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THE FINITENESS OBSTRUCTION OF THE HOMOTOPY MIXING OF TWO CW COMPLEXES

Zdzislaw Wojtkowiak*

The aim of this paper is to compute the finiteness obstruction of the homotopy mixing of two CW-complexes. We begin with the following definition due to G. Mislin.

Definition 1. A CW complex X is of type FP if the singular chain complex of the universal cover of X is $Z[\pi_1(X)]$-chain homotopy equivalent to a complex of finite length $0 \to P_n \to P_{n-1} \to \dots\dots \to P_0 \to 0$ where each P_i is a finitely generated and projective $Z[\pi_1(X)]$-module.

If X is of type FP, the finiteness obstruction of C.T.C. Wall is defined by

$$w(X) = \sum_{i=0}^{n} (-1)^i [P_i] \in \tilde{K}_0(Z[\pi_1(X)]).$$

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$w(X)$ does not depend on the choice of P_* and it is clear that it vanishes if X is finite. If $\pi_1(X)$ is finitely presented then the converse is also true ([3] Theorem 13).

Now I give a definition of a homotopy mixing I shall use. It will be a sort of a fibrewise homotopy mixing. Let $F \to E \to B_{(*)}$ be a fibration with a nilpotent fibre F. A fibration $F_1 \to E_1 \to B(**)$ is called a fibrewise P-localization of the fibration (*) if there is a map over B of the fibration (*) into the fibration (**) which on fibres induces the ordinary P-localization(see [2] ch.I.§8.). The fibrewise localization of a fibration $\tilde{X} \to X \to K(\pi_1(X);1)$ I shall denote by $(\tilde{X})_P \to X_P^f \to K(\pi_1(X);1)$.

Suppose that X and Y are two spaces with fundamental groups isomorphic to π. I assume that isomorphisms are fixed so the fundamental groups I shall denote simply by π. Let $P \cup R$ be a partition of all primes and let $h:(X)_0^f \to (Y)_0^f$ be a homotopy equivalence which induces the identity on π.

The homotopy pullback of the following diagram

$$X_P^f \to (X)_0^f \xrightarrow{h} (Y)_0^f \leftarrow Y_R^f$$

we shall call a fibrewise mixing of X and Y at P and R.

Let us denote by Z the homotopy pullback of this diagram. Then the universal cover $\tilde{Z}$ is the usual Zabrodsky mixing of $\tilde{X}$ and $\tilde{Y}$.

Now we shall state our main results. We shall consider only the case if π is finite.

Theorem 1. If X and Y are of type FP then Z is also of type FP.

To formulate our next result we shall use some results from algebraic K-theory. Consider the following cartesian diagram:

$$\begin{array}{ccc} Z[\pi] & \to & Z_R[\pi] \\ \downarrow & & \downarrow \\ Z_p[\pi] & \to & Q[\pi] \end{array}$$

Lemma 1. There is an associated Mayer-Vetoris exact sequence:

$$K_1(Z_p[\pi]) \oplus K_1(Z_R[\pi]) \to K_1(Q[\pi]) \xrightarrow{\partial P} \widetilde{K}_0(Z[\pi]) \to \widetilde{K}_0(Z_p[\pi]) \oplus \widetilde{K}_0(Z_R[\pi])$$

Proof follows by combining together two exact sequences from [1] Chapter IX (6.3) Theorem and using modules which have projective resolutions of length ≤ 1.

The homotopy equivalence $H : (X)_0^f \to (Y)_0^f$ induces a $Q[\pi]$-chain homotopy equivalence $C_*(\tilde{h}) : C_*(\widetilde{X}_0^f) \otimes Q \to C_*(\widetilde{Y}_0^f) \otimes Q$. The singular complexes $C_*(\widetilde{X}_0^f) \otimes Q$ and $C_*(\widetilde{X}) \otimes Q$ are chain homotopy equivalent (resp. $C_*(\widetilde{Y}_0^f) \otimes Q$ and $C_*(\widetilde{Y}) \otimes Q$). Hence h induces a chain equivalence $H : C_*(\widetilde{X}) \otimes Q \to C_*(\widetilde{Y}) \otimes Q$. We can suppose that $C_*(\widetilde{X}) \otimes Q$ (resp. $C_*(\widetilde{Y}) \otimes Q$) is a cellular chain complex of $\widetilde{X}$ (resp. $\widetilde{Y}$) because the cellular chain complex and the singular one are natural chain homotopy equivalent. Hence if X and Y are finite H is a chain homotopy equivalence between based complexes and its torsion $\tau(H) \in K_1(Q[\pi])$ is defined. Now we can state our second result.

Theorem 2. Suppose that X and Y are finite complexes with finite fundamental group π. Then

$$w(Z) = \partial_p(\tau(H))$$

If X and Y are homologically nilpotent i.e. π acts nilpotently on homology of universal covers we can prove much more.

The group ring $Q[\pi]$ splits in the following way $Q[\pi] = (Q[\pi])^{\pi} \oplus Q[\pi]/_{(\Sigma)}$ where $\Sigma = \sum_{g \in \pi} g$.

