Abstract:

Coding theory began as an engineering problem suggested by the Shannon, Golay and Hamming works and has developed by using more and more sophisticated mathematical techniques. This subject overlaps with so many other: group theory, number theory, switching functions, combinatorial geometries, association schemes, etc.

We present in this paper a summary of some combinatorial properties of the family of regular codes, recently developed in (4), based on the coefficients of the dual weight enumerator of its cosets and the Krawtchouk polynomials. From the Goethals-van Tilborg's characterization (3) and from the Delsarte's work (1) we prove that for every $s$-weight linear code $C(n,k)$, whose orthogonal code $C^\perp$ is regular, we can define a $s$-classes association scheme such that the rows of its eigenmatrix $P$ are the coefficients of the dual weight enumerators of the orthogonal code and its cosets.

1. ASSOCIATION SCHEME AND DESIGN STRUCTURE

An association scheme with $n$-classes on a set $X$ consists of a partition of the set $X \times X$ into $n+1$ classes $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ satisfying the following conditions:

1: $\Gamma_0 = \{(x,x) \in X \times X\}$

2: Given $x \in X$, the number $v_i = \# \{y \in X : (x,y) \in \Gamma_i\}$ depends only on $y$. 
3: Given \((x,y)\in \Gamma_k\), the number 
\[ p_{ij}^k = \# \{ z \in X : (x,z) \in \Gamma_i \text{ and } (y,z) \in \Gamma_j \} \]
depends only on \(i,j\) and \(k\).

(\# denote the cardinal of the set)

If we denote by \(D_i\) the adjacency matrix of the graph \((X,\Gamma_i)\)
(the second condition asserts that each graph \((X,\Gamma_i)\) is regular)
which entries \(D_i(x,y)\) are equal to 1 if \((x,y)\in \Gamma_i\) and 0 otherwise;
we observe that the \((x,y)\)-entry in the matrix product \(D_i \cdot D_j\) is \(p_{ij}^k\)
if \((x,y)\in \Gamma_k\). Moreover the diagonal entries are equal to zero unless \(i=j\), in which case they are equal to \(v_i\). Then we can write:

\[ D_i \cdot D_j = D_j \cdot D_i = \sum_{k=0}^{n} p_{ij}^k D_k \]

where \(p_{ij}^0 = \delta_{ij}\) and \(p_{10}^k = p_{k1}^0 = \delta_{ik}\).

This shows that the commuting symmetric matrices \(D_0, D_1, \ldots, D_n\)
span a \((n+1)\)-dimensional real algebra called the Bose-Mesner
algebra of the scheme, which is semisimple (ref (1), Th. 2.1 and
Th. 2.2) and hence admits a basis of mutually orthogonal idempo-
tent matrices \(J_0, J_1, \ldots, J_n\); that is \(J_i \cdot J_k = \delta_{ik} J_i\). Respect to
the basis \(\{J_0, J_1, \ldots, J_n\}\), every \(D_k\) can be written as an expression
of the form:

\[ D_k = \sum_{i=0}^{n} p_k(i) J_i \quad \text{for } k=0,1,\ldots,n \]

where \(p_k(i)\) are the eigenvalues of \(D_k\), because \(D_k \cdot J_i = p_k(i) J_i\).
The square matrix \(P\) of order \(n+1\) whose \((i,k)\)-entry is \(p_k(i)\),
\(0 \leq i,k \leq n\), is called the eigenmatrix of the scheme.

