativity up to homotopy. We want to find a canonical free cochain complex with cohomology isomorphic to the singular cohomology and which is commutative as graded algebra. If we deal with differentiable manifolds and we take $\mathbb{R}$ as field of coefficients, the problem admits a solution because we can consider the DeRham complex of differential forms. However, over Z_p there is no such a commutative cochain complex due to the existence of Steenrod operations which are precisely the obstructions to commutativity at the cochain level. The problem can also be solved for general spaces over Q by considering rational differential forms (see [50]) and so we can compute the rational cohomology of a space X from a commutative differential graded algebra $A(X)$ which characterizes the rational homotopy type of X.

Rational homotopy theory is now a very active field, mainly since the work of Sullivan [50]. There is no place in this paper to give a detailled review of rational homotopy theory, however, I would like to present just an idea of the proof of the realizability theorem. I will not follow the original treatment by Quillen but a more topological one, given by Halperin and Stasheff ([45]).

The starting point is that we know how to realize the free algebras over Q. Since ΩS^{2n+1} and S^{2n+1} are rationally equivalent to $K(Q,2n)$ and $K(Q,2n+1)$ respectively, we know that $H^*(K(Q,2n);Q) \cong Q[x]$, dim x =2n and $H^*(K(Q,2n+1);Q) \cong E(x)$, dim x = 2n+1. Hence, any free algebra over Q, i.e. any algebra of the form $Q[x_1,\ldots,x_r] \cong E(y_1,\ldots,y_s)$, dim $x_i = 2n_i$, dim $y_j = 2m_j+1$, appears as the cohomology of a product of some Eilenberg-MacLane spaces, namely $K(Q,2n_1) \times \ldots \times K(Q,2n_r) \times K(Q,2m_1+1) \times \ldots \times K(Q,2m_s+1)$. Notice the big difference from the case of integral coefficients: The free algebra $Z[x]$ is realizable only if dim x = 2,4. After having models for the free algebras we have to try to put relations in these algebras in order to realize any Q-algebra. Since the free algebras are realized by products of Eilenberg-MacLane spaces, we have a good control of the maps and we can use fibrations to kill specific relations. The procedure works well because we can use the Hirsch theorem ([15], [53], [13], [31]) to compute the rational cohomology of the total space of a fibration.

Let us consider a simple example ([44]). Assume that we want to realize the algebra $A = E(x)$, dim x = 2n (notice that E(x) is not free!). Of course, we know that S^{2n} realizes this algebra, but we will show in this example how the general method works. Let us consider a vector space Z_o generated by the generators of A, i.e. $Z_o = Qx$. If ΛZ_o denotes the free algebra over Z_o, we can consider the projection $\Lambda Z_o \longrightarrow A$. Let K be the kernel and define Z_1 as the vector space generated by the generators of K (i.e. by the relations of A). Then, $Z_1 = Qy$ with y corresponding to $x^2 \in \Lambda Z_o$. Z_o and Z_1 are graded vector spaces if we consider in Z_o the gradation of A and in Z_1 the gradation of K with a shift downward by one. Then, $Z = Z_o \oplus Z_1$ is a bigraded vector space with a basis x,y, deg x = (2n,0), deg y = (4n-1,1). We define a differential of degree (1,-1) on ΛZ by $dx=0$, $dy=x^2$. In this way we have constructed a free commutative graded differential algebra $(\Lambda Z,d)$ whose cohomology is $Q[x]/x^2 = A$. Moreover, the algebra $(\Lambda Z,d)$ has been constructed in such a way that we can translate it to a tower of fibrations which produces a space E with $H^*(E;Q) \cong A$. In this case, since Z has only two steps, the tower of fibrations reduces to a single fibration

$$\begin{array}{ccc} K(Q,4n-1) & & \\ \downarrow & & \\ E & & \\ \downarrow & & \\ K(Q,2n) & \xrightarrow{\;x^2\;} & K(Q,4n) \end{array}$$

and the Hirsch theorem proves that $H^*(E;Q) \cong H(\Lambda Z,d) \cong E(x)$.

The method sketched above can be applied to any algebra A but it will require in general infinitely many steps, according to how long is the resolution $(\Lambda Z,d)$ of A. This is the case if we consider the algebra $E(x,y)$, $\dim x = \dim y = 2n$. The first step would be $\Lambda Z_o = Q[x,y]$. Then $\Lambda Z_1 = E(u,v,w)$ with u,v,w corresponding to the relations $x^2=0$, $yx=0$, $y^2=0$ respectively. The differential acts as follows $du=x^2$, $dv=yx$, $dw=y^2$. Then, the first fibration has the form

$$\begin{array}{ccc} K(Q^3,2n-1) & & \\ \downarrow & & \\ E_1 & & \\ \downarrow & & \\ K(Q^2,2n) & \xrightarrow{\;(x^2,yx,y^2)\;} & K(Q^3,4n) \end{array}$$

But $H^*(E_1)$ is not yet isomorphic to A because u,v,w are not free. For instance, they verify the relations $xv=yu$, $xw=yv$. We take $\Lambda Z_2 = Q[r,s]$, with $dr=yu-xv$, $ds=yv-xw$ and we construct a fibration over E_1 corresponding to ΛZ_2. Since there are relations between r and s, we have to go on until we get a resolution $(\Lambda Z,d)$ of A and, corresponding to it, a tower of fibrations which defines a space X with $H^*(X;Q) \cong A$. For more details on the definition of $(\Lambda Z,d)$, see [30], [45].