This splitting induces the corresponding splitting of chain complexes and chain maps. Hence

$$C_*(\widetilde{X}) \otimes Q = (C_*(\widetilde{X}) \otimes Q)^{\pi} \oplus A_*(X) \text{ and } H = H_1 \oplus H_2,$$

where $(C_*(\widetilde{X}) \otimes Q)^{\pi}$ is a Q-module and $A_*(X)$ is a $Q[\pi]_{/(\Sigma)}$-module. If X and Y are homologically nilpotent then $H_*(C_*(\widetilde{X}) \otimes Q)$ and $H_*(C_*(\widetilde{Y}) \otimes Q)$ are trivial π-modules. Hence complexes $A_*(X)$ and $A_*(Y)$ are acyclic and the Reidemeister torsions $\tau(X)$ of $A_*(X)$ and $\tau(Y)$ of $A_*(Y)$ are defined. We have

$$\tau(H) = (\tau(H_1);\tau(H_2)) \in K_1(Q) \times K_1(Q[\pi]/_{(\Sigma)}),$$

$$\tau(H_1) = \prod_i \det H_{2i}(H) \Big/ \prod_i \det H_{2i+1}(H) = \det H \text{ and}$$

$\tau(H_2) = \tau(X) - \tau(Y)$. Hence follows our third theorem.

Theorem 3. If X and Y are homologically nilpotent and finite with finite fundamental group π then

$$w(Z) = \partial_p(\det(H),1) + \partial_p(1,\tau(X) - \tau(Y)).$$

One can show that $\partial_p(\det(H);1) \in \operatorname{im}(K_1(Z/_{|\pi|}) \to \widetilde{K}_0(Z[\pi]))$.

Now I give a sketch of proofs. Let $C(H)_*$ be a mapping cone of the following map:

$$C_*(\widetilde{X}) \otimes Z_p \oplus C_*(\widetilde{Y}) \otimes Z_R \longrightarrow C_*(\widetilde{X}) \otimes Q \oplus C_*(\widetilde{Y}) \otimes Q \xrightarrow{H-id} C_*(\widetilde{Y}) \otimes Q \quad (1)$$

and let $K_*(\widetilde{Z})$ be a mapping cone of

$$C_*(\widetilde{Z}) \otimes Z_p \oplus C_*(\widetilde{Z}) \otimes Z_R \longrightarrow C_*(\widetilde{Z}) \otimes Q.$$

One can show that $C(H)_*$ and $C_*(\widetilde{Z})$ map into $K_*(\widetilde{Z})$ inducing an isomorphism on homology. It follows from (1) that $H_i(C(H))$ is a finitely generated $Z[\pi]$-module for every i and $H^N(C(H);M) = 0$ for N big enough and M an arbitrary $Z[\pi]$-module. Therefore one can find a complex P_* of finite length such that every P_i is finitely generated and projective and a map $f : P_* \to C(H)_*$ inducing an isomorphism on homology.

Hence it follows that Z is of type FP. Let

$d_1 : C(H)_* \to C_*(\widetilde{X}) \otimes Z_P$ and $d_2 : C(H)_* \to C_*(\widetilde{Y}) \otimes Z_R$ be given by $d_1(y,x_1,y_1) = (-1)^{n-1}x_1$, $d_2(y,x_1,y_1) + (-1)^{n-1}y_1$ where $y \in C_n(\widetilde{Y}) \otimes Q$, $x_1 \in C_{n-1}(\widetilde{X}) \otimes Z_P$ and $y_1 \in C_{n-1}(\widetilde{Y}) \otimes Z_R$. Then $s_1 = (d_1 \otimes id_{Z_P}) \circ (f \otimes id_{Z_P})$ and $s_2 = (d_2 \otimes id_{Z_R}) \circ (f \otimes id_{Z_R})$ are chain homotopy equivalences. Let $s : C(H)_* \otimes Q \to C_*(\widetilde{Y}) \otimes Q$ be given by $s(y, x_1, y_1) = (-1)^n y$. Then s is a chain homotopy between $H \circ (d_1 \otimes id_Q)$ and $d_2 \otimes id_Q$. Hence $H \circ (s_1 \otimes id_Q)$ are also chain homotopic. We can assume that $P_* \otimes Z_P$ and $P_* \otimes Z_R$ are free and based. Hence $P_* \otimes Q$ has two bases and let $\tau(P_* \otimes Q)$ be the torsion of the identity with respect to these bases. It follows immediately from the definition of ∂_P that

$$\partial_P(\tau(P_* \otimes Q)) = w(P_*) = w(Z).$$

From the diagram

$$\begin{array}{ccc} P_* \otimes Q & \xrightarrow{\ id\ } & P_* \otimes Q \\ \downarrow{\scriptstyle s_1 \otimes id_Q} & & \downarrow{\scriptstyle s_2 \otimes id_Q} \\ C_*(\widetilde{X}) \otimes Q & \xrightarrow{\ H\ } & C_*(\widetilde{Y}) \otimes Q \end{array}$$

which commutes up to homotopy we obtain that

$$\tau(P_* \otimes Q) = \tau(s_2 \otimes id_Q)^{-1} \cdot \tau(H) \cdot \tau(s_1 \otimes id_Q).$$

But $\tau(s_1 \otimes id_Q) \in K_1(Z_P[\pi])$ and $\tau(s_2 \otimes id_Q) \in K_1(Z_R[\pi])$. Hence $\partial_P(\tau(P_* \otimes Q)) = \partial_P(\tau(H))$ and the proofs of Theorems 1 and 2 are finished.

The details will appear elsewhere.

References

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Uniwersytet Warszawski
Instytut Matematyki
Pałac Kultury i Nauki 9p
00-901 Warszawa

and

University of Oxford
Mathematical Institute
24/29 St. Giles
Oxford OX1 3LB.