The matrix \(Q\) defined by \(Q := |X| P^{-1}\), whose \((i,k)\)-entry
will be denoted by \(q_k(i)\), is called the dual eigenmatrix of the
scheme. Note that from -2- we obtain

\[ J_k = |X|^{-1} \sum_{i=0}^{n} q_k(i) D_i \quad \text{for } k=0,1,\ldots,n \]

For every association scheme defined on a set \(X\) which has
\(P\) and \(Q\) as eigenmatrix and dual eigenmatrix respectively, holds:

\[ P^t = \Delta_v \cdot Q \cdot \Delta_u^{-1} \]

where \(\Delta_v\) and \(\Delta_u\) are diagonal matrices having the same order that
\(P\) and \(Q\) and which diagonal entries are the valences \(v_i\) and multi-
plicities \(u_i\); that is, \(v_i\) is defined as 2; and \(u_i\) is the dimension
of the subspace \(V_i\) which has associated the eigenvalues \(p_k(0)\),
Moreover:

\[
p_k(0) = \nu_k \quad \text{and} \quad q_k(0) = \psi_k
\]

being \( p_k(0) \) the eigenvalue of \( D_k \) which is associated to the eigenvector \((1,1,\ldots,1)\).

We define now a combinatorial structure, called design, over a subset \( Y \) of a point set \( X \) of an association scheme with \( n \)-classes.

The **inner distribution** of \( Y \) is the \((n+1)\)-tuple \( a = (a_0, \ldots, a_n) \) which coordinate \( a_i \) is:

\[
a_i := |Y|^{-1} \sum_{x \in Y} \sum_{y \in Y} D_i(x,y)
\]

and the **dual distribution** of \( Y \) is the \((n+1)\)-tuple \( b = (b_0, b_1, \ldots, b_n) \) that consists of the inner distribution of \( Y^\perp := \{x \in X : (x|y) = 0 \ \forall y \in Y\} \) where \((.|.)\) is the scalar product.

A subset \( Y \) of a point set \( X \) of an association scheme with \( n \)-classes such that their inner and dual distribution satisfy:

\[
s = \# \{ a_k \neq 0 ; k \neq 0 \}
\]

and

\[
b_1 = b_2 = \ldots = b_\tau = 0 \quad \text{and} \quad b_{\tau+1} \neq 0
\]

is called a **design of degree** \( s \) and **maximum strength** \( \tau \).

**Theorem 1** (Delsarte)

Let \( Y \subseteq X \) be a design of degree \( s \) and maximum strength \( \tau \) which satisfies \( \tau = 2s-2 \) or \( 2s-1 \) or \( 2s \). Then we can define an association scheme with \( s \)-classes \( \Gamma_0, \Gamma_1, \ldots, \Gamma_s \)

\[
\Gamma_i^Y = Y \cap Y^\perp_i
\]

where \( \Gamma_j \) is the \( j \)-th class of the association scheme on \( X \) which associated coordinate \( a_i \) of the inner distribution of \( Y \) is non-zero. Moreover, the dual eigenmatrix \( Q = (q_k(i)) \) of the association scheme \((Y, \Gamma^Y)\) is given by the formula:

\[
q_k(i_j) = \begin{cases} 
  p_k(i_j) & \text{if } k = 0, 1, \ldots, s-1 \\
  \alpha(i_j) - \psi_{s-1}(i_j) & \text{if } k = s
\end{cases}
\]
where \( a(z) \) is the annihilator polynomial of \( Y \), that is:
\[
a(z) = \prod_{j=1}^{n} \left( 1 - \frac{z}{i_j} \right),
\]
being \( i_j \) the nonzero coordinates in the inner distribution of \( Y \); and \( \psi_{s-1}(z) = \sum_{k=0}^{s-1} P_k(z) \), being \( P_k(z) \) the Krawtchouk polynomial (see \(-18\)) of degree \( k \) in the variable \( z \).

**Remark:** this theorem corresponds to theorem 5.25 and corollary 5.26 of Delsarte’s work (1), and it is a crucial result for our theorem 5.