5. Completion

In chapter 2 we showed that if p is an odd prime and $Z_p[x]$ is realizable then dim x =2n and n divides p-1 but we did not show if this condition is also sufficient. We also know many necessary conditions for a polynomial algebra to be realizable, but only very few of these algebras have been realized. In general (we exclude the rational case) we have always tried to obtain necessary conditions for realizability, like compatibility with Steenrod operations, secondary operations or K-theory operations, but nothing has been said about constructive methods. In the present chapter we will consider a method for constructing spaces with interesting cohomology and in the next chapter we will see how this method is crucial to understand the problem of realizing a wide class of algebras including polynomial algebras. This constructive method was introduced by Sullivan ([52], see also [51]).

In [51] Sullivan defines two new tools to deal with mod p problems in topology: the localisation and the profinite completion. Localisation is a functor which associates to each simply-connected space X and to each prime p a space X_p and a map $X \longrightarrow X_p$ which in homology and homotopy acts like the classical p-localisation of abelian groups and such that it is universal with this property. If X is a CW complex, then X_p is defined by replacing the cells of X by "local cells" i.e. by Moore Spaces $M(Z_{(p)},n)$. Localisation theory is today a standard tool in algebraic topology which allows to decompose a homotopy problem in its p-components. For a good reference, see [23].

Completion is a functor which associates to each simply-connected space X and to each prime p a space $X_p^{\hat{}}$ whose homotopy groups are Ext-p-complete as abelian groups, and a map

$X \longrightarrow X^{\wedge}_p$ which is a mod p homology equivalence and such that the all construction is universal. It is possible to give an unified treatment of localisation and completion for not necessarily simply-connected spaces ([14]).

The usefulness of completion comes from the fact that it allows to obtain more "flexible" spaces without changing the mod p cohomology. Here "flexible" is used in the sense that they admit more automorphisms. For instance, let us consider a loop space ΩX. The units of $[S^1,S^1] \cong Z$ act on ΩX, but the group of units of Z is very small. If we take the p-adic completion of ΩX, $\Omega X^{\wedge}_p$, we have the group of units of $[S^{1\wedge}_{p}, S^{1\wedge}_{p}] \cong Z^{\wedge}_p$ acting on $\Omega X^{\wedge}_p$ and this group is much larger, for instance it contains (p-1)-roots of unity!

Hence, the group of (p-1)-roots of unity acts on any completed loop space (see [18] for the details). We can consider a concrete example: let us take $\Omega X = K(Z,2) = CP^{\infty}$. We have $H^*(\Omega X; Z_p) = Z_p[x]$, dim x = 2. Let us take the p-adic completion of ΩX (p an odd prime), $\Omega X^{\wedge}_p = K(Z^{\wedge}_p, 2)$. We still have the same mod p cohomology. On $\Omega X^{\wedge}_p$ acts the group G of (p-1)-roots of unity. If we denote by Y the orbit space, we can compute $H^*(Y;Z_p)$ by using the appropriate spectral sequence and we see that it is isomorphic to the subalgebra of $H^*(\Omega X^{\wedge}_p; Z_p) \cong Z_p[x]$ invariant by the action of G. Hence, $H^*(Y;Z_p) \cong Z_p[y]$, dim y = 2(p-1), where y corresponds to $x^{p-1} \in H^*(\Omega X^{\wedge}_p; Z_p)$. More in general, if r divides p-1, we have a cyclic subgroup of order r in G and we can consider the orbit space of $\Omega X^{\wedge}_p$ under the action of this subgroup. We have proved:

<u>Theorem</u> *If $p > 2$, then $Z_p[x]$ is realizable if and only if dim $x = 2n$ and n divides $p-1$.*

Instead of CP^{∞} we can consider the space $S\langle 3\rangle$, the 3-connective cover of S^3. It is a loop space, hence, we can complete it and take the orbit space under the action of a cyclic group of order dividing p-1, in the same way as above. We get:

Theorem *If* $p>2$ *and* r *divides* $p-1$*, then there is a space* X *such that* $H^*(X;Z_p)\cong Z_p[x]\otimes X(\beta x)$*,* $\dim x = 2pr$*, where* β *is the Bockstein homomorphism.*

The converse is an open problem (see [9] and also section 7.4 of these notes).

It is easy to generalize the first example above to products $CP^{\infty}\times\ldots\times CP^{\infty}$. After completion, the group $GL(n,Z_p^{\wedge})$ acts on this space and the spaces of orbits under actions of convenient subgroups of this group show interesting examples of realizable algebras. We have in this way a constructive method to realize cohomology algebras.

Theorem *Let* A *be an algebra isomorphic to the algebra of invariants of* $Z_p[x_1,\ldots,x_n]$*,* $\deg x_i = 2$*, under the action of a finite subgroup of* $GL(n,Z_p^{\wedge})$ *of order prime to p. Then* A *is realizable.*

It is a natural question to ask if this method can produce cohomology algebras which are polynomial algebras. We have seen that in the case of one generator we obtain all polynomial algebras which are realizable and it is not difficult to find examples with more than one generator. Actually, it is a purely algebraic problem: we loof for finite groups of order prime to p such that the algebra of invariants is a polynomial algebra. Fortunately, these groups are completely characterized. The algebra of invariants of a polynomial algebra under the linear action of a group is a polynomial algebra if and only if the group belongs to a well known class of finite groups called groups generated by *pseudo-reflections* (i.e. by endomorphisms φ such that $1-\varphi$ has rank one). Hence, corresponding to each group in this class (over $Z_p^{\wedge}$) with order prime to p there is a polynomial algebra that can be realized by the method explained above (we call it '*Sullivan's method*') and conversely, if a polynomial algebra is realizable by Sullivan's method then it corresponds to some group generated by pseudo-reflections. Fortunately, these groups are completely classified ([40]) (over C, but it is not difficult to pass to $Z_p^{\wedge}$) and so we can make a list of realizable polynomial algebras that can be realized by Sullivan's method. This list was computed in [17] and it is usually called the Clark-Ewing list. It is as follows (the type is the set of degrees of the generators):

