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FIBRATIONS ARISING IN THE STUDY OF P-TORSION IN HOMOTOPY

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The purpose of this article is to highlight two significant results related to odd torsion in the homotopy theory of spheres and Moore spaces which has come out of the work of Cohen, Moore, and Neisendorfer and to survey those parts of the proof related to the existence and properties of certain fibrations. This article is designed as a supplement to the author's two other surveys (LN 788 and LN 842) and as a guide to the original papers (see references to the three authors) including some modification of their point of view.

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§1. Modular spheres and related fibrations.

For each integer q we can consider a map of degree q defined $S^{m-1} \xrightarrow{q} S^{m-1}$ and the corresponding cofibre sequence defining the space $S^m(q)$ namely

$$S^{m-1} \xrightarrow{q} S^{m-1} \longrightarrow S^m(q) \longrightarrow S^m \xrightarrow{q} S^m.$$

(1.1) Definition. The mod q m-sphere is the space $S^m(q) = S^{m-1} \cup_q e^m$ which is the middle term of the above cofibre sequence. The space $S^m(q)$ is an example of a Moore space since $\tilde{H}_{m-1}(S^m(q)) = \mathbb{Z}/q$ is its only nonvanishing reduced integral homology group. The three authors Cohen, Moore, and Neisendorffer, use the notation $P^m(q)$ for $S^m(q)$. Since this space is to be thought of as a modular sphere, we prefer the notation involving the letter S, and in this way we avoid confusion with projective spaces.

Observe that $S^m(q)$ is defined for $m \geq 2$, i.e. there is no modular circle and that the suspension $S(S^m(q)) = S^{m+1}(q)$ so that $S^m(q)$ is a suspension for $m \geq 3$ and a double suspension for $m \geq 4$. With the modular sphere we define the modular homotopy groups.

(1.2) Definition. The mod q homotopy groups of a (pointed) space X is $\pi_m(X, \mathbb{Z}/q) = [S^m(q), X]$ the set of homotopy classes of maps with the group structure defined from coH-space struture of the suspension $S^m(q)$ for $m \geq 3$.

As usual the modular homotopy groups are functors from the category of pointed spaces to groups and further abelian groups for $m \geq 4$. For q odd and $m \geq 4$ the abelian group $\pi_m(X, \mathbb{Z}/q)$ is a $\mathbb{Z}/q$ - module. Since $[S^m(2), S^m(2)] = \mathbb{Z}/4$ we will consider only q that are odd, and moreover q is usually an prime power $q = p^r$ where $p > 2$. For some considerations the prime p = 3 is special and requires addition arguments. These modifications are taken up in the article of Neisendorfer, see [9].

(1.3) Remark. Given a map $f : X \longrightarrow Y$ there is a canonical fibre sequence

$$\Omega X \xrightarrow{\Omega f} \Omega Y \longrightarrow F_f \longrightarrow X \xrightarrow{f} Y$$

where F_f is the homotopy fibre of f. Briefly recall f is factored $X \longrightarrow X' \longrightarrow Y$ where $X \longrightarrow X'$ is a homotopy equivalence and $X' \longrightarrow Y$ is a fibration. The space X' is the subspace of (x,u) in $X \times \mathrm{Map}([0,1],Y)$ such that $f(x) = u(0)$, and F_f is the subspace of X' consisting of (x,u) with $u(1) = *$, the base point.

(1.4) Notations. From two mappings in the cofibre sequence defining the mod q spheres we have two fibre sequences

$$S^m\{q\} \longrightarrow S^m \xrightarrow{q} S^m \quad \text{and} \quad F^m\{q\} \longrightarrow S^m(q) \longrightarrow S^m.$$

If $u : T \longrightarrow X$ and $f : X \longrightarrow Y$ are two maps, then the composite fu is null homotopic if and only if u factors $T \longrightarrow F_f \longrightarrow X$ up to homotopy, and moreover, the factorizations $T \longrightarrow F_f$ of f are in bijective correspondence with the orbits of the action of the group $[T,\Omega Y]$ acting on the set $[T,F_f]$. A basic example of this factorization for us is $S^m(q) \longrightarrow S^m\{q\} \longrightarrow S^m$.

