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CUP-I PRODUCT AND HIGHER HOMOTOPIES IN THE DE RHAM COMPLEX

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In this note we give a description of the Steenrod squares starting with the analogue of the cup-i product in the deRham complex of Cartan-Miller ([3], [8]).

The integration of forms over simplices, which induces the isomorphism on cohomology in the deRham theorem, is not an algebra map. It is proved that it induces a morphism of coalgebras. Therefore the cohomology of a pullback fibration can be computed in terms of the deRham complex in some cases.

1. Main results.

Let R be a commutative ring with a unit. Denote by S the category of simplicial sets and by M_R, dM_R^* and $dM_R^{*,*}$ the category of R-modules, differential graded R-modules and differential bigraded R-modules respectively.

Cartan and Miller ([3], [8]) defined the deRham functor $A = A^{*,*} = A_R^{*,*}: S \to dM_R^{*,*}$ which to each simplicial set X assigns a bigraded algebra $\{A^{p,q}(X)\}$, $(A^{p,q}(X) = 0$ for $p > q)$, of compatible R-polynomial forms on the simplices of X.

Denote by $\Gamma_R = \Gamma_R(t)$ the divided power algebra over R on a single generator t. $\Gamma_R = \bigoplus_{q\geq 0} \Gamma_q$, where Γ_q denotes the free R-module on a single generator $\gamma_q = \gamma_q(t)$. The pairing $\gamma:\Gamma_p \otimes \Gamma_q \to \Gamma_{p+q}$ defined by setting

$$\gamma_p\gamma_q = \frac{(p+q)!}{p!q!}\gamma_{p+q}$$

gives Γ_R the structure of a commutative ring with unit $\gamma_0 = 1$.

Denote by $C = C^{*,*}:\mathcal{S} \to d\mathcal{M}_R^{*,*}$ the functor $C^{p,q}(X) = C^p(X;\Gamma_q)$ = normalized cochains with the coefficients in the module Γ_q.

In [3] and [8] it is shown that

Proposition 1. The integration of forms over simplices induces a transformation

$$\rho:A \to C$$

such that the induced map

$$\rho^*:H^p(A^{*,q}(X)) \to H^p(X;\Gamma_q)$$

is an isomorphism for $p < q$, and that the diagram

$$\text{(1)}\qquad \begin{array}{ccc} H^{p_1}(A^{*,q_1}(X)) \otimes H^{p_2}(A^{*,q_2}(X)) & \xrightarrow{\wedge} & H^{p_1+p_2}(A^{*,q_1+q_2}(X)) \\ \downarrow \rho^*\otimes\rho^* & & \downarrow \rho^* \\ H^{p_1}(X;\Gamma_{q_1}) \otimes H^{p_2}(X;\Gamma_{q_2}) & \xrightarrow{\cup} & H^{p_1+p_2}(X;\Gamma_{q_1+q_2}) \end{array}$$

commutes, where $\wedge$ denotes the map induced by the wedge product of forms and $\cup$ denotes the cup product followed by the pairing $\gamma:\Gamma_{q_1} \otimes \Gamma_{q_2} \to \Gamma_{q_1+q_2}$. The map $\rho:A \to C$ is not multiplicative. We will show that ρ extends to a morphism of coalgebras

$$B(\rho):BA \to BC\ ,$$

where BA denotes the bar construction on A. Maps like ρ which are not multiplicative on the level of algebras but extend to a morphism on the level of the bar construction define morphisms in an extended category DASH (Differential graded algebras-strongly homotopy multiplicative), [6]. We prove the following two theorems:

<u>Theorem 1</u>. The transformation of functors

$$\rho : A \to C$$

extends to the category DASH.

It follows that if the Eilenberg-Moore spectral sequence for the fibre square

$$\begin{array}{ccc} E_f & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

converges, then $H^*(E_f;R)$ can be computed in terms of $A(X) \otimes BA(Y) \otimes A(E)$.