Nr	Type	Primes
1	$(4,6,\dots,2(n+1))$	$p \nmid (n+1)!$
2a	$(2m,4m,\dots,2(n-1)m,2qn)$	$p \nmid n!$, $p\equiv 1(m)$, $m>1$, $m=qr$
2b	$(4,2m)$	$m>2$, $p\equiv \pm1(m)$
3	$(2m)$	$p\equiv 1\ (m)$
4	$(8,12)$	$p\equiv 1\ (3)$
5	$(12,24)$	$p\equiv 1\ (3)$
6	$(8,24)$	$p\equiv 1\ (3)$
7	$(24,24)$	$p\equiv 1\ (12)$
8	$(16,24)$	$p\equiv 1\ (4)$
9	$(16,48)$	$p\equiv 1\ (8)$
10	$(24,48)$	$p\equiv 1\ (12)$
11	$(48,48)$	$p\equiv 1\ (24)$
12	$(12,16)$	$p\equiv 1,3\ (8)$; $p\neq 3$
13	$(16,24)$	$p\equiv 1\ (8)$
14	$(12,48)$	$p\equiv 1,19\ (24)$
15	$(24,48)$	$p\equiv 1\ (24)$
16	$(40,60)$	$p\equiv 1\ (15)$
17	$(40,120)$	$p\equiv 1\ (20)$
18	$(60,120)$	$p\equiv 1\ (15)$
19	$(120,120)$	$p\equiv 1\ (60)$
20	$(24,60)$	$p\equiv 1,4\ (15)$
21	$(24,120)$	$p\equiv 1,49\ (60)$
22	$(24,40)$	$p\equiv 1,9\ (20)$
23	$(4,12,20)$	$p\equiv 1,4\ (5)$
24	$(8,12,28)$	$p\equiv 1,2,4\ (7)$
25	$(12,18,24)$	$p\equiv 1\ (3)$
26	$(12,24,36)$	$p\equiv 2\ (3)$
27	$(12,24,60)$	$p\equiv 1,4\ (15)$

Nr	Type	Primes
28	(4,12,16,24)	$p\neq 2,3$
29	(8,16,24,40)	$p\equiv 1\ (4)$, $p\neq 5$
30	(4,24,40,60)	$p\equiv 1,4\ (5)$
31	(16,24,40,48)	$p\equiv 1\ (4)$, $p\neq 5$
32	(24,36,48,60)	$p\equiv 1\ (3)$
33	(8,12,20,24,36)	$p\equiv 1\ (3)$
34	(12,24,36,48,60,84)	$p\equiv 1\ (3)$, $p\neq 7$
35	(4,10,12,16,18,24)	$p\neq 2,3,5$
36	(4,12,16,20, 24,28,36)	$p\neq 2,3,5,7$
37	(4,16,24,28, 36,40,48,60)	$p\neq 2,3,5,7$

These are the irreducible types. Of course, any product of types from this list is also realizable and any polynomial algebra which is realizable by Sullivan's method is a product of types in this list. The list represents a very important advance in the realizability problem because before it only a very few of the above polynomial algebras were known to be realizable. At the end of their paper, Clark-Ewing say ([17]): *We see no reason not to conjecture that the list of types constructed above, and their products, with the exceptional primes determined, is the complete list of polynomial algebras realizable as cohomology rings, but the evidence for this is very slender.*

6. The theorem of Adams and Wilkerson

In 1980 appeared a paper by Adams-Wilkerson ([5]) which has an extraordinary importance for the realizability problem. This paper developes a program iniciated by Wlkerson in [60] and the results that they prove give an almost complete answer to the question of realizability of polynomial algebras. In this section we will present a review of the Adams-Wilkerson paper, showing the main ideas that it contains.

In [60] Wilkerson says: *It seemed that the best chance to obtain a complete classification was to apply some constructive methods, rather than the argument by contradiction or special cases typical of most earlier work.* He presents a program in two steps to solve the problem. Assume A is realizable, i.e. $A \cong H^*(X;Z_p)$. The two steps are:

1) Embed A in a polynomial algebra on two-dimensional generators $Z_p[x_1,\ldots,x_n] \cong H^*((CP^\infty)^n;Z_p)$;

2) classify these embeddings.

Let us denote by $\underline{C}$ the category of unstable A_p-algebras which are

i) zero in odd degrees;

ii) integral domains;

iii) of finite transcendence degree.

If p is odd and a polynomial algebra over Z_p is realizable then it belongs to the category $\underline{C}$. Adams-Wilkerson develope in $\underline{C}$ a theory of extensions that allows us to consider the concepts of *algebraic extension, separable extension, Galois extension, algebraically closed object, algebraic closure,* etc. In this framework they prove:

1) Any object in $\underline{C}$ can be embedded in an algebraic closure. There is also a canonical embedding.

2) The objects of $\underline{C}$ that are algebraically closed are those isomorphic to $H^*((CP^\infty)^n;Z_p) \cong Z_p[x_1,\ldots,x_n]$, dim $x_i = 2$.

This solves the first step in Wilkerson's program: we can embed the realizable polynomial algebras in polynomial algebras on two-dimensional generators. In order to classify these embeddings, we want to know when the extension $A \longrightarrow Z_p[x_1,\dots,x_n]$ is Galois. Adams-Wilkerson find a necessary and sufficient condition in order that this holds, but for the main applications it is enough to consider the following sufficient condition:

3) If $A \in \underline{C}$ is generated as a Z_p-algebra by a finite number of generators $x_1,\dots,x_n$ of degrees $2d_1,\dots,2d_n$ where each d_i is prime to p, then the canonical embedding of A in its algebraic closure is a Galois extension.

