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FORMAL GROUPS AND RING STRUCTURES FOR CERTAIN PERIODIC COHOMOLOGY THEORIES

Urs Würgler

The purpose of this talk is to report on some results concerning a classification problem for a special kind of cohomology theories. For the beginning, however, I would like to consider a different and perhaps more concrete problem which may serve as motivation for the rest.

Let $K^*(-)$ denote ordinary complex K-theory. We consider it as a $\mathbb{Z}/2$-graded theory defined on the category $\underline{CW}_*$ of pointed spaces of the homotopy type of a CW-complex. Recall that $K^0(S^0) \cong \mathbb{Z}$, $K^1(S^0) = 0$ and that there is a natural equivalence $\beta: K^0(X) \xrightarrow{\sim} K^0(S^2 \wedge X)$, the Bott isomorphism. $K^*(-)$ is usually considered as a multiplicative theory, the product being induced by the tensor product operation of complex vector bundles. We may ask the following

Question 1: Are there products in $K^*(-)$ different from the ordinary one and, if so, can one describe the set Prod(K) of all isomorphism classes of such products in some reasonable way ?

Note that all products we consider here are assumed to be with unit, associative and commutative in the graded sense. Moreover two products $\mu, \mu': K^*(X) \otimes K^*(X) \to K^*(X)$ are called isomorphic if there is an isomorphism $\theta: K^*(-) \to K^*(-)$ of cohomology theories with values in the category Ab of abelian groups (an additive isomorphism) such that the following diagram commutes:

$$\begin{array}{ccc} K^*(-)\otimes K^*(-) & \xrightarrow{\mu} & K^*(-) \\ \downarrow{\scriptstyle \theta\otimes\theta} & & \downarrow{\scriptstyle \theta} \\ K^*(-)\otimes K^*(-) & \xrightarrow{\mu'} & K^*(-) \end{array}$$

To answer the question above one could certainly try to construct elements $\mu \in K^0(BU \wedge BU)$ with appropriate properties and then determine the set of all such elements. Here, however, we will adopt a different point of view.

Let A be an ungraded commutative ring with unit. For any such ring A we consider the set C(A) of all isomorphism classes [T] of $\mathbb{Z}/2$-graded multiplicative cohomology theories $T^*(-)$ with coefficient ring $T^*(S^0)$ of the form $T^0(S^0) = A$, $T^1(S^0) = 0$ ($\mathbb{Z}/2$-graded ring theories with coefficients A for short). Clearly, $[K] \in C(\mathbb{Z})$. Using this notation we ask the following- unprecise- question:

Question 2: Given a ring A, can one describe the set C(A) or at least some interesting subsets of C(A) in an explicit way?

Of course, this is just the classification problem for $\mathbb{Z}/2$-graded ring theories with coefficients A.

Now we remark that there is a connection between Question 1 and Question 2. Put $A = \mathbb{Z}$ and let $C_K(\mathbb{Z})$ denote the subset of

$C(\mathbb{Z})$ whose elements are all isomorphism classes $[T] \in C(\mathbb{Z})$ with the property that $T^*(-)$ is additively isomorphic to $K^*(-)$, i.e. $T^*(-) \cong K^*(-)$ as $\mathbb{Z}/2$-graded cohomology theories with values in the category $\underline{Ab}$. Suppose $[T] \in C_K(\mathbb{Z})$, let $\theta: T^*(X) \xrightarrow{\sim} K^*(X)$ be an additive equivalence and suppose $\alpha: T^*(X) \otimes T^*(X) \to T^*(X)$ is a product on $T^*(-)$. Then $\theta \circ \alpha \circ (\theta^{-1} \otimes \theta^{-1}): K^*(X) \otimes K^*(X) \longrightarrow K^*(X)$ defines a product on $K^*(-)$. Moreover, different equivalences θ and isomorphic products on $T^*(-)$ produce isomorphic products on $K^*(-)$ and one sees easily that there is a bijection

$$(*) \qquad C_K(\mathbb{Z}) \xrightarrow{\sim} \mathrm{Prod}(K)$$

defined by $\alpha \mapsto \theta \circ \alpha \circ (\theta^{-1} \otimes \theta^{-1})$. This leads us to study the problem raised by Question 2 in more detail.