(1.5) Remark. Let $h = gf$ be the composite of $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$. Then we have a commutative diagram from the homotopy fibres of the three maps f, g, and h.

$$\begin{tikzcd}
& F_f \arrow[dl] \arrow[dr] & & \\
F_h \arrow[rr] \arrow[d] & & X \arrow[d, "f"'] \arrow[dr] & \\
F_g \arrow[rr] & & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

The sequence $F_f \longrightarrow F_h \longrightarrow F_g$ is a fibre sequence, i.e. F_f is homotopy fibre of $F_h \longrightarrow F_g$.

(1.6) <u>Example</u>. We apply this to the previous factorization of the map $S^m(q) \longrightarrow S^m$ resulting from pinching the bottom cell to a point. We obtain a new homotopy fibre $E^m\{q\}$ and a commutative diagram of fibrations.

$$\begin{tikzcd}
\Omega S^m \arrow[dr] \arrow[dd, dashed, "\text{composite is } \Omega q"'] & E^m\{q\} \arrow[d] \arrow[dr] & & \\
& F^m\{q\} \arrow[r] \arrow[dl] & S^m(q) \arrow[d] \arrow[dr] & \\
\Omega S^m \arrow[rr] & & S^m\{q\} \arrow[r] & S^m
\end{tikzcd}$$

Since $\pi_m(S^m(q)) = \mathbb{Z}/q$ in the fibration $F^m\{q\} \longrightarrow S^m(q) \longrightarrow S^m$, the homotopy exact sequence yields an isomorphism

$$\mathbb{Z} = \pi_m(S^m) \longrightarrow \pi_{m-1}(F^m\{q\}).$$

Hence up to sign we have a natural map which can be choosen so that its composite with $F^m\{q\} \longrightarrow \Omega S^m$ is the suspension morphism $E : S^{m-1} \longrightarrow \Omega S^m$. This leads to the following commutative diagram where F_E is the homotopy fibre of E.

$$\begin{tikzcd}
F_E \arrow[r] \arrow[d] & E^m\{q\} \arrow[d] \\
S^{m-1} \arrow[r] \arrow[dr, "E"'] & F^m\{q\} \arrow[d] \\
& \Omega S^m
\end{tikzcd}$$

(1.7) <u>Remark</u>. The double suspension of odd spheres $E^2 : S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$, equal to the composite

$$S^{2n-1} \xrightarrow{E} \Omega S^{2n} \xrightarrow{\Omega E} \Omega^2 S^{2n+1},$$

will play a basic role in the next sections, and its fibre is denoted $C(n) \longrightarrow S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$. Looping the previous diagram for $m = 2n+1$, and composing with the suspension map, we obtain

$$\begin{tikzcd}
C(n) \arrow[r] \arrow[d] & \Omega E^{2n+1}\{q\} \arrow[d] \\
S^{2n-1} \arrow[r] \arrow[dr, "E^2"'] & \Omega F^{2n+1}\{q\} \arrow[d] \\
& \Omega^2 S^{2n+1}.
\end{tikzcd}$$

§2. Theorem I. The partial inverse of the double suspension map.

In this section all spaces are localized at an odd prime p.

(2.1) Remark. It is a basic result which was essentually known from Serre's thesis [15], [16] that for loops on an even sphere the suspension map $E : S^{2n-1} \longrightarrow \Omega S^{2n}$ is a homotopy split monomorphsim and ΩS^{2n} has the homotopy type of $S^{2n-1} \times \Omega S^{4n-1}$.

Here being localized at an odd prime is essentual. This reduces the question of odd torsion in even dimensional spheres to that of odd dimensional spheres, and it leads to the study of the double suspension $E^2 : S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$. In (1.7) we introduced the fibre C(n) of the map E^2, and we have a commutative diagram (recall everything is localized at an odd p).

$$\begin{tikzcd}
C(n) \arrow[r] \arrow[d] & \Omega E^{2n+1}\{p^r\} \arrow[d] \\
S^{2n-1} \arrow[r] & \Omega F^{2n+1}\{p^r\} \arrow[d] \\
\Omega^2 S^{2n+1} \arrow[ur] \arrow[r, "\Omega^2 p^r"'] & \Omega^2 S^{2n+1}
\end{tikzcd}$$

Now we are in a position to bring in a basic result of Cohen, Moore, and Neisendorfer contained in [2] and [3].

