Pub. Mat. UAB
Vol. 26 Nº 3 Des. 1982

COHOMOLOGY OPERATIONS AND H-SPACES

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1. INTRODUCTION

The theory of cohomology operations and the theory of H-spaces were interlocked throughout their various stages of development:

The first systematic approach to the theory of (high order) cohomology operations is due to J. F. Adams ([Adams]). In that celabrated paper a solution was given to a question whose one formulation is the following: What spheres support continuous multiplications with units (i.e. H-structures)?

The cohomology operations of the simplest type are the Bockstein operations. These were tied together by Browder ($[\text{Browder}]_{1,2,3}$) to form the Bockstein spectral sequence which was used to study the cohomology of finite dimensional H-spaces.

$[\text{Zabrodsky}]_{1,2,3}$, $[\text{Kane}]_{1}$, $[\text{Lin}]_{1,2,3}$ and others used high order operations to farther analyze the cohomology of finite H-spaces. In particular, $[\text{Lin}]_{1,2}$ proved the classical "loop space congecture": The homology of the loop space of a finite dimensional H-space is torsion free .

$[\text{Hubbuck}]_{1,2,3}$ used k-theory operations to study the cohomology and topology of finite H-spaces. He found restrictions on their possible types and their Pontrjagin rings. Among other theorems he proved ($[\text{Hubbuck}]_{1}$) that a homotopy commutative finite H-space has the homotopy type of a torus.

Finally [Kane]$_{2,3}$ recently used BP operations to study the cohomology of H-spaces.

Going in the other direction, the theory of H-spaces was used in the constructions and evaluations of high order operations.

In the following lectures I shall try to demonstrate by some examples these relations between the two theories.

2. BASIC DEFINITIONS

We usually assume spaces to be of the homotopy type of CW complexes with a (non-degenerate) base point. Maps and homotopies are base point preserving. Thus, an H-space could be assumed to be a space X with a multiplication μ so that the base point x_0 is an actual unit: $\mu(x,x_0) = x = \mu(x_0,x)$.

The definition of a cohomology operation has various degrees of abstractions. One of the most general form is the following:

A cohomology operation Φ consists of three spaces and two maps $\Phi = \langle K_0, E, K_1, r, h\rangle$: $r\colon E \to K_0$, $h\colon E \to K_1$:

$$\begin{tikzcd}
E \arrow[r, "h"] \arrow[d, "r"'] & K_1 \\
K_0 &
\end{tikzcd}$$

Φ defines a "natural transformation" from $\operatorname{im}([\ \ , E] \to [\ \ , K_0])$ to the family of subsets of $[\ \ , K_1]$. In a more direct terms: For any space X Φ defines a function from a subset of the set $[X, K_0]$ of homotopy classes of maps $X \to K_0$ to the set of subsets of $[X, K_1]$ in the following way: The domain of Φ is the set $\operatorname{im}(r_* : [X, E] \to [X, K_0])$ where r_* is the left composition with r: $r_*([\hat{f}] = [r \circ \hat{f}]$ ($[u]$ the homotopy class of u). Hence, $[f] \in [X, K_0]$ is in the domain of Φ if and only if $[f]$ "lifts" to $[\hat{f}] \in [X, E]$, $r \circ \hat{f} \sim f$ (see diagram D1). The value $\Phi([f])$ is then the set $\{[h \circ \hat{f}] \mid r \circ \hat{f} \sim f\} \subset [X, K_1]$

(D1)
$$\begin{tikzcd} & E \arrow[r, "h"] \arrow[d, "r"] & K_1 \\ X \arrow[ur, "\hat{f}"] \arrow[r, "f"] & K_o & \end{tikzcd}$$

In case $E = K_o$, $r = 1$ the operation is called primary and is simply the right composition with h. The domain of Φ is then all of $[X, K_o]$ and its values are singletons, i.e.: elements of $[X, K_1]$.

This is a general formulation which is not very useful if one does not restrict oneself to some special cases. Normally we consider cohomology operations related to (generalized) cohomology theories (hence the name). All cohomology operations here will be given in terms of Ω-spectra:

An Ω spectrum is a sequence $\underline{E}_* = \{\underline{E}_n, \varphi_n\}_{n=o}^{\infty}$ where $\underline{E}_n$ are spaces and φ_n are homotopy equivalences, $\varphi_n \colon E_n \xrightarrow{\approx} \Omega E_{n+1}$.

The cohomology theory E^* associated with the Ω-spectrum $\underline{E}_*$ is the sequence of functors $\{E^n\} = [\ \ , \underline{E}_n]$, that is: For a space X - $E^n(X) = [X, \underline{E}_n]$. For a map $f \colon X \to Y$ $E^n(f) \colon E^n(Y) \to E^n(X)$ is the right composition with f: $E^n(f)[u] = [u \circ f]$ $(u \colon Y \to \underline{E}_n)$. As $\underline{E}_n$ are double loop spaces (and much more) $E^n(X)$ are abelian groups and $E^n(f)$ are homomorphisms.

3. PRIMARY OPERATIONS. STABLE OPERATIONS

An elementary primary operation of type m,n in the cohomology theory E^* is an element $[\alpha] \in [\underline{E}_m, \underline{E}_n]$, $\alpha: \underline{E}_m \to \underline{E}_n$. It defines a primary operation $\Phi = \langle K_o = E_m = E, K_1 = E_n, r = 1, h = \alpha\rangle$ which is obviously the left composition with α. The set of all primary operations of type m,n is the set $[\underline{E}_m, \underline{E}_n]$. As an operation α is a function $E^m(X) \to E^n(X)$.

A stable elementary primary operation of degree k is a sequence $\alpha = \{\alpha_n \in [\underline{E}_n, \underline{E}_{n+k}]\}_{n=o}^{\infty}$ (for $k < o$ we consider E_t = point for $t < o$). These are related by the following (homotopy) commutative diagram:

(D2)
$$\begin{tikzcd}
\underline{E}_n \arrow[r, "\alpha_n"] \arrow[d, "\varphi_n"'] & \underline{E}_{n+k} \arrow[d, "\varphi_{n+k}"] \\
\Omega\, \underline{E}_{n+1} \arrow[r, "\Omega\alpha_{n+1}"] & \Omega\, \underline{E}_{n+1+k}
\end{tikzcd}$$

In this case $\alpha_n: E^n(X) \to E^{n+k}(X)$ are homomorphisms.

The set of stable cohomology operations in the theory E^* forms a graded ring: One can add any two operations of the same degree as $[\underline{E}_n, \underline{E}_{n+k}]$ is an abelian group. The product $\alpha'' \cdot \alpha'$ is given by: $(\alpha''\cdot\alpha')_n = \alpha''_{n+k} \circ \alpha'_n$ if α' is of degree k. The degree of $\alpha'' \cdot \alpha'$ is the sum of the degrees of α' and α''. These definitions are consistent with the defining relations of a stable operation (D2).