Observe that any $\mathbb{Z}/2$ - graded ring theory $T^*(-)$ with coefficients an ungraded ring A is automatically a complex-orientable theory, i.e. the canonical complex line bundle η_∞ over $\mathbb{C}P_\infty$ is $T^*(-)$-orientable. This follows immediately from [1], p.399.Let $m: \mathbb{C}P_\infty \times \mathbb{C}P_\infty \longrightarrow \mathbb{C}P_\infty$ be the classifying map of the bundle $\eta_\infty \times \eta_\infty$ and let $x \in T^0(\mathbb{C}P_\infty)$ be an Euler class of η_∞ (a $\mathbb{C}$- orientation of $T^*(-)$). Then, as is well known, the formal power series $F(x_1, x_2) := m^*(x) \in A[\![x_1, x_2]\!]$ is a one-dimensional commutative formal group law on A (a formal group on A for short), where $x_i = pr_i^*(x) \in T^*(\mathbb{C}P_\infty \times \mathbb{C}P_\infty)$. Now formal groups corresponding to different $\mathbb{C}$ - orientations of the same theory are isomorphic and isomorphic theories with the same coefficient ring produce isomorphic formal groups, so if we associate to any $\mathbb{Z}/2$-graded ring theory $T^*(-)$ with coefficients A its formal group we get a map

$$\Phi : C(A) \longrightarrow FG(A)$$

where FG(A) denotes the set of (strict) isomorphism classes of formal groups over A. We will use this map to get an answer to Question 2 in some particular cases.

Suppose first that A = k is a field. If the characteristic of k is 0, classical results imply that C(k) consists of only one element, namely $H^{**}(-;k)$. For fields of positive characteristic, however, the situation changes. First we have:

Theorem 1: Let k be a field of characteristic p> 2. Then the map $\Phi: C(k) \longrightarrow FG(k)$ is a bijection.

Remark: For p=2 we have only partial results. In this case, the map Φ is surjective but not injective. Difficulties arise from the fact that all elements of C(k) different from $H^{**}(-,k)$ are non-commutative.

Formal groups over fields of positive chracteristic are rather well understood (see for example the book [3]). In particular, there is an important isomorphism invariant for such formal groups F, their height $ht_F \in \mathbb{N} \cup \{\infty\}$. Briefly, $ht_F = n$ if $[p]_F(x) = ax^{p^n}$ + terms of higher order, $a \neq 0$, and $ht_F = \infty$ if $[p]_F(x) = 0$. Let $FG(k)^n$ denote the subset of FG(k) of formal groups of height n and put $C(k)^n = \Phi^{-1}(FG(k)^n)$. Then $FG(k) = \cup_{n=1}^{n=\infty} FG(k)^n$ and $C(k) = \cup_{n=1}^{n=\infty} C(k)^n$. The next theorem tells us that $\mathbb{Z}/2$ - graded ring theories with coefficients k and formal groups of equal height are very strongly related, in fact they only differ by their multiplicative structure:

Theorem 2: Let p be any prime and suppose $T_1^*(-)$, $T_2^*(-)$ are $\mathbb{Z}/2$-graded ring theories with coefficients k, a field of characteristic p. Then $T_1^*(-)$ and $T_2^*(-)$ are isomorphic as cohomology theories with values in the category of k-vectorspaces if and only if their formal groups are of the same height.

Recall from [2],[4] that for any prime p and any positive integer n the $\mathbb{Z}/2$-graded version of the n-th Morava K-theory with coefficients k, $K(n)^*(-;k)$, represents an element of $C(k)^n$. For $n = \infty$ we set $K(\infty)^*(-;k) = H^{**}(-;k)$. Note also that $K(1)^*(-;\mathbb{F}_p) = K^*(-;\mathbb{F}_p)$. Using the same argument which lead to the bijection (*) we get from theorems 1 and 2 the

Corollary 3: Let k be a field of characteristic p>2. Then for all $n \in \mathbb{N} \cup \{\infty\}$ there are bijections

$$C(k)^n \xrightarrow{\sim} \mathrm{Prod}(K(n)^*(-,k)) \xrightarrow{\sim} FG(k)^n.$$

It should be noted that for $FG(k)^n$, there are several more or less explicit descriptions available (see e.g. [3]). Let us recall very briefly one of them. Consider the power series

$$\log_n(x) = \sum_{i \geq 0} p^{-i} x^{p^{in}} \in \mathbb{Q}[\![x]\!]$$

and put $\overline{F}_n(x,y) = \log_n^{-1}(\log_n(x) + \log_n(y))$. $\overline{F}_n(x,y)$ is a formal group over $\mathbb{Z}_{(p)}$. $F_n(x,y)$, its reduction mod p, is defined over $\mathbb{F}_p$ and so over every field of characteristic p. Let $\overline{k}_{sep}$ be a separable closure of k and $S_n = \mathrm{Aut}_{\overline{k}_{sep}}(F_n)$ the automorphism group of F_n over $\overline{k}_{sep}$. A classical result of Dieudonné-Lubin tells us that S_n is isomorphic to the group of units of the maximal order in the central division algebra D_n of invariant