Assume now that $A \in \underline{C}$ is integrally closed. Then, if $A \in K$ is the canonical embedding of A in its algebraic closure and A verifies the hypothesis of 3), then A is equal to the algebra of invariants of $K \cong H^*((CP^\infty)^n;Z_p)$ under the action of the Galois group G(K/A). Hence, we can apply the methods of chapter 5 and we can construct a space X with $H^*(X;Z_p) \cong A$. This proves the following:

<u>Theorem</u> *Let A be a commutative graded algebra such that*

i) A is an unstable A_p-algebra;

ii) A is an integral domain;

iii) A is zero in odd degrees;

iv) A is integrally closed;

v) A is generated by finitely many generators, all of degree prime to p.

Then A is realizable by Sullivan's method.

In particular, assume $H^*(X;Z_p)$ is a polynomial algebra with generators in dimensions prime to p. Then $H^*(X;Z_p)$ verifies the hypothesis of the theorem and we can conclude that $H^*(X;Z_p)$ is realizable by Sullivan's method. But we have seen in chapter 5 that there is a list containing all polynomial algebras that appear by applying Sullivan's method. Hence:

Theorem *Let A be a polynomial algebra over* Z_p *with generators in degrees prime to p. Then A is realizable if and only if A is a product of algebras in the Clark-Ewing list.*

This result represents an outstanding advance in the realizability problem. Observe that, befor the work of Adams-Wilkerson, even the case of two generators was open and now we have a very general result which determines the polynomial algebras which can appear as cohomology algebras over Z_p when the prime p is big enough. It remains only to study the *'sporadic'* cases in which the prime p divides the dimension of some generator.

It is interesting to notice that if the dimensions of the generators of a polynomial algebra over Z_p are prime to p then to admit an action of the Steenrod algebra is the unique obstruction to realizability. Notice also that the work of Adams-Wilkerson is of a purely algebraic nature.

After this (superficial) review of the Adams-Wilkerson result, it is natural to ask if it settles the realizability problem for polynomial algebras over the integers. There are classical examples of spaces whose integral cohomology is a polynomial algebra, for example

$$H^*(BU(n);Z) \cong Z[x_1,\ldots,x_n], \quad \dim x_i = 2i;$$
$$H^*(BSp(n);Z) \cong Z[y_1,\ldots,y_n], \quad \dim y_i = 4i;$$
$$H^*(BSU(n);Z) \cong Z[x_2,\ldots,x_n], \quad \dim x_i = 2i.$$

Hence, the types $(2,4,6,\ldots,2n)$, $(4,8,\ldots,4n)$, $(4,6,\ldots,2n)$ and products of them are realizable over Z. The classical conjecture is that only these types are realizable.

Conjecture *If* $H^*(X;Z)$ *is a polynomial algebra, then* $H^*(X;Z) \cong H^*(Y;Z)$ *where Y is a product of* $BU(n)$, $BSp(m)$, $BSU(k)$, $n,m,k = 1,2,\ldots$

If a type is realizable over Z then it is realizable over Z_p

for all p. Hence, we can apply to the above conjecture the Adams-Wilkerson theorem. If we do it we see that, unfortunately, the Adams-Wilkerson theorem does not settle the conjecture. As Adams-Wilkerson point out, we can consider the type

$$(4,4,4,8,8,8,12,12,16,16,20,24,24,28)$$

which belongs to the Clark-Ewing list for every prime $p > 3$, but it is not one of the types allowed by the conjecture. It seems very likely that the Adams-Wilkerson theorem plus some further work at the small primes (p=2,3) could suffice to settle the conjecture. But this work has not yet been done. However, there is a special case in which it is possible to prove that the conjecture holds:

Theorem *([6]) If $H^*(X;Z) \cong Z[x_1,\ldots,x_r]$, $\dim x_i = n_i$, $n_1 < \ldots < n_r$, then $(n_1,\ldots,n_r)$ is equal to one of the following types*

a) $(2,4,6,\ldots,2n)$;

b) $(4,8,\ldots,4n)$;

c) $(2,4,8,\ldots,4n)$;

d) $(4,6,\ldots,2n)$.

In other words, if there are no two generators of the same degree, then the conjecture holds. The theorem is proved using the result of Adams-Wilkerson and some classical results like [54], [49], [16].

7. Other results

In this chapter we discuss several results about realizability of cohomology algebras which are not included in the topics studied in the other sections of these notes.

7.1 Generalized cohomology

In chapter 3 we have seen how to use K-theory to obtain necessary conditions for realizability. Instead of using K-theory it is possible to consider other generalized cohomology theories and to use the operations in these theories to obtain new restrictions on the action of the Steenrod algebra on $H^*(X;Z_p)$. The case of BP-theory has been studied by Kane ([32]) and he has obtained interesting results which improve what was obtained by using K-theory operations.