(2.2) Theorem. There exists a space $Y = Y(n,p^r)$ and a map $Y(n,p^r) \longrightarrow \Omega E^{2n+1}\{p^r\}$ such that this map product with the previous diagram followed with loop space multiplication gives a commutative diagram where the composite of the horizontal arrows in each case is a homotopy equivalence

$$\begin{tikzcd}
C(n) \times Y(n,p^r) \arrow[r] \arrow[d] & \Omega E^{2n+1}\{p^r\} \times \Omega E^{2n+1}\{p^r\} \arrow[r] \arrow[d] & \Omega E^{2n+1}\{p^r\} \arrow[d] \\
S^{2n-1} \times Y(n,p^r) \arrow[r] & \Omega F^{2n+1}\{p^r\} \times \Omega F^{2n+1}\{p^r\} \arrow[r] & \Omega F^{2n+1}\{p^r\}.
\end{tikzcd}$$

In the next section we make some remarks about the proof of this theorem and give some indication of the structure of the space $Y(n,p^r)$ which plays a basic role in the study of exponents. Now from (2.2) we derive the partial inverse property of the double suspension E^2 localized at an odd prime.

(2.3) Theorem I. Associated to the double suspension $E^2 : S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$ is a map $\pi : \Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$ such that $\pi E^2 = p$ and $E^2\pi = \Omega^2 p$ where p denotes a map of degree p.

The proof of this basic theorem can be derived immediately from the following diagram which results from the diagram preceding (2.2) and the splitting of S^{2n-1} off from $\Omega F^{2n+1}\{p^r\}$ by (2.2).

$$\begin{tikzcd}
S^{2n-1} \arrow[rr, "1"] \arrow[dr] & & S^{2n-1} \\
& \Omega F^{2n+1}\{p^r\} \arrow[ur] \arrow[dr] & \\
\Omega^2 S^{2n+1} \arrow[ur] \arrow[rr, "\Omega^2 p"'] & & \Omega^2 S^{2n+1}
\end{tikzcd}$$

The map π is the composite of $\Omega^2 S^{2n+1} \longrightarrow \Omega F^{2n+1}\{p^r\} \longrightarrow S^{2n-1}$. The desired properties of composition of this map with E^2 can be seen immediately from this diagram.

The following corollary of the existence of the partial inverse to E^2 was conjectured by Barratt.

(2.4) Corollary. For the p primary part of the homotopy of the odd spheres we have $p^n \pi_j(S^{2n+1})_{(p)} = 0$ if $i > 2n+1$.

The corollary follows by induction on n using the next sequence of abelian p-groups where two morphisms between two adjacent groups compose two ways to multipication by p.

$$0 = \pi_j(S^1)_{(p)} \underset{\pi}{\overset{E^2}{\rightleftarrows}} \pi_{j+2}(S^3)_{(p)} \underset{\pi}{\overset{E^2}{\rightleftarrows}} \pi_{j+4}(S^5)_{(p)} \cdots \underset{\pi}{\overset{E^2}{\rightleftarrows}} \pi_{j+2n}(S^{2n+1})_{(p)} \cdots$$

§3. Remarks on the splitting of loop spaces.

The homotopy of loop spaces $\pi_*(\Omega X)$ is a graded abelian group with a Lie bracket $[a,b]$ called the Samelson or Whitehead product defined

$$[\ ,\ ] : \pi_i(\Omega X) \times \pi_j(\Omega X) \longrightarrow \pi_{i+j}(\Omega X).$$

For $a = [u] \in \pi_i(\Omega X)$ and $b = [v] \in \pi_j(\Omega X)$ we make the commutator which is null homotopic on the wedge $S^i \vee S^j$ defining $w : S^{i+j} \longrightarrow \Omega X$. We define $[a,b] = [w]$.

$$\begin{tikzcd} S^i \vee S^j \arrow[r] & S^i \times S^j \arrow[r, "uvu^{-1}v^{-1}"] \arrow[d] & \Omega X \\ & S^{i+j} = S^i \wedge S^j \arrow[ur, "w"'] & \end{tikzcd}$$

(3.1) Remark. The Hurewicz morphism $\phi : \pi_*(\Omega X) \longrightarrow H_*(\Omega X)$ satisfies $\phi([a,b]) = [\phi(a),\phi(b)]$ where $[\ ,\ ]$ on $H_*(\Omega X)$ is the Lie bracket (with sign) associated with the loop product on $H_*(\Omega X)$.

