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SECONDARY OPERATIONS, K-THEORY AND H-SPACES

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Abstract Let X be an H-space with $H_*(X, Z_{(p)})$ a free $Z_{(p)}$-module of finite type and whose rational Pontrjagin ring is an associative, graded commutative algebra. Recently it has been proved that the ring $H_*(X, Z_{(p)})$ is isomorphic to a tensor product of quasi-monogenic algebras. The proof uses K-theoretic techniques. In this note we relate the proof to results obtained by the more familiar unstable secondary operations.

1. Let (X,μ) be an H-space. The fundamental theorem on unstable secondary cohomology operations in $H^*(X, Z/pZ)$ is this.

Theorem 1.1 Let $\bar{x} \in QH^{2n}(X, Z/pZ)$ have representative $\bar{x} \in H^{2n}(X, Z/pZ)$ with $\bar{\mu}^*(\bar{x}) \in B \otimes B$ where B is a sub-Hopf algebra of $H^*(X, Z/pZ)$ which is stable under the action of $\mathcal{A}(p)$. Suppose that $\bar{x} = \theta(\bar{y})$ where $\theta \in \mathcal{A}(p)$ and that $\beta P^n\theta = \sum a_i b_i$ is an unstable relation holding in dimension $2n - |\theta| + 1$. Assume that $b_i(\bar{y}) \in \bar{B}.\bar{B}$. Then there exists a secondary operation X defined on $\bar{y}$ such that

$$\bar{\mu}^{(p-1)}(X(\bar{y})) = \bar{x} \otimes \bar{x} \otimes \dots \otimes \bar{x} + \sum \text{Image } a_i$$

in $Q(H^*(X, Z/pZ)/\!/B) \otimes Q(H^*(X, Z/pZ)/\!/B) \otimes \dots \otimes Q(H^*(X, Z/pZ)/\!/B)$.

There are several variants of this theorem. The prototype can be found in [9] and the primary reference is [7]. Successful applications have been made in investigating torsion in the homology of finite H-spaces, particularly in numerous papers of Kane and Lin. The theorem has also been used to investigate the cohomology of H-spaces with little or no torsion [6,8]. In this note a related result is made explicit which is proved using complex K-theory under the additional assumptions that (a) $H_*(X,Z_{(p)})$ is a free $Z_{(p)}$-module of finite type, (b) the rational Pontrjagin ring $H_*(X,Q)$ is both associative and graded commutative. Condition (a) is a major restriction. The result proved is essentially a lemma of K-theory and condition (a) is needed primarily to give a direct translation from K-theory to cohomology. Without condition (a) a version of the K-theoretic lemma can be proved but the conclusion then has to be translated back to cohomology through a spectral sequence. The condition (b) is not significant. It is satisfied for some μ by all H-spaces which occur naturally and in any case a more elaborate argument will lead to a related conclusion without it.

Before stating Theorem 1.2, we indicate the way in which Theorem 1.1 is usually employed and how we wish to strengthen it. Let $p = 2$ and $\overline{x} \in H^{4n}(X,Z/2Z)$ be indecomposable. We wish to conclude that there exists $\overline{z} \in H^{8n}(X,Z/2Z)$ such that in $Q(H^*(X,Z/2Z)) \otimes Q(H^*(X,Z/2Z))$ we have $\overline{\mu}(\overline{z}) = \overline{x} \otimes \overline{x}$. First we must find an $\mathfrak{A}(2)$ sub-Hopf algebra B with $\overline{\mu}(\overline{x}) \in B \otimes B$, so for simplicity we assume that $\overline{x}$ is primitive so that we may choose $\overline{B} = 0$. Without more information on the action of $\mathfrak{A}(2)$, we must set θ = Identity. A suitable Adem relation is then $Sq^1 Sq^{4n} = Sq^{4n}(Sq^1) + Sq^2(Sq^1 Sq^2 Sq^{4n-4})$ for $Sq^1(\overline{x}) = 0$ and $Sq^1 Sq^2 Sq^{4n-4}(\overline{x}) \underline{=} 0$ hold automatically if condition (a) is satisfied. Theorem 1.1 then implies that there exists $\overline{z} = X(\overline{x})$ such that $\overline{\mu}(\overline{z}) = \overline{x} \otimes \overline{x}$ + Image Sq^2 in $QH^*(X,Z/2Z) \otimes QH^*(X,Z/2Z)$, since Sq^{4n} can give no contribution for dimensional reasons. The problem we face is to remove the indeterminacy associated with Sq^2.