Example: The Steenrod Algebra. Let $\underline{E}_n = K(Z/pZ,n)$ - the Eilenberg MacLane spaces, p-a prime,(E_* is then called the Eilenberg MacLane spectrum $\underline{K}(Z/pZ)$). The ring of elementary stable cohomology operations is called the Steenrod algebra $a(p)$. For $p = 2$ $a(2)$ is generated by operations Sq^i of degree i ($Sq^o = 1$) subject to relations known as the Adem relations. Reference: [Steenrod-Epstein].

A non elementary primary cohomology operation in E^* is a map $\alpha: E(0) \to E(1)$ where $E(i) = \prod_{j=1}^{s_i} \underline{E}^{(i)}_{n_j}$ $i = 0,1$. One can easily see how to define a non elementary stable primary operation. Such an operation is given by a matrix whose entries (α_{ij}) are elementary stable operations with the property: degree $\alpha_{i_1,j}$ - degree $\alpha_{i_2,j}$ is independent of j.

4. SECONDARY OPERATIONS ASSOCIATED WITH A RELATION

Fix the cohomology theory E^*. By $E(i)$ we always denote a product of terms $\prod_{j=1}^{s_i} \underline{E}^{(i)}_{k_j}$. ($E(i,j)$ will denote other types of spaces as will be seen in the sequel).

Given (non elementary and not necessarily stable) operations $\alpha_0 \colon E(0) \to E(1)$, $\alpha_1 \colon E(1) \to E(2)$. A relation among primary operations is a relation of the type $\alpha_1 \circ \alpha_0 \sim *$. (*-the constant map). (If α_i are stable, and therefore given by martices, this relation describes ordinary relations in the ring of stable operations).

The relation $\alpha_1 \circ \alpha_0 \sim *$ induces a commutative diagram:

(D3)

$$\begin{tikzcd}
 & \Omega E(2) \arrow[d, "j_1"] & \\
E(0,1) \arrow[ur, "\alpha_{0,1,2}"] \arrow[d, "r_0"'] & E(1,2) \arrow[d, "r_1"] & \\
E(0) \arrow[ur, "\alpha_{0,1}"] \arrow[r, "\alpha_0"] & E(1) \arrow[r, "\alpha_1"] & E(2)
\end{tikzcd}$$

where $E(i,i+1)$ is the homotopy fiber of α_i, $i = 0,1$. j_1 is the inclusion of the fiber of r_1. $\alpha_{0,1}$ exists since $\alpha_1 \circ \alpha_0 \sim *$. $\alpha_{0,1,2}$ is induced by $\alpha_{0,1}$. $\alpha_{0,1}$ and $\alpha_{0,1,2}$ are uniquely determined by the choice of the homotopy $E(0) \times I \to E(2)$, $* \sim \alpha_1 \circ \alpha_0$. The operation $\Phi = < E(0), E(0,1), \Omega E(2), r_0, \alpha_{0,1,2} >$ is called a secondary operation associated with the relation $\alpha_1 \circ \alpha_0 \sim *$.

The above operation Φ depends on the choice of $\alpha_{o,1}$, or as remarked, on the choice of the homotopy $* \sim \alpha_1 \circ \alpha_0$. The difference between choices of such homotopies is given by a map $w: E(0) \to \Omega E(2)$. The difference between the two maps $\alpha'_{o,1,2}$ and $\alpha''_{o,1,2}$ induced by the two choices of homotopies is then given by $\alpha''_{o,1,2} - \alpha'_{o,1,2} = w \circ r_0$.

Given a space X and a cohomology class $x \in [X, E(0)]$. (x is actually a "vector" of cohomology classes $x_{n_j}^{(o)} \in \underline{E}^{n_j}(X)$). x is in the domain of Φ (for any Φ, induced by any null homotopy $* \sim \alpha_1 \circ \alpha_0$) if and only if $\alpha_0 x = 0$. The value $\Phi(x)$ is then $[\alpha_{o,1,2} \circ \tilde{x}]$ where $\tilde{x} : X \to E(0,1)$ is a "lifting" of $x: X \to E(0)$, $r_0 \circ \tilde{x} \sim x$. If Φ', Φ'' correspond to two different homotopies $* \sim \alpha_1 \circ \alpha_0$ whose difference, as above, is $w: E(0) \to \Omega\, E(2)$ then $[\alpha''_{o,1,2} \circ \tilde{x}] - [\alpha'_{o,1,2} \circ \tilde{x}] = [w \circ r_0 \circ \tilde{x}] = [w \circ x]$. Hence, $\Phi''(x)$ is obtained by translating $\Phi'(x)$ by $w \circ x$ where $w \in [E_0, \Omega\, E(2)]$ is a primary operation. This could be formulated as follows:

A relation $\alpha_1 \circ \alpha_0 \sim *$ among primary operations induces secondary operations $\{\Phi\}$. Any two such operations differ by a primary operation.

5. MASSEY PRODUCTS - TODA BRACKETS. HIGH ORDER OPERATIONS

Let $\alpha_0 : E(0) \to E(1)$, $\alpha_1 : E(1) \to E(2)$, $\alpha_2 : E(2) \to E(3)$ be primary operations and suppose $\alpha_1 \circ \alpha_0 \sim *$, $\alpha_2 \circ \alpha_1 \sim *$. Again, $E(i)$ are products $\prod_{j=1}^{s_i} E^{(i)}_{n_j}$. Extend diagram (D3) to obtain:

(D4)
$$\begin{tikzcd}
 & \Omega E(2) \arrow[r, "\Omega\alpha_2"] \arrow[d, "j_1"] & \Omega E(3) \arrow[d, "j_2"] & \\
E(0,1) \arrow[ur, "\alpha_{0,1,2}"] \arrow[d, "r_0"] & E(1,2) \arrow[ur, "\alpha_{1,2,3}"] \arrow[d, "r_1"] & E(2,3) \arrow[d, "r_2"] & \\
E(0) \arrow[r, "\alpha_0"] \arrow[ur, "\alpha_{0,1}"] & E(1) \arrow[r, "\alpha_1"] \arrow[ur, "\alpha_{1,2}"] & E(2) \arrow[r, "\alpha_2"] & E(3)
\end{tikzcd}$$

$E(i,i+1)$ - the homotopy fiber of α_i, j_i - the inclusion of the fibre. $\alpha_{0,1}$, $\alpha_{1,2}$, $\alpha_{1,2,3}$, $\alpha_{0,1,2}$ exist as $\alpha_1 \circ \alpha_0 \sim *$, $\alpha_2 \circ \alpha_1$ *. They are uniquely determined by choices of homotopies $* \sim \alpha_1 \circ \alpha_0$, $* \sim \alpha_2 \circ \alpha_1$.

