Pub. Mat. UAB
Vol. 26 Nº 3 Des. 1982

NOTE ON BLOCK INVARIANTS

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In [6], the T-torsions of an N-dimensional CW complex K are introduced. Using these block invariants, we can list the generators and their order of the 2-primany component of cohomotopy groups $\pi^{N-1}(K)$ and $\pi^{N-2}(K)$. Now, we generalize these block invariants and discuss their property. Especially, some necessary conditions are obtained for the existence of a cross-section.

1. Let K be an N-dimensional CW complex, $n = N - 1$, $m = N - 2$. We have an exact couple which is based on the cohomotopy exact sequence of the triple (K, K^q, K^{q-1}):

$$\cdots \to \pi^r(K, K^q) \xrightarrow{i} \pi^r(K, K^{q-1}) \xrightarrow{j} \pi^r(K^q, K^{q-1}) \to$$
$$\to \pi^{r+1}(K, K^q) \to \ldots,$$

where i and j are the homomorphisms induced by the inclusions

$$(K, K^{q-1}) \to (K, K^q) \quad \text{and} \quad (K^q, K^{q-1}) \to (K, K^{q-1})$$

respectively, and k is the coboundary operator of the triple (K, K^q, K^{q-1}). Define precisely,

$$A = \sum_{r,q} A^{r,q},$$

$$A^{r,q} = \begin{cases} \pi^r (K, K^q), & r > \frac{N+1}{2}, \\ 0, & r \leqslant \frac{N+1}{2}; \end{cases}$$

$$C = \sum_{r,q} C^{r,q},$$

$$C^{r,q} = \begin{cases} \pi^r(K^q, K^{q-1}), & r > \frac{N+1}{2}, \\ \ker(\pi^{r+1}(K, K^q) \xrightarrow{i} \pi^{r+1}(K, K^{q-1})), & r = \left[\frac{N+1}{2}\right],^{1)} \\ 0, & r < \left[\frac{N+1}{2}\right], \end{cases}$$

and the homomorphisms

$$i^{r,q} : A^{r,q} \to A^{r,q-1},$$

$$j^{r,q} : A^{r,q} \to C^{r,q+1},$$

$$k^{r,q} : C^{r,q} \to A^{r+1,q}$$

are the appropriate i, j and k for $r > \frac{N+1}{2}$, when $r = \left[\frac{N+1}{2}\right]$, $k^{r,q}$ is the inclusion, and the remaining homomorphisms i, j and k are all null homomorphisms.

It is clear that $\langle A, C;\ i, j, k \rangle$ is an exact couple.

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1) $\left[\frac{N+1}{2}\right]$ stands for the largest integer not exceeding $\frac{N+1}{2}$.

Consider the first derived couple $\langle \Gamma, H; i, j, b \rangle$ of the cohomotopy exact couple $\langle A, C; i, j, k \rangle$. By definition,

$$\Gamma^{r,q} = \operatorname{Im} i^{r,q},$$

$$H^{r,q} = \frac{\ker d^{r,q}}{\operatorname{Im} d^{r-1,q-1}},$$

where $d^{r,q} = j^{r+1,q} \circ k^{r,q} : C^{r,q} \to C^{r+1,q+1}$ and homomorphisms

$$i^{r,q} : \Gamma^{r,q} \to \Gamma^{r,q-1},$$

$$j^{r,q} : \Gamma^{r,q} \to H^{r,q+1},$$

$$b^{r,q} : H^{r,q} \to \Gamma^{r+1,q+1}$$

are induced by $i^{r,q-1}$, $j^{r,q}(i^{r,q})^{-1}$ and $k^{r,q}$ respectively.

The group $\pi^r(K^q, K^{q-1})$ can be interpreted as the q-th cochain group of K with coefficient group $\pi_q^s(S^r)$ [1]. Thus $H^{r,q} \approx H^q(K; \pi_q^s(S^r))$. The homomorphism

$$d_0 = j \circ b : H^q(K; \pi_q^s(S^{r-1})) \to H^{q+2}(K; \pi_{q+2}^s(S^r))$$

is a cohomology operation [4]; in particular,

$$d_0 = j \circ b : H^q(K; \pi_q^s(S^q)) \to H^{q+2}(K; \pi_{q+2}^s(S^{q+1}))$$

and

$$d_0 = j \circ b : H^q(K; \pi_q^s(S^{q-1})) \to H^{q+2}(K; \pi_{q+2}^s(S^q))$$

are Steenrod squares from integral coefficients to coefficients mod 2 and from coefficients mod 2 to coefficients mod 2.[3].