The Brown-Peterson operations are well known since the work of Novikov and Quillen (see [3]). If $E = (e_1, e_2, \ldots)$ is a sequence of non-negative integers with only finitely many non-zero terms, we can associate to E a certain Steenrod operation $P^E \in A_p$ (see [36]) and a certain BP-operation r_E. Kane proves that there is a commutative diagram

$$\begin{array}{ccc} BP^*(X) & \xrightarrow{\;r_E\;} & BP^*(X) \\ \downarrow & & \downarrow \\ H^*(X;Z_p) & \xrightarrow{\;\chi(P^E)\;} & H^*(X;Z_p) \end{array}$$

This diagram relates the Brown-Peterson operations to the Steenrod operations in ordinary cohomology. In this way we can translate the relations between the operations r_E to new relations in the action of the Steenrod algebra on $H^*(X;Z_p)$. Kane proves

Theorem ([32]) *Let p be odd. Let X be a CW complex of finite type such that $H^*(X)$ is torsion free. Suppose either $2s \leqslant (p-3)p^t$ or $ps \leqslant (p-1)p^t$. If $P^{p^s}, \ldots, P^{p^{s+t}}$ act trivially on $H^*(X;Z_p)$ then P^n acts trivially for $n \geqslant p^s$.*

In particular, this theorem implies the results of chapter 2 (if $p>2$ and there is no torsion) and it is more precise than the results we proved using K-theory operations because in the case of K-theory we usually need to know that some groups $H^i(X;Z_p)$ are zero in order to conclude $P^n = 0$, in contrast with the above theorem which concerns only the action of the Steenrod algebra. Notice also that the method works only if there is no torsion. If $H^*(X)$ has p-torsion, the result is false as we can see by considering the space X with $H^*(X;Z_p) \cong Z_p[x] \otimes E(\beta x)$, dim x = 2p, constructed in chapter 5.

We can also mention [35] which contains another application of generalized cohomology to realizability.

7.2 Integral cohomology of H_0-spaces

An H_0-space is a space X such that $H^*(X;Q)$ is a free algebra. We can consider the problem of determine what are the possible values for the integral cohomology af an H_0-space. For example, consider the case in which $H^*(X;Q) \cong Q[x]$. If $H^*(X;Z)$ has no torsion then it has a base $\{x_i\}$, $x_i \in H^{2ni}(X;Z)$, and the algebra structure of $H^*(X;Z)$ is determined by a sequence of integers (μ_i) defined by $x_1^i = \mu_i x_i$. We can ask what sequences (μ_i) can be realized by the integral cohomology ring of some space. The problem is open, but some necessary conditions are known ([24]). On the other hand, it is proved in [11] that if $(\lambda_i)_{i \geqslant 2}$ is any sequence of integers, $\lambda_i \geqslant 0$, then the algebra corresponding to $\mu_i = i!\,\lambda_2^{i-1}\ \lambda_3^{i-2} \ldots \lambda_{i-1}^2\ \lambda_i$ is realizable.

In [11], Aguadé-Zabrodsky study the general case in which $H^*(X;Q) \cong Q[x_1,\dots,x_r] \otimes E(y_1,\dots,y_s)$ and they prove that $H^*(X;Z)$ cannot in general be 'too small' in the sense that it has to contain an algebra with divided powers. Of course, some restriction is needed in order to prove this result, because $H^*(X;Z)$ could be even a polynomial algebra, but this case has to be considered as 'sporadic'. The result is as follows

Theorem ([11]) *Let X be an H_0-space such that $H^*(X;Q) \cong Q[x_1,\dots,x_r] \otimes E(y_1,\dots,y_s)$, $\dim x_i = 2n_i$, $\dim y_i = 2m_i+1$, and such that $H^*(X;Z)$ has p-torsion only for finitely many primes p. Assume $r < \psi(n_1,\dots,n_r)$. Then, there exists an infinite set of primes P and a monomorphism*

$$\Gamma(X_1,\dots,X_r) \longrightarrow H^*(X;Z_{(P)}).$$

Here $\Gamma(X_1,\dots,X_r)$ denotes an algebra with divided powers and ψ is a function defined by

$$\psi(n_1,\dots,n_r) = (n_1\dots n_r)^{-1}\,\varphi(n_1\dots n_r)\,\left(\Sigma\,(1/n_i)\right)^{-1}$$

where φ is the Euler function. Roughly speaking, the theorem says that 'in the general case' and for infinitely many primes, the integral cohomology of an H_0-space has to have at least as much divisibility as an algebra with divided powers.

7.3 Realizability of invariant subalgebras

The standard method to realize cohomology algebras has been studied in chapter 5 and can be described as follows: One has a space X and a self-map $T:X \longrightarrow X$ and in some cases it is possible to construct a space *'to the right'* of X, $X \longrightarrow \bar{X}$, such that $H^*(\bar{X};Z_p)$ is the subalgebra of $H^*(X;Z_p)$ invariant under T. This is the case when we have a completed loop space and the action of a (p-1)-root of the unity. Zabrodsky has studied this method in general ([62]) and he proves the following realizability result

<u>Theorem</u> *([62]) Let* $T:X \longrightarrow X$ *be a self map. Assume that for every* n *there exist integral polynomials* $P_{1,n}$, $P_{2,n}$ *such that* $P_{i,n} \mid P_{i,n+1}$, $i=1,2$ *and such that* $P_{1,n}$ *and* $P_{2,n}$ *are relatively prime. Further suppose*

i) $(P_{1,n}\, P_{2,n})\, H^n(T;Z) = 0$;

ii) for every n*, the multiplicative closure of the characteristic roots of* $P_{1,n} \otimes Z_p$ *contains no characteristic root of* $P_{2,n} \otimes Z_p$.

Then, $A = \bigoplus_n \operatorname{im} P_{2,n}\; H^n(T;Z_p) = \bigoplus_n \ker P_{1,n}\; H^n(T;Z_p)$ *is realizable, i.e. there exists a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle S} \\ X & \xrightarrow{\;f\;} & Y \end{array}$$

such that $H^*(f;Z_p)$ *is injective and its image is* A.