The class $[\alpha_{1,2,3} \circ \alpha_{0,1}] \in [E(0), \Omega E(3)]$ is a primary operation. Two different choices of the homotopy $* \sim \alpha_1 \circ \alpha_0$ will yield two maps $\alpha'_{0,1}$, $\alpha''_{0,1}$. These maps are related by $[\alpha_{1,2,3} \circ \alpha''_{0,1}] - [\alpha_{1,2,3} \circ \alpha'_{0,1}] = [\Omega\alpha_2 \circ w_0]$, where $w_0 \in [E(0), \Omega E(2)]$ measures the difference between the two choices of homotopies $* \sim \alpha_1 \circ \alpha_0$. (Note that the difference $\alpha_{1,2,3} \circ \alpha''_{0,1} - \alpha_{1,2,3} \circ \alpha'_{0,1}$ is independent of the choice of the homotopy $* \sim \alpha_2 \circ \alpha_1$ and its induced map $\alpha_{1,2,3}$.)

Similarly, two distinct choices of the homotopies $* \sim \alpha_2 \circ \alpha_1$ (with a difference measured by a map $w_1 \colon E(1) \to \Omega\, E(3)$) yield two maps $\alpha'_{1,2,3}, \alpha''_{1,2,3} \colon E(1,2) \to \Omega\, E(3)$ related by $[\alpha''_{1,2,3}]-[\alpha'_{1,2,3}] = [w_1 \circ r_1]$. It follows that $[\alpha''_{1,2,3} \circ \alpha''_{0,1}]-[\alpha'_{1,2,3} \circ \alpha'_{0,1}] = [\Omega\, \alpha_2 \circ w_0] + [w_1 \circ r_1]$ and the coset of $[\alpha_{1,2,3} \circ \alpha_{0,1}]$ in

$$[E(0), \Omega\, E(3)] \,/\, (\Omega\, \alpha_2)_* [E(0), \Omega\, E(2)] + r_1^*[E(1), \Omega\, E(3)]$$

is independent of any choices of homotopies, is denoted by $<\alpha_0, \alpha_1, \alpha_2>$ and is called the *Massey product or Toda bracket* of $\alpha_0, \alpha_1, \alpha_2$.

Note that $0 \in <\alpha_0, \alpha_1, \alpha_2>$ if and only if one can choose $\alpha_{0,1}$ and $\alpha_{1,2,3}$ so that $\alpha_{1,2,3} \circ \alpha_{0,1} \sim *$.

Example: Reexamine the Steenrod algebra $A(2)$ generated by Sq^i of degree i. By the Adem relations Sq^i for $i \neq 2^j$ could be described as a sum $\sum_k Sq^{t_k^{(i)}} \cdot Sq^{s_k^{(i)}}$ $t_k^{(i)}, s_k^{(i)} > 0$. It follows that as a ring $A(2)$ is generated by Sq^{2^j}. However, the main result of [Adams] is that Sq^{2^j} for $j > 3$ could be decomposed in terms of Massey products. More precisely: There exist primary operations:

$\alpha_0 \colon E(0) \to E(1)$, $E(0) = K(Z/2Z,N)$, $N \geq 2^j$

$\alpha_1 \colon E(1) \to E(2)$, $\alpha_2 \colon E(2) \to E(3) = K(Z/2Z,N+2^j+1)$

$E(1), E(2)$ have the properties:

$\pi_i(E(1)) \neq 0$ only if $N < i \leq 2^{j-1}$.

$\pi_i(E(2)) \neq 0$ only if $N + 1 < i \leq N + 2^j$.

$\pi_i(E(j))$ are $Z/2Z$ vector spaces

$\alpha_1 \circ \alpha_0 \sim *$, $\alpha_2 \circ \alpha_1 \sim *$ and $Sq^{2^j} \in <\alpha_0, \alpha_1, \alpha_2>$.

This implies the following:

There is no space X so that:

$H^i(X, Z/2Z) \neq 0$ only if $i=0, N, N+2^j, j > 3,$

$H^N(X, Z/2Z) = Z/2Z = H^{N+2^j}(X, Z/2Z)$ and $Sq^{2^j} x_N \neq 0$, where $x_N \in H^N(X, Z/2Z)$ is the generator. Indeed, suppose such a space X exists. One obtains the following extension of (D4):

(D5)

$$\begin{tikzcd}
 & & \Omega E(2) \arrow[r, "\Omega\alpha_2"] \arrow[d, "j_1"] & \Omega E(3) \arrow[d, "j_2"] & \\
 & & E(1,2) \arrow[d, "r_1"] \arrow[ur, "\alpha_{1,2,3}"] & E(2,3) \arrow[d, "r_2"] & \\
X \arrow[r, "x_N"] & K(Z/2Z,N) = E(0) \arrow[r, "\alpha_0"] \arrow[ur, "\alpha_{0,1}"] & E(1) \arrow[r, "\alpha_1"] \arrow[ur, "\alpha_{1,2}"] & E(2) \arrow[r, "\alpha_2"] & E(3)
\end{tikzcd}$$

$Sq^{2^j} \in \langle\alpha_0, \alpha_1, \alpha_2\rangle$ means that one can choose $\alpha_{1,2,3}$, $\alpha_{0,1}$ so that $[\alpha_{1,2,3} \circ \alpha_{0,1}] = Sq^{2^j}$. As $H^i(X, Z/2Z) = 0$ for $N < i < N+2^j$ by simple obstruction theory $[X, E(1)] = 0$ and $[X, \Omega E(2)] = 0$ and therefore $[X, E(0,1)] = 0$, $\alpha_{0,1} \circ x_n \sim *$, $Sq^{2^j} x_N = [\alpha_{1,2,3} \circ \alpha_{0,1} \circ x_N] = 0$. A contradiction.

Now suppose $\alpha_0, \alpha_1, \alpha_2$ are given, $\alpha_1 \circ \alpha_0 \sim *$, $\alpha_2 \circ \alpha_1 \sim *$ and suppose $0 \in \langle\alpha_0, \alpha_1, \alpha_2\rangle$. One can extend (D4) to obtain:

$$\begin{tikzcd}
 & \Omega^2 E(3) \arrow[d, "\Omega j_2"] & & \\
 & \Omega E(2,3) \arrow[d, "\Omega r_2"] & & \\
E(0,1,2) \arrow[uur, "\alpha_{0,1,2,3}"] \arrow[d, "r_{o,1}"'] & \Omega E(2) \arrow[r, "\Omega\alpha_2"] \arrow[d, "j_1"] & \Omega E(3) \arrow[d, "j_2"] & \\
E(0,1) \arrow[d, "r_o"'] \arrow[uur, "\alpha_{0,1,3}"] \arrow[ur, "\alpha_{o,1,2}"] & E(1,2) \arrow[d, "r_1"] \arrow[ur, "\alpha_{1,2,3}"] & E(2,3) \arrow[d, "r_2"] & \\
E(0) \arrow[r, "\alpha_o"] \arrow[ur, "\alpha_{o,1}"] & E(1) \arrow[r, "\alpha_1"] \arrow[ur, "\alpha_{1,2}"] & E(2) \arrow[r, "\alpha_2"] & E(3)
\end{tikzcd}$$

$E(0,1,2)$ - the homotopy fiber of $\alpha_{o,1,2}$. If $\alpha_{o,1}$, $\alpha_{1,2,3}$ are chosen so that $\alpha_{1,2,3} \circ \alpha_{o,1} \sim *$. $\Omega\alpha_2 \circ \alpha_{o,1,2} \sim \alpha_{1,2,3} \circ \alpha_{o,1} \sim *$ and $\alpha_{o,1,2}$ lifts to the homotopy fiber of $\Omega\alpha_2$, $\alpha_{o,1,3}\colon E(0,1) \to \Omega\, E\, (2,3)$, this map induces a map $\alpha_{o,1,2,3}\colon E(o,1,2) \to \Omega^2 E(3)$.

