Pub. Mat. UAB
Vol. 26 Nº 3 Des. 1982

PERIODIC SURFACE DIFFEOMORPHISMS WHICH BOUND, BOUND PERIODICALLY

John Ewing
Allan Edmonds

§0. Introduction

The computation of the bordism group of orientation preserving diffeomorphisms on closed surfaces was recently completed - first by Bonahon ([2]),[3]) and then by Edmonds and Ewing [6]. In both cases the key proposition turns out to be a remarkably simple looking statement.

Proposition

Let S be a closed, oriented surface and $f: S \to S$ an orientation preserving diffeomorphism such that $f^m = 1$. Suppose there is a 3-manifold W and a diffeomorphism $F: W \to W$ such that $\partial(W) = S$ and $F|\partial W = f$. Then there is another 3-manifold W′ and periodic diffeomorphism $F': W' \to W'$ with $(F')^m = 1$ such that $\partial(W') = S$ and $F'|\partial(W') = f$.

In other words; a periodic surface diffeomorphism which bounds, bounds periodically.

Bonahon's elegant proof of this proposition uses the full force of modern 3-dimensional topology, calling on (among other things) Mostow Rigidity and Thurston's Hyperbolization Theorem. The purpose of this note is to provide a quite different proof which uses an elementary form of the G-signature Theorem known as the Eichler Trace Formula (ca. 1930) and an elementary theorem in number theory due to Carl Ludwig Siegel (ca. 1949).

§1. The Theorem of Siegel

We first record the theorem in number theory which we require. It is, in fact, a result about Dirichlet series which is closely connected with Dirichlet's famous theorem about the non-vanishing of L-series.

Suppose m is a positive integer and $\psi : \mathbb{Z} \to \mathbb{Z}$ is a function satisfying:

(i) $\psi(a) = \psi(b)$ if $a \equiv b \mod m$

(ii) $\psi(a) = 0$ if $(a,m) > 1$

(iii) $\psi(-a) = -\psi(a)$ for all a.

We can consider the Dirichlet series

$$\Psi(s) = \sum_{\nu=1}^{\infty} \frac{\psi(\nu)}{\nu^s}$$

which is easily seen to converge for $Re(s) > 0$.

<u>Theorem</u> [4]. If $\Psi(1) = 0$ then $\psi \equiv 0$.

The proof of this is remarkably simple and takes two slim pages. (An alternative proof is provided in [6] and a more general result can be found in [1]).

§2. <u>The G-signature</u>

We can now begin proving the main proposition. The first question we ought to ask is: How do we use the fact that $(S,f) = \partial(W,F)$? The answer is: to show the G-signature vanishes.

Here's a quick review of the definition. We define a skew-Hermitian form β on $H^1(S;\mathbb{C}) = H^1(S;\mathbb{Z}) \otimes \mathbb{C}$ by

$$\beta(x_1 \otimes \alpha_1, x_2 \otimes \alpha_2) = \alpha_1 \overline{\alpha}_2 \langle x_1 \cup x_2, [S] \rangle.$$

Now the signature of this form (the number of eigenvalues in the lower half plane - the number of eigenvalues in the upper half plane) is zero; that's not too interesting. But we also have that automorphism $f^* : H^1(S;\mathbb{C}) \to H^1(S;\mathbb{C})$ and the G-signature measures how β and f^* interact; it <u>is</u> interesting. Specifically, for $k = 0,1,\ldots,m-1$ let V_k denote the eigenspace of f^* corresponding to the eigenvalue ζ^k where $\zeta = e^{2\pi i/m}$. then the G-signature is defined by

$$\text{sign}(f,S) = \sum_{k=0}^{m-1} \text{sign}(\beta|V_k)\zeta^k.$$

It is, of course, an algebraic integer.

The important thing about the signature of a manifold is that it vanishes when the manifold bounds. The important thing about the G-signature of a manifold is that it also vanishes when the manifold bounds equivariantly. To see this, for example, when (S,f) is a periodic boundary we simply note that if $(S,f) = \partial(W,F)$ then $\mathrm{im}(H^1(W;\mathbb{C}) \to H^1(S;\mathbb{C}))$ is an f^*-invariant subspace which is its own orthogonal complement (by Poincare duality.)

But now we simply observe that nowhere have we used the fact that F is periodic; exactly the same argument shows that if $(S,f) = \partial(W,F)$ then $\mathrm{sign}(f,S) = 0$, whether or not F is periodic.

§3. The Fixed Point Data

Before proceeding further we ought to think about how to conclude our argument. How do we show (S,f) bounds periodically? The answer is easy and classical: we look at the fixed point data.

Suppose f has isolated fixed points $P_1,\dots,P_t$. The "type" of each fixed point P is measured by the behaviour of df on the tangent space at P; if df is multiplication by ζ^k where $\zeta = e^{2\pi i/m}$ then we say P has type ζ^k. The collection of fixed points and types is called the fixed point data.

Now let n_k denote the number of fixed points of type ζ^k. (Of course, $n_k = 0$ if $(k,m) > 1$.)