Zabrodsky's method can be dualized ([63]) to obtain realizations of subgroups of $\pi_* X$ corresponding to splittings of the mod p characteristic polynomial of $T_*: \pi_* X \longrightarrow \pi_* X$ where $T:X \longrightarrow X$ is a self map. In this way Zabrodsky realizes some polynomial algebras which do not appear in the Clark-Ewing list (from the Adams-Wilkerson theorem we know that p has to divide the dimensions of the generators). For example:

1) $X = BSU(n)$, $T = \Psi_\lambda$, *the Adams-Sullivan map, where* λ *represents an element of order* m *dividing* $p-1$ *in* Z_p^*. *We obtain a space with cohomology* $Z_p[x_1,\dots,x_r]$, $\dim x_i = 2mi$, $r=[n/m]$.

2) $X = BE_8$, $T = \Psi_2$, $p=5$. *We obtain a space whose cohomology is a polynomial algebra over* Z_5 *of type* $(16,24,40,48)$.

3) $X = (BF_4)_{1/2}$, $\Psi: X \longrightarrow X$ *the map constructed by Friedlander in [21],* $p=3$. *Then* $H^*(\bar{X};Z_3) \cong Z_3[x,y]$, $\dim x = 12$, $\dim y = 16$.

7.4 The algebra $Z_p[x]\otimes E(y)$

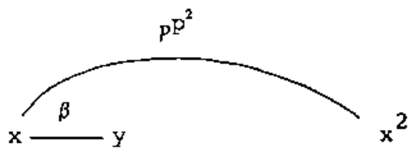
The methods described in these notes in order to obtain necessary conditions for realizability do not give very much information in the case in which the spaces involved have torsion. Actually, very few is known about realizability of algebras which are non-zero in odd degrees. As an example we can consider the algebra $Z_p[x]\otimes E(y)$ that has been studied in [8], [9]. Let us assume p odd (see [9] for the case p=2). Just by using the Steenrod algebra it is possible to prove that $\dim x = 2n$ with $n=p^i r$ where r divides p-1. On the other hand, if n divides p-1, we can realize $Z_p[x]$ and so we can also realize $Z_p[x]\otimes E(y)$. We see that there is a big gap between this sufficient condition for realizability and the necessary condition above. If we assume that the Bockstein homomorphism operates trivially on $Z_p[x]\otimes E(y)$, the classical methods of secondary operations and K-theory operations work well on this algebra and we can prove

Theorem *([9]) If $H^*(X;Z_p)\cong Z_p[x]\otimes E(y)$, $\dim x = 2n$ and β operates trivially, then n divides p-1.*

Hence, in this case the realizability problem can be completely solved. If $\beta y \neq 0$, then it is an immediate consequence of the Adem relations that n must divide p-1. Conversely, if n divides p-1, we can realize $Z_p[x]\otimes E(y)$ with $\beta y\neq 0$ by considering the space $X = K(G,1)$, where G is the semidirect product $G = Z_p \ltimes Z_n$ (see [9] for the details). The most interesting case is the case in which $\beta x\neq 0$, i.e. the case $Z_p[x]\otimes E(\beta x)$. In chapter 5 we have seen that if n divides p(p-1), this algebra is realizable. It seems very likely that this condition is necessary as well

Conjecture *([18], [9]) If $H^*(X;Z_p)\cong Z_p[x]\otimes E(\beta x)$, $\dim x = 2n$, then n divides p(p-1).*

It is possible to show that the conjecture is true in some cases. Assume, for example, $n = p^2(p-1)$. Then, the first block in $H^*(X;Z_p)$ looks like



Hence, there is a nontrivial secondary operation connecting y and x^2. Since we can take suspensions until we are in the stable range, we get a nontrivial element of ${}_p\pi^s_{2p^2(p-1)-2}$. But this group is known to be zero ([56]) and so the space X cannot exist.

This example shows that the knowledge of the stable homotopy groups gives results about realizability of cohomology algebras. For more examples, see [9], [19], [55].

7.5 Polynomial algebras on two generators

The realizability problem for the algebra $Z_p[x,y]$, p odd, has been studied in several papers: [37], [20], [59], but the results obtained are partial. Since the publication of [5] (see chapter 6), it is possible to give a complete characterisation of those polynomial algebras on two generators which arise as cohomology algebras.

Theorem *([Aguadé, unpublished]) If $p>2$, a polynomial algebra on two generators is realizable over Z_p if and only if its type belongs to the following list:*

a) $(2m,4m)$, $p\equiv 1(m)$;

b) $(4,2m)$, $p\equiv \pm 1(m)$ *or* $p=3$, $m=6$;

c) $(2m,2n)$, $p\equiv 1(n)$, $p\equiv 1(m)$;

d) $(8,12)$, $p\equiv 1(3)$;

e) $(12,24)$, $p\equiv 1(3)$;

f) $(8,24)$, $p\equiv 1(12)$;

g) $(24,24)$, $p\equiv 1(12)$;

h) $(16,24)$, $p\equiv 1(4)$;

i) $(16,48)$, $p\equiv 1(8)$;

j) $(24,48)$, $p\equiv 1(12)$;

k) $(48,48)$, $p\equiv 1(24)$;

l) $(12,16)$, $p\equiv 1,3(8)$;

m) $(16,24)$, $p\equiv 1(8)$;

n) $(24,48)$, $p\equiv 1(24)$;

o) $(40,60)$, $p\equiv 1(5)$;

p) $(60,120)$, $p\equiv 1(60)$;

q) $(24,60)$, $p\equiv 1,4(15)$;

r) $(24,120)$, $p\equiv 1,49(60)$;

s) $(24,40)$, $p\equiv 1,9(20)$.

