Pub. Mat. UAB
Vol. 26 Nº 3 Des. 1982

PROFINITE CHERN CLASSES FOR GROUP REPRESENTATIONS

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Introduction

Let $\rho : G \to GL_n\mathbb{C}$ be a complex representation of the discrete group G. If one wishes to study ρ from an algebraic topologist's point of view, one forms the induced map $B\rho : BG \to BGL_n\mathbb{C}$ of classifying spaces, which gives rise to an n-dimensional complex vector bundle $\xi(\rho)$ over $BG = K(G,1)$. The Chern classes of this vector bundle $\xi(\rho)$,

$$c_j(\rho) \in H^{2j}(G;\mathbb{Z}) ,$$

are called the Chern classes of ρ. These cohomology classes may be used to obtain information on $H^*(G;\mathbb{Z})$, or to study the representation ρ itself. For instance, if ρ factors through $GL_n\mathbb{R}$, the associated complex vector bundle over BG will be invariant under complex conjugation, and by a well known property of Chern classes this implies that $c_j(\rho) = (-1)^j c_j(\rho)$, that is, $2c_j(\rho) = 0$ for j odd. More generally, there is an obvious action of field automorphisms of $\mathbb{C}$

on the vector bundles of the form $\xi(\rho)$, and it is our objective to study the behavior of Chern classes under this action. Using Sullivan's computation of the "Galois action" on $H^*(BGL_n\mathbb{C};\mathbb{Z}/m\mathbb{Z})$ (cf. [10]) we will be able to understand this action on the Chern classes reduced modulo m . A different approach is described in Grothendieck's paper [6], using p-adic Chern classes defined in an algebraic geometry setting (see also Soulé [9]); results on ordinary Chern classes follow then by means of the comparison theorem, relating the etale homotopy type of a complex variety with its ordinary homotopy type and its profinite completion. If one is interested in results concerning finite groups, then a more direct approach is possible by identifying the Galois action on the representation ring with certain Adams operations (see [5]).

For our approach, it turns out to be natural to work with profinite Chern classes

$$\hat{c}_j(\rho) \in H^{2j}(G;\hat{\mathbb{Z}})$$

They are defined as the images of the ordinary Chern classes $c_j(\rho)$ under the map induced by the coefficient homomorphism $\mathbb{Z} \to \hat{\mathbb{Z}}$, $\hat{\mathbb{Z}} = \lim\limits_{\leftarrow} \mathbb{Z}/n\mathbb{Z}$ the ring of profinite integers. For $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ a field automorphism of $\mathbb{C}$ and $\rho : G \to GL_n\mathbb{C}$ a representation, one defines ρ^σ by $\sigma_* \circ \rho$, where $\sigma_* : GL_n\mathbb{C} \to GL_n\mathbb{C}$ is obtained by applying σ to the entries of a matrix. We show first that $c_j(\rho)$ depends only on χ_ρ , the character of ρ . Therefore, $c_j(\rho) = c_j(\rho^\sigma)$ if σ fixes the values of χ_ρ . On the other hand, we show that $c_j(\rho^\sigma) = \hat{\sigma}^j c_j(\rho)$, where $\hat{\sigma}$ is a unit in $\hat{\mathbb{Z}}$ which is determined by the action of σ on the roots of unity in $\mathbb{C}$. Our main theorem then results from an analysis of these relations. It involves certain numbers $\overline{E_K}(j)$ which are defined for a number field K and which were introduced in [5]:

$$\overline{E_K}(j) = \max\{m \mid j \equiv 0 \mod \exp(\mathrm{Gal}(K(\xi_m)/K))\}$$

where ξ_m denotes a primitive m-th root of unity, and $\exp(\mathrm{Gal}(K(\xi_m)/K))$ is the exponent of the Galois group of $K(\xi_m)$ over K .

__Main Theorem.__ Let $\rho : G \to GL_n\mathbb{C}$ be a representation with character χ_ρ . Suppose $K \subset \mathbb{C}$ is a number field such that $\chi_\rho(g) \in K$ for all $g \in G$. Then the following holds:

A) $\overline{E_K}(j)\, \hat{c}_j(\rho) = 0 \in H^{2j}(G;\hat{\mathbb{Z}})$ for all $j > 0$.