(3.2) Remark. Returning to the decomposition in the thesis of Serre mentioned in (2.1), we use the Lie bracket of the suspension map with itself $[E,E] : S^{4n-2} \longrightarrow \Omega S^{2n}$ which extends to a map $g : \Omega S^{4n-1} \longrightarrow \Omega S^{2n}$. Multiplying this map g with the suspension map E, we obtain a map

$$S^{2n-1} \times \Omega S^{4n-1} \longrightarrow \Omega S^{2n}$$

which is a homology isomorphism with field coefficients for all characteristics except 2 since $[\phi(E),\phi(E)] = \phi(E)\phi(E) - (-1)\phi(E)\phi(E) = 2\phi(E)^2$ in $H_*(\Omega S^{2n})$. Inverting the prime 2 on the spaces, we have a map

$$S^{2n-1}[1/2] \times \Omega S^{4n-1}[1/2] \longrightarrow \Omega S^{2n}[1/2]$$

which is a homotopy equivalence.

Now assume that all spaces are localized at an odd prime p.

Using the same idea of generating maps of spaces of the form $S^m\{p^r\}$, $S^m\{p^{r+1}\}$, and (wedges of mod p^r spheres) into a loop space by looking at its mod p^r homotopy as a Lie algebra, the three authors Cohen, Moore, and Neisendorfer derived the following decompositions, the verification of which took up the bulk of the two papers [2] and [3]. These decompositions where surveyed in Husemoller [6], [7], and [8].

(3.3) <u>Theorem</u>. There is a homotopy equivalence
$$S^{2n+1}\{p^r\} \times \Omega(\bigvee_{0 \leqq k} S^{4n+2nk+3}(p^r)) \xrightarrow{\simeq} \Omega S^{2n+2}(p^r).$$

(3.4) <u>Theorem</u>. The space $Y(n,p^r)$ in theorem (2.2) which enters into the following commutative diagram where the horizontal arrows are homotopy equivalences
$$\begin{array}{ccc} C(n) \times Y(n,p^r) & \xrightarrow{\simeq} & \Omega E^{2n+1}\{p^r\} \\ \downarrow & & \downarrow \\ S^{2n-1} \times Y(n,p^r) & \xrightarrow{\simeq} & \Omega F^{2n+1}\{p^r\} \end{array}$$
has the form up to homotopy $Y(n,p^r) = \prod_{1 \leqq i} S^{2p^i n-1}\{p^{r+1}\} \times \Omega S[n]$ where $S[n]$ is an infinite wedge of mod p^r spheres in dimensions $n(i) \geqq 4n-1$ with $n(i) \longrightarrow +\infty$.

§4. Ω^k-exponents. Theorem II on the Ω^2-exponent of a mod p^r sphere.

In this section all spaces are localized at an odd prime p.

(4.1) Definition. A space X has an Ω^k-exponent n provided $p^n.\mathrm{id}$ is null homotopic and $p^{n-1}.\mathrm{id}$ is not null homotopic on the loop space $\Omega^k X$.

Observe that if a k-connected space X has an Ω^k-exponent n, then $p^n \pi_*(X) = 0$.

(4.2) Example. In (2.3) we saw that $p^n \pi_{2n+1+j}(S^{2n+1}) = 0$ for $j > 0$. Since $\pi_m(S^m) = \mathbb{Z}_{(p)}$ for a sphere localized at p, we can consider $S^{2n+1}\langle 2n+1\rangle$ the $(2n+1)$ - connected cover where just the bottom homotopy group is killed. Then $p^n \pi_*(S^{2n+1}\langle 2n+1\rangle) = 0$. By a result of B. Gray [5] there are elements of order p^n in these homotopy groups, and it is the case that $S^{2n+1}\langle 2n+1\rangle$ has Ω^{2n}-exponent n. Neisendorfer and Selick have shown that $S^{2n+1}\langle 2n+1\rangle$ does not have an Ω^k- exponent for $k \leq 2n-2$.

