Pub. Mat. UAB
Vol. 26 Nº 2 Juny 1982

RINGS WITH CHAIN CONDITIONS ON PROJECTIVE IDEALS

by

JOHN KOEHL

Department of Mathematics
Louisiana State University

Rebut el 2 d'octubre del 1981

1. Introduction

A ring R is said to satisfy *right a.c.c*[*f.g.*] P if R satisfies the ascending chain condition on (finitely generated) projective right ideals; *right d.c.c.*[*f.g.*]P is defined similarily.

In section two we show that if R satisfies right d.c.c.P and if U is a two-sided ideal which is also a minimal projective right ideal, then either $U^2 = 0$ or $U^2 = U$(2.3). If R is commutative and U is finitely generated, then $U^2 = U$ and is generated by an idempotent element (2.5). We also show that if R satisfies right a.c.c.P or right d.c.c.P then every projective right ideal is countably generated (2.9). The polynomial (also power series) ring in infinitely many variables over a field are examples of non-Noetherian rings satisfying right a.c.c.P.

The symbols EP are used to denote *enough projectives*, every nonzero right ideal contains a nonzero projective right ideal; and MEP denotes the condition that every nonzero right ideal contains a nonzero finitely generated projective right ideal.

2. Structure of Minimal Projective Ideals

If R is a commutative ring satisfyin d.c.c.P then R has minimal projective ideals. The next theorem describes how these ideals are related to the other projective ideals. We use the following known result.

2.1 Lemma. *Let* R *and* S *be rings, let* P *be a projective* R*-module and let* Q *be an* (R,S)*- bimodule that is projective as an* S*-module. Then* $P \otimes_R Q$ *is a projective* S*-module.*

Proof. See, for example, Faith [3, 11.15, p.430].■

2.2 Theorem. *Suppose* Q *is a two-sided ideal in a ring* R *and is a minimal projective right ideal. Then given any projective right ideal* P *of* R, *either* $PQ = 0$ or $Q \subseteq P$.

Proof. Since P and Q are projective right ideals and Q is an (R,R)-bimodule, $P \otimes Q$ is projective. Since P is projective and Q is an ideal of R, $P \otimes Q = PQ$ so PQ is also projective. But $PQ \subseteq P$ and $PQ \subseteq Q$. Since Q is a minimal projective right ideal either $PQ = 0$ or $Q = PQ \subseteq P$.■

2.3 Corollary. *If* P *is a minimal projective right ideal which is also a two-sided ideal, then either* $P^2 = P$ *or* $P^2 = 0$.■

2.3-A Remark. Note that both of the possibilities mentioned in 2.3 actually occur. For example, let $R = \begin{pmatrix} F & 0 \\ F & F \end{pmatrix}$ be the ring of 2×2 lower triangular matrices over a field F. It is well known that R is semihereditary. The ideal $P = \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix}$ satisfies $P^2 = 0$. However, if R is commutative and P is finitely generated, then $P^2 = P$ (2.5).

2.4 Remark. G. Michler [8] has shown that if $P^2 = P$ in a left perfect ring R, then $P = ReR$ where e is an idempotent in R and is central modulo the radical J.

2.5 Lemma. *In a commutative ring* R *a minimum projective ideal* P *is idempotent. If* P *is also finitely generated, then* P *is generated by an idempotent element.*

Proof. Since P is a projective module, the dual basis lemma guarantees that there exists a set of elements $\{p_\alpha\}$, p_α in P, and a set of homomorphisms $\{f_\alpha\}$ with f_α in $\mathrm{Hom}_R(P,R)$ such that $f_\alpha(P_\alpha) = 0$ for almost all α, and $x = \Sigma p_\alpha f_\alpha(x)$ for each x in P. Since $P \subseteq R$, a commutative ring, we see that $x = \Sigma f_\alpha(p_\alpha x)$ for each x in P. Thus $P^2 = 0$ implies $P = 0$. Since $P \neq 0$, we see by 2.3 that $P^2 = P$.