B) The bounds $\overline{E_K}(j)$ on the orders of $\hat{c}_j(\rho)$ are best possible in the obvious sense.

__Remarks.__ The numbers $\overline{E_K}(j)$ can be described in a very explicit way in terms of invariants attached to K (cf. [5]). For instance, if j is even and $K = \mathbb{Q}$, one has

$$\overline{E_{\mathbb{Q}}}(j) = \mathrm{den}(B_j/2j)$$

with $B_2 = 1/6$, $B_4 = 1/30$ etc. the Bernoulli numbers. Note also that the numbers $\overline{E_K}(j)$ agree with Grothendieck's bounds [6] and they are also equal to the numbers $w_j(K)$ defined in Cassou-Noguès' paper [3], (see also [7]) .

1. Representations and traces

A representation $\rho : G \to GL_n\mathbb{C}$ defines a G-action on $\mathbb{C}^n$. We write $V = V(\rho)$ for the corresponding $\mathbb{C}[G]$-module. As usual, we define the complex representation ring $R(G)$ to be the ring additively generated by isomorphism classes of finite dimensional $\mathbb{C}[G]$-modules, with relations of the form $[W] = [V] + [W/V] \in R(G)$ for every short exact sequence $V \to W \to W/V$ of finite-dimensional $\mathbb{C}[G]$-modules; $[V]$ denotes the image of V in $R(G)$. The multiplication in $R(G)$ is defined using the tensor product over $\mathbb{C}$ of $\mathbb{C}[G]$-modules. If $V = V(\rho)$ and if we choose a composition series $V_1 \subset V_2 \subset \ldots \subset V_n = V$, we see that $[V] = \Sigma[V_j/V_{j-1}] \in R(G)$ with $V_j/V_{j-1} \cong V(\rho_j)$, ρ_j an irreducible representation; this means that from the point of view of $R(G)$, every representation is semi-simple. The Jordan-Hölder Theorem states that the irreducible representations ρ_j are uniquely determined by ρ (up to equivalence and order). Thus $R(G)$ has an additive basis consisting of the elements of the form $[V_\alpha]$, V_α a simple $\mathbb{C}[G]$-module of finite dimension.

The character χ_ρ of ρ is the function $G \to \mathbb{C}$ defined by $\chi_\rho(g) = \operatorname{trace}(\rho(g))$, $g \in G$. Of course, χ_ρ depends on $V(\rho)$ only, and we sometimes write $\chi_{V(\rho)}$ for χ_ρ . If $V \to W \to W/V$ is a short exact sequence of finite dimensional $\mathbb{C}[G]$-modules, then $\chi_W = \chi_V + \chi_{W/V}$. Therefore $\rho \mapsto \chi_\rho$ gives rise to an additive homomorphism

$$\chi : R(G) \longrightarrow \mathbb{C}^G$$
$$[V] \longmapsto \chi_V$$

into the ring $\mathbb{C}^G$ of $\mathbb{C}$-valued functions on G . Since $\chi_{V\otimes W} = \chi_V \cdot \chi_W$, χ actually defines a homomorphism of rings. The image $\chi(R(G))$ is denoted by $R_\chi(G)$ and we call it the character ring of G .

<u>Theorem 1.</u> The map $\chi : R(G) \to R_\chi(G)$ is an isomorphism of rings.

<u>Proof.</u> Let $\rho_1, \rho_2 : G \to GL_n\mathbb{C}$ be two completely reducible representations. Then $\chi_{\rho_1} = \chi_{\rho_2}$ implies $V(\rho_1) \cong V(\rho_2)$ as $\mathbb{C}[G]$ - modules: this is a consequence of the double centralizer Theorem,cf. Bourbaki [2; chapitre VIII, § 12, Prop. 3]. If $x \in R(G)$ is an arbitrary element, we can write x in the form $x = \Sigma[V_i] - \Sigma[W_j]$ with V_i and W_j simple $\mathbb{C}[G]$-modules for all i and j . Suppose now that $\chi(x) = 0$. Then $\Sigma\chi([V_i]) = \Sigma\chi([W_j])$ and therefore $\oplus V_i \cong \oplus W_j$ because the representations $\oplus V_i$ and $\oplus W_j$ are semi-simple. We infer $x = \Sigma[V_i] - \Sigma[W_j] = 0$ and thus χ is injective.