<u>Theorem 2</u>. There are natural maps $\overset{i}{\wedge}$, called wedge-i products, such that the diagram

$$\begin{array}{ccc} A_R^{p_1,q_1}(X) \otimes A_R^{p_2,q_2}(X) & \xrightarrow{\overset{i}{\wedge}} & A_R^{p_1+p_2-i,q_1+q_2}(X) \\ \downarrow \rho\otimes\rho & & \downarrow \rho \\ C^{p_1}(X;\Gamma_{q_1}) \otimes C^{p_2}(X;\Gamma_{q_2}) & \xrightarrow{U_i} & C^{p_1+p_2-i}(X;\Gamma_{q_1+q_2}) \end{array}$$

is homotopy commutative. Here $\cup_i$ denotes the cup-i product of Steenrod ([9], [10]) followed by the pairing $\gamma:\Gamma_{q_1} \otimes \Gamma_{q_2} \to \Gamma_{q_1+q_2}$ of the coefficient modules.

Corollary. Let X be a simplicial set and let a_1, a_2 be cocycles in $A_{Z_2}^{p_1q_1}(X)$, $A_{Z_2}^{p_1q_2}(X)$ respectively with

$$e_{q_1}\rho(a_1) = e_{q_2}\rho(a_2) \text{ in } C^p(X;Z_2) \ ,$$

where e_q denotes the map $C^*(X;\Gamma_q) \to C^*(X;Z_2)$ induced by the coefficient map which sends γ_q to the nonzero element in Z_2.

For $i \leq p$, $a_1 \overset{i}{\wedge} a_2$ is a cocycle in $A_{Z_2}^{2p-i,q_1+q_2}(X)$ whose class is related to the Steenrod squares by the identity

$$e_{q_1+q_2}\rho(a_1 \overset{i}{\wedge} a_2) = \frac{(q_1+q_2)!}{(q_1)!(q_2)!} Sq^{p-1} \ e_{q_1}(\rho(a_1)).$$

2. Method of acyclic models.

The theorems are proved by using the method of acyclic models for contravariant functors defined on the category of simplicial sets [5].

Denote by $\Delta[n]$ the standard n-simplex considered as a simplicial set, and let i_n be the n-simplex $\langle 0,1,\ldots,n \rangle$ of $\Delta[n]$. For any simplicial set X, and an n-simplex x, there is a unique simplicial map $f_x:\Delta[n] \to X$ such that $f_x(i_n) = X$.

A unit for a functor $F^*:\mathcal{S} \to dM_R^*$ consists of an R-module W together with a transformation $\eta:W \to F^0$ with $d\eta = 0$.

A functor $F^*:\mathcal{S} \to dM_R^*$ with the unit (W,η) is called acyclic on models if for each $n \geq 0$ there is a map

$\varepsilon_n : F^0(\Delta[n]) \to W$ of R-modules and maps

$$h_n : F^p(\Delta[n]) \to F^{p-1}(\Delta[n]) \text{ for } p \geq 1$$

such that

$$h_n d + dh_n = 1 - \eta\varepsilon_n ,$$

$$\varepsilon_n \eta = 1.$$

Given a contravariant functor $L : S \to M_R$, define a new contravariant functor $\hat{L} : S \to M_R$ by setting $\hat{L}(X) = \bigcup_{\substack{x \in X_n \\ n \geq 0}} (L(\Delta[n]), x)$. An element f in $\hat{L}(X)$ is a rule which assigns to each element $x \in X_n$ an element $f_x \in L(\Delta[n])$. We identify the element f with the corresponding pair (f_x, x).

There is a transformation $\phi : L \to \hat{L}$ defined as follows: Given $\omega \in L(X)$ set $\phi(\omega)$ equal to the set of pairs $(f_x(\omega), x)$. A transformation $T : K \to L$ induces a transformation $\hat{T} : \hat{K} \to \hat{L}$ by the rule $\hat{T}(f_x, x) = (T(f_x), x)$. Note that $\hat{T}\phi = \phi T$.

A contravariant function $L : S \to M_R$ is called <u>corepresentable</u> if there is a transformation $\psi\ \hat{L} \to L$ such that $\psi \circ \phi = 1$.

3. <u>The bigraded deRham complex</u>.

Let R be a commutative ring with unit. Denote by $\Gamma(t,dt) = \Gamma_R(t) \otimes E_R(dt)$ the Koszul complex where $E_R(dt)$ is the exterior algebra over R on a single generator dt. $\Gamma(t,dt)$ is graded by setting degree of t to be 0 and degree of dt to be 1. Define $d\gamma_p(t) = \gamma_{p-1}(t)dt$, $d(dt) = 0$. Then $\Gamma(t,dt)$ is a commutative differential graded algebra over R.