(4.3) Proposition. On the H-space $S^m\{p^r\}$ the multiple $p^r.\mathrm{id}$ is null homotopic. For the H-space structure on $C(n)$ induced from the fact that E^2 is an H-space map $p.\mathrm{id}$ is null homotopic.

This proposition and the previous result (4.2) are proved in section 5, [4]. The proof uses essentually the existence of the partial inverse of E^2 and also the result of Selick [14] that $\Omega^2 S^3\langle 3\rangle$ is a homotopy retract of $\Omega^2 S^{2p+1}\{p\}$.

Before stating the second main result, we point out a negative result of a very simple nature contained in [4] as Proposition 3.5. This result is a consequence of the fact that $H_*(\Omega SX)$ is the tensor Hopf algebra on the coalgebra $H_*(X)$ when $H_*(X)$ is flat over the ground ring.

(4.4) <u>Proposition</u>. If X is a space with $\bar{H}_*(X, \mathbb{Z}/p) \neq 0$, then SX does not have an Ω-exponent.

The remainder of this article centers around the following result.

(4.5) <u>Theorem II</u>. The mod p^r sphere $S^m(p^r)$ has an Ω^2-exponent equal to $r+1$.

The fact that $p^{2r+1}.id$ is null homotopic on $\Omega^2 S^m(p^r)$ was shown by Cohen, Moore, and Neisendorfer in [4]. The fact that $r+1$ is the exponent was shown by Neisendorfer [11].

The idea of the proof is to show that $\Omega S^m(p^r)$ decomposes as a sequential $\varinjlim$ of finite products of spaces where $p^{r+1}.id$ is null homotopic on each factor. For even dimension $m = 2n+2$ we have by (3.3) a homotopy equivalence

$$\Omega S^{2n+2}(p^r) \longrightarrow S^{2n+1}\{p^r\} \times \Omega(\bigvee_{0\leq k} S^{4n+2nk+3}(p^r)).$$

By (4.3) the multiple $p^r.id$ is null homotopic on $S^{2n+1}\{p^r\}$. The other factor is a wedge of odd dimensional mod p^r spheres. Recall at this point the following result of Neisendorfer [9].

(4.6) <u>Proposition</u>. There is a homotopy equivalence between $S^m(p^r) \wedge S^n(p^r)$ and $S^{m+n}(p^r) \vee S^{m+n-1}(p^r)$.

This together with the classical Hilton-Milnor theorem

$$\begin{aligned} \Omega S(X \vee Y) &\simeq \Omega SX \times \Omega S(\bigvee_{0\leq k} X^{\wedge k} \wedge Y) \\ &\simeq \Omega SX \times \Omega S(Y \vee (X \wedge Y) \vee (X \wedge X \wedge Y) \vee \ldots) \end{aligned}$$

shows that the second term in the above decomposition of $\Omega S^{2n+2}(p^r)$ itself decomposes as $\varinjlim$ of finite products of $\Omega S^{n(i)}(p^r)$ where $n(i) \longrightarrow +\infty$.

(4.7) <u>Remark</u>. Any wedge $\bigvee_i S^{m(i)}(p^r)$ where $m(i) \rightarrow +\infty$ or is finite in number has an Ω^2-exponent of $r+1$.

This means that the burden of the proof of (4.5) Theorem II rests on finding a good decomposition of $\Omega S^{2n+1}(p^r)$ for an odd dimensional mod p^r sphere $S^{2n+1}(p^r)$. Unfortunately this loop space does not lend itself to the direct analysis that we find for $\Omega S^{2n+2}(p^r)$.

The idea is to use the second factor of $Y(n,p^r)$ given in (3.4), namely $\Omega S[n]$, and deloop the three maps from $\Omega S[n]$ coming in the diagram given in (3.4)

$$\Omega S[n] \longrightarrow \Omega E^{2n+1}\{p^r\} \longrightarrow \Omega F^{2n+1}\{p^r\} \longrightarrow \Omega S^{2n+1}(p^r)$$

yielding three fibre sequences

$$\begin{tikzcd}
 & \Omega E^{2n+1}\{p^r\} \arrow[r] \arrow[d] & V^{2n+1}\{p^r\} \arrow[d] \arrow[dr] & & E^{2n+1}\{p^r\} \arrow[d] \\
\Omega S[n] \arrow[ur] \arrow[r] \arrow[dr] & \Omega F^{2n+1}\{p^r\} \arrow[r] \arrow[d] & U^{2n+1}\{p^r\} \arrow[d] \arrow[r] & S[n] \arrow[ur] \arrow[r] \arrow[dr] & F^{2n+1}\{p^r\} \arrow[d] \\
 & \Omega S^{2n+1}(p^r) \arrow[r] & T^{2n+1}\{p^r\} \arrow[ur] & & S^{2n+1}(p^r)
\end{tikzcd}$$