Since χ is surjective by definition, the assertion of the theorem follows.

2. Galois action

Let $\sigma \in Gal(\mathbb{C}/\mathbb{Q})$ be an automorphism of $\mathbb{C}$. By applying σ to the entries of a matrix, one obtains an induced group automorphism $\sigma_* : GL_n\mathbb{C} \to GL_n\mathbb{C}$. If $\rho : G \to GL_n\mathbb{C}$ is a representation, we write ρ^σ for the composite representation $\sigma_* \circ \rho$. As usual, we denote the group of automorphisms of $\mathbb{C}$ over $K \subset \mathbb{C}$ by $Gal(\mathbb{C}/K)$.

__Theorem 2.__ Let $\rho : G \to GL_n\mathbb{C}$ be a representation and let $\mathbb{Q}(\chi_\rho)$ denote the subfield of $\mathbb{C}$ generated by the traces of the matrices $\rho(g)$, $g \in G$. If $\sigma \in Gal(\mathbb{C}/\mathbb{Q}(\chi_\rho))$ then

$$[V(\rho)] = [V(\rho^\sigma)] \in R(G)$$

__Proof.__ Note that for σ an automorphism of $\mathbb{C}$ over $\mathbb{Q}(\chi_\rho)$, $\chi_{\rho^\sigma}(g) = \sigma(\chi_\rho(g)) = \chi_\rho(g)$ for all $g \in G$. Therefore, $\chi([V(\rho)]) = \chi([V(\rho^\sigma)]$ and we infer from Theorem 1 that $[V(\rho)] = [V(\rho^\sigma)]$.

__Remark.__ If $\rho : G \to GL_n\mathbb{C}$ is a representation of a finite group G, then it is well known that the representations ρ and ρ^σ are actually equivalent for every $\sigma \in Gal(\mathbb{C}/\mathbb{Q}(\chi_\rho))$.

For an infinite group, this need not be so. For example, if $\rho : \mathbb{Z} \to GL_4\mathbb{C}$ is given by

$$1 \longmapsto \left(\begin{array}{cc|cc} i & 1 & & \\ 0 & i & \multicolumn{2}{c}{0} \\ \hline & & -i & 0 \\ \multicolumn{2}{c|}{0} & 0 & -i \end{array}\right)$$

then $\mathbb{Q}(\chi_\rho) = \mathbb{Q}$ and, taking σ to be complex conjugation, one easily checks that $V(\rho) \not\cong V(\rho^\sigma)$ although $\chi_\rho = \chi_{\rho^\sigma}$.

Let $K \subset \mathbb{C}$ be a number field and let $\mu(\mathbb{C})$ denote the group of roots of unity in $\mathbb{C}$. The following numbers $w_j(K)$ have been considered by Soulé in [9]:

$$w_j(K) = \text{card}\{x \in \mu(\mathbb{C}) \mid \sigma^j x = x \quad \text{for all} \quad \sigma \in \text{Gal}(\mathbb{C}/K)\}$$

We want to show that $w_j(K) = \overline{E_K}(j)$, $\overline{E_K}(j)$ being defined as in the introduction (see also [5]). Let $\mu_m \subset \mu(\mathbb{C})$ denote the group of m-th roots of unity. Then $\mu_m \subset K(\xi_m)$ where ξ_m denotes a primitive root of unity in $\mathbb{C}$. The obvious map

$$\text{Gal}(\mathbb{C}/K) \longrightarrow \text{Aut}\ \mu_m$$

factors through the surjective restriction map $\text{Gal}(\mathbb{C}/K) \longrightarrow \text{Gal}(K(\xi_m)/K)$. Since $\text{Gal}(K(\xi_m)/K)$ acts faithfully on μ_m, the assertion

" $\sigma^j x = x$ for all $x \in \mu_m$ and all $\sigma \in \text{Gal}(\mathbb{C}/K)$ "

is therefore equivalent to the assertion

" $j \equiv 0 \bmod \exp(\mathrm{Gal}(K(\xi_m)/K))$ "

where $\exp(\mathrm{Gal}(K(\xi_m)/K))$ denotes the exponent of the group $\mathrm{Gal}(K(\xi_m)/K)$. Using the fact that all finite subgroups of $\mu(\mathbb{C})$ are cyclic we infer that $w_j(K)$ agrees with

$$\overline{E_K}(j) = \max\{m \mid j \equiv 0 \bmod \exp(\mathrm{Gal}(K(\xi_m)/K))\}$$

for every number field K and every $j > 0$.