If P is finitely generated, theorem 76, page 50 of Kaplansky [8] says that P contains an idempotent element. Since P is a minimum projective, the idempotent element must generate P.■

2.6 Theorem. *If* R *is a commutative semiprime ring with* EP *that satisfies* d.c.c.P, *then* R *is a direct sum of a finite number of fields.*

Proof. Since R satisfies d.c.c.P, R has minimal projective ideals. Say I_1 is a minimal projective ideal. Then I_1 is simple since R has EP, and hence I_1 is generated by an idempotent e_1. By Jacobson [6, Proposition 1, p.65], I_1 is a field. Let $R_1 = (1 - e_1)R$. Then R_1 has EP and satisfies d.c.c.P. Thus $R_1 = 0$ or contains a minimal projective I_2 where I_2 is a field generated by an idempotent e_2. Then $R_2 = (1-e_1-e_2)R = 0$ or contains a minimal projective I_3 which is generated by an idempotent. Since R has d.c.c.P, R contains no infinite direct sum of projective ideals so the above process must terminate after a finite number of steps. Hence $R = \Sigma \oplus I_n$, a finite sum, where each I_n is a field.■

2.7 Corollary. *If* R *is a commutative Noetherian semiprime ring that satisfies* d.c.c.P, *then* R *is a direct sum of a finite number of fields.*

Proof. Since R is Noetherian, minimal projective ideals are finitely generated and hence generated by an idempotent. Thus minimal projectives are summands of the ring. The remainder of the proof is the same as the proof of 2.6.■

We can also say something about the projective right ideals in a ring satisfying right a.c.c.P. We first state a theorem due to Kaplansky [6].

2.8 Theorem. *Every projective module is a direct sum of countably generated modules.*■

2.9 Lemma. *If* R *satisfies right* a.c.c.P *or right* d.c.c.P *then every projective right ideal is countably generated.*

Proof. By Kaplansky's theorem, each projective (right ideal) is a direct sum of countably generated projectives. If R satisfies right a.c.c.P or right d.c.c.P the number of independent summands of a projective right ideal is finite. Thus each projective right ideal is a sum of a finite number of countably generated modules and hence is countably generated.■

3. Inheritance Properties of Chain Conditions on Projectives

3.1 Theorem. *Let* R *be any ring. If* M_R *satisfies the descending chain condition on (finitely generated) projective submodules then each homomorphic image of* M *also satisfies this condition.*

Proof. Let $f: M \longrightarrow N$ be an R-epimorphism of right R-modules and let $P_1 \supseteq P_2 \supseteq P_3 \supseteq \dots$ be a sequence of projective submodules of N. Then $f^{-1}(P_1) \longrightarrow P_1 \longrightarrow 0$ splits so $P_1 \hookrightarrow f^{-1}(P_1) \subseteq M$. Thus P_1 satisfies the descending chain condition on (finitely generated) projective submodules. Hence there exists an n such that $P_n = P_{n+k}$ for $k = 1,2,3,\ldots$. It follows that N satisfies d.c.c. on (finitely generated) projective submodules.■

We use the following theorems of H. Bass [1] and J.E. Björk [2] several times in the proof of the following result and in the next section.

3.2 Theorem (Bass). *Let* R *be a ring,* J *its radical. Then the following are equivalent:*

1) R *is left perfect; i.e., every left* R*-module has a projective cover.**

2) J *is left* T*-nilpotent and* R/J *is semisimple.*

3) *A direct limit of projective left* R*-modules is projective.*

4) R *satisfies the descending chain condition on principal right ideals.*

5) R *has no infinite sets of orthogonal idempotents, and every nonzero right* R*-module has nonzero socle.*■

3.3 Theorem (Björk). *A ring* R *is left perfect if and only if* R *satisfies the descending chain condition on finitely generated right ideals.*■

* Projective cover is the dual of injective hull.

3.4 Theorem. *If* R *is a right perfect ring and* M_R *satisfies* a.c.c.(f.g)P *then each homomorphic image of* M *also satisfies this condition.*

Proof. Let $f\colon M \longrightarrow N$ be an R-epimorphism and suppose that $P_1 \subseteq P_2 \subseteq P_3 \subseteq \dots$ is an ascending chain of projective submodules of N. Let $P = \cup P_i$. Then P is flat. By Bass [1], P is projective. Thus $f^{-1}(P) \longrightarrow P \longrightarrow 0$ splits and P embeds in $f^{-1}(P) \subseteq M$. Hence P satisfies a.c.c.(f.g.)P and the above sequence of projective modules must terminate after a finite number of steps. It follows that N satisfies a.c.c.(f.g.)P.■

4. Study of Some Particular Classes of Rings under the Assumption of Chain Conditions on Projective Ideals.

We can characterize semiprime right semihereditary rings which satisfy right d.c.c.f.g.P. More generally, a ring in which principal right ideals are projective is a *right* P.P. *ring*.