Similarly if p is odd, $n \neq 1 \bmod p$ and $\overline{x} \in PH^{2n}(X,Z/pZ)$ is indecomposable, one can deduce from Theorem 1.1 that there exists $\overline{z} \in H^{2np}(X,Z/pZ)$ such that
$\overline{\mu}^{(p-1)}(\overline{z}) = \overline{x} \otimes \overline{x} \otimes \ldots \otimes \overline{x}$ + Image P^1 in
$Q(H^*(X,Z/pZ)) \otimes Q(H^*(X,Z/pZ)) \otimes \ldots \otimes Q(H^*(X,Z/pZ))$. We wish to remove the indeterminacy P^1.

<u>Theorem 1.2</u> Let $\overline{x} \in H^{2n}(X,Z/pZ)$ be indecomposable and $n \neq 1 \bmod p$. Then there exists $\overline{z} \in H^{2np}(X,Z/pZ)$ such that in $Q(H^*(X,Z/pZ)) \otimes Q(H^*(X,Z/pZ)) \otimes \ldots \otimes Q(H^*(X,Z/pZ))$,
$\overline{\mu}^{(p-1)}(\overline{z}) = \overline{x} \otimes \overline{x} \otimes \ldots \otimes \overline{x}$, (p factors).

If $n = 1 \bmod p$, then the conclusion of Theorem 1.2 remains true unless $n = 1 + p^{\alpha_1} + p^{\alpha_2} + \ldots + p^{\alpha_s}$ where $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_s$, see [5].

§2 The proof of Theorem 1.2 is contained in [5] and we will try and make this more explicit. Throughout this note we will use the notation that if $\overline{x} \in H^{2n}(X,Z/pZ)$, then $x \in H^{2n}(X,Z_{(p)})$ is a class whose reduction mod p is $\overline{x}$. If $\overline{x}^p$ in Theorem 1.2 is zero, then $x^p = 0 \bmod p$ in $H^{2np}(X,Z_{(p)})$ and the reduction mod p of the class $p^{-1}x^p$ is a suitable choice for $\bar{z}$, as can be verified by an elementary Hopf algebraic computation. Therefore we will assume that $\bar{x}^p \neq 0$. We will first dispose of the case when $n = 0$ mod p which will complete the proof of Theorem 1.2 for $p = 2$.

Let $T^*: H^*(X,A) \to H^*(X,A)$ be the homomorphism induced by a p-th power map with respect to some order of multiplication, $T: X \to X$, where $A = Z/pZ$, $Z_{(p)}$ or Q. Proposition 2.2 of [5] implies the following lemma.

<u>Lemma 2.1</u> Let $\bar{x} \in H^{2n}(X,Z/pZ)$ where $n = 0$ mod p and $\bar{x}^p \neq 0$. Then there exists $\bar{w} \in H^{2np}(X,Z/pZ)$ such that $T^*(\bar{w}) = \bar{x}^p + \bar{y}^p$ where $\bar{y}$ is decomposable.

This implies what is required if $n = 0$ mod p. We choose x, w and y in $H^*(X,Z_{(p)})$ representing $\bar{x},\bar{w},\bar{y}$. Thus $T^*(w) = pw + x^p + y^p + pz$ with z decomposable. One now applies the purely Hopf algebraic techniques of [3]. By Lemma 2.2 of [3] we can choose a multiplicative basis for $H^*(X,Z_{(p)})$, $\{x_i\}$ say, such that $T^*(x_i) = px_i + \sum r_i z_i^p$ for each i where $r_i \in Z_{(p)}$. It follows that we can choose w', differing from w by a decomposable element such that $T^*(w') = pw' + x^p + v'$, where $v \in \{\bar{H}^*(X,Z_{(p)})\}^{p+1}$. Now as in Lemma 2.3 of [3] we deduce that there exists an element $v' \in \{\bar{H}^*(X,Q)\}^{p+1}$ such that in $H^*(X,Q)$, $T^*(w' + (p-p^p)^{-1}x^p + v') = p(w' + (p-p^p)^{-1} + v')$. Thus $w' + (p-p^p)^{-1}x^p + v'$ is primitive in $H^*(X,Q)$. Direct computations then verify that $\bar{\mu}^{(p-1)}(w') = x \otimes x \otimes \ldots \otimes x$ mod p in $Q(H^*(X,Z_{(p)}) \otimes Q(H^*(X,Z_{(p)}) \otimes \ldots \otimes Q(H^*(X,Z_{(p)}))$ and this implies Theorem 1.2 in this case.