Corollary 1. Let $K \subset \mathbb{C}$ be a number field. Then the torsion subgroup of the multiplicative group K^* is cyclic of order $\overline{E_K}(1)$.

Proof. The torsion subgroup of K^* is $\mu(\mathbb{C}) \cap K$. Its order is obviously equal to the largest number m such that $\mu_m \subset K$, which is the same as $\overline{E}_K(1)$ or $w_1(K)$.

3. Chern classes

We write $c(\rho) = \Sigma\, c_j(\rho) \in H^*(G;\mathbb{Z})$ for the total Chern class of a representation $\rho : G \to GL_n\mathbb{C}$. Clearly, $c(\rho)$ depends on $V = V(\rho)$ only, and we sometimes write $c(V)$ for $c(\rho)$. Let $V \to W \to W/V$ be a short exact sequence of finite-dimensional $\mathbb{C}[G]$-modules. Then $c(W) = c(V) \cdot c(W/V)$ since every short exact sequence of vector bundles over a CW-complex is split. Taking Chern classes thus defines a map

$$c : R(G) \longrightarrow H^*(G;\mathbb{Z})$$

$$[V] \longmapsto c([V]) := c(V)$$

which is a homomorphism of the underlying abelian group of $R(G)$ into the multiplicative group of units of the graded ring $H^*(G;\mathbb{Z})$.

<u>Theorem 3.</u> Let $\rho_1, \rho_2 : G \to GL_n\mathbb{C}$ be two representations with $\chi_{\rho_1} = \chi_{\rho_2}$. Then

$$c(\rho_1) = c(\rho_2) \in H^*(G;\mathbb{Z})$$

<u>Proof.</u> By Theorem 1, $\chi_{\rho_1} = \chi_{\rho_2}$ implies that $[V(\rho_1)] = [V(\rho_2)]$. Therefore $c(\rho_1) = c([V(\rho_1)]) = c([V(\rho_2)]) = c(\rho_2)$.

The first Chern class of a representation $\rho : G \to GL_n\mathbb{C}$ can be described in a very explicit way. Let $\det : GL_n\mathbb{C} \to \mathbb{C}^* = GL_1\mathbb{C}$ denote the determinant map. Then $\det \rho$ is a one-dimensional representation and, by a well-known property of vector bundles,

$$c_1(\rho) = c_1(\det \rho) \in H^2(G;\mathbb{Z}) .$$

Consider the coefficient sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 0$$

with exp the exponential map. From the associated exact cohomology sequence we obtain a boundary map

$$H^1(G;\mathbb{C}^*) \longrightarrow H^2(G;\mathbb{Z})$$

and by composing with the canonical isomorphisms

$$\mathrm{Hom}(G,\mathbb{C}^*) \cong \mathrm{Hom}(H_1(G),\mathbb{C}^*) \cong H^1(G;\mathbb{C}^*)$$

we get a natural homomorphism

$$\delta : \mathrm{Hom}(G,\mathbb{C}^*) \longrightarrow H^2(G;\mathbb{Z}) .$$

It is well known that $\delta(\det \rho) = c_1(\rho)$.

If we think of $H^2(G;\mathbb{Z})$ as the group of equivalence classes of central extensions of G by $\mathbb{Z}$, the element $c_1(\rho)$ can be represented by

$$E(\rho) : \mathbb{Z} \rightarrowtail X(\rho) \twoheadrightarrow G ,$$

which is the extension induced via $\det \rho : G \longrightarrow \mathbb{C}^*$ from the extension $\mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$. Note that $E(\rho)$ is split if $\det \rho$ factors through a free abelian group (this is clear since $E(\rho)$ is induced from an abelian extension).