The simplicial differential graded algebra A over R is defined by setting $A_n = \Gamma_R(X_0,\ldots,X_n) \otimes E_R(dX_1,\ldots,dX_n)$ with $dX_0 + dX_1+\ldots+dX_n = 0$. The face and degeneracy maps d_i , s_i are defined by

$$(d_i\omega)(X_0,\ldots,X_{n-1}) = \omega(X_0,\ldots,X_{i-1},0,X_i,\ldots,X_{n-1}),$$
$$(s_i\omega)(X_0,\ldots,X_{n+1}) = \omega(X_0,\ldots,X_{i-1},X_i+X_{i+1},X_{i+2},\ldots,X_{n+1}).$$

A_n is given a bigrading by setting $A_n^{p,q}$ equal to the R-submodule of A_n freely generated by the monomials

$$\gamma_{\alpha_0}(X_0)\ldots\gamma_{\alpha_n}(X_n)dX_{i_1} \wedge dX_{i_2}\wedge\ldots\wedge dX_{i_p}$$

with

$$\alpha_i \geq 0 \ , \ 1 \leq i_1 < i_2<\ldots<i_p \leq n$$

and

$$q = \alpha_0+\ldots+\alpha_n + p.$$

Note that $A^{p,q} = 0$ for $p > q$.

For each pair (p,q), $A^{p,q}$ is a simplicial R-module. For a simplicial set X, $A^{*,*}(X) = \mathrm{Hom}(X,A^{*,*})$ is a commutative bigraded differential algebra over R. If we interpret an element in A_n as a differential form on the euclidean space with coordinates $(X_0,\ldots,X_n)$ then the integration of differential forms over the simplices $X_0 + X_1+\ldots+X_n = t$ defines a transformation $\rho:A \to C$ which induces an isomorphism $\rho^*:H^p(A^{*,q}(X)) \to H^p(X;\Gamma_q)$ for $p < q$.

A unit $\eta:\Gamma \to A^{0,*}(X)$ is defined by setting $\eta(\gamma_q) \in A^{0,q}(X)$ equal to the map $X \to A^{0,q}$ of simplicial sets which assigns to each n-simplex x of X the element

$$\sum_{\alpha_0+\alpha_1+\ldots+\alpha_n=q} \gamma_{\alpha_0}(x_0)\ldots\gamma_{\alpha_n}(x_n) \text{ in } A_n^{0,q}.$$

The transformation $\eta:\Gamma \to A^{0,*}(X)$ commutes with the products.

Proposition 2. The functor $C^{p,q}:\mathcal{S} \to \mathcal{M}_R$ is corepresentable for all p and q. The functor $C^{*,q}:\mathcal{S} \to d\mathcal{M}_R^*$ is acyclic on models.

Proof. A modification of the proof of the covariant version of this statement from [5] will prove this statement.

Proposition 3. The functor $A^{p,q}:\mathcal{S} \to \mathcal{M}_R$ is corepresentable for $p < q$. The functor $A^{*,q}:\mathcal{S} \to d\mathcal{M}_R^*$ is acyclic on models for all q.

Proof. The acyclicity of $A^{*,q}$ is equivalent to the axiom (a) and the corepresentability follows from the axiom (b). The axioms (a) and (b) are stated in [3]. The statement of Proposition 1 follows from the Proposition 2 and Proposition 3 and from the acyclic model theorem [1].

4. Higher homotopies and differential forms.

The transformation of functors

$$\rho:A \to C$$

induced by integration gives an isomorphism of R-modules

$$\rho^*:H^p(A^{*,q}(X)) \to H^p(X;\Gamma_q)$$

which commutes with the products in the sense that the diagram(1) in the above deRham theorem is commutative. But a similar diagram on the cochain level

$$\begin{array}{ccc} C^{*,r}(X) \otimes C^{*,s}(X) & \xrightarrow{U} & C^{*,r+s}(X) \\ \uparrow & & \uparrow \\ \rho \otimes \rho & & \rho \\ A^{*,r}(X) \otimes A^{*,s}(X) & \xrightarrow{\wedge} & A^{*,r+s}(X) \end{array}$$

is only homotopy commutative. To prove the existence of this and of the higher homotopies is the goal of this section.