The operation $\Phi' = \langle E(0), E(0,1,2), \Omega^2 E(3), r_o \circ r_{o,1}, \alpha_{o,1,2,3}\rangle$ is called a third order operation associated with the relation $0 \in \langle\alpha_o, \alpha_1, \alpha_2\rangle$.

One can proceed inductively to define a k-fold Massey product $\langle\alpha_0, \alpha_1, \ldots \alpha_k\rangle$. α_i-primary. This is defined whenever $\langle\alpha_0, \alpha_1, \ldots \alpha_{k-1}\rangle$ is defined and contains 0 and $\alpha_k \circ \alpha_{k-1} \sim *$. If $0 \in \langle\alpha_0, \alpha_1 \ldots \alpha_k\rangle$ one can define a $k+1$ order cohomology operation.

6. COHOMOLOGY OPERATIONS AND H-SPACES

We shall demonstrate how one uses the theory of cohomology operations to study the cohomology of H-spaces.

If X is any space, (assume connected for simplicity) then $H^*(X, Z/2Z)$ is a ring (more precisely, an algebra over $Z/2Z$). $x \in H^*(X, Z/2Z)$ is said to be indecomposable if x cannot be written as $x = \sum_i x_i' \cdot x_i''$ where x_i', x_i'' are of positive dimensions.

Suppose X is an H-space. $x \in H^m(X, Z/2Z)$ is called a primitive element if x is represented by an H-map $X \to K(Z/2Z, m)$. We shall prove the following:

Theorem: Let X be a connected H-space. Suppose $H^*(X, Z/2Z)$ is an exterior algebra on generators of $\dim \equiv 1 \pmod 4$, i.e.: $H^*(X, Z/2Z)$ is a free commutative graded algebra with generators of dimension $4k_i+1$, $i=1,2\ldots.$ Then if $x \in H^{4k+1}(X, Z/2Z)$ is a primitive element (hence $x: X \to K(Z/2Z, 4n+1)$ is an H-map) then $Sq^{4n} x \neq 0$ (and is again primitive). Consequently:

(i) X cannot be finite dimensional.

(ii) Consider the Pontrjagin ring $H_*(X, Z/2Z)$ of X. (I.e.: This is the ring structure $H_*(X, Z/2Z) \otimes H_*(X, Z/2Z) \to H_*(X, Z/2Z)$ induced by the multiplication $\mu: X \times X \to X$).

If $H_*(X, Z/2Z)$ is an associative algebra then $H^*(\Omega X, Z/2Z)$ is a polynomial algebra on generators of dimensions $\equiv 0 \pmod 4$.

Given an H-space X satisfying the hypothesis of the theorem. Then:

a) $H^*(X, Z)$ is 2-torsion free.

(the proof uses the Bockstein spectral sequence on the Hopf algebra $H^*(X, Z/2Z)$). Consequently, $Sq^1 H^*(X, Z/2Z) = 0$).

b) A primitive element x in $H^*(X, Z/2Z)$ is not decomposable. Consequently, all primitive elements of $H^*(X, Z/2Z)$ are of dimension $\equiv 1 \bmod 4$.

c) If $H_*(X, Z/2Z)$ is an associative algebra $H^*(X, Z/2Z)$ is then primitively generated, i.e.: One can choose the primitives of $H^*(X, Z/2Z)$ as free algebra generators.

The conclusions (i) and (ii) follow from the theorem as follows:

i) $Sq^{4n}\colon H^{4n+1}(X, Z/2Z) \to H^{8n+1}(X, Z/2Z)$ is injective on primitives. If $H^*(X, Z/2Z)$ is not trivial there exists a non zero class $x \in H^{4n+1}(X, Z/2Z)$ of lowest positive dimension. This class has to be primitive. The set $\{x, Sq^{4n}x, Sq^{8n}\,Sq^{4n}x, \ldots (Sq^{2^t n}\,Sq^{2^{t-1} n} \ldots Sq^{4n}x) \ldots\}$ is an infinite set of non zero cohomology classes of increasing dimensions.

ii) Here one uses spectral sequences to compute $H^*(\Omega X, Z/2Z)$, e.g.: The Eilenberg Moore spectral sequence. One can see that $H^i(\Omega X, Z/2Z) = 0$ if $i \not\equiv 0 \bmod 4$ and that $Sq^{4n}y = y^2 \neq 0$ for any primitive element in $H^{4n}(\Omega X, Z/2Z)$. This implies (ii).

To prove the theorem we need the following properties of the Steenrod algebra (see [Steenrod-Epstine]):

Non Stability: $Sq^i x = \begin{cases} x^2 & \text{if } x \in H^i(X, Z/2Z) \\ 0 & \text{if } x \in H^j(X, Z/2Z),\ j > i \end{cases}$.

Preservation of algebra filtrations. (Follows from the Cartan formula): Let $F^r H^*(X, Z/2Z)$ be the ideal of $H^*(X, Z/2Z)$ generated by r fold products $x_1 \cdot x_2 \cdot \ldots \cdot x_r$ of elements x_i of positive dimensions. Then $F^r H^*(X, Z/2Z)$ are $A(2)$ invariant, i.e.: if $\alpha \in A(2)$, $x \in F^r H^*(X, Z/2Z)$ then $\alpha x \in F^r H^*(X, Z/2Z)$.

A(2) preserve primitive elements: If X is an H-space and $x \in H^*(X, Z/2Z)$ is primitive then αx is primitive for any $\alpha \in A(2)$.

Consider the following Adem relation in $A(2)$:

(R) $Sq^2\, Sq^{4n} + Sq^{4n+1} Sq^1 = Sq^{4n+2}$

(R) defines a secondary operation as follows:

Let

$E(0) = K(Z/2Z, 4n+1)$, $E(1) = K(Z/2Z, 4n+2) \times K(Z/2Z, 8n+1)$

$E(2) = K(Z/2Z, 8n+3)$.

α_o: $E(0) \to E(1)$ is given by

$$\begin{tikzcd}
 & K(Z/2Z, 4n+2) \\
K(Z/2Z, 4n+1) \arrow[ur, "Sq^1"] \arrow[r, "\alpha_o"] \arrow[dr, "Sq^{4n}"'] & K(Z/2Z, 4n+2) \times K(Z/2Z, 8n+1) \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
 & K(Z/2Z, 8n+1)
\end{tikzcd}$$

α_1: $E(1) \to E(2)$ is given by

$\alpha_1 = Sq^{4n+1} \circ p_1 + sq^2 \circ p_2$ where $+$ is the addition induced by the loop multiplication on $[\ \ , E(2)]$.