Of the three fibrations with base spaces $V^{2n+1}\{p^r\}$, $U^{2n+1}\{p^r\}$, and $T^{2n+1}\{p^r\}$ the first two split immediately from (3.4). By a careful study of the homological properties of the third fibration where $H_*(\Omega S[n]) \longrightarrow H_*(\Omega S^{2n+1}(p^r))$ is injective over $\mathbb{Z}/p$, one deduces the following, see [4, proposition 1.4].

(4.8) <u>Proposition</u>. The loops on an odd dimensional mod p^r sphere $\Omega S^{2n+1}(p^r)$ is homotopically equivalent to the product $T^{2n+1}\{p^r\} \times \Omega S[n]$.

Combining this with remarks related to (4.6) and (4.7) above, we see that the following results.

(4.9) <u>Proposition</u>. For $m \geq 3$ the loop space $\Omega S^m(p^r)$ has the same homotopy type as $\varinjlim$ of finite products of $S^{2m(i)+1}\{p^r\}$ and $T^{2n(j)+1}\{p^r\}$ where $m(i), n(j) \geq n$ for $m = 2n+1$ or $2n+2$ and $m(i), n(j) \longrightarrow +\infty$.

This reduces the proof of (4.5) to the study of $T^{2n+1}\{p^r\}$.

§5. <u>The Ω-exponent of</u> $T^{2n+1}\{p^r\}$.

In view of (4.9) the spaces $S^n(p^r)$ will have an Ω^{k+1}-exponent if and only if the spaces $T^{2m+1}\{p^r\}$ have an Ω^k-exponent. Now we give the argument that $p^{2r+1}.\mathrm{id}$ is null homotopic on $\Omega T^{2n+1}\{p^r\}$. which is in [4].

(5.1) <u>Theorem</u>. On the space $\Omega T^{2n+1}\{p^r\}$ the multiple $p^{2r+1}.\mathrm{id}$ is null homotopic.

<u>Proof</u>. For this apply the construction (1.5) to the composite $\Omega S[n] \longrightarrow \Omega E^{2n+1}\{p^r\} \longrightarrow \Omega S^{2n+1}(p^r)$. We obtain the diagram

$$\begin{tikzcd}
 & \Omega V^{2n+1}\{p^r\} \arrow[d] \arrow[dr] & & \\
 & \Omega T^{2n+1}\{p^r\} \arrow[r] \arrow[dl] & \Omega S[n] \arrow[d] \arrow[dr] & \\
\Omega^2 S^{2n+1}\{p^r\} \arrow[rr] & & \Omega E^{2n+1}\{p^r\} \arrow[r] & \Omega S^{2n+1}(p^r)
\end{tikzcd}$$

Since $V^{2n+1}\{p^r\}$ has the same homotopy type as $C(n) \times \prod_{1\le i} S^{2p^i n-1}\{p^{r+1}\}$, we deduce that $p^{r+1}.\mathrm{id}$ is null homotopic on $\Omega V^{2n+1}\{p^r\}$ the fibre of the vertical fibration in the diagram. Since $p^r.\mathrm{id}$ is null homotopic on $\Omega^2 S^{2n+1}\{p^r\}$, the base of this fibration, it follows that $p^{2r+1}.\mathrm{id}$ is null homotopic on the total space $\Omega T^{2n+1}\{p^r\}$ of the fibration. This proves the theorem.

The result of Neisendorfer that $p^{r+1}.\mathrm{id}$ is null homotopic on $\Omega T^{2n+1}\{p^r\}$ is much deeper and involves returning to an analysis of $T^{2n+1}\{p^r\}$ similar to what is used to prove theorem (3.4).

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