We shall use the following notation:

$$A_{(i)} = A \otimes \ldots \otimes A \text{ (i-times)} ,$$

$$(A \otimes \ldots \otimes A)_i^{p,q} = \bigoplus_{\substack{p_1+\ldots+p_i=p \\ q_1+\ldots+q_i=q}} (A^{p_1,q_1} \otimes \ldots \otimes A^{p_i,q_i}) ,$$

$$(A \otimes \ldots \otimes A)_i^{p} = (A \otimes \ldots \otimes A)_i^{p,*} .$$

If i = 2,3 then the tensor products are mostly written out.

The main point of this section is the proof of the following

<u>Lemma 1</u>. Let $\rho_1 = \rho: A \to C$ be the transformation of functors induced by the integration, and let

$$\phi_\wedge : A^{p_1,q_1} \otimes A^{p_2,q_2} \to A^{p_1+p_2,q_1+q_2}$$

be the wedge product on A and

$$\phi_U : C^{p_1,q_1} \otimes C^{p_2,q_2} \to C^{p_1+p_2,q_1+q_2}$$

by the cup product on C. Then there exists natural transformations of functors

$$\rho_i = \rho_i^p : A_{(i)}^{p,q} \to C^{p+1-i,q}$$

of degree (-i+1) such that

$$d\rho_i + (-1)^i \rho_i d = \sum_{j=1}^{i-1} (-1)^j \{\phi_U(\rho_j \otimes \rho_{i-j}) -$$

$$- \rho_{i-1}(A_{(j-1)} \otimes \phi_\Lambda \otimes A_{(i-j-1)})\}.$$

Remark. The transformations ρ_i in the two lowest degrees satisfy the identities

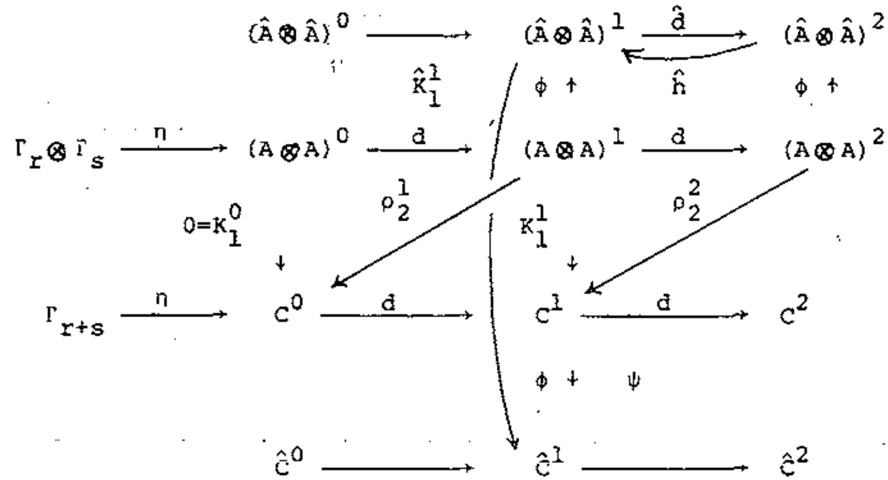
$$d\rho_1 - \rho_1 d = 0$$

$$d\rho_2 + \rho_2 d = \rho_1\phi_\Lambda - \phi_U(\rho_1 \otimes \rho_1).$$

The second identity implies that the map ρ^*, induced by $\rho = \rho_1$ on the cohomology, commutes with the products.

Proof. The statement of the lemma is proved inductively by the method of acyclic models and it is based on three fundamental facts: 1. The function $A^{p,q}$ is acyclic on models and 2. $C^{p,q}$ is corepresentable. Furthermore; 3. On the elements of degree zero from $A^{0,2}(X), X \in S$, $\rho_1\phi_\Lambda - \phi_U(\rho_1 \otimes \rho_1)$ is equal to zero and 4. The functor $(A \otimes \ldots \otimes A)_i^{p,q}$ is acyclic on models in the dimension bounded by the filtration from above. The associated functor is denoted by $(\hat{A} \otimes \ldots \otimes \hat{A})_i^{p,q}$. The proof is a simple extension of the argument from [5] to the deRham complex A. We give the details here for completeness and also because the first part of the argument is used in the proof of the lemma in the next section.