On ker $d_o^{r,q}$, we can define a secondary cohomology operation

$$h_o^1 : \ker d_o^{r,q} \to H_1^{r+1,q+3} = \frac{\ker d^{r+1,q+3}}{\mathrm{Im}\ d^{r,q+1}}$$

as follows.

If $\alpha \in \ker d_o^{r,q}$, then $j^{r+1,q+1} \circ b^{r,q}(\alpha) = 0$, hence $b^{r,q}(\alpha) \in \mathrm{Im}\ i^{r+1,q+2}$. Let $\beta_1 \in \Gamma^{r+1,q+2}$ imply $i^{r+1,q+2}(\beta_1) = b^{r,q}(\alpha)$. Then $j^{r+1,q+2}(\beta_1) \in \ker d^{r+1,q+3}$. It is easily to verify that the class $h_o^1(\alpha) = \{j^{r+1,q+2}(\beta_1)\} \in H_1^{r+1,q+3}$ is independent of the choice made by the element β_1. Thus we obtain the homomorphism h_o^1.

In [5], the following theorems are proved.

Theorem 1. The secondary cohomology operation

$$h_o^1 : \ker d_o^{q,q} \to H_1^{q+1,q+3}$$

is the Adem's secondary operation Φ.

Theorem 2. For $N > 3$, we have a short exact sequence

$$0 \to \frac{H^{n+1}(K;Z_2)}{Sq^2 H^{n-1}(K;Z)} \xrightarrow{i} \pi^n(K) \xrightarrow{j} H^n(K;Z) \to 0 \tag{1}$$

When $N > 5$, for $\pi^m(K)$ we have

$$0 \to \Gamma \to \pi^m(K) \to \ker Sq^2 \ (\subset H^m(K;Z)) \to 0 \tag{2}$$

$$0 \to \text{Coker } \Phi \to \Gamma \to \text{Coker } Sq^2 \to 0$$

where Coker $\Phi = \dfrac{H^{m+2}(K;Z_2)}{Sq^2 H^m(K;Z_2) + \text{Im}\Phi}$ and coker $Sq^2 = \dfrac{H^{m+1}(K;Z_2)}{Sq^2 H^{m-1}(K;Z)}$.

This theorem makes it clear that, using cohomology groups and cohomology operations, we may determine the structure of the cohomotopy groups $\pi^n(K)$ and $\pi^m(K)$ within an extended limit of precision.

2. For determining these groups exactly, we first note

Corollary 3. $\pi^n(K)$ and $H^n(K;Z)$ have the same rank and odd primary components.

Now, we consider the 2-primary component of $\pi^n(K)$.

Denote the p-primary component of group G by $G_{(p)}$ and $_mG = \{g: mg = 0\}$. Assume

$$H^n(K;Z)_{(2)} = \underbrace{Z_{2^{l_1}} + \dots + Z_{2^{l_1}}}_{s_1} + \underbrace{Z_{2^{l_2}} + \dots + Z_{2^{l_2}}}_{s_2} + \dots + \underbrace{Z_{2^{l_r}} + \dots + Z_{2^{l_r}}}_{s_r},$$

$$l_1 > l_2 > \dots > l_r.$$

In [6], the cohomology operations

$$T^{(1)}(k) : {}_{2^{1_k}}H^n(K;Z) \to \text{Coker } Sq^2 = \frac{H^{n+1}(K;Z_2)}{Sq^2 H^{n-1}(K;Z)},$$
$$k = 1, \ldots, r,$$

are defined. Each operation has the properties:

(i) For $\{z\} \in {}_{2^{1_k}}H^n(K;Z)$, we can choose $e \in \pi^n(K)$ such that $j\,e = \{z\}$. Because $2^{1_k}\{z\} = 0$, then $j(2^{1_k}e) = 2^{1_k}j(e) = 2^{1_k}\{z\} = 0$. Let $F \in \text{Coker } Sq^2$ be an element such that $iF = 2^{1_k}e$. Then

$$T^{(1)}(k)(\{z\}) = F.$$

(Notice that the element F is uniquely determined in Coker Sq^2)

$$\text{(ii)} \qquad T^{(1)}(k) \Big| {}_{2^{1_t}}H^n(K;Z) = 0, \quad k > t.$$

Using these operations, we can determine the cohomotopy group $\pi^n(K)_{(2)}$ from (1). In fact[2,6], we can reconstructe the group $\pi^n(K)_{(2)}$ by $H^n(K;Z)_{(2)}$, Coker Sq^2 and $T^{(1)}(k)$ as follows. Let $e_1^{n+1}, \ldots, e_u^{n+1}$ be a basis of Coker Sq^2, and $e_1^n, \ldots, e_{s_1}^n$; $e_{s_1+1}^n, \ldots, e_{s_1+s_2}^n; \ldots; e^n_{\sum_{i=1}^{r-1} s_i+1}, \ldots, e^n_{\sum_{i=1}^{r} s_i}$ be the generators of $H^n(K;Z)_{(2)}$, the order of $e^n_{\sum_{i=1}^{k-1} s_i+j}$ be 2^{1_k},

$j = 1,\dots,s_k$, $k = 1,\dots,r$. Then we can construct a group $E_{T(1)}$ $(\mathrm{Coker}\ Sq^2, H^n(K;Z)_{(2)})$ as follows. First, we have the set $\mathrm{Coker}\ Sq^2 x H^n(K;Z)_{(2)}$. Note the order of the generator $e^n_{k-1 \atop \sum_{i=1} s_i+j}$, $j = 1,\dots, s_k$, is 2^{l_k}, so the element of $H^n(K;Z)_{(2)}$ has the form

$$\sum_{k=1}^{r} \sum_{j=1}^{s_k} \alpha_j^k \, e^n_{k-1 \atop \sum_{i=1} s_i+j}, \quad 0 \leqslant \alpha_j^k < 2^{l_k}.$$

Let $F \in \mathrm{Coker}\ Sq^2$, then the element of $\mathrm{Coker}\ Sq^2 x H^n(K;Z)_{(2)}$ has the form

$$(F, \sum_{k=1}^{r} \sum_{j=1}^{s_k} \alpha_j^k \, e^n_{k-1 \atop \sum_{i=1} s_i+j}), \quad 0 \leqslant \alpha_j^k < 2^{l_k}.$$

Using the operation $T^{(1)}(k)$, define an addition of pairs as follows.

$$(F, \sum_{k,j} \alpha_j^k \, e^n_{k-1 \atop \sum_{i=1} s_i+j}) + (F'; \sum_{k,j} {}'\alpha_j^k \, e^n_{k-1 \atop \sum_{i=1} s_i+j}) =$$

$$= (F+F'+\sum_{k,j} \epsilon_j^k T^{(1)}(k) \, e^n_{k-1 \atop \sum_{i=1} s_i+j}, \sum_{k,j} (\alpha_j^k + {}'\alpha_j^k - \epsilon_j^k 2^{l_k}) \, e^n_{k-1 \atop \sum_{i=1} s_i+j}).$$

where

$$\epsilon_j^k = \begin{cases} 0, & \text{if} \quad \alpha_j^k + {}'\alpha_j^k < 2^{l_k}, \\ 1, & \text{if} \quad \alpha_j^k + {}'\alpha_j^k \geqslant 2^{l_k}. \end{cases}$$

This addition is associative, makes Coker $Sq^2 \times H^n(K;Z)_{(2)}$ a group. That is the group $E_{T(1)}(\text{Coker } Sq^2, H^n(K;Z)_{(2)})$. Now, let

$$T^{(1)}(k)\, e^n_{\sum_{i=1}^{k-1} s_i + j} = \sum_{\xi=1}^{u} \alpha^k_{j\xi}\, e^{n+1}_\xi, \quad j = 1,\ldots,s_k,\ k = 1,\ldots,r,$$

then $\tilde{e}^{n+1}_i = (e^n, 0)$, $i = 1,\ldots,u$, $\tilde{e}^n_{\sum_{i=1}^{k-1} s_i + j} = (0, e^n_{\sum_{i=1}^{k-1} s_i + j})$,

