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GROUP RING OBJECTS DEFINED BY MOD p DECOMPOSITIONS OF BU AND ALGEBRAIC K-THEORY

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Let k be a coefficient (commutative) ring, and let $(cc)_k$ denote the category of positively graded, (co)commutative coalgebras over k. Let $\underline{C}$ denote any full subcategory of the category (sp) of topological spaces which is closed under taking finite products. For X and Y in $\underline{C}$ the homology product for $H_*(X,k)$ (or just $H_*(X)$ when there is no ambiquity) is defined

$$\alpha : H_*(X) \otimes H_*(Y) \longrightarrow H_*(X \times Y),$$

and we assume that it is an isomorphism which is always the case when k is a field. In this situation the natural diagonal map $\Delta : X \longrightarrow X \times X$ induces $H_*(\Delta)$

$$\begin{array}{ccc} H_*(X) & \xrightarrow{H_*(\Delta)} & H_*(X \times X) \\ & \searrow^{\Delta H_*(X)} & \uparrow \alpha \\ & & H_*(X) \otimes H_*(X) \end{array}$$

which composed with the inverse of α to define a coalgebra structure $\Delta H_*(X)$ on $H_*(X)$. We view homology as a functor

$$H_* : \underline{C} \longrightarrow (cc)_k .$$

Since the tensor product of commutative coalgebras is the product in the category $(cc)_k$ of commutative coalgebras over k, the Künneth isomorphism α can be interpreted further to say that $H_* : \underline{C} \longrightarrow (cc)_k$ preserves products. There is a corresponding assertion for the respective categories spaces with base points and augmented coalgebras.

A basic structure on a space (with base point) is that of a "product" map $\mu : X \times X \longrightarrow X$ making X into an H-space. We will only be interested in homotopy associative H-spaces, e.g. topological groups and loop spaces $\Omega(Y)$. In these cases $H_*(X)$ has a multiplciation which is the composite

$$H_*(X) \otimes H_*(X) \xrightarrow{\alpha} H_*(X \times X) \xrightarrow{H_*(\mu)} H_*(X).$$

When X is connected, or more generally, when the semigroup $\pi_0(X)$ of connected components is a group, $H_*(X)$ becomes a group object over the category $(cc)_k$, that is, a Hopf algebra. Hence an H-space is thought of very much like an object which is a "homotopy group".

Now in K-theory, both topological and algebraic, one encounters spaces with two H-space structures usually given by some direct sum and tensor product operation on bundles or modules. The main examples are $\mathbb{Z} \times BU$, $\mathbb{Z} \times BO$, and $\mathbb{Z} \times F(\psi^q - 1)$ in topological K-theory, $BGL(A)^+$ in algebraic K-theory, and $B(\underline{S}_\infty)^+$ in stable homotopy where $\underline{S}_\infty = \bigcup_n \underline{S}_n$ is the infinite symmetric group. Here BG^+ is the Quillen plus construction, see Hausmann and Husemoller [2]. In these cases $H_*(X)$ becomes a ring object over the category $(cc)_k$. In Husemoller [5] we studied this situation and introduced the concept of group ring of a group over $(cc)_k$ and identified the homology of several spaces as group rings over a specific group object, for example,

$$H_*(\mathbb{Z} \times BU) \cong \text{group ring of } H_*(BU(1)).$$

over the category $(cc)_k$.

In this article we extend these results to the factors of BU localized at a prime. While this article was being prepared the author was a guest of the Max Planck Institut fur Mathematik in Bonn where he profited from discussions with D. Quillen on this work. He would like to express his appreciation for their support.

§1. Decomposition of BU and BSO at a prime.

The following theorem has been known since the 1960's. Peterson [7] was the first to publish a proof of it.

(1.1) Theorem. There is a space Y_p localized at an odd prime such that

$$BU_{(p)} \quad \text{and} \quad Y_p \times \Omega^2 Y_p \times \ldots \times \Omega^{2(p-2)} Y_p$$

have the same homotopy type, and moreover, Y_p does not decompose into a product. Further,

$$BSO_{(p)} \quad \text{and} \quad Y_p \times \Omega^4 Y_p \times \ldots \times \Omega^{2p-6} Y_p$$

have the same homotopy type.

Peterson proved the above theorem in terms of a characterization of $BSO_{(p)}$ and $BU_{(p)}$ in terms of k-invariants, see [7 ,theorems 2.1,2.2]. He constructed the space Y_p using Sullivan's bordism with "singularities" theory. The orginal proof of this theorem seems to be due to Adams, Anderson, and Atiyah all independently.

This space $BSO_{(p)}$ was of wide interest because Sullivan in his thesis proved that $BSO_{(p)}$ and the surgery space $(F/PL)_{(p)}$ are of the same homotopy type. In the paper of Peterson he also gives a proof of this assertion.