<u>Corollary 2.</u> Let $\rho : G \to GL_n\mathbb{C}$ be a representation such that $\det \rho : G \to \mathbb{C}^*$ factors through a free abelian group. Then $c_1(\rho) = 0$.

We will apply this Corollary to the canonical representation $\iota_n : GL_n K \to GL_n \mathbb{C}$ for $K \subset \mathbb{C}$ a number field. In this case, the torsion subgroup $T(K^*) \subset K^*$ is cyclic of order $\overline{E_K}(1)$ (see Corollary 1) and $K^*/T(K^*)$ is a free abelian group (it maps into the free abelian group generated by the prime ideals of $\mathcal{O}(K)$, where $\mathcal{O}(K)$ is the ring of integers of K, and the kernel of this map is free abelian).

<u>Corollary 3.</u> Let $\rho : G \to GL_n \mathbb{C}$ be a representation with $\det \rho(g) \in K$ for all g, where K is a number field. Then $\overline{E_K}(1)\ c_1(\rho) = 0$.

<u>Proof.</u> We have only to note that $\overline{E_K}(1) \cdot \det(\rho) : G \to K^*$ factors through a free abelian subgroup of K^* (isomorphic to $K^*/T(K^*)$).

<u>Corollary 4.</u> Let $K \subset \mathbb{C}$ be a number field and let $\iota_n : GL_n K \to GL_n \mathbb{C}$ denote the canonical representation. Then

$$c_1(\iota_n) \in H^2(GL_n K; \mathbb{Z})$$

has order $\overline{E_K}(1)$ for all $n \geq 1$.

<u>Proof.</u> We know that $\overline{E_K}(1)\ c_1(\iota_n) = 0$ from Corollary 3. On the other hand, using the restriction map induced via the obvious inclusions

$$\mu(C) \cap K \hookrightarrow K^* = GL_1(K) \to GL_n K$$

an easy computation shows that

$$\operatorname{res}(c_1(\iota_n)) \in H^2(\mu(\mathbb{C}) \cap K;\mathbb{Z}) \cong \mathbb{Z}/\overline{E_K}(1)\,\mathbb{Z}$$

is a generator. Therefore, $c_1(\iota_n)$ has order precisely $\overline{E_K}(1)$.

4. Proof of the Main Theorem

Let $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/m\mathbb{Z}$ denote the ring of profinite integers. It is well known that for an arbitrary CW-complex X the canonical map

$$H^*(X;\hat{\mathbb{Z}}) \longrightarrow \varprojlim H^*(X;\mathbb{Z}/m\mathbb{Z})$$

is an isomorphism (this may be seen using the natural compact topology on the groups $H^j(X;\mathbb{Z}/m\mathbb{Z})$ cf. Sullivan [10]). Therefore, the kernel of the canonical map

$$H^*(X;\mathbb{Z}) \longrightarrow H^*(X;\hat{\mathbb{Z}})$$

consists of all elements $x \in H^*(X;\mathbb{Z})$ which are infinitely divisible (x is called infinitely divisible, if for all natural numbers n , there exists a $y(n)$ such that $x = ny(n)$). We write $\hat{c}_j(\rho) \in H^{2j}(G;\hat{\mathbb{Z}})$ for the image of $c_j(\rho)$; note that $\hat{c}_j(\rho)$ and $c_j(\rho)$ have the same orders in case $H^{2j}(G;\mathbb{Z})$ does not contain any infinitely divisible elements.

A group G is called geometrically finite if the classifying space $K(G,1)$ is of the homotopy type of a finite complex (this is equivalent to saying that G is finitely presentable and of type FF in the sense of Serre [8]) . The $\hat{\mathbb{Z}}$-cohomology of an arbitrary group may be detected by maps from geometrically finite groups as follows.

<u>Theorem 4.</u> Let G be an arbitrary group. Then there exists a family $\{f_\alpha : G_\alpha \to G\}$ with each group G_α geometrically finite, such that

$$\{f_\alpha^*\} : H^*(G;\hat{\mathbb{Z}}) \longrightarrow \Pi\, H^*(G_\alpha;\hat{\mathbb{Z}})$$

is injective.