The proof uses [5], [37], [59], [20] and [63], this last only to show that the type (12,16), p=3, is realizable (see section 7.3).

8. Problems

8.1 Determine the polynomial algebras over Z_p which are realizable. We have seen in chapter 6 that if p does not divide the dimensions of the generators, the problem is solved.

In particular, it would be interesting to have more examples of realizable polynomial algebras in which p divides the degree of some generator, like in the examples constructed by Zabrodsky (see 7.3). Is it possible to find restrictive conditions to be verefied by these algebras? Is it true that if there is a generator in dimension 2pr and the algebra is realizable then there are at least p-1 generators?

8.2 Prove the conjecture of p. 30 about realizability of polynomial algebras over the integers. To do it, study the realizability of polynomial algebras over Z_2 and Z_3.

8.3 Generalize the theorem of Adams-Wilkerson ([5]). For example, is there a counterexample to the theorem if we supress the condition that the algebra has to be integrally closed? Is there a good method to realize non integrally closed algebras? By taking push-outs?

8.4 Study the realizability of algebras with non-trivial Bockstein. Prove or disprove the conjecture of page 36 about $Z_p[x] \otimes E(y)$. Is it possible to develope methods like in [5] for algebras with odd-dimensional terms? What about using $S(3)$ instead of CP^∞?

8.5 Let us consider algebras A over Z_p such that $x^p=0$ for all $x \in A$. Is there any algebra of this form which is not realizable? Notice that, since all p-powers vanish, the Steenrod operations, secondary operations, etc. may act trivially on A. It could be conjectured that any algebra with $x^p=0$ for all x is realizable. Prove or disprove it. Notice that the 'free' algebras in this category, i.e. $Z_p[x]/x^p$ are realizable. Is there a proof of the conjecture which is similar to the method used in the rational case (cf. section 4)?

8.6 In 7.2 we have seen that in many cases the integral cohomology algebra has to have a lot of divisibility. Is there a converse to this result? If an algebra A over Z has a big divisibility, does it imply that it is realizable? For example, given any Z-algebra A, we define $\bar{A}$ as equal to A additively, but in $\bar{A}$ we put the new product

$$x \cdot y = \binom{\dim x + \dim y}{\dim x} xy,$$

i.e. we put artificially a big divisibility in the algebra A. It has been conjectured that the new algebra $\bar{A}$ is always realizable. Prove or disprove it.

8.7 Study the relation between the odd-dimensional part and the even -dimensional part of the cohomology algebra. Let A be an algebra which is zero in odd degrees. We say that A is *semi-realizable* if there is a space X such that $H^{even}(X;Z_p)$ A^{even}. We can say that A is *strongly semi-realizable* if the space X can be chosen without p-torsion. Study the relation between semi-realizability, strong semi-realizability and realizability. For example, $Z_p[x]$, $\dim x = 2p$ is semi-realizable, but not realizable. On the other hand, the Adams-Wilkerson theorem proves that if A is a polynomial algebra on generators in dimensions prime to p, then realizability is equivalent to semi-realizability. Is there a strongly semi-realizable algebra which is not realizable?

References

1. Adams, J.F. *H-spaces with few cells* Topology 1 (1961), 67-72.
2. Adams, J.F. *On the non-existence of elements of Hopf invariant one* Ann. Math. 72 (1960), 20-104.
3. Adams, j.F. *Stable homotopy and generalized homology* Chicago, 1971.
4. Adams, J.F., Atiyah, M. *K-theory and the Hopf invariant* Quart. J. Math. 17 (1966), 31-38.
5. Adams, J.F., Wilkerson, C. *Finite H-spaces and algebras over the Steenrod algebra* Ann. Math. 111 (1980), 95-143.
6. Aguadé, J. *A note on realizing polynomial algebras* Israel J. Math. 38 (1981), 95-99.
7. Aguadé, J. *Com és d'algebraica la Topologia Algebraica?* Pub. Math. UAB 23 (1981), 81-98.
8. Aguadé, J. *The realizability of certain algebras as cohomology rings* Rev. Univ. Santander 2 (1979), 463-464.
9. Aguadé, J. *Cohomology algebras with two generators* Math. Z. 177 (1981), 289-296.
10. Aguadé, J. *Fiberings of spheres by spheres mod p* Quart. J. Math. 31 (1980), 129-137.
11. Aguadé, J., Zabrodsky, A. *Algebraic and geometric models for H_o-spaces* (to appear).
12. Atiyah, M. *K-theory* Benjamin 1967.
13. Borel, A. *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts* Ann. Math. 57 (1953),115-207.
14. Bousfield, A.K., Kan, D.M. *Homotopy limits, completions and localisations* Lecture Notes in Math. 304 (1972).
15. Brown, E.H. *Twisted tensor products I* Ann. Math. 69 (1963),193-196.
16. Clark, A. *On π_3 of finite dimensional H-spaces* Ann. Math. 78 (1963), 193-196.
17. Clark, A., Ewing, J. *The realisation of polynomial algebras as cohomology rings* Pacific J. Math. 50 (1974), 425-434.