(R) implies $[\alpha_1 \circ \alpha_0] = Sq^{4n+1}Sq^1 + Sq^2Sq^{4n} = Sq^{4n+2} = 0$ (the latter vanishes by the non stability condition $Sq^{4n+2}[1_{E(o)}] = 0$ as $1_{E(o)}: E(0) \to E(0)$ is an element of $H^{4n+1}(E(0), Z/2Z)$).

We shall investigate the value of a secondary operation Φ associated with $* \sim \alpha_1 \circ \alpha_0$ on a primitive class $x \in H^{4n+1}(X, Z/2Z)$ in the domain of Φ. (We shall conclude that there is no such class and therefore $Sq^{4n}x \neq 0$ for any primitive element of dimension 4n+1.)

<u>H deviations</u>: Let X,μ, Y,μ' be H-space.

Given a map $f: X \to Y$ there exists a map $D_f: X \wedge X \to Y$ called the H-deviation of f with the following properties. (Compare with [Zabrodsky]$_4$, Chapter 1 where D_f is denoted by $H\,D(f, \mu, \mu')$):

i) The two maps $X \times X \to Y$ given by $x, y \to f(x \cdot y)$ and $x, y \to D_f(x,y) \cdot [f(x) \cdot f(y)]$ are homotopic (here $(\ )\cdot(\ )$ denotes both products μ and μ') or in a functional notation: $f \circ \mu \sim \mu' \circ \{D_f \circ \wedge \times [\mu \circ (f \times f)]\} \circ \Delta_{X \times X}$ where $\Delta_{X \times X}(x,y) = (x,y, x,y)$ is the diagonal map and $\wedge : X \times X \to X \wedge X$ - the projection.

ii) f is an H-map if and only if $D_f \sim *$.

iii) Let X_0, X_1 be H-spaces and X_2 - a loop space. Given maps $f_0: X_0 \to X_1$, $f_1: X_1 \to X_2$. Suppose $D_{f_1} \circ (D_{f_0} \wedge 1): X_0 \wedge X_0 \wedge X_1 \to X_2$ is null homotopic. Then $[D_{f_1 \circ f_0}] = [D_{f_1} \circ (f_0 \wedge f_0)] + [f_1 \circ D_{f_0}]$. In particular: If f_1 is an H-map $D_{f_1 \circ f_0} \sim f_1 \circ D_{f_0}$ and if f_0 is an H-map $D_{f_1 \circ f_0} \sim D_{f_1} \circ (f_0 \wedge f_0)$.

Now consider again the operation Φ associated with (R) and the following commutative diagrams:

$$\begin{tikzcd}
K(Z/2Z,\ 8n+3) \arrow[d, "B\hat{j}"'] & & \\
\hat{E} \arrow[r, "\hat{\alpha}"] \arrow[d, "B\hat{r}"'] & B\ E(1,2) \arrow[d, "Br_1"] & \\
K(Z/2Z,\ 4n+2) \arrow[r, "B\alpha_0"] & K(Z/2Z,\ 4n+3) \times K(Z/2Z,\ 8n+2) \arrow[r, "B\alpha_1"] & K(Z/2Z,\ 8n+4)
\end{tikzcd} \tag{D7}$$

$B\alpha_0$ is given by $p_1 \circ B\alpha_0 = Sq^1 : K(Z/2Z,\ 4n+2) \to K(Z/2Z,\ 4n+3)$, $p_2 \circ B\alpha_0 = Sq^{4n} : K(Z/2Z,\ 4n+2) \to K(Z/2Z,\ 8n+2)$.

$B\alpha_1 = Sq^{4n+1} \circ p_1 + Sq^2 \circ p_2$, hence $\Omega\ B\alpha_0 \sim \alpha_0$, $\Omega\ B\alpha_1 \sim \alpha_1$. $\hat{E}$ - the homotopy fiber of Sq^{4n+2}, $B\ E(1,2)$ - the homotopy fiber of $B\alpha_1$, $\Omega\ B\ E(1,2) = E(1,2)$ as in the (D4) diagram defining Φ. Loop the above diagram and observe that $\Omega\ \hat{E} \simeq K(Z/2Z,\ 4n+1) \times K(Z/2Z,\ 8n+2)$ and therefore $\hat{r} = \Omega\ B\ \hat{r} : \Omega\ \hat{E} \to K(Z/2Z,\ 4n+1)$ admits a left inverse $\chi : K(Z/2Z,\ 4n+1) \to \Omega\ \hat{E}$, $\hat{r} \circ \hat{\chi} \sim 1$. One can see that the choices of such inverses (also called cross sections) are in 1-1 correspondence with liftings $\alpha_{0,1} : E(0) = K(Z/2Z,\ 4n+1) \to E(1,2)$ of diagram D4 for Φ. Thus, looping (D7) one obtains:

$$\begin{tikzcd}
K(Z/2Z,\ 8n+2) \arrow[r, "1", "\simeq"'] \arrow[d, "\Omega B\hat{j}=\hat{j}"'] & \Omega E(2) \arrow[d, "j_1"] \\
\Omega\ \hat{E} \arrow[r, "\Omega\ \hat{\alpha}"] \arrow[d, shift right, "\hat{r}"'] & E(1,2) \arrow[d, "r_1"] \\
E(0)= K(Z/2Z,\ 4n+1) \arrow[u, shift right, "\hat{\chi}"'] \arrow[ur, "\alpha_{0,1}"] \arrow[r, "\alpha_1"] & E(1)
\end{tikzcd} \tag{D8}$$

$$\alpha_{0,1} = \Omega\hat{\alpha} \circ \hat{\chi}$$

$$\hat{r} \circ \hat{\chi} \sim 1_{E(o)}$$

If $\tilde{\chi}$ (any choice!) is an H-map one can use some obstruction theory to show that then $\tilde{\chi}$ is indeed a loop map. Hence, $B\hat{r}$ admits a cross section $B\tilde{\chi}$, $B\hat{r} \circ B\tilde{\chi} \sim 1_{BE(o)}$. This will imply that $Sq^{4n+2}: K(Z/2Z, 4n+2) \to K(Z/2Z, 8n+4)$ is null homotopic which is false. It follows that χ is not an H-map and $D_{\chi}: E(0) \wedge E(0) \to \Omega\hat{E}$ is not null homotopic. Now,

$$[E(0) \wedge E(0) = K(Z/2Z, 4n+1) \wedge K(Z/2Z, 4n+1), \Omega \hat{E} \approx K(Z/2Z, 4n+1) \times K(Z/2Z, 8n+2)]$$
$$\approx H^{4n+1}(K(Z/2Z, 4n+1) \wedge K(Z/2Z, 4n+1), Z/2Z) +$$
$$H^{8n+2}(K(Z/2Z, 4n+1) \wedge K(Z/2Z, 4n+1), Z/2Z).$$