It remains to prove Theorem 1.2 when p is odd and n is congruent to neither 0 or 1 mod p. Let $\overline{x}_n \in H^{2n}(X,Z/pZ)$ with n as described be indecomposable. Then by adding a decomposable element to $\overline{x}_n$, we may assume that it is primitive and so in Theorem 1.2 we may assume that $\overline{x} = \overline{x}_n$ is primitive. We have also seen it is sufficient to consider the case where $\overline{x}^p = P^n(\overline{x})$ is non zero. The Adem relation $P^1P^{r-1} = rP^r$ implies that Theorem 1.2 follows from the next proposition.

Proposition 2.2 (p odd). Let $\overline{x} \in PH^{2n}(X,Z/pZ)$ with $n \neq 0$ or 1 mod p. Suppose that in $PH^*(X,Z/pZ)$, $P^2(\overline{s}) = \overline{x}^p \neq 0$. Then there exists $\overline{z} \in H^{2np}(X,Z/pZ)$ such that in $Q(H^*(X,Z/pZ)) \otimes Q(H^*(X,Z/pZ)) \otimes \ldots \otimes Q(H^*(X,Z/pZ))$, $\overline{\mu}^{(p-1)}(\overline{z}) = \overline{x} \otimes \overline{x} \otimes \ldots \otimes \overline{x}$.

The proof is essentially contained in Proposition 3.3 of [5] but we shall provide the details. We recall some basic facts about Z/2Z graded complex K-theory. First we note that for Hopf algebraic reasons there is no loss of generality in assuming for the proof that $H_{2i+1}(X,Z_{(p)}) = 0$ for all i, so that $K^1(X,Z_{(p)}) \cong 0$. The grade zero term $K(X,Z_{(p)})$ is filtered as a ring by the CW-filtration with associated graded ring $\oplus H^{2i}(X,Z_{(p)})$, where $K(X,Z_{(p)})_{2i}/K(X,Z_{(p)})_{2i+1} \cong H^{2i}(X,Z_{(p)})$. Also $K(X,Z_{(p)})$ is canonically isomorphic to a direct sum $\overline{\oplus}\ K^{\alpha}(X,Z_{(p)})$, $1 \leq \alpha \leq p-1$ and $K^{\beta}(X,Z_{(p)})$. $K^{\gamma}(X,Z_{(p)}) \subset K^{[\beta+\gamma]}(X,Z_{(p)})$ where $[\beta+\gamma] + s(p-1) = \beta + \gamma$. The filtration induced on $K^{\alpha}(X,Z_{(p)})$ has as associated graded group $\oplus\ H^{2\alpha+2i(p-1)}(X,Z_{(p)})$, $i \geq 0$.

The following standard properties of the Adams operators $\psi^p : K(X, Z_{(p)}) \to K(X, Z_{(p)})$ will be used [1,2].

(a) ψ^p is a natural filtration preserving ring homomorphism;

(b) $\psi^p(K^\alpha(X, Z_{(p)})) \subset K^\alpha(X, Z_{(p)})$;

(c) If $\xi \in K^\alpha(X, Z_{(p)})$ has exact filtration $2n$ where $\alpha = n \bmod(p-1)$, then there exist $\xi_i \in K^\alpha(X, Z_{(p)})_{2n+2i(p-1)}$ such that $\psi^p(\xi) = \sum p^{n-i}\xi_i$, $0 \le i \le n$, where $\xi_o = \xi$ and we may choose $\xi_n = \xi^p$. Further if ξ has image $x \in H^{2n}(X, Z_{(p)})$ and ξ_i has image $x_i \in H^{2n+2i(p-1)}(X, Z_{(p)})$ then $P^i\overline{x} = \overline{x}_i$ in $H^*(X, Z/pZ)$. We will write $t : K(X, Z_{(p)}) \to K(X, Z_{(p)})$ for the homomorphism induced by $T : X \to X$.