$j = 1,\ldots,s_k$, $k = 1,\ldots,r$, is a set of generators of $E_{T(1)}(\text{Coker } Sq^2, H^n(K;Z)_{(2)})$, and the relations are

$$\begin{cases} 2\tilde{e}^{n+1}_i = 0, \quad i = 1,\ldots,u, \\ 2^{l_k}\tilde{e}^n_{\sum_{i=1}^{k-1} s_i + j} = \sum_{\xi=1}^{u} \alpha^k_{j\xi}\, \tilde{e}^{n+1}_\xi, \quad j = 1,\ldots,s_k,\ k = 1,\ldots,r. \end{cases}$$

It is not hard to show[6]

$$\pi^n(K)_{(2)} \cong E_{T(1)}(\text{Coker } Sq^2, H^n(K;Z)_{(2)}).$$

So we have

Theorem 4. Let K be an N-dimensional CW complex. $H^n(K;Z)_{(2)}$, Coker Sq^2 and their generators are as above, then the group $\pi^n(K)_{(2)}$ has a presentation

$$< \tilde{e}_1^{n+1},\ldots,\tilde{e}_u^{n+1},\tilde{e}_1^n,\ldots,\tilde{e}^n_{\sum_{i=1}^{r} s_i} \mid 2\tilde{e}_i^{n+1} = 0,\ i = 1,\ldots,u,$$

$$2^{l_k}\tilde{e}^n_{\sum_{i=1}^{k-1} s_i+j} = \sum_{\xi=1}^{u} \alpha^k_{j\xi}\tilde{e}_\xi^{n+1},\ j=1,\ldots,s_k,\ k=1,\ldots,r>$$

where $\tilde{e}_i^{n+1}$, $\tilde{e}^n_{\sum_{i=1}^{k-1} s_i+j}$ are corresponding to the generators e_i^{n+1}, $e^n_{\sum_{i=1}^{k-1} s_i+j}$ respectively, and $(\alpha^k_{j\xi})$ is the matrix representation of $T^{(1)}(k)$ with respect to the ordered bases

$$\{ e_1^n,\ldots,e^n_{\sum_{i=1}^{r} s_i} \} \text{ and } \{ e_1^{n+1},\ldots,e_u^{n+1}\} .$$

For the group $\pi^m(K)_{(2)}$, we have similar results. But the situation is more complicated. In fact, instead of $T^{(1)}(k)$, we have three groups of homomorphisms, namely

$$T_2^{(2)}(k):\ {}_{2^{l_k}}\ker Sq^2 \to \operatorname{Coker} Sq^2 ,$$

$$T_{2,0}^{(2)}(k):\ \ker T_2^{(2)}(K) \to \operatorname{Coker} \Phi ,$$

and

$$T_1^{(2)}:\ \operatorname{Coker} Sq^2 \to \operatorname{Coker} \Phi ,$$

where

$$\ker Sq^2(\subset H^m(K;Z)),$$

$$\operatorname{Coker} Sq^2 = \frac{H^{m+1}(K;Z_2)}{Sq^2H^{m-1}(K;Z)},$$

$$\operatorname{Coker} \Phi = \text{a factor group of } H^{m+2}(K;Z_2).$$

Using these homomorphisms, we can obtain a presentation of $\pi^m(K)_{(2)}$, see [6].

3. In theorem 4, we obtain a presentation of the group $\pi^n(K)_{(2)}$. But, for a fixed integer $t > 0$, we do not know how many copies of Z_{2t} are contained in $\pi^n(K)_{(2)}$.

Now we introduce the $T^{(1)}$- torsions as follows.

Definition. Let K be an N-dimensional CW complex. Assume $H^n(K;Z)_{(2)}$ and Coker Sq^2 are as above. Then we have

$$T^{(1)}(k) : {}_{2}1_kH^n(K;Z)_{(2)} \to \operatorname{Coker} Sq^2.$$

Define

$$\sigma_k = \dim \bigcup_{i\leqq k} \operatorname{Im} T^{(1)}(i) - \dim \bigcup_{i<k} \operatorname{Im} T^{(1)}(i),\ k = 1,\ldots,r,$$

$$\sigma_{r+1} = u - \sum_{k=1}^{r} \sigma_k.$$

and call them the $T^{(1)}(k)$ torsions of K.