Let us denote $K_1 = \rho_1\phi_\Lambda - \phi_U(\rho_1 \otimes \rho_1)$. We define $\rho_2^0 = 0$ and ρ_2^1 so that $\rho_2^1 d = 0$ and extended arbitrarily to $(A \otimes A)^1$. Then we consider the diagram



Since $K_1^0 = \rho_1\phi_\Lambda - \phi_U(\rho_1 \otimes \rho_1) = 0$ on the elements of degree zero we have $\rho_2^1 d = K_1^0$. We define

$$\rho_2^2 = \psi\hat{K}_1^1\hat{h}\phi.$$

Since

$$K_1^1 d = (\rho_1\phi_\Lambda - \phi_U(\rho_1 \otimes \rho_1))d = \rho_1\phi_\Lambda d - \phi_U d(\rho_1 \otimes \rho_1) =$$

$$= d(\rho_1\phi_\Lambda - \phi_U(\rho_1 \otimes \rho_1)) = 0$$

on the elements of degree zero, we can conclude that

$$\rho_2^2 d = \psi\hat{K}_1^1\hat{h}\phi d = \psi\hat{K}_1^1\hat{h}\hat{d}\phi = \psi\hat{K}_1^1(1-\hat{d}\hat{h})\phi =$$

$$= \psi\hat{K}_1^1\phi - \psi\hat{K}_1^1\hat{d}\hat{h}\hat{\phi} = \psi\hat{K}_1^1\phi = K_1^1.$$

Hence we get the formula

$$\rho_2^2 d = \rho_1 \phi_\Lambda - \phi_U(\rho_1 \otimes \rho_1).$$

Suppose that we have proved the identity
$d\rho_2^\ell + \rho_2^{\ell+1} d = K_1^\ell = \rho_1\phi_\Lambda - \phi_U(\rho_1 \otimes \rho_1)$ on the elements of degree $\ell < n-1$. We define ρ_2^n by setting

$$\rho_2^n = \psi \hat{w}_1^{n-1} \hat{h} \phi, \; w_1^{n-1} = K_1^{n-1} - d\rho_2 .$$

Using a similar argument to that above we get

$$\begin{aligned} w_1^{n-1} d &= (\rho_1\phi_\Lambda - \phi_U(\rho_1 \otimes \rho_1))d - d\rho_2^{n-1} d = \\ &= d(\rho_1\phi_\Lambda - \phi_U(\rho_1 \otimes \rho_1)) - \\ &- d(\rho_1\phi_\Lambda - \phi_U(\rho_1 \otimes \rho_1) - d\rho_2^{n-2}) = 0 . \end{aligned}$$

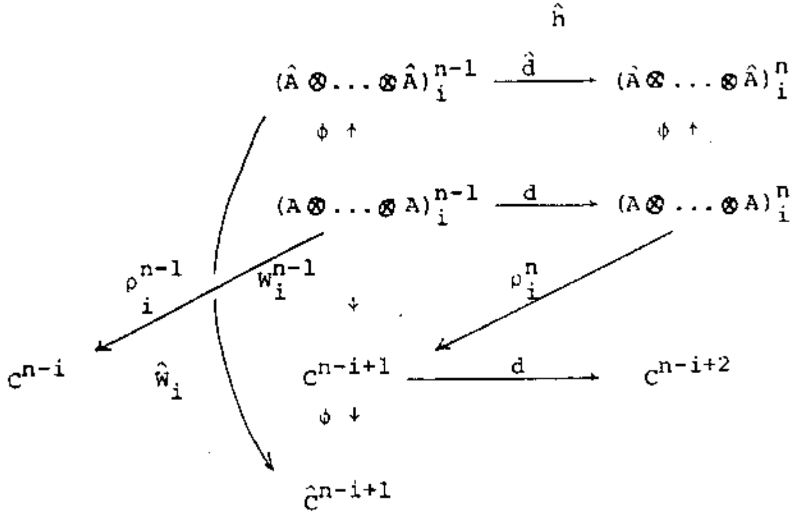
From here we get

$$\begin{aligned} \rho_2^n d &= \psi \hat{w}_1^{n-1} \hat{h} \phi d = \psi \hat{w}_1^{n-1} \hat{h} \hat{d} \phi = \\ &= \psi \hat{w}_1^{n-1} (1 - \hat{d}\hat{h}) \phi = \psi \hat{w}_1^{n-1} \phi = w_1^{n-1} , \end{aligned}$$