The proof of the decomposition of $BU_{(p)}$ (and similarly $BSO_{(p)}$) which is most transparent is essentually in Atiyah and Tall [1]. The space BU_p, which is BU completed at p, represents the K-functor on finite complexes tensored with $\mathbb{Z}_p$, the ring of p-adic integers. The Adams operations ψ^k for k integral extend to ψ^α for α any p-adic integer, and the group of units Γ in $\mathbb{Z}_p$ acts as a compact automorphism group of a compact $\mathbb{Z}_p$-algebra valued functor. Now Γ is the direct product of C, a cyclic group of order p-1, with $1 + p\mathbb{Z}_p$. The space Y_p represents the subfunctor $(K(X) \otimes \mathbb{Z}_p)^C$ of $K(X) \otimes \mathbb{Z}_p$ left fixed under C. The other p-2 summands correspond to χ-eigen-spaces as χ runs over the cyclic character group of C. Since ψ^{α} and $\psi^{\alpha+1}$ are related under the Bott isomorphism between BU_p and $\Omega^2 BU_p$, and since C has the form $\{\alpha, \alpha+1, \ldots, \alpha+p-2\}$, we see that the even loop spaces $\Omega^{2i} Y_p$ for $i = 0, \ldots, p-2$ represent the other factors.

(1.2) Remark. Since Y_p represents a subring valued functor, it has a homotopy ring structure, and we can ask the question: Is the ring object $H_*(\mathbb{Z} \times Y_p, \mathbb{Z}_{(p)})$ a group ring object over $(cc)_{\mathbb{Z}_{(p)}}$? This we answer affirmatively in the next section by relating it to the algebraic K-theory of Quillen.

(1.3) Remark. In Husemöller [3] we have a decomposition of the (additive) Hopf algebra

$$H_*(BU, \mathbb{Z}_{(p)}) = \bigotimes_{p \nmid i} B_{(p)}[x_i, 2i].$$

It is easily seen by the nature of the action of $\psi^{\alpha}(x) = \alpha^n x$ for primative elements x of degree n that

$$H_*(Y_p, \mathbb{Z}_{(p)}) = \bigotimes_{(p-1) | i} B_{(p)}[x_i, 2i] = B[x, 2(p-1)]$$

in the notation of [3].

§2. The space Y_ℓ and algebraic K-theory; $H_*(\mathbb{Z} \times Y_\ell, \mathbb{Z}_{(\ell)})$ as a group ring object.

Let ℓ be an odd prime and consider $BU_{(\ell)}$. In Quillen [8] and [9] fundamental use was made of the Brauer lifting to define the horizontal maps in the following commutative diagram.

$$\begin{array}{ccc} BGL(\bar{F})^+ & \longrightarrow & BU \\ \uparrow & & \uparrow \\ BGL(F)^+ & \longrightarrow & F(\psi^q - 1) \end{array}$$

where F is the field of q elements, $q = p^a$ where p is a prime, and $\bar{F}$ is the algebraic closure of F. These horizontal maps induce isomorphisms in homology with coefficients in a field of characterisitic ℓ.

Given ℓ we are free to consider any p different from ℓ for the analysis. Quillen proves that if p has order $\ell-1$ in $\mathbb{F}_\ell^*$ then there is a subfield k of $\bar{F}$ such that

$$H_*(BGL(k)^+, \mathbb{F}_\ell) \longrightarrow H_*(Y_\ell, \mathbb{F}_\ell) = B[x, 2\ell] \text{ is an isomorphism.}$$

For this see[9, pp. 577-8] and again we have used the notation of [3]. By [5,Theorem 3] the ring object $H_*(Y_\ell)$ is a group ring object with field coefficients. Since all the modules over $\mathbb{Z}_\ell$ are free, the same is true for homology with values in this ring. In summary we have the following theorem.

(2.1) Theorem. The ring object $H_*(Y_\ell, \mathbb{Z}_{(\ell)})$ is a group ring object over the category $(cc)_{\mathbb{Z}_{(\ell)}}$.

Bibliography

1. M.F. Atiyah and D.O. Tall, Group representations, λ-rings, and the J-homomorphism, Topology, 8 (1969), pp. 253-98.

2. J-C. Hausmann and D. Husemoller, Acyclic maps, L'Enseignement mathematique, XXV (1979), pp. 53-75.

3. D. Husemoller, The structure of the Hopf algebra $H_*(BU)$ over a $\mathbb{Z}_{(p)}$- algebra, Am. J. of Math., XCIII (1971), pp. 329-49.

4. D. Husemoller, On the homology of the fibre of $\psi^q - 1$, Proceedings of the Seattle Algebraic K-theory Conference I, Springer LN 341, (1973).

5. D. Husemoller, Homology of certain H-spaces as group ring objects, Academic Press 1976, a volume dedicated to S. Eilenberg.

6. J. Milnor and J.C. Moore, On the structure of Hopf algebras, Annals of Math., 81 (1965), pp 211-64.

7. F.P. Peterson, The Mod p homotopy type of BSO and F/PL, Bol. Soc. Mat. Mexicana (2), 14 (1969) pp. 22-7.

8. D. Quillen, The Adams conjecture, Topology, 10 (1971), pp. 67-80.

9. D. Quillen, On the cohomology and K-theory of the general linear groups over a finite field, Annals of Math., 96 (1972), pp 552-86.

10. D. Sullivan, Genetics of homotopy theory and the Adams conjecture, Annals of Math, 100 (1974) pp. 1-79.

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