18. Cooke, G.E. *Constructing spaces with interesting Z/p-cohomology via ε-actions on loop spaces* Amer. J. Math. 101 (1979), 515-542.
19. Cooke, G.E., Smith, L. *On realizing modules over the Steenrod algebra* J. Pure Applied Alg. 13 (1978), 71-100.
20. Ewing, J. *Sphere bundles over spheres as loop spaces mod p* Duke Math. J. 40 (1973), 157-162
21. Friedlander, E.M. *Exceptional isogenies and the classifying spaces of simple Lie groups* Ann. Math. 101 (1975), 510-520
22. Goncalves, D.C. *Mod 2 homotopy associative H-spaces* in 'Geometric applications of homotopy theory', Lecture Notes in Math. 657 (1978), 198-216.
23. Hilton, P., Mislin, G. Roitberg, J. *Localization of nilpotent groups and spaces* North-Holland 1975.
24. Hoffman, P. *Cohomology realisations of* $Q[x]$ Quart. J. Math. 24 (1973), 251-253.
25. Hoffman, P. *On the realizability of Steenrod modules* Quart. J. Math. 20 (1969), 403-407.
26. Hoffman, P., Zabrodsky, A. *Thin lens spaces* Canad. Math. Bull. 21 (1978), 31-35.
27. Hubbuck, J.R. *Two lemmas on primary cohomology operations* Proc. Camb. Phil. Soc. 68 (1970), 631-636.
28. Hubbuck, J.R. *Generalized cohomology operations and H-spaces of low rank* Trans. Amer. Math. Soc. 141 (1969), 335-360.
29. Hubbuck, J.R. *Finitely generated cohomology Hopf algebras* Topology 9 (1970), 205-210.
30. Jozefiak, J.T. *Tate resolutions for commutative graded algebras over a local ring* Fund. Math. 74 (1972), 209-231.
31. Kahn, D.W. *The existence and applications of anticommutative cochain algebras* Illinois J. Math. 7 (1963), 376-395.
32. Kane, R. *Brown-Peterson operations and Steenrod modules* Quart. J. Math. 30 (1979), 455-467.
33. Liulevicius, A. *The factorisation of cyclic reduced powers by secondary cohomology operations* Memoirs Amer. Math. Soc. 42 (1962).
34. Mahammed, N., Piccinini, R. Suter, U. *Some applications of topological K-theory* North-Holland 1980.

35. Mahowald, M., Milgram, R.J. *Operations which detect Sq^4 in connective K-theory and their applications* Quart. J. Math. 27 (1976), 415-432.

36. Milnor, J. *The Steenrod algebra and its dual* Ann. Math. 67 (1958), 150-171

37. Nakagawa, R., Ochiai, S. *On the dimensions of the generators of a polynomial algebra over the mod p Steenrod algebra* Proc. Japan Acad. 43 (1967), 932-936.

38. Papastavridis, S. *Polynomial algebras which are modules over the mod p Steenrod algebra* Bull. Soc. Math. Grèce 17 (1976), 24-27.

39. Quillen, D. *Rational homotopy theory* Ann. Math. 90 (1969), 209-295.

40. Shephard, G.C., Todd, J.A. *Finite unitary reflection groups* Canad. J. Math. 6 (1954), 274-304.

41. Shimada, N. *Triviality of the mod p Hopf invariants* Proc. Japan Acad. 36 (1960), 68-69.

42. Sigrist, F., Suter, U. *Eine Anwendung der K-Theorie in der Theorie der H-Räume* Comm. Math. Helv. 47 (1972), 36-52.

43. Smith. L. *On the characteristic zero cohomology of the free loop space* (preprint).

44. Stasheff, J.D. *Rational homotopy-obstruction and perturbation theory* Proceedings, Vancouver 1977, Lecture Notes in Math. 673 (1978).

45. Stasheff, J.D., Halperin, S. *Obstructions to homotopy equivalence* Advances in Math. 32 (1979), 233-279.

46. Steenrod, N.E. *The cohomology algebra of a space* L'Enseign. Math. 7 (1961), 153-178.

47. Steenrod, N.E. *Polynomial algebras over the algebra of cohomology operations* Proceedings, Neuchâtel 1970, Lecture Notes in Math. 196 (1971).

48. Steenrod, N.E., Epstein, D.B.A. *Cohomology operations* Annals of Math. Studies, Princeton 1962.

49. Sugawara, N., Toda, H. *Squaring operations on truncated polynomial algebras* Japanese J. Math. 38 (1969), 39-50.

50. Sullivan, D. *Infinitesimal computations in topology* I.H.E.S. 47.

51. Sullivan, D. *Geometric Topology, part I: Localisation, periodicity and Galois symmetry* M.I.T. 1970.

52. Sullivan, D. *Genetics of homotopy theory and the Adams conjecture* Ann. Math. 100 (1974), 1-79.
53. Thom, R. *Operations en cohomologie réelle* Séminaire Cartan, 1954-55.
54. Thomas, E. *Steenrod squares and H-spaces II* Ann. Math. 81 (1969), 473-495.
55. Toda, H. *Note on cohomology rings of certain spaces* Proc. Amer. Math. Soc. 14 (1963), 89-95.
56. Toda, H. *An important relation in homotopy groups of spheres* Proc. Japan Acad. 43 (1967), 839-842.
57. Wang, J.S.P. *On the cohomology of the mod 2 Steenrod algebra and the non-existence of elements of Hopf invariant one* Ill. J. Math. 11 (1967), 480-490.
58. Wilkerson, C. *Spheres which are loop spaces mod p* Proc. Amer. Math. Soc. 39 (1973), 616-618.
59. Wilkerson, C. *K-theory operations in mod p loop spaces* Math. Z. 132 (1973), 29-44.
60. Wilkerson, C. *Classifying spaces, Steenrod operations and algebraic closure* Topology 16 (1977), 227-237.
61. Yamanoshita, T. *On the mod p Hopf invariant* Proc. Japan Acad. 36 (1960), 97-98.
62. Zabrodsky, A. *Endomorphisms in the homotopy category* (preprint).
63. Zabrodsky, A. *On the realization of invariant subgroups of* $\pi_* X$ (preprint).