<u>Proof.</u> Let $X = K(G,1) = \bigcup X_\alpha$ with each X_α a finite and connected CW-complex. Choose acyclic maps $g_\alpha : K(G_\alpha,1) \to X_\alpha$ with G_α geometrically finite (the construction of such maps g_α may be found in Baumslag-Dyer-Heller [1]). Define $f_\alpha : G_\alpha \to G$ to be the map of fundamental groups induced from $K(G_\alpha,1) \to X_\alpha \to X$. Using the compactness of the groups $H^j(X_\alpha;\hat{\mathbb{Z}})$ one may prove that the canonical map $H^*(X,\hat{\mathbb{Z}}) \to \varprojlim H^*(X_\alpha;\hat{\mathbb{Z}})$ is an isomorphism (cf. Sullivan [10]). The natural map $H^*(X;\hat{\mathbb{Z}}) \longrightarrow \Pi\, H^*(X_\alpha;\hat{\mathbb{Z}})$ is thus injective, and the assertion of the theorem follows since $g_\alpha : H^*(X_\alpha;\mathbb{Z}) \to H^*(G_\alpha,\mathbb{Z})$ is an isomorphism for every α .

We will consider $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/m\mathbb{Z}$ in the following way as a $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$-module. Let $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. If σ acts on μ_m (the m-th roots of unity) by the k-power map then we define

$\sigma(m) : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ to be multiplication by k . We put $\hat{\sigma} = \lim_{\leftarrow} \sigma(m) : \hat{\mathbb{Z}} \to \hat{\mathbb{Z}}$; note that $\hat{\sigma} \in \hat{\mathbb{Z}}^*$ is the element whose reduction mod m is $\sigma(m) = \bar{k} \in (\mathbb{Z}/m)^*$. The map $\sigma \mapsto \hat{\sigma}$ defines the desired action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on $\hat{\mathbb{Z}}$. The induced map on $H^j(\ ;\hat{\mathbb{Z}})$ will be denoted by σ too, for it is also multiplication by $\hat{\sigma} \in \hat{\mathbb{Z}}^*$.

The group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts on the etale homotopy type of a complex variety which is defined over $\mathbb{Q}$. This action may be used to define an induced action on the profinite completion of the classical homotopy type of the variety. By a limit argument, one obtains an action $\psi^\sigma : (BGL\mathbb{C})^\wedge \longrightarrow (BGL\mathbb{C})^\wedge$ for $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. The notation ψ^σ is chosen in view of the following proposition, which is due to Sullivan [10].

<u>Proposition 1.</u> Let $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ and $\psi^\sigma : (BGL\mathbb{C})^\wedge \longrightarrow (BGL\mathbb{C})^\wedge$ the induced map via etale homotopy theory. Then

$$(\psi^\sigma)^* = \hat{\sigma}^j : H^{2j}((BGL\mathbb{C})^\wedge;\hat{\mathbb{Z}}) \to H^{2j}((BGL\mathbb{C})^\wedge;\hat{\mathbb{Z}})$$

Using this proposition, we obtain the following.

<u>Theorem 5.</u> Given a representation $\rho : G \to GL_n\mathbb{C}$ and $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. Then, for $j \geq 1$

$$\hat{c}_j(\rho^\sigma) = \hat{\sigma}^j \, \hat{c}_j(\rho) \in H^{2j}(G;\hat{\mathbb{Z}})$$

Proof. We consider first the case of a geometrically finite G. Using techniques of etale homotopy (see for instance Deligne-Sullivan [4]) it follows that the map

$$K(G,1)^{\wedge} \xrightarrow{(B\rho)^{\wedge}} (BGL_n\mathbb{C})^{\wedge} \xrightarrow{\text{can}} (BGL\mathbb{C})^{\wedge} \xrightarrow{\psi^{\sigma}} (BGL\mathbb{C})^{\wedge}$$

is homotopic to

$$K(G,1)^{\wedge} \xrightarrow{(B\rho^{\sigma})^{\wedge}} (BGL_n\mathbb{C})^{\wedge} \xrightarrow{\text{can}} (BGL\mathbb{C})^{\wedge}$$

In view of Proposition 1 this implies that $\hat{c}_j(\rho^{\sigma}) = \hat{\sigma}^j \hat{c}_j(\rho) \in H^{2j}(G;\hat{\mathbb{Z}})$. If G is an arbitrary group, we apply Theorem 4 to reduce to the case of a geometrically finite group.