In the particular case where \( X \) is an additive finite abelian group, useful in coding theory, the \((n+1)\)-classes \( \Gamma_0, \Gamma_1, \ldots, \Gamma_n \) are invariant under translations; that is:
\[
(x, y) \in \Gamma_i \implies (x+z, y+z) \in \Gamma_i \quad \forall z \in X
\]
Thus we have a partition of \( X \) into \((n+1)\) classes \( X_0, X_1, \ldots, X_n \) defined by
\[
(x, y) \in \Gamma_i \iff (y-x) \in X_i \quad \text{for } i=0,1,\ldots,n
\]

In the other way, let \( S \) denote the square matrix of order \(|X|\) with \( \chi_y(x) \) its entries indexed by the elements \( x, y \in X \) (\( \chi_y \) is the associated character to \( y \) on \( X \)). The orthogonality relations satisfied by them can be expressed by the matrix equation
\[
\mathbf{S}^* S = S \mathbf{S} = |X| \mathbf{I}
\]
where \( \mathbf{S}^* \) is the conjugate transpose of \( S \). They follow from
\[
\chi_y(x) = \chi_y(-x)
\]
and from the relations
\[
\sum_{x \in X} \chi_y(x) = \begin{cases} |X| & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}
\]
the columns of \( S \) are the eigenvectors of all the matrices in the Bose-Mesner algebra of the association scheme (ref. (1), (2)), hence we can write a new partition of \( X \) into \((n+1)\) classes \( X'_0, X'_1, \ldots, X'_n \) where \( X'_i \) is the set of indices \( z \in X \) for which the corresponding column of \( S \) is in the \( i \)-th eigenspace \( V_i \). In this way, we have:
\[
X'_0 = \{0\} \quad \text{and} \quad |X'_i| = \nu_i
\]
From these partition of \( X \) we can define a new partition \( \Gamma'_0, \Gamma'_1, \ldots, \Gamma'_n \) on \( X \times X \) as follows:
\[
(x, y) \in \Gamma'_i \iff (y-x) \in X'_i
\]
resulting that the \((n+1)\) classes \( \Gamma'_i \) form an association scheme on
X, called the dual scheme from the constructed one by the classes \( \Gamma_i \) (rf(2)).

The respective eigenmatrices and parameters satisfy (rf. (1) th 2.8):

\[ P = Q^T \quad \text{and} \quad Q = P^T \]

\[ v_k = u_k^T \quad \text{and} \quad \mu_k = \nu_k^T \]

Let \( X = (F_q^n, +) \) be the set of all the \( n \)-tuples \( x = (x_1, \ldots, x_n) \) from a Galois field \( F_q \) of order \( q = p^r \) for a prime number \( p \). We make \( X \) a metric space by defining the Hamming distance \( d_H(x, y) \) between two \( n \)-tuples to be the number of coordinates in which they are different.

For \( i = 0, 1, \ldots, n \) we define \( \Gamma_i \) by the set of all pairs of \( n \)-tuples at distance \( i \), that is:

\[ \Gamma_i = \{(x, y) \in X \times X : d_H(x, y) = i\} \]

which constitutes an association scheme, called the Hamming scheme. This one is a self-dual association scheme (\( P = Q \)) and its eigenmatrix elements are given by:

\[ p_k(i) = q_k(i) = P_k(i) \]

where \( P_k(i) \) denotes the Krawtchouk polynomial of degree \( k \) in the variable \( i \), which is defined in (5) by

\[ P_k(i) = i^n \sum_{j=0}^k (-1)^j (q-1)^{k-j} i^j (n-k)^{k-j} \]

for two prefix numbers: \( n \) and \( q \).

From the expressions -5-, -17- and -18- we can write the parameters of a Hamming scheme by

\[ \nu_k = v_k = \binom{n}{k} (q-1)^k \]

Remark: For the design structure defined by 7- over a subset \( Y \) of the point set \( X \) of a Hamming scheme; that is, \( Y \) can be considered as a linear code, the inner and dual distribution coincide with the enumerator weight coefficients \( (A_0, A_1, \ldots, A_n) \) of \( Y \) and the enumerator weight coefficients \( (B_0, B_1, \ldots, B_n) \) of its dual code \( Y^\perp \) respectively.
2.- REGULAR CODES

A linear code \( C(n,k) \) over \( F_q \) is a \( k \)-dimensional subspace of the \( n \)-dimensional vectorial space that consists on all the \( n \)-tuples with the elements of \( F_q \): \( V=\{u=(u_1,u_2,\ldots,u_n): u_i \in F_q \} \). The orthogonal code \( C^\perp(n,n-k) \) is the \( (n-k) \)-dimensional subspace of \( V \) consisting on all the \( n \)-tuples \( v \in V \) which inner product with every codeword of \( C \) is zero.