Obviously, the $T^{(1)}(k)$ torsions are all homotopy numerical invariants

Using these invariants, we can list the generators and their order of cohomotopy group $\pi^n(K)_{(2)}$ as follows[6].

Theorem 5. Let k be an N-dimensional CW complex. Then

$$\operatorname{rank} \pi^n(K) = \operatorname{rank} H^n(K;Z)$$

$$\pi^n(K)_{(p)} = H^n(K;Z)_{(p)},\ p>2,$$

and $\pi^n(K)_{(2)}$ is determined by $H^n(K;Z)_{(2)}$, dimension of Coker Sq^2 over Z_2 and $T^{(1)}(k)$ torsions. More precisely, if $H^n(K;Z)_{(2)}$, and Coker Sq^2 are as above, then

$$\pi^n(K)_{(2)} = \bigoplus_{k=1}^{r} \underbrace{Z_{2^{l_k+1}}+\dots+Z_{2^{l_k+1}}}_{\sigma_k} + \bigoplus_{k=1}^{r} \underbrace{Z_{2^{l_k}}+\dots+Z_{2^{l_k}}}_{s_k-\sigma_k} \underbrace{+Z_2+\dots+Z_2}_{\sigma_{r+1}},$$

where σ_k is the $T^{(1)}(k)$ torsion of K.

In a similar way, we can define the $T_2^{(2)}(k)$ torsions, $T_{2,0}^{(2)}(k)$ torsions and the relative $T_1^{(2)}$ torsions of K. As an application of these invariants, we can list the generators and their order of $\pi^m(K)_{(2)}$, see [6].

4. We mention that the operations $T_2^{(1)}(k)$ are only defined for $n = N - 1$, and operations $T^{(2)}(k)$ for $m = N - 2$. Now we generalize the definition of operations $T^{(1)}(k)$ for all s as follows.

At first, we recall the definition of $T^{(1)}(k)$.

$$\begin{array}{ccccc} \pi^{n-1}(K^{n-1},K^{n-2}) & \xrightarrow{k} & \pi^n(K,K^{n-1}) & \xrightarrow{j} & \pi^n(K^n,K^{n-1}) \\ & & \uparrow i & & \\ \pi^{n-1}(K^n \quad ,K^{n-1}) & \xrightarrow{k} & \pi^n(K,K^n \quad) & \xrightarrow{j} & \pi^n(K^{n+1},K^n) \end{array} \tag{3}$$

Let $\{z\} \in {}_2 1_k H^n(K;Z)$ and $z \in \pi^n(K^n,K^{n-1})$ be one of its representations (see (3)), then there exists an element $\alpha \in \pi^{n-1}(K^{n-1},K^{n-2})$ such that

$$jk(\alpha) = 2^1 k z.$$

So α is a cocycle mod 2.

Note that there is also an element $\beta \in \pi^n(K,K^{n-1})$ such that

$$j(\beta) = z.$$

Now $k(\alpha) - 2^1 k\beta \in \pi^n(K,K^{n-1})$ satisfies the equation

$$j(k(\alpha) - 2^1 k_\beta) = 2^1 k_z - 2^1 k_z = 0,$$

so there is $\gamma \in \pi^n(K,K^n)$ such that

$$i\gamma = k\alpha - 2^1 k\ \beta.$$

Then, by definition, $T^{(1)}(k)(\{z\}) = \{\gamma\} \in \text{Coker } Sq^2$.

Now we analyse this process. First we obtain a cocycle α mod 2, it is from the class $\{z\} \in {}_{2}l_kH^n(K;Z)$. Secondary, we obtain the class $\{\gamma\} \in \text{Coker } Sq^2$ from the mod 2 class α. We know that[3] the second step is exactly equivalent to the action of the Steenrod squares. So, if we can generalize the first step, then we can generalize the operation $T^{(1)}(k)$.

Now we do this as follows.