The first summand is zero ($E(0) \wedge E(0)$ is $8n+1$ connected) the second equals $Z/2Z$. Hence, the only non trivial map in $[E(0) \wedge E(0), \Omega \hat{E}]$ is given by $K(Z/2Z, 4n+1) \wedge K(Z/2Z, 4n+1) \xrightarrow{w_o} K(Z/2Z, 8n+2) \xrightarrow{\hat{j}} \Omega \hat{E}$ and $D_{\tilde{\chi}} = \hat{j} \circ w_o$. ($w_o$ is also denoted by $\iota_{4n+1} \otimes \iota_{4n+1}$). By the properties of H-deviations [property (iii)] $[D_{\alpha_{o,1}}] = [\Omega \hat{\alpha} \circ D_{\tilde{\chi}}] = [\Omega \hat{\alpha} \circ \hat{j} \circ w_o] = [j_1 \circ w_o]$. And again, this is true for any choice of $\alpha_{o,1}$.

Now consider the (D4) diagram for Φ and its evaluation on a primitive class $x \in H^{4n+1}(X, Z/2Z)$, $x \in \ker Sq^{4n}$, ($x \in \ker Sq^1$ by remark a)).

(D9)

$$\begin{tikzcd}
 & \Omega E(1) \arrow[r, "\Omega\alpha_1"] \arrow[d, "j_o"'] & \Omega E(2) \arrow[d, "j_1"] \\
 & E(0,1) \arrow[ur, "\alpha_{o,1,2}"] \arrow[d, "r_o"'] & E(1,2) \arrow[d, "r_1"] \\
X \arrow[r, "x"] \arrow[ur, "\tilde{\chi}"] & K(Z/2Z, 4n+1) = E(0) \arrow[r, "\alpha_o"] \arrow[ur, "\alpha_{o,1}"] & E(1)
\end{tikzcd}$$

$Sq^{4n}x=0$, $Sq^1 x=0$ implies $\alpha_o \circ x \sim *$ and $\tilde{x}$ exists. Now, x, r_o, r_1, j_o, j_1 are H-maps hence:

$0 = [D_x] = [D_{r_o \circ \tilde{x}}] = [r_o \circ D_{\tilde{x}}]$, hence,

$D_{\tilde{x}}: X \to E_{o,1}$ lifts to a map $w: X \wedge X \to \Omega\, E(1)$, $D_{\tilde{x}} \sim j_o \circ w$.

$[j_1 \circ D_{\alpha_{o,1,2}}] = [D_{j_1 \circ \alpha_{o,1,2}}] = [D_{\alpha_{o,1}} \circ (r_o \wedge r_o)] = [j_1 \circ w_o \circ (r_o \wedge r_o]$.

Now, $E(0) \wedge E(0)$ is $8n+1$ connected, $\pi_i(\Omega\, E(1)) = 0$ for $i > 8n$, hence $[E(0) \wedge E(0), \Omega\, E(1)] = 0$ and consequently $j_{1*}: [E(0) \wedge E(0), \Omega\, E(2)] \to [E(0) \wedge E(0), E(1,2)]$ is injective and

$$D_{\alpha_{o,1,2}} \sim w_o \circ (r_o \wedge r_o).$$

Now, $D_{\alpha_{o,1,2}} \circ (D_{\tilde{x}} \wedge 1) \sim w_o \circ (r_o \wedge r_o) \circ (D_{\tilde{x}} \wedge 1) =$ $w_o \circ (r_o \circ D_{\tilde{x}} \wedge r_o) \sim *$ as $r_o \circ D_{\tilde{x}} \sim D_x \sim *$. Hence the conditions in property (iii) of H-deviation hold and

$[D_{\alpha_{o,1,2} \circ \tilde{x}}] = [\alpha_{o,1,2} \circ D_{\tilde{x}}] + [D_{\alpha_{o,1,2}} \circ (\tilde{x} \wedge \tilde{x})] = [\alpha_{o,1,2} \circ j_o \quad w] +$

$+ [w_o \circ (r_o \wedge r_o) \circ (\tilde{x} \wedge \tilde{x})] = [\Omega\alpha_1 \circ w] + [w_o \circ (x \wedge x)]$.

$[w_o] = \iota_{4n+1} \otimes \iota_{4n+1}$, $[w_o \circ (x \wedge x)] = x \otimes x \in H^*(X \wedge X, Z/2Z)$.

Now, the image of $x \otimes x$ in $H^*(X \times X, Z/2Z)$ is of algebra filtration 2 (and not of filtration > 2) as $x \otimes x = (x \otimes 1) \cdot (1 \otimes x)$, and x is indecomposable.

On the other hand, consider $w \in [X \wedge X, \Omega\, E(1)] \approx$ $H^{4n+1}(X \wedge X, Z/2Z) + H^{8n}(X \wedge X, Z/2Z)$. For dimension reasons the image of w in $[X \times X, \Omega\, E(1)]$ must have algebra filtration at least 4: The image of $H^*(X \wedge X, Z/2Z) \to H^*(X \times X, Z/2Z)$ has filtration ≥ 2. As all generators in

$H^*(X \times X, Z/2Z)$ are of congruency $\equiv 1 \pmod 4$ elements of dimension $\equiv 0 \bmod 4$ have filtration ≥ 4. Elements of dimension $\equiv 1 \bmod 4$ and of filtration > 1 must have filtration ≥ 5.

As the Steenrod algebra preserve filtration (and $H^*(X \wedge X, Z/2Z) \to H^*(X \times X, Z/2Z)$ is injective) $[\Omega\alpha_1 \circ w]$ has filtration ≥ 4 and consequently $[D_{\alpha_{0,1,2}} \circ \tilde{x}] \equiv x \otimes x \bmod F^4H^*(X \times X, Z/2Z))$. In particular, $D_{\alpha_{0,1,2}} \circ \tilde{x} \not\simeq *$ and $\alpha_{0,1,2} \circ \tilde{x} \not\simeq *$. As this holds, for all choices of $\alpha_{0,1}$, $0 \notin \Phi(x)$. Moreover, one can use Hopf algebra properties of $H^*(X, Z/2Z)$ and the above evaluation of $D_{\alpha_{0,1,2}} \circ \tilde{x}$ to conclude that the elements in $\Phi(x)$ are all generators. This is impossible for $\Phi(x) \subset H^{8n+2}(X, Z/2Z)$ and there are no algebra generators in these dimensions.

The conclusion is therefore that there are no primitive elements in $H^*(X, Z/2Z)$ in the domain of Φ. As $Sq^1x = 0$ for every $x \in H^*(X, Z/2Z)$ $Sq^{4n}x \neq 0$ for every primitive element x in $H^{4n+1}(X, Z/2Z)$.

Remark: There are H-spaces with this type of cohomology: If Sp is the simplectic group then $Sp \approx \Omega^2X$. Both X and the universal covering space of Ω^2Sp are H-spaces with cohomology of the type described in the theorem.