The weight enumerator of a code \( C \) is the polynomial in the variable \( z \):

\[
A_C(z) = \sum_{w(u)} z^w(u) = \prod_{i=0}^{n} A_i \cdot z^i
\]

where \( w(u) \) denotes the weight of the codeword \( u \), that is the number of nonzero coordinates \( u_i \), and \( A_i \) is the number of codewords of weight \( i \).

Let \( B_C(z)=A_{C^\perp}(z) = \sum_{j=0}^{n} B_j \cdot z^j \) be the weight enumerator of \( C^\perp \). Between the weight enumerator of a linear code and the one of its orthogonal code, there exist the following relation:

\[
B_C(z) = |C|^{-1} (1+(q-1)z)^n A_C(\frac{1-z}{1+(q-1)z})^{-21}
\]

called the MacWilliams identity; or equivalently:

\[
A_C(z) = |C^\perp|^{-1} \sum_{j=0}^{n} B_j \cdot (1-z)^j (1+(q-1)z)^{n-j}^{-22}
\]

where the right hand is called the dual form of the weight enumerator of \( C \), (ref. (5)).

In order to generalize expression -22- for the weight enumerator of any coset \( C_j := C+u_j \) of \( C \), we write:

\[
A_{C_j}(z) = \prod_{i=0}^{n} A_i(C_j) \cdot z^i
\]

where \( A_i(C_j) \) denotes the number of vectors of weight \( i \) into the coset \( C_j \), for \( j=0,1,\ldots,q^{n-k}-1 \).

Remark: \( V=C_0 \cup C_1 \cup \ldots \cup C_{q^{n-k}-1} \) is the partition given by the equivalence relation \( R_C=\{(u,v) \in V \times V: u-v \in C\} \), and the leader \( u_j \) is a minimum weight vector into \( C_j \). Obviously, \( C_0=C \) itself.

Taking the complex algebra of all the polynomials in the variables \( X_{q^i,j} \) for \( q_i \in F_q \) and \( 1 \leq j \leq n \), denoted by \( A \), we can
define the two following applications:

\[ f: V \longrightarrow A \]
\[ a=(a_1, \ldots, a_n) \longmapsto f(a)= \prod_{i=1}^{n} x_{a_i}^{i} \]

and

\[ g: V \longrightarrow A \]
\[ a=(a_1, \ldots, a_n) \longmapsto g(a)= \sum_{b \in V} x_{a}^{b}.f(b) \]

where \( x_{a} \) denotes a character defined on the additive group \((V,+).\)

From definitions of \( f \) and \( g \), and from the characters properties, we can write (ref. (4)):

**Lemma 2**

For any coset \( C_j \) of a linear code \( C(n,k) \) the following identity holds:

\[ \sum_{u \in C_j} g(u) = |C_j| \sum_{b \in C^\perp} x_{a}^{b}.f(b) \]

Identifying \( x_{a}, i=1 \) and \( x_{a_i}, i \neq 0 \), into -24- and -25-; the duality of the MacWilliams identity -21- and our lemma 2 proves (ref(4)):

**Theorem 3**

The weight enumerator of any coset \( C_j:=C+u_j \) of a linear code \( C(n,k) \) can be written under the dual form:

\[ A_{C_j}(z)=|C^\perp|^{-1} \sum_{h=0}^{n} B_{h}(C_j)(1-z)^{h}(1+(q-1)z)^{n-h} \]

where the coefficients \( B_{h}(C_j) \) are defined as:

\[ B_{h}(C_j)= \sum_{v \in C^\perp, \overline{w(v)}=h} x_{u_j}^{v} \]

**Remarks:** When the coset \( C_j \) is the code itself, this theorem is equivalent to MacWilliams identity -22- because we have, from the properties of characters: \( B_{h}(C)=B_{h} \).