Let $\{z\} \in {}_{2}l_kH^s(K;Z)$ and $z \in C^s(K;Z)$ be one of its representations, then there exists $\alpha \in C^{s-1}(K;Z)$ such that

$$\delta\alpha = 2^{lk} z.$$

Now α is a cocycle mod 2, but it is not unique. Obviously, the indeterminacy is an integral cocycle. So we can define

$$\partial_k : {}_{2}l_k H^s(K;Z) \to \frac{H^{s-1}(K;Z_2)}{\mu H^{s-1}(K;Z)}$$

as

$$\partial_k(\{z\}) = \{\alpha\} ,$$

where μ is the natural homomorphism.

In this way, we can generalize $T^{(1)}(k)$, and define the cohomology operation

$${}^{s}T^{(1)}(k) : {}_{2}l_kH^s(K;Z) \to \text{Coker } Sq^2$$

as

$$ {}^{s}T^{(1)}(k) = Sq^2 \cdot \partial_k . $$

Obviously, we have

Theorem 6. Let K be an N-dimensional complex and $n = N - 1$. Then on ${}_2 1_k H^n(K;Z)$, we have operations ${}^{n}T^{(1)}(k)$ and $T^{(1)}(k)$. But they are coincide:

$$ {}^{n}T^{(1)}(k) = T^{(1)}(k) : {}_2 1_k H^n(K;Z) \to \text{Coker } Sq^2 . $$

The operation $T^{(1)}(k)$ has properties (i) and (ii) (see §2)

Now we consider the corresponding properties for operation ${}^{s}T^{(1)}(k)$.

Theorem 7. The operation

$$ {}^{s}T^{(1)}(k) = Sq^2 \cdot \partial_k : {}_2 1_k H^s(K;Z) \to \text{Coker } Sq^2 $$

has the property (ii), that is if $\{z\} \in {}_2 1_k H^s(K;Z)$ and $k>t$, then

$$ {}^{s}T^{(1)}(k) \quad (\{z\}) = 0. $$

Proof. It is sufficient to prove

$$ \partial_k(\{z\}) = 0. $$

Let $\alpha \in C^{s-1}(K;Z)$ imply

$$\delta \alpha = 2^{l_t} z,$$

where $z \in C^s(K;Z)$ is a representation of $\{z\}$. Then we have

$$\delta(2^{l_k - l_t}\alpha) = 2^{l_k} z.$$

By definition, $\partial_k\{z\} = \{2^{l_k - l_t}\alpha\}$. Now $2^{l_k-l_t}\alpha$ as a cocycle mod 2 is 0, so $\partial_k\{z\} = 0$.

In general, the kernel of

$$j : \pi^s(K) \to H^s(K;Z)$$

is not Coker Sq^2(cf. (2)). But on ker j, we can define

$$j' : \ker j \to \text{Coker } Sq^2.$$

Let $e \in \pi^s(K)$ and the order of $j(e)$ be 2^{l_k}. Then $2^{l_k}e \in \ker j$. Using the method of [6], we can prove

$$j'(2^{l_k}e) = Sq^2 \circ \partial_k(j(e)).$$

5. Using the operations

$${}^sT^{(l)}(k) : {}_{2^{l_k}}H^s(K;Z) \to \text{Coker } (Sq^2 : H^{s-1}(K;Z) \to H^{s+1}(K;Z_2))$$

we define

$$s\sigma_k = \dim \bigcup_{i \leqslant k} \text{Im } {}^sT^{(l)}(i) - \dim \bigcup_{i < k} \text{Im } {}^sT^{(l)}(i), \; k=1,\ldots,r,$$

and call them the ${}^{s}T^{(1)}(k)$ torsions of K.

Let K be an N-dimensional CW complex and let K^n be its n-skelton. Let $n > 0$ and let K_n be the complex formed by shinking K^{n-1} to a point e° which is not in $K-K^{n-1}$ and is the single 0-cell of K_n. Now we discuss the relation between ${}^{s}T^{(1)}(k)$ torsions of K and of K^n, K_n.

We first consider K^n.

Theorem 8.

${}^{s}T^{(1)}(k)$ torsion of K^n = ${}^{s}T^{(1)}(k)$ torsion of K,

if $s \leq n-1$,

${}^{s}T^{(1)}(k)$ torsion of $K^n = 0$, if $s > n-1$.