4.1 Lemma. *Let* R *be a semiprime right* P.P. *ring. Then* R *satisfies right* d.c.c.f.g.P *if and only if* R *is semisimple.*

Proof. Since R is right P.P., each principal right ideal is projective. Thus right d.c.c.f.g. P implies the descending chain condition on principal right ideals. By Bass' theorem, R/J, J = radical R, is semisimple and J is left

T-nilpotent. If $J \neq 0$, there exists a minimal right ideal $I \subseteq J$. Now either $I^2 = 0$ or there is an e in I, $e \neq 0$, and $e^2 = e$. Since J is left vanishing, J contains no nonzero idempotents. Thus $J = 0$. Hence R is semisimple.

The converse implication is clear since semisimple rings are Artinian.■

4.2 Corollary. *If* R *is a regular ring satisfying right* d.c.c.f.g.P *then* R *is semisimple.*

Proof. Regular rings are semiprime right P.P. (in fact, semihereditary) and thus by 4.1 are semisimple if they satisfy right d.c.c.f.g.P.■

Definition. A ring R is said to satisfy $(\text{a.c.c.})^{\oplus}$ if it contains no infinite set of independent right ideals. The a.c.c. on right annulets of R is abbreviated $(\text{a.c.c.})^{\perp}$. If a ring satisfies both $(\text{a.c.c.})^{\oplus}$ and $(\text{a.c.c.})^{\perp}$, then the ring is said to be *right Goldie.*

We say a right R-module M is *uniform* if X,Y nonzero submodules of M implies $X \cap Y \neq 0$. A ring is (*right*) *uniform* if R_R is uniform.

To prove our next theorem we need the following theorems of A.W. Goldie [4,5].

4.3 Theorem. *A ring* R *is semiprime right Goldie if and only if its quotient ring* Q(R) *is semisimple.*■

4.4 Lemma. *A uniform semiprime right Goldie ring* R *is a right Ore domain.*■

Note that in this case, as with all domains, right a.c.c.f.g.P implies a.c.c. on principal right ideals.

4.5 Theorem. *If* R *is a ring satisfying right* d.c.c.f.g.P, *then* R *is a right Ore ring and* R = Q(R).

Proof. If a is a regular element in R, but not invertible, then a^nR properly contains $a^{n+1}R$ for n = 1,2, 3,.... But a regular implies $a^nR \approx R$ is projective for each n, contradicting d.c.c.f.g.P. Thus each regular element in R is right invertible. Hence R = Q(R).■

4.6 Theorem. *Let* R *be a semiprime right Goldie ring. Then* R *satisfies right* d.c.c.f.g.P *if and only if* R *is semisimple.*

Prof. If R satisfies d.c.c.f.g.P then R = Q(R) by 4.5. Thus by 4.3, R is semisimple.

The converse is clear.■

A domain satisfying the right Ore condition is a right Goldie domain. The converse of this follows immediately from the following two lemmas proven by A.W. Goldie [4,5].

4.7 Lemma. *If* R *satisfies* (a.c.c.)$^{\oplus}$ *then every* *nonzero right ideal of* R *contains a uniform right ideal.*

Proof. Let I be a nonzero right ideal. If I is not uniform, then I contains nonzero I_1, J_1 with $I_1 \cap J_1 = 0$. If I_1 is not uniform there exists nonzero I_2, J_2 contained in I_1 with $I_2 \cap J_2 = 0$. Continue this process to obtain $J_1 \subset J_1 \oplus J_2 \subset J_1 \oplus J_2 \oplus J_3 \subset \cdots$. Since R satisfies (a.c.c.)$\oplus$, the above sequence must terminate. If it terminates at the n^{th} step, I_n is uniform.■

4.8 Lemma. *Let* R *be a ring with uniform right ideal* U. *If* U *has a nonzero element which is not a left divisor of* 0 *then* R *is a right uniform ring.*

Proof. Let I, J be right ideals such that $I \cap J = 0$, and let u in U be an element with $u^{\perp} = 0$. Then $uI \cap uJ = 0$. Hence I = 0 or J = 0.■

4.9 Lemma. *If* R *is a right* (M)EP *ring satisfying right* a.c.c.(f.g.)P, *then* R *satisfies* (a.c.c.)$\oplus$.

Proof. Let $\{A_i\}_{i \in I}$ be a collection of nonzero independent right ideals in R. Then each A_i contains a nonzero projective right ideal P_i. The sum $\sum_{i \in I} P_i$ is direct. Hence by a.c.c.P, I is a finite set. Thus each collection of independent right ideals in R is finite; i.e., R satisfies (a.c.c.)$\oplus$.