which gives the desired result

$$d\rho_2^{n-1} + \rho_2^n d = \rho_1 \phi_\Lambda - \phi_U(\rho_1 \otimes \rho_1).$$

Next we assume that the chain homotopy ρ_j has been defined, and satisfies the above formula, for $j < i$ in all dimensions (obviously bounded by the filtration degree) and that ρ_i has been defined and satisfies the formula in dimensions $< n$. Then we consider the diagram



where

$$W_i = (-1)^i d\rho_i^{n-1} + \sum_{j=1}^{i-1} (-1)^j \{\phi_U(\rho_j \otimes \rho_{i-j}) -$$

$$-\rho_{i-1}(A_{(j-1)} \otimes \phi_\Lambda \otimes A_{(i-j-1)})\}.$$

From the induction hypothesis it follows that $W_i^{n-1}d = 0$. Then we define

$$\rho_i^n = \psi\hat{W}_i^{n-1}\hat{h}\phi \ ,$$

and we get

$$\begin{aligned}
\rho_i^n d &= \psi\hat{W}_i^{n-1}\hat{h}\phi d \\
&= \psi\hat{W}_i^{n-1}\hat{h}\hat{d}\phi \\
&= \psi\hat{W}_i^{n-1}(1-\hat{d}\hat{h})\phi \\
&= \psi\hat{W}_i^{n-1}\phi - \psi\hat{W}_i^{n-1}\hat{d}\hat{h}\phi \\
&= \psi\hat{W}_i^{n-1}\phi \\
&= W_i^{n-1}
\end{aligned}$$

which completes the proof of the theorem.

5. $\overset{i}{\Lambda}$-product on forms.

In previous section the chain homotopy $\rho_2 : (A \otimes A)^n \to C^{n-1}$ was used to prove the existence of higher homotopies and thus to extend the map $\rho_1 : A \to C$ to a broader category DASH.

In this section the chain homotopy ρ_2 is used to construct an analogue of the cup-i product on differential forms, called wedge-i product. This operation of the wedge-i product leads to the definition of the Steenrod square in a similar way as the cup-i product determines those operations in the original work of Steenrod. For the definition and fundamental properties of the cup-i product we refer to [9].

In order to distinguish the homotopies of the previous section from the chain homotopies used here we change the notation. Let $\mu = \rho_1$, $\mu_1 = \rho_2$. Then μ_i, $i > 2$, defined below is not related to the ρ_i's of Lemma 1.

Now we define the morphism of R-modules

$$T : (A \otimes A)^k \to (A \otimes A)^k ,$$

$$T : (C \otimes C)^k \to (C \otimes C)^k$$

by $T(a \otimes b) = (-1)^{|a||b|} b \otimes a$. It follows that

(i) $T^2 = I = \text{identity}$,

(ii) $Td = dT$,

(iii) $T(\mu \otimes \mu) = (\mu \otimes \mu)T$.

<u>Lemma 2</u>. Let $\mu : A \to C$ be the transformation of functors

induced by the integration, and let

$$\phi_{U_i}:(C^{*,r}\otimes C^{*,s})^n \to C^{n-i,r+s} \ , \ i \geq 0 \ ,$$

be the cup-i product followed by $\gamma:\Gamma_r \quad \Gamma_s \to \Gamma_{r+s}$ and let

$$\phi_\Lambda:(A^{*,r}\otimes A^{*,s})^n \to A^{n,r+s}$$

be the exterior product.