Proof. Obviously,

$$i^*_{s+1} : H^{s+1}(K;Z_2) \cong H^{s+1}(K;Z_2), \quad \text{if} \quad s+1 < n,$$

and

$$i^*_r : H^r(K;Z) \cong H^r(K^n;Z), \quad \text{if } r = s,\ s-1,$$

where i^*_m is induced by the identical map $i : K^n \to K$.

Whence

${}^{s}T^{(1)}(k)$ torsion of K^n = ${}^{s}T^{(1)}(k)$ torsion of K, if $s < n-1$. On considering the normal form of the incidence matrix for

$$\delta : C^n \to C^{n+1}.$$

where C^m is the group of integral m-cochains in K, we see that

$$i_n^* : H^n(K, Z_2) \to H^n(K^n, Z_2)$$

is monomorphism, and

$$H^n(K^n, Z_2) = i_n^* H^n(K, Z_2) + \mu H_\circ^n ,$$

where $H_\circ^n$ is a free summand of $H^n(K^n, Z)$ which arises from the basic cochain $c \in C^n$ such that $\delta c = (2l+1)c'$ $(l \geqslant 0)$ and

$$\mu : H^n(K^n, Z) \to H^n(K^n, Z_2)$$

is the natural homomorphism.

Since

$$i_r^* : H^r(K,Z) \cong H^r(K^n,Z), \text{ if } r = n-1,\ n-2,$$

then we have

${}^{n-1}T^{(l)}(k)$ torsion of K^n = ${}^{n-1}T^{(l)}(k)$ torsion of K. But

$$\dim \left(\frac{H^n(K^n,Z_2)}{Sq^2 H^{n-2}(K^n,Z)}\right) - \sum_k {}^{n-1}\sigma_k \text{ of } K^n \geqslant \dim\left(\frac{H^n(K,Z_2)}{Sq^2 H^{n-2}(K,Z)}\right) - \sum_k {}^{n-1}\sigma_k$$

of K.

When $s+1 > n$, then

$$H^{s+1}(K^n, Z_2) = 0,$$

whence

$$\text{Coker}\ (\ Sq^2 : H^{s-1}(K^n,Z) \to H^{s+1}(K^n,Z_2)) = 0.$$

We have

$$^{s}T^{(1)}(k) \text{ torsion of } K^n = 0, \text{ if } s+1 > n.$$

Now we consider K_n.

Theorem 9.

$$^{s}T^{(1)}(k) \text{ torsion of } K_n = 0, \text{ if } s < n+1,$$

$$^{s}T^{(1)}(k) \text{ torsion of } K_n = {}^{s}T^{(1)}(k) \text{ torsion of } K,$$

$$\text{if } s \geqslant n+1.$$

Proof. When $s < n$, then

$$H^s(K_n, Z) = 0,$$

whence

$$^{s}T^{(1)}(k) \text{ torsion of } K_n = 0, \text{ if } s < n.$$

If $s = n$, then $H^n(K_n, Z)$ is a free Abelian group since K_n has no (n-1)-dimensional torsion. So in this case

$$^{n}T^{(1)}(k) \text{ torsion of } K_n = 0.$$

Let $\iota : K \to K_n$ be the identification map. Since $\iota | K-K^{n-1}$ is a homeomorphism onto $K_n - e^\circ$ which maps each cell of $K-K^{n-1}$ on a cell of $K_n - e^\circ$, it follows that

$$\iota_n^* : H^n(K_n, Z) \to H^n(K, Z) \tag{4}$$

is an epimorphism, and

$$\iota_s^* : H^s(K_n, G) \to H^s(K, G) \tag{5}$$

is a isomorphism if $s > n$. In the commutative diagram

$$\begin{array}{ccc} H^n(K_n,Z) & \xrightarrow{Sq^2} & H^{n+2}(K_n,Z_2) \\ \downarrow \iota^* & & \downarrow \iota^* \\ H^n(K,Z) & \xrightarrow{Sq^2} & H^{n+2}(K,Z_2), \end{array}$$

note (4) is an epimorphism and (5) is an isomorphism, the homomorphism ι^* induces isomorphism

$$\iota^* : \frac{H^{n+2}(K_n,Z_2)}{Sq^2H^n(K_n,Z)} \cong \frac{H^{n+2}(K,Z_2)}{Sq^2H^n(K,Z)}.$$