$1/n$ and rank n^2 over $\mathbb{Q}_p$. Let Γ be the Galois group $Gal(\bar{k}_{sep}:k)$. Then Γ acts on S_n (by acting on the coefficients of power series) and there is a bijection

$$(**) \qquad FG(k)^n \xrightarrow{\sim} H^1(\Gamma, S_n) \ .$$

This bijection together with the fact that formal groups of infinite height over a ring of prime characteristic are isomorphic to the additive formal group imply the following

Corollary 4: *If k is a separably closed field of odd characteristic and* $n<\infty$ *or if k is an arbitrary field of positive characteristic and* $n=\infty$, *then, up to isomorphism,* $K(n)^*(-,k)$ *is the only* $\mathbb{Z}/2$ *-graded ring theory with coefficients k and formal group of height n.*

If $n = 1$, S_1 is isomorphic to the group $\mathbb{Z}_p^*$ of p-adic units. If $k = \mathbb{F}_{p^r}$ is a finite field, Γ is topologically generated by the Frobenius homomorphism and one obtains $H^1(\Gamma, \mathbb{Z}_p^*) \cong \mathbb{Z}_p^*$. So corollary 3 implies a bijection

$$C(\mathbb{F}_{p^r}) \xrightarrow{1\sim} Prod(K^*(-,\mathbb{F}_{p^r})) \xrightarrow{\sim} \mathbb{Z}_p^* \ .$$

For more general rings A we have only very partial results to offer for the moment and the question seems to be difficult. To end this talk, let me just describe some results for the case $A = \mathbb{Z}$. This will be enough to answer our initial Question 1. Let P denote the set of all primes and let $F(x,y)$ be a formal group over $\mathbb{Z}$. Define the height function of F, $ht_F: P \to \mathbb{N} \cup \{\infty\}$, by setting $ht_F(p)$ = height of F mod p over $\mathbb{F}_p$. It is an isomorphism invariant of F. Using this notion we get some sort of global version of Theorem 2.

Theorem 5: Let $T_1^*(-)$ and $T_2^*(-)$ be $\mathbb{Z}/2$ -graded ring theories with coefficients $\mathbb{Z}$ and formal groups F_1 resp. F_2. Then $T_1^*(-)$ and $T_2^*(-)$ are additively isomorphic if and only if $ht_{F_1}(p) = ht_{F_2}(p)$ for all primes p.

We do not know if the map $\Phi: C(\mathbb{Z}) \longrightarrow FG(\mathbb{Z})$ is surjective or injective in general although we have some partial results which we will not describe here. However, if we restrict our attention to the subset $C_K(\mathbb{Z})$ of $C(\mathbb{Z})$, we can be more precise.

Let $FG(\mathbb{Z})^1$ be the set of all isomorphism classes of formal groups F over $\mathbb{Z}$ of height 1 at any prime, i.e. $ht_F(p) = 1$ for all p, and let Φ_K denote the restriction of Φ to $C_K(\mathbb{Z})$.

Theorem 6: There are bijections

$$\mathrm{Prod}(K) \xrightarrow{\sim} C_K(\mathbf{Z}) \xrightarrow[\sim]{\Phi_K} FG(\mathbf{Z})^1 \xrightarrow{\sim} \prod_{p \in P} \mathbb{Z}_p^* \quad .$$

One may ask what all these new products on $K^*(-)$ described by theorem 6 are good for. It turns out that there are interesting connections between them and characteristic classes $c_\chi \in H^{**}(BU, \mathbf{Q})$ associated to certain integral Hirzebruch genera (i.e. ring homomorphisms)

$$\chi : \Omega_*^U \longrightarrow \mathbf{Z} \subset \mathbf{Q}$$

which can be described in terms of Riemann-Roch relations. Also, to any exotic product on $K^*(-)$ their corresponds a set of "exotic Adams operations" with interesting properties.

References

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Mathematisches Institut
der Universität
Sidlerstrasse 5

CH-3012 Bern