We turn to the proof of Proposition 2.2. For dimensional reasons we may choose primitive elements in $H^*(X, Z_{(p)})$, x, s and u to represent $\overline{x}$, $\overline{s}$ and $\overline{u} = P^1(\overline{s})$ in $H^*(X, Z/pZ)$. We then choose K-theory representatives ξ, ζ and η to represent x, s, u always lying in the appropriate K^α summand of K. Now $P^1(\overline{s}) = 2^{-1}\overline{x}^p$ and so working always modulo $K(X, Z_{(p)})_{2np+2}$, we have $\psi^p(\eta) = p^{np-(p-1)}\eta + 2^{-1}p^{np-p}\xi^p + p^{np-(p-1)}w_{np}$ where $w_{np} \in K(X, Z_{(p)})_{2np}$. We will assume that the proposition is false so there will exist no $\overline{z} \in H^{2np}(X, Z/pZ)$ with $T^*(\overline{z}) = \overline{x}^p$. Let $t(\eta) = p\eta + w'_{np}$ and consider the equation $\psi^p(t(\eta)) = t\psi^p(\eta)$. This gives $p^{np-p+2}\eta + 2^{-1}p^{np-p+1}\xi^p + p^{np-p+2}w_{np} + p^{np}w'_{np}$ $= p^{np-p+2}\eta + p^{np-p+1}w'_{np} + 2^{-1}p^{np}\xi^p + p^{np-p+1}t(w_{np})$. Thus $t(w_{np}) = pw_{np} + 2^{-1}(1-p^{p-1})\xi^p - (1-p^{p-1})w'_{np}$. Our assumption that the proposition is false then implies that $w'_{np} \not\equiv 0 \bmod p$ and so for any choice of η, $t(\eta) \not\equiv 0 \bmod p$.

Now without affecting anything done above, we assume that

ζ has been chosen so that $t(\zeta) = p\zeta + v_{np}$ where $v_{np} \in K(X,Z_{(p)})_{2np}$. The argument needed to show that this is possible is very similar to that used above based on Lemma 2.2 of [3]. We work temporarily mod $K(X,Z_{(p)})_{2np-2(p-1)+2}$. Let $\psi^p(\zeta) = p^{np-2(p-1)}\zeta + p^{np-2p+1}\eta'$ so that in $K(X,Z_{(p)})_{2np-2(p-1)}/K(X,Z_{(p)})_{2np-2(p-1)+2} \cong H^{2np-2(p-1)}(X,Z_{(p)})$ the reduction mod p of η' is $\bar{u}$. Let $\psi^k(\zeta) = k^{np-2(p-1)}\zeta + \eta''$. After simplifying the explicit expression for $\psi^p\psi^k(\zeta) = \psi^k\psi^p(\zeta)$ we obtain $k^{np-2(p-1)}(1-k^{(p-1)})\eta' = p(1-p^{p+1})\eta''$. If we choose k to be a generator of $(Z/p^2Z)^*$, so that p^2 does not divide $1-k^{(p-1)}$, we deduce that $\eta'' = \alpha\eta$ mod p where α is a unit.

Now returning to $K(X,Z_{(p)})/K(X,Z_{(p)})_{2np+2}$, it is clear that by modifying, if necessary, η'' in filtrations greater than $2np-2(p-1)$, we may assume that $\psi^k(t(\zeta)) = k^{np-2(p-1)}\zeta + \eta''$. The last equation we consider is $\psi^k(t(\zeta)) = t\psi^k(\zeta)$ mod p. This gives $k^{np}v_{np} = k^{np-2(p-1)}v_{np} + t(\eta'')$ and so $t(\eta'') = 0$ mod p. But mod p, η'' is up to units a choice for η and we have proved that $t(\eta) \neq 0$ mod p. This contradiction establishes the proposition and completes the proof of Theorem 1.2.

The elimination of the indeterminacy described in Theorem 1.2 appears to be the main reason why the proof of Theorem 1.1 of [5] succeeds where earlier attempts have failed. It is perhaps worth mentioning that there can be no direct cohomological analogue of this result. Finally we remark that [5] leads to significant simplifications in the proofs of the results of [4], in particular to the unstable characterization of the H-space $BU_p^{\oplus}$ at odd primes.

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