The same proof works for MEP rings satisfying right a.c.c.f.g.P.■

4.10 Theorem. *If* R *is a domain satisfying* a.c.c.f.g.P, *then* R *is an Ore domain.*

Proof. By Goldie, it suffices to show that R satisfies (a.c.c.)$^{\oplus}$. But since domains are MEP rings, domains satisfying right a.c.c.f.g.P also satisfy (a.c.c.)$^{\oplus}$.■

4.11 Lemma. *Any right nonsingular ring satisfying* (a.c.c.)$^{\oplus}$ *also satisfies* (a.c.c.)$^{\perp}$.

Proof. See, for example, Faith [2, 9.12.2,p.396].■

4.12 Corollary. *If* R *is a right nonsingular ring with* EP *and satisfying right* a.c.c.P, *then* R *is right Goldie.*

Proof. By 4.9, R satisfies (a.c.c.)$^{\oplus}$ and hence also satisfies (a.c.c.)$^{\perp}$.■

4.13 Lemma. *Let* R *be a right semihereditary ring.*

1. *if* R *satisfies right* a.c.c.f.g.P *then* R *is right Noetherian.*
2. *If* R *satisfies right* d.c.c.f.g.P *then* R *is left perfect.*

Prof. Since all finitely generated right ideals are projective, 1. follows since right a.c.c.f.g.P is equivalent to a.c.c. on finitely generated right ideals and 2. follows from Bass' theorem.■

For completeness, we next prove a couple of known lemmas giving conditions on rings which assure us the ring will satisfy a.c.c.P. The first lemma says much more, it says the ring is Noetherian.

4.14 Lemma. *If* R *satisfies* (a.c.c.)$\oplus$ *and* a.c.c. *on essential right ideals, then* R *is right Noetherian.*

Proof. Let A be an essential right ideal of R. Suppose A is generated by $\{x_i\}_{i\in I}$. Then by (a.c.c.)$\oplus$ there is a finite subset X of $\{x_i\}$ with $T = \sum_{x_i \in X} \oplus x_i R$ an essential submodule of A. Then T is essential in R and $T \subseteq T_1 \subseteq T_2 \subseteq \ldots$, Where $T_r = T + x_1R + x_2R + \ldots + x_rR$. This is a sequence of essential submodules of R and must terminate, say at T_k. Then A is generated by $\{x_1, x_2 \ldots, x_k\} \cup X$.

Now if I is any right ideal of R, either I is essential or there exists $I_1 \neq 0$ with $I \cap I_1 = 0$. Then either $I \oplus I_1$ is essential or there exists I_2 with $(I \oplus I_1) \cap I_2 = 0$. This process must terminate after a finite number of steps since R satisfies (a.c.c.)$\oplus$. Thus for some n, $I \oplus I_1 \oplus I_2 \oplus \ldots \oplus I_n$ is essential and hence finitely generated. But I is a summand of a finitely generated right ideal and hence is finitely generated. Thus R is right Noetherian.■

4.15 Lemma. *If* R *is a semiprime right Goldie ring, then each essential projective right ideal is finitely generated.*

Proof. Let P be an essential projective right ideal. If $\{x_i\}$ is any generating set for P then by the dual basis lemma there exists a family $\{f_i\}$ of elements in $P^* = \mathrm{Hom}_R(P,R)$ such that for each p in P, $p = \Sigma x_i f_i(p)$ with $f_i(p) = 0$ for almost all i.

Since R is semiprime right Goldie, the right quotient ring of R, $Q(R) = Q$, is also the injective hull of R. Hence each f in P^* can be extended to f' in $\mathrm{Hom}_R(Q,Q)$. But then $f'(x) = qx$ where $q = f'(1)$ in Q.

P is essential so there exists a regular element x in P. Now' $x = \Sigma x_i f_i(x) = \Sigma x_i q_i x$, with $q_i x = 0$ for almost all i. Hence almost all $q_i = 0$ so almost all $f_i = 0$, say $f_n = 0$ for $n > N$. It follows that P is finitely generated by $x_1, x_2, \ldots, x_N$.