Then there exist natural transformations of functors

$$\mu_i = \mu_i^n:(A^{*,r}\otimes A^{*,s})^n \to C^{n-i,r+s} \ , \ i \geq 1 \ ,$$

such that

$$d\mu_1 + \mu_1 d = \mu\phi_\Lambda - \phi_U(\mu\otimes\mu)$$

and $$d\mu_i + (-1)^{i+1}\mu_i d = \mu_{i-1}(T+(-1)^{i+1}I) - \phi_{U_{i-1}}(\mu\otimes\mu)$$

for $i \geq 2$.

Definition of the wedge-i product.

By an explicit construction similar to that in [4] it can be shown that there is a transformation of functors

$$\tau:C^{p,q} \to A^{p,q} \text{ for } p < q$$

such that

(i) $\tau d = d\tau$,

(ii) $\mu\tau = I$,

(iii) $\tau\mu - I = sd + ds$,

where $s:A \to A$ is of degree -1,

(iv) $\tau\eta = \eta$.

The wedge-i product ($\overset{1}{\Lambda}$-product) is the map

$$\overset{i}{\Lambda} = \phi_{\overset{\Lambda}{i}} : (A^{*,r} \otimes A^{*,s})^n \to A^{n-i,r+s} ,$$

$$\overset{i}{\Lambda} = \tau\mu_i(T + (-1)^i I) \text{ for } i \geq 1 ,$$

$$\overset{0}{\Lambda} = \Lambda = \text{the wedge product on forms.}$$

<u>Proof of Lemma 2</u>. The statement is proved inductively by the method of acyclic models. Recall that $\mu = \rho_1$ and that for $\mu_1 = \rho_2$, by Lemma 1, holds the identity

$$d\mu_1^k + \mu_1^{k+1}d = \mu\phi_\Lambda - \phi_U(\mu \otimes \mu).$$

If composed with T we get

$$d\mu_1^k T + \mu_1^{k+1}Td = \mu\phi_\Lambda T - \phi_U T(\mu \otimes \mu).$$

Therefore we have

$$d\mu_1^k(T-I) + \mu_1^{k+1}(T-I)d = -\phi_U(T-I)(\mu \otimes \mu) \text{ for } k \geq 0$$

since $\mu\phi_\Lambda(T-I) = 0$. From [9] we have the identity

$$d\phi_{U_i} + (-1)^{i+1}\phi_{U_i}d = (-1)^i\phi_{U_{i-1}}(T-I) \text{ for } i \geq 0.$$

Next we want to prove the existence of

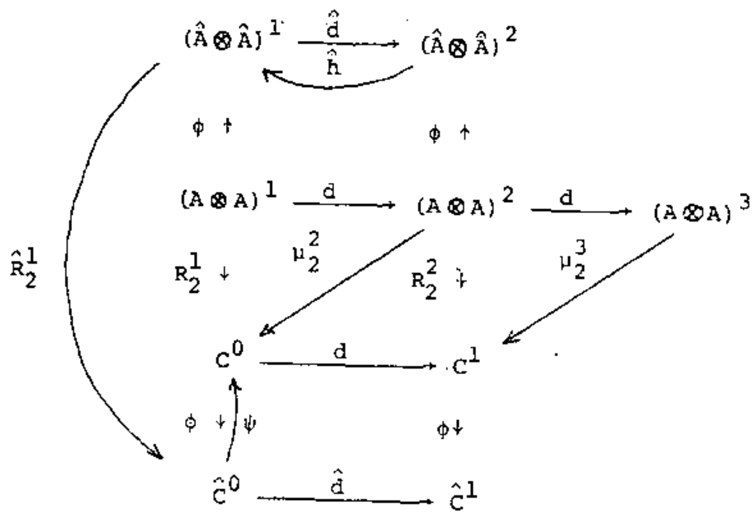
$$\mu_2^k : (A \otimes A)^k \to C^{k-2} ,$$

$$\mu_2^k = 0 \text{ for } k = 0, 1, \text{ and}$$

$$d\mu_2^k - \mu_2^{k+1}d = \mu_1(T-I) - \phi_{U_1}(\mu \otimes \mu).$$

Let us denote the right hand side of the last formula by R_2^k.

We consider the diagram



Then we define μ_2^2 by

$$-\mu_2^2 = \psi\hat{R}_2^1\hat{h}\phi .$$

From the fact that

$$R_2^1 d