The coefficients \( B_{h}(C_j) \) are easily obtained if \( C \) is a regular code, through the Krawtchouk polynomials.

Let "s" be the number of distinct nonzero weights in the orthogonal code \( C^\perp \) of a given code \( C \); s is usually called external distance of \( C \). A linear code \( C \) is named \( r \)-regular.
0 ≤ s ≤ r, if and only if the weight enumerators of their cosets $C_j$, which have minimum weight $i ≤ r$, depends only on $i$. When $r = s$, $C$ is called completely regular.

Goethals-van Tilborg give the characterization of the regular codes as a function of the minimum weight $d$ and the external distance $s$ of the code (ref. (3) th. 7) as follows:

1) if $s < d ≤ 2s - 1$, then $C$ is $(d - s)$-regular
2) if $d > 2s - 1$, then $C$ is completely regular

**Theorem 4**

For every coset $C_j := C + u_j$ having minimum weight $i$, $i ≤ r ≤ t$, of a $r$-regular $t$-error correcting code ($d ≥ 2t + 1$), we have:

$$B_h(C_j) = \left(\binom{n}{i}(q-1)i\right)^{-1} P_i(h) B_h$$

where $P_i(h)$ is the Krawtchouk polynomial of degree $i$ in the variable $h$ (ref. -18-) and $B_h$ is the number of the $n$-tuples of weight $h$ into $C^1$.

**Proof:**

By the duality of -27- we can write:

$$\sum_{h=0}^{n} B_h z^h = |C|^{-1} \sum_{m=0}^{n} A_m(C_j) (1-z)^m (1+(q-1)z)^{n-m}$$

and since $A_m(C_j)$ depends only on $i$, we have for $i ≤ r$:

$$\sum_{C_j}^{i} \left( \sum_{m=0}^{n} A_m(C_j) (1-z)^m (1+(q-1)z)^{n-m} \right)$$

$$= \left(\binom{n}{i}(q-1)i \sum_{m=0}^{n} A_m(C_j) (1-z)^m (1+(q-1)z)^{n-m} \right)$$

$$= \left(\binom{n}{i}(q-1)i \sum_{h=0}^{n} B_h(C_j) z^h \right) |C|$$

By the character property:

$$\sum_{u \in V} \chi_V(u) = P_i(w(v))$$

where $P_i(w(v))$ is the Krawtchouk polynomial of degree $i$ in the variable $w(v)$; we can write, in the other way:
\[
\sum_{h=0}^{\Sigma^n} B_h(C_j).z^h = \sum_{h=0}^{\Sigma^n} \left( \sum_{v \in \mathcal{C}^1} u_j. \chi_v(u).w(v) \right).z^h
\]

From -31-, the equality between -32- and -33- holds and so -30- is proved. ##

Remark: The particular case when \( s \leq t+1 \) is very interesting since the coefficients \( B_h(C_j) \) can be found for every coset of a regular code. (If \( s > t \), following the above theorem, and if \( i = t+1 \) following the next corollary).