Theorem 6. Let $\rho : G \to GL_n\mathbb{C}$ be a representation with $\mathbb{Q}(\chi_{\rho}) \subset K \subset \mathbb{C}$, K a number field. Then, for all $j > 0$,

$$\overline{E_K}(j)\ \hat{c}_j(\rho) = 0 \in H^{2j}(G;\hat{\mathbb{Z}})$$

Proof. Let $x = \hat{c}_j(\rho)$. The reduction mod m of x, $red_m(x)$, generates a cyclic subgroup of $H^{2j}(G;\mathbb{Z}/m\mathbb{Z})$ on which $\sigma \in Gal(\mathbb{C}/\mathbb{Q})$ acts by $red_m(x) \longmapsto red_m\, c_j(\rho^{\sigma}) = \sigma(m)^j\, red_m(x)$. If we choose σ to be an automorphism over K, we infer from Theorem 3 that $\hat{c}_j(\rho) = \hat{c}_j(\rho^{\sigma})$ and thus $\sigma(m)^j\, red_m(x) = red_m(x)$. The order of the element $red_m(x)$ therefore divides $\overline{E_K}(j) = \text{card}\{z \in \mu(\mathbb{C}) \mid \sigma^j z = z$ for all $\sigma \in Gal(\mathbb{C}/K)\}$. Hence $\overline{E_K}(j)\ red_m(x) = 0$ for all m and, since $H^{2j}(G;\hat{\mathbb{Z}}) = \varprojlim H^{2j}(G;\mathbb{Z}/m\mathbb{Z})$, we infer that $\overline{E_K}(j)x = 0$.

This completes the proof of part A) of the Main Theorem. It remains to show that the bounds $\overline{E_K}(j)$ are best possible. This can be seen using the calculations performed in [5]. We recall (Theorem 4.12 of [5]) that $\overline{E_K}(j)$ is the best possible bound for the order of the Chern classes c_j of K-representations of finite groups, with the single exception when j is even and K formally real; in this latter case the best possible such bound is $\frac{1}{2}\overline{E_K}(j)$. It suffices therefore to prove the following.

<u>Theorem 7.</u> Let K be a formally real number field and $j > 0$ even. Then there exists a finite 2-group G and a representation $\rho : G \to GL(\mathbb{C})$ with $\mathbb{Q}(\chi_\rho) \subset K$ and

$$\frac{1}{2}\overline{E_K}(j)\ c_j(\rho) \neq 0$$

<u>Proof.</u> The construction of such a ρ can be performed in essentially the same way as the construction of ρ in the course of the proof of Proposition 4.11 (b) of [5]. One thus obtains a representation of a generalized quaternion group with $\mathbb{Q}(\chi_\rho) \subset K$ and with Schur index equal to two with respect to $\mathbb{Q}(\chi_\rho)$, such that $\frac{1}{2}\overline{E_K}(j)\ c_j(\rho) \neq 0$.

<u>Remark.</u> If $\rho : G \to GL_n\mathbb{C}$ is a semi-simple representation and $K \supset \mathbb{Q}(\chi_\rho)$ a subfield of $\mathbb{C}$, then there is a finite extension L of K in $\mathbb{C}$ such that ρ is equivalent to a representation defined over L. This interesting observa-

tion was communicated to me by P. Menal. We plan to use this fact in a later paper to show that for a very general ρ the actual Chern classes $c_j(\rho)$ (rather than $\hat{c}_j(\rho)$) are of finite order bounded by $\overline{E}_K(j)$, if K is a number field containing $\mathbb{Q}(\chi_\rho)$.

This expository paper is based on lectures delivered at the Universitat Autònoma de Barcelona. The final form of the results will be published in a joint paper with B. Eckmann.

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May 1982

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