4.16 Corollary. *If* R *is semiprime right Goldie with* EP *then each projective right ideal is finitely generated.*

Proof. If R has EP then each projective right ideal is a summand of an essential projective and hence is finitely generated.

5. Examples

Right Noetherian rings clearly satisfy right a.c.c.P.

The following example shows that right a.c.c.P rings need not be Noetherian.

5.1. Example.

The ring $R = F((x_1, x_2, \ldots, x_n \ldots)) = \bigcup_{n=1}^{\infty} F((x_1, \ldots, x_n))$ of formal power series in infinitely many commuting variables over a field F is a local ring, so projective ideals are free, hence principal (since R is a commutative domain). It suffices to show that R has a.c.c. on principal ideals. But this holds in any UFD (See, for example, Kaplansky [8, Theorem 179, p.132]).■

Also the ring $R = F[x_1, x_2, \ldots]$ satisfies a.c.c. P since projective ideals are principal and R satisfies a.c.c. on principal ideals.

Left perfect rings satisfy right d.c.c.f.g. P since they satisfy d.c.c. on all finitely generated right ideals (See [2]). The next example shows that the converse is not necessarily true.

5.2 Example. Let $S = K((x_1, x_2, x_3))$ be the ring of formal power series in three indeterminates over a commutative field K, let $I = (x_1x_2,\ x_1x_3,\ x_2x_3,\ x_2 + x_3)$ and define R to be S/I. Then R is local so projective ideals are free. Hence the only projective ideals in R are R and 0. Thus R satisfies a.c.c.P and d.c.c. P. However R is not perfect since $(x_1) \supseteq (x_1^2) \supseteq (x_1^3) \supseteq \ldots$ is a nonterminating sequence of principal ideals in R.■

Let R be a ring with a right quotient ring Q. If

P is an ideal in R, define $P^{-1} = \{q$ in $Q \mid qP \subseteq R\}$. We say that P is invertible if $PP^{-1} = P^{-1}P = R$. Let $\mathcal{P}$ denote the collection of all invertible ideals in R.

5.3 Theorem. *If* R *satisfies* a.c.c.$\mathcal{P}$, *then each invertible ideal may be written as a product of maximal invertible ideals.*

Proof. Let A be an invertible ideal. If A is maximal, we are done. If not, then A is contained in a maximal invertible ideal B. Then $A \subseteq B^{-1}A = C$. C is an invertible ideal since $(A^{-1}B)C = A^{-1}BB^{-1}A = R$. If C is maximal, we are done since $A = BC$ is a product of maximal invertible ideals. If not, continue. By a.c.c. $\mathcal{P}$ the process must terminate and we have A as a product of maximal invertible ideals.∎

5.4 Corollary. *If* R *is a commutative domain with* a.c.c.f.g. P *then each projective ideal is a product of maximal projective ideals.*

Proof. In a commutative domain, the invertible ideals are the finitely generated projective ideals.∎

REFERENCES

1. Bass, H., Finitistic dimension and a homological generalization of semiprimary rings, Trans. Amer. Math. Soc., 95, 466-488 (1960).

2. Björk, J. E., Rings satisfying a minimum condition on principal ideals, J. Reine Angew. Math. 236, 466-488 (1960).

3. Faith, C., *Algebra: Rings, Modules, and Categories* I, Grundlehren der Math. Wiss., Bd. 190, Springer-Verlag, New York and Berlin, 1973.

4. Goldie, A.W., The structure of prime rings under ascending chain conditions, Proc. Lond. Math. Soc. VIII, 589-608 (1958).

5. Goldie, A. W., Semiprime rings with maximun condition, Proc. Lond. Math. Soc., 201-220 (1960).

6. Jacobson, N., *Structure of Rings*, Colloquium Publication, Vol. 37. American Math. Soc., Providence 1956 (1964 revised).

7. Kaplansky, I., Projective Modules, Ann. Math. 68, 372-377 (1958)

8. Kaplansky, I., *Commutative Rings*, Allyn and Bacon, Inc. Boston. (1970).

9. Michler, G., Idempotent ideals in perfect rings, Canad. J. Math., 21, 301-309 (1969).

10. Zariski, O., and Samuel, P., *Commutative algebra*, Vol I, Van Nostrand, Princeton and New York, (1958).