Corollary: (ref. (4))

a) With the same assumptions of the theorem 3 we have:

\[
\sum_{j=0}^{M-1} B_h(C_j) = \begin{cases} 
0 & \text{if } h \neq 0 \\
M=q^n-k & \text{if } h = 0
\end{cases}
\]

b) With the same assumptions of the theorem 4 we have:

\[
\sum_{h=0}^{\Sigma^n} B_h(C_j) = \begin{cases} 
0 & \text{if } 0 \leq i < r \\
|C^\perp|=q^n-k & \text{if } i = 0
\end{cases}
\]

3. REGULAR CODES AND ASSOCIATION SCHEMES

A linear code is said to be \textit{projective} if its generator matrix has not a linear dependence between any two columns. Let \( C(n,k) \) be a linear projective code with "s" distinct nonzero weights \( w_1, w_2, \ldots, w_s \) and such that the minimum weight in the orthogonal code \( d' \) satisfies: \( d' \geq 2s-1 \); that is \( C^\perp \) is completely regular (ref. 29-), then we can define two dual association scheme on the additive group associated to code \( C \).
Let $X = (C, +)$ be the additive group associated to the projective code $C(n, k)$, that is, the set of all the codewords with the componentwise modulo $q$ addition. We define the partition $\Gamma_0, \Gamma_1, \ldots, \Gamma_s$ by:

$$\Gamma_0 = \{(x, x) \in X \times X\} \quad \text{and} \quad \Gamma_i = \{(x, y) \in X \times X : w(x - y) = w_i\}$$

which constitutes an association scheme with $s$-classes on $X$ where $v_i = |X_i| = A_{w_i}$ for $0 \leq i \leq s$ being $X_i$ a class of the partition induced by $\Gamma_i$ on $X$. (ref. (1) and (2)).

In the other way, we consider $C = V / C^\perp$ as a quotient of additive groups and consequently the decomposition of $V$ into $N = q^k$ cosets of $C^\perp$, that is $C = Y_0 \cup Y_1 \cup \ldots \cup Y_N$ where $Y_0 = C^\perp$ and $Y_j = C^\perp + \mathbf{v}_j$ for $j \neq 0$. Since $C^\perp$ is a completely regular code all its cosets, having the same minimum weight at most $s$, have the same weight enumerator. Moreover, since $d' \geq 2s - 1$, $C^\perp$ is a $(s-1)$-error correcting code and each codeword of $C$ with weight at most $s - 1$ belong to distinct cosets $Y_j$ of $C^\perp$. In this way, we can write a new partition $\Gamma', \Gamma'', \ldots, \Gamma'_s$ over $X$ as follows:

$$\Gamma'_0 = \{(Y_j, Y_j) \in C \times C\} \quad \text{and} \quad \Gamma'_i = \{(Y_j, Y_k) \in C \times C : w_H(Y_j - Y_k) = i\}$$

where $w_H(Y_j - Y_k)$ denotes the minimum weight in the coset $Y_j - Y_k = C^\perp + (\mathbf{v}_j - \mathbf{v}_k)$. This partition defines an association scheme with $s$-classes on $C$, this one considered as the set of all the cosets $Y_i$ of $C^\perp$, and constitutes the dual scheme of the one defined by $s-1$-classes. In this case, $\nu_i = |X_i| = \binom{n}{i}(q-1)^i$ for $0 \leq i \leq s - 1$ and $\mu_s = q^k - \sum_{k=0}^{s-1} \mu_k$

Using the theorem 1, we can show an important combinatorial result, explained in the following:

**Theorem 5:**

For every linear projective code $C(n, k)$ with $s$ distinct nonzero weights such that the minimum weight $d'$ of the orthogonal code satisfies: $2s - 1 \leq d' \leq 2s - 1$, we can define an association scheme with $s$-classes on $C$ such that the rows of its eigenmatrix $P$ coincide with the coefficients of the dual weight enumerators of $C^\perp$ and of its cosets.
proof:

According with definition -7- and remark of page 5; this code C is a design of degree "s" and maximum strength $\tau=d'-1$, which satisfies the assumptions of theorem 1. Then $(C,L^C)$ is an association scheme whose dual eigenmatrix Q is obtained by -9-.

From -29- we claim that $C^L$ is completely regular and moreover that is a $(s-1)$-error correcting code ($d'>2t+1$ being $t=s-1$ the capacity of error correction). In this way, we can apply the theorem 4 for every coset of $C^L$ having minimum weight $\leq t$ and -34- for them which minimum weight is $t+1$.