Definition

If $n_a = n_{m-a}$ for all a, $1 \le a < m$, then we say the fixed point data cancels.

The following lemma is easy and well-known [4].

Lemma

The pair (S,f) bounds periodically if and only if the fixed point data of f, and all its powers, cancels.

The argument in one direction is elementary. If $(S,f) = \partial(W,F)$, $F^m = 1$, then the fixed point set of F consists of 1-dimensional submanifolds which can intersect $S = \partial(W)$ only in a pair of "canceling" fixed points.

The argument in the other direction is by induction on the number of fixed points. Given a pair of canceling fixed points we can remove a small disc about each and attach a handle equivariantly to obtain a cobordant pair with fewer fixed points. The induction is completed by using the fact that free actions in dimension 2 always bound. A more detailed argument can be found in [6].

4. The Eichler Trace Formula

We now know what we have to prove - that the fixed point data cancels. And we know what we have to work with - that the invariant sign(f,S) vanishes. What is obviously required is a means of connecting the fixed point data and the invariant sign(f,S). That is precisely what the Eichler Trace Formula does.

In this setting it says the following.

Eichler Trace Formula [7]. Suppose S is a closed, oriented surface and $f: S \to S$ is a periodic diffeomorphism of period m. Let n_k be the number of fixed points of type ζ^k where $\zeta = e^{2\pi i/m}$. Then

$$\text{sign}(f,S) = \sum_{k=1}^{m-1} n_k \frac{\zeta^k + 1}{\zeta^k - 1} .$$

Where does Siegel's Theorem come in? The expression in the Trace Formula above is actually a Dirichlet series in disguise.

To see this, we note that

$$\frac{\zeta^k + 1}{\zeta^k - 1} = \frac{1}{i} \operatorname{ctn} \left(\pi \frac{k}{m}\right)$$

Now $\pi \operatorname{ctn}(\pi z)$ has a particularly nice partial fraction decomposition:

$$\pi \operatorname{ctn} (\pi z) = \frac{1}{z} + \sum_{\nu=1}^{\infty} \left(\frac{1}{z+\nu} + \frac{1}{z-\nu}\right).$$

Setting $z = {}^k/m$ we see that

$$\frac{\zeta^k + 1}{\zeta^k - 1} = \frac{m}{\pi i} \left[\frac{1}{k} + \sum_{\nu=1}^{\infty} \left(\frac{1}{k+m\nu} + \frac{1}{k-m\nu}\right)\right]$$

$$= \frac{m}{\pi i} \sum_{\nu=0}^{\infty} \left(\frac{1}{k+m\nu} + \frac{-1}{m-k+m\nu}\right)$$

(One must be careful about conditional convergence here and below, but the argument to justify rearranging terms is simple and standard.)

If we define $\varepsilon_k : \mathbb{Z} \to \mathbb{Z}$ by

$$\varepsilon_k(a) = \begin{cases} 1 & a \equiv k \mod m \\ -1 & a \equiv -k \mod m \\ 0 & \text{otherwise} \end{cases}$$

then clearly

$$\frac{\zeta^k + 1}{\zeta^k - 1} = \frac{m}{\pi i} \sum_{\nu=0}^{\infty} \frac{\varepsilon_k(\nu)}{\nu} .$$

We can now use this in the Trace Formula. We let

$$\psi = \sum_{k=1}^{m-1} n_k \varepsilon_k$$

and can then write

$$\operatorname{sign}(f,S) = \frac{m}{\pi i} \sum_{\nu=0}^{\infty} \frac{\psi(\nu)}{\nu} .$$

Now when k and m are relatively prime the ε_k are functions which satisfy the conditions of Siegel's Theorem mentioned in section 1. Hence, the linear combination ψ satisfies these conditions as well.

If (S,f) bounds then we know that $\operatorname{sign}(f,S) = 0$. Applying Siegel's Theorem, we conclude that ψ is identically zero. But for any a, $1 \leq a < m$, there are at most two ε_k's which are non-zero on a; we see that

$$0 = \psi(a) = n_a - n_{m-a},$$

and so the fixed point data cancels. The proof is complete.

References

1. A. Baker, B.J. Birch, E.A. Wirsing, On a problem of Chowla, J. of Number Theory, 5(1973), pp.224-236.

2. F. Bonahon, Cor bordisme des diffeomorphismes des surfaces, C.R. Acad. Sci. Paris, Ser. A-B, 290 (1980), pp. A765-A767.

3. ______, Cobordism of automorphisms of surfaces, Republications, Univ. de Paris-Sud, Bât. 425, Orsay.

4. S. Chowla, A special infinite series, Norske Vid. Sel. For., 37(1964), No. 16, pp.85-87.

5. P.E. Conner and E.E. Floyd, Differentiable Periodic Maps, Berlin-Heidelberg-New York, Springer, 1964.

6. A. Edmonds and J. Ewing, Remarks on the cobordism group of surface diffeomorphisms, to appear in Math. Ann.

7. H.M. Farkas and I. Kra, Riemann Surfaces, Berlin-Heidelberg-New York, Springer, 1980.

Indiana University
Bloomington, Indiana 47405
USA