7. H-SPACES AND COHOMOLOGY OPERATIONS

We shall show here how the theory of cohomology operations uses H-space theory.

Consider the Adem relation

(R_1) $Sq^2Sq^2 + Sq^1Sq^2Sq^1 = 0$

R_1 induces a secondary operation Φ_1 described by the (D4) type diagram as follows:

$$\begin{tikzcd}
\Omega E(1) \arrow[r, dash, "\Omega\alpha_1"] \arrow[d, "j_0"'] & \Omega E(2) = K(Z/2Z,N+3) \arrow[d, "j_1"] & \\
E(0,1) \arrow[ur, "\alpha_{0,1,2}"] \arrow[d, "r_0"'] & E(1,2) \arrow[d, "r_1"] & \\
K(Z/2Z,N) = E(0) \arrow[r, "\alpha_0"] \arrow[ur, "\alpha_{0,1}"] & E(1) = K(Z/2Z,N+2) \times K(Z/2Z,N+3) \arrow[r, "\alpha_1"] & E(2) = K(Z/2Z,N+4)
\end{tikzcd} \tag{D10}$$

α_0 is given by $p_1 \circ \alpha_0 = Sq^2$, $p_2 \circ \alpha_0 = Sq^2Sq^1$

α_1 is given by $[\alpha_1] = [Sq^2 \cdot p_1] + [Sq^1 \circ p_2]$

$\alpha_1 \circ \alpha_0 \sim *$ by (R_1).

Consider the composition $Sq^{4n}_{\circ}$ as in the last chapter

$$K(Z,\ 4n+1) \xrightarrow{\rho} K(Z/2Z,\ 4n+1) \xrightarrow{Sq^{4n}} K(Z/2Z,\ 8n+1)$$

where ρ is induced by the reduction $Z \to Z/2Z$.

Sq_0^{4n} is in the domain of Φ_1 (for $N = 8n+1$). Indeed, by (R) of the previous chapter

$$[p_1 \circ \alpha_0 \circ Sq_0^{4n}] = [p_1 \circ \alpha_0 \circ Sq^{4n} \circ \rho] = [Sq^2 \circ Sq^{4n} \circ \rho] =$$

$$= [Sq^{4n+2} \circ \rho + Sq^{4n+1} \circ Sq^1 \circ \rho].$$

Now, $Sq^{4n+2}[\rho] = 0$ as ρ is of dimension $4n+1$ (using the non stability condition of the Steenrod algebra). $[Sq^1 \circ \rho] = 0$ as $H^{4n+2}(K(Z,4n+1),M) = 0$ for any coefficients module M.

Consequently, $[p_1 \circ \alpha_0 \circ Sq_0^{4n}] = 0$, $[p_2 \circ \alpha_0 \circ Sq_0^{4n}] = [Sq^2 Sq^1 \circ Sq^{4n} \circ \rho]$. Using Adem relations one has $Sq^2 Sq^1 Sq^{4n} = Sq^{4n+2} Sq^1$ and as $Sq^1[\rho] = 0$, $[p_2 \circ \alpha_0 \circ Sq^{4n}] = 0$ and $[Sq_0^{4n}] \in \operatorname{Ker} \alpha_0$.

We shall evaluate $\Phi_1[Sq_0^{4n}]$:

(D11)

$$\begin{tikzcd}
 & K(Z/2Z,8n+2) \arrow[r, "i_1"] \arrow[d, "\hat{j}_0"'] & \Omega E(1) \arrow[r, "\Omega\alpha_1"] \arrow[d, "j_0"'] & \Omega E(2) \arrow[d] \\
 & \Omega\hat{E}_0 \arrow[r, "\Omega\hat{\alpha}_1^{(0)}"] \arrow[d, shift right, "\hat{r}_0"'] & E(0,1) \arrow[ur, "\alpha_{0,1;2}"] \arrow[d, dash] & E(1,2) \arrow[d] \\
K(Z,4n+1) \arrow[r, "Sq_0^{4n}"] & K(Z/2Z,8n+1) \arrow[u, shift right, "\hat{x}_0"'] \arrow[r, "\alpha_0"] \arrow[ur, "\hat{\alpha}_0"] \arrow[dr, to path={|- (\tikztotarget)}, "Sq_0^{4n+2}"] & E(1) \arrow[r, "\alpha_1"] \arrow[ur, "\alpha_{0,1}"] \arrow[d, "p_1"] & E(2) \\
 & & K(Z/2Z,8n+3) &
\end{tikzcd}$$

$\hat{j}_0$, $\hat{r}_0$, $\hat{x}_0$, $\Omega\hat{E}_0$ are analogous to $\hat{j}$, $\hat{r}$, $\hat{x}$, $\Omega\hat{E}$ in (D8) and share similar properties. All spaces and maps except for $\hat{x}_0$ and $\hat{\alpha}_0$ are loop spaces and loop maps.

As in the previous chapter:

$D_{\hat{\alpha}_0} = [\Omega\hat{\alpha}_1^{(0)} \circ D_{\hat{x}_0}] = [\Omega\hat{\alpha}_1^{(0)} \circ j_0 \circ \hat{w}_0] = [j_0 \circ i_1 \circ \hat{w}_0]$ where

$\hat{w}_0 = \rho \otimes \rho \in [K(Z,4n+1) \wedge K(Z,4n\underline{+}1), K(Z/2Z,8n+2)]$.

As $\alpha_{0,1,2}$ is an H-map $D_{\alpha_{0,1,2}\circ\hat{\alpha}_0} = [\alpha_{0,1,2}] \circ D_{\hat{\alpha}_0} =$

$= [\alpha_{0,1,2}\circ j_0\circ i_1\circ\hat{w}_0] = [\Omega\alpha_1\circ i_1\circ\hat{w}_0] = Sq^2 \circ \hat{w}_0$ $(\Omega\alpha_1\circ i_1 = Sq^2;\ K(Z/2Z,8n+2) \to$ $\to K(Z/2Z,8n+4))$.

Using the Cartan formula ([Steenrod Epstein]) one obtains for any lifting $\hat{\alpha}_0$ of Sq_0^{4n}:

$D_{\alpha_{0,1,2}\circ\hat{\alpha}_0} = Sq^2\hat{w}_0 = Sq^2(\rho\otimes\rho) = Sq^2\rho \otimes \rho + \rho \otimes Sq^2\rho$ $(Sq^1\rho=0)$.