If we arrange those results in a $(s+1)\times(s+1)$ matrix B whose entries $(i,w_i)$ are the coefficients $B_{w_i}(C+v_j)$, where this coset has minimum weight $i$, $i=0,1,\ldots,s$ and $w_0=0$; we can write:

$$B = C^{-1}.Q^T.A$$

where A and C are diagonal matrices whose entries are, respectively; $a_{ii}=A_{w_i}$ (the coefficients of the weight enumerator of $C=(C^L)^L$) and

$$c_{ii} = \begin{cases} \binom{n}{i}(q-1)^i \text{ for } 0\leq i \leq s-1 \\ q^{n-(n-k)} \sum_{j=0}^{s-1} c_{jj} \end{cases}$$

(the number of cosets of $C$ with minimum weight $i$).

From definitions -36- and -37- we assure that the right hand in -38- coincide with the transpose of the right hand in -4-. Then, $B=P$ and this proof is end. ##

example 1:

For the perfect binari Golay code with a weight enumerator (rf. (5)):

$$AG(z)=1+253z^7+506z^8+1288z^{11}+1288z^{12}+506z^{15}+253z^{16}+z^{23}$$

from which we know it is completely regular and that its orthogonal code has three nonzero weights:

$$AG^L(z)=1+506z^8+1288z^{12}+253z^{16}$$

we can obtain easily all the coefficients $B_h(G_j)$ of the dual weight enumerator of all the cosets $G_j:=G+u_j$ through theorem 4 and from the three-term recurrence of the Krawtchouk polynomial (rf. (4)), and we have the next table where B is the central piece:
Consequently, let $C$ be this three weight linear code, which orthogonal code is the binary Golay code with minimum weight $d'=7$. By theorem 5, we can define an association scheme with 3-classes on $C$ whose eigenmatrix $P$ is:

$$P = \begin{pmatrix}
1 & 506 & 1288 & 253 \\
1 & 154 & -56 & -99 \\
1 & 26 & -56 & 29 \\
1 & -6 & 8 & 3
\end{pmatrix}$$

Remark: We refer to the reader to example 6.1 in (1) where this eigenmatrix is constructed in a different way.

Example 2:

In the same work (1) Delsarte gives an association scheme, with two classes, on the orthogonal code at the ternary Golay code $G_3(11,6)$, where the corresponding eigenmatrix $P$ is given by:

$$P = \begin{pmatrix}
1 & 132 & 110 \\
1 & 24 & -25 \\
1 & -3 & 2
\end{pmatrix}$$

Applying the inverse reasoning to example 1, we can obtain from theorem 5, since the ternary Golay code is perfect (rf.(5)), that is completely regular (rf.(3)); that the two non-zero weights of the ternary Golay code are $w_1=6$ and $w_2=9$. Those results are obtained from -30- and from the Krawtchouk polynomials $P_1(x)=22-3x$ and $P_2(x)=220-\frac{129}{2}x+\frac{9}{2}x^2$ (rf.(5)) (e.g. $B_w = 132$; and $24=\frac{1}{w_1(w_1-1)}.P_1(w_1).132$ in second column of $P$, then $w_1=6$).

In this way, our theorem 3 permits give the dual weight enumerator for the ternary Golay code and for its 243 cosets:
For $w(u_j)=1$; there are $\binom{11}{1}2=22$ cosets with the same minimum weight, we have:

$$A_{G_3}^+u_j(z)=\frac{1}{243}(1+2z)^{11}+132(1-z)^6(1+2z)^5+110(1-z)^9(1+2z)^2$$

And for $w(u_j)=2$; there are $\binom{11}{2}2^2=220$ cosets with the same minimum weight, we have:

$$A_{G_3}^+u_j(z)=\frac{1}{243}(1+2z)^{11}+24(1-z)^6(1+2z)^5-25(1-z)^9(1+2z)^2$$

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