Then from the commutative diagram

$$\begin{array}{ccc} {}_{2^l k}H^{n+1}(K_n,Z) & \xrightarrow{{}^{n+1}T^{(l)}(k)} & \dfrac{H^{n+2}(K_n,Z_2)}{Sq^2H^n(K_n,Z)} \\ \iota^* \downarrow \cong & & \iota^* \downarrow \cong \\ {}_{2^l k}H^{n+1}(K,Z) & \xrightarrow{{}^{n+1}T^{(l)}(k)} & \dfrac{H^{n+2}(K,Z_2)}{Sq^2H^n(K,Z)}, \end{array}$$

we have

${}^{n+1}T^{(l)}(k)$ torsion of K_n = ${}^{n+1}T^{(l)}(k)$ torsion of K.

If $s \geqslant n+2$, then from (5) is an isomorphism, we have

${}^{s}T^{(l)}(k)$ torsion of K_n = ${}^{s}T^{(l)}(k)$ torsion of K.

Let $\pi : E \to B$ be a map of E onto B. Map $F : B \to E$ is called a section of π if $\pi F : B \to B$ is the identity of B.

Theorem 10. If $\pi : E \to B$ has a section, then

$$\sum_{i=1}^{k} {}^s\sigma_i \text{ of } E \geqslant \sum_{i=1}^{k} {}^s\sigma_i \text{ of } B, \; k = 1, 2, \ldots$$

Proof. Let $F : B \to E$ be a section of π, then from $\pi F = 1_B$, we have

$$\pi_s^* : H^s(B,G) \to H^s(E,G)$$

is a monomorphism. In fact, we have

$$H^s(E,G) = \ker F_s^* + \operatorname{Im} \pi_s^*$$

$$= \ker F_s^* + \pi_s^* H^s(B,G).$$

Whence

$$\frac{H^{s+2}(E,Z_2)}{Sq^2 H^s(E,Z)} = \frac{\ker F_{s+2}^* + \pi_{s+2}^* H^{s+2}(B,Z_2)}{Sq^2(\ker F_s^* + \pi_s^* H^s(B,Z)} =$$

$$= \frac{\ker F_{s+2}^*}{Sq^2 \ker F_s^*} + \frac{\pi_{s+2}^* H^{s+2}(B,Z_2)}{Sq^2 \pi_s^* H^s(B,Z)} .$$

Hence from the commutative diagram

$$\begin{array}{ccc} H^s(B,Z) & \xrightarrow{Sq^2} & H^{s+2}(B,Z_2) \\ \downarrow \pi_s^* & & \downarrow \pi_{s+2}^* \\ H^s(E,Z) & \xrightarrow{Sq^2} & H^{s+2}(E,Z_2) \end{array}$$

we obtain

$$\pi^* : \frac{H^{s+2}(B,Z_2)}{Sq^2H^s(B,Z)} \longrightarrow \frac{H^{s+2}(E,Z_2)}{Sq^2H^s(E,Z)}$$

is a monomorphism. Then from the commutative diagram

$$\begin{array}{ccc} {}_2{}^1{}_k H^{s+1}(B,Z) & \xrightarrow{{}^{s+1}T^{(1)}(k)} & \dfrac{H^{s+2}(B,Z_2)}{Sq^2H^s(B,Z)} \\ \downarrow \pi^* & & \downarrow \pi^* \\ {}_2{}^1{}_k H^{s+1}(E,Z) & \xrightarrow{{}^{s+1}T^{(1)}(k)} & \dfrac{H^{s+2}(E,Z_2)}{Sq^2H^s(E,Z)} \end{array}$$

we have

$$\sum_{i=1}^{k} {}^s\sigma_i \text{ of } E \geqslant \sum_{i=1}^{k} {}^s\sigma_i \text{ of } B, \; K = 1,2,\ldots$$

6. Remark. In the previous paragraphes we have discussed the generalization of $T^{(1)}(k)$ torsions and obtained their properties. We can use the same stratege to generalize the $T^{(2)}(k)$-torsions, and obtaine the similar properties. The details will be published elsewhere.

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