$u\colon K(Z,4n+1) \to K(Z/2Z,8n+4)$ is being given algebraically by $[u] = [\rho]\cdot Sq^2[\rho]$ (or "geometrically" by the composition

$$K(Z,4n+1) \xrightarrow{\Delta} K(Z,4n+1) \times K(Z,4n+1) \xrightarrow{\rho\times\rho} K(Z/2Z,4n+1) \times K(Z/2Z,4n+1)$$
$$\xrightarrow{Sq^2\times 1} K(Z/2Z,4n+3) \times K(Z/2Z,4n+1) \xrightarrow{\wedge} K(Z/2Z,4n+3) \wedge K(Z/2Z,4n+1)$$
$$\xrightarrow{\otimes} K(Z/2Z,8n+3)]$$

where $\otimes$ represents the generator of $H^{8n+3}(K(Z/2Z,4n+3) \wedge K(Z/2Z,4n+1), Z/2Z = Z/2Z)$. Then $D_u = Sq^2\rho \otimes \rho + \rho \otimes Sq^2\rho$ (D_u of a cohomology class u of an H-space X is the reduced coproduct in the Hopf algebra $H^*(X,Z/2Z)$).

It follows easily that if $v = [\alpha_{0,1,2}\circ\hat{\alpha}_0] \in \Phi_1(Sq_0^{4n})$ is any element $v-u$ is primitive. Now, one can show that $\hat{\alpha}_0$ can be chosen so that $v=u$ and $[\rho]\cdot Sq^2[\rho] \in \Phi_1(Sq_0^{4n})$.

(Outline of proof: looping (D1Ī) twice one obtains $\Omega^2(Sq_0^{4n}) \sim *$ and $\Omega^2\hat{\alpha}_0 \sim \Omega^2 j_0\circ z$ for some $z\colon K(Z/2Z,4n-1) \to \Omega^3E(1) = K(Z/2Z,8n) \times K(Z/2Z,8n+1)$. z must be an H-map as $\Omega^2\hat{\alpha}_0 \sim \Omega^2 j_0\circ z$ is an H-map, $0 = [\Omega^2 j_0] \circ D_z$, and as $[K(Z/2Z,4n-1) \wedge K(Z/2Z,4n-1), \Omega^3E(0) = K(Z/2Z,8n-2)] = 0$ $(\Omega^2 j_0)_*$ on $[K(Z/2Z,4n-1) \wedge K(Z/2Z,4n-1), \Omega^3E(1)]$ is injective, $D_z = 0$. Any H-map between Eilenberg MacLane spaces is an r-loop map for any r and $z \sim \Omega^2\hat{z}$ for some

$\hat{z}$: $K(Z/2Z,4n+1) \to \Omega E(1)$. Use $\hat{z}$ to change the homotopy $* \sim \alpha_0 \circ Sq_0^{4n}$ and then, for the new $\hat{\alpha}_0$ one has $\Omega^2[\alpha_{0,1,2} \circ \hat{\alpha}_0] = 0$, $* \sim \Omega^2 v \sim \Omega^2(v-u)$ (as $\Omega^2 u \sim *$, since $\Omega^2 \wedge \sim *$ in the "geometric" definition of u). But one can see that Ω^2: $H^{8n+4}(K/Z,4n+1), Z/2Z) \to H^{8n+2}(K(Z,4n-1),Z/2Z)$ is injective on primitives (Eilenberg Moore spectral sequence) and $u \sim v$).

Consequently:

$\Phi_1(Sq_0^{4n}) = [\rho] \circ Sq^2[\rho] + \text{im } Sq^2 + \text{im } Sq^1$.

Corollary I: Let $x \in H^{4n+1}(X,Z)$ be any class, X - any space. If $Sq_0^{4n}x = Sq^{4n}\rho x = 0$ then $\rho x \cdot Sq^2 \rho x = Sq^2 y_1 + Sq^1 y_2$ for some $y_1 \in H^{8n+2}(X,Z/2Z)$, $y_2 \in H^{8n+3}(X,Z/2Z)$.

Proof of I: Consider the following

$$\begin{tikzcd}
 & & \Omega E(1) \arrow[r, "\Omega\alpha_1"] \arrow[d, "j_0"'] & \Omega E(2) \\
 & & E(0,1) \arrow[ur, "\alpha_{0,1,2}"] \arrow[d] & \\
X \arrow[r, "x"] \arrow[uurr, dashed, "y"] & K(Z,4n+1) \arrow[r, "Sq_0^{4n}"] \arrow[ur, "\hat{\alpha}_0"] & K(Z/2Z,8n+1) &
\end{tikzcd}$$

$\hat{\alpha}_0$, $\alpha_{0,1,2}$ as in D_{11}, $\hat{\alpha}_0$ chosen so that $\alpha_{0,1,2} \circ \hat{\alpha}_0 = [\rho]\, Sq^2[\rho]$. As $Sq_0^{4n}x = 0$ $\hat{\alpha}_0 \circ x = j_0 \circ y$ for some $y\colon X \to \Omega E(1) \approx K(Z/2Z,8n+2) \times K(Z/2Z,8n+3)$. Put $y_i = \rho_i y$ and then $\rho x \cdot Sq^2 \rho x = [\alpha_{0,1,2} \circ \hat{\alpha}_0 \circ x] = [\Omega\alpha_1 \circ y] = Sq^2 y_1 + Sq^1 y_2$.

Corollary II: There is no space X with $H^*(X,Z/2Z)$ being the exterior algebra on x and $Sq^2 x$, dim x = 4n+1. (I.e. X satisfies: $H^i(X,Z/2Z) \neq 0$ only if i=0, 4n+1, 4n+3, 8n+4, and in these dimensions $H^i(X,Z/2Z) \approx Z/2Z$ with non zero elements 1, x, $Sq^2 x$ and $x \cdot Sq^2 x$ for i=0, 4n+1, 4n+3 and 8n+4 respectively)

Proof of Corollary II: In such a space $Sq_{o}^{4n}x = 0$ (as $H^{8n+1}(X,Z/2Z) = 0$) but $0 \neq x \circ Sq^{2}x = Sq^{2}y_{1} + Sq^{1}y_{2}$ is impossible for there are no elements in the dimensions of y_1 and y_2.

<u>Corollary III</u>: ([Hilton-Whitehead]), (4.11) P.435). If $\iota \in \Pi_{4n+1}(S^{4n+1}) = Z$ is a generator and $o \neq \eta \in \Pi_{4n+2}(S^{4n+1}) = Z/2Z$ $(n > 0)$ then $<\iota, \eta> \neq 0$ where $< >$ is the Whitehead product.

<u>Proof of Corollary III</u>. If $<\iota, \eta> = 0$ one can form a space X which is a S^{4n+1} fibration over S^{4n+3} with η as the first attaching map, i.e.: $X \approx S^{4n+1} \cup e^{4n+3} \cup e^{8n+4}$, $X/S^{4n+1} \approx S^{4n+3} \vee S^{8n+4}$ and S^{4n+1} is the homotopy fiber of $X \to X/S^{4n+1} \overset{\approx}{\to} S^{4n+3} \vee S^{8n+4} \to S^{4n+3}$. But such a space will have the cohomology of the space described in Corollary II.

8. CONCLUDING REMARKS

This is by no means the end of the road for the two theories and their partnership. A work in progress ([Harper-Zabrodsky]) attempts to generalize all that was said in chapter 7 for odd primes - p. Here one requires p-th order operations which naturally are far harder to define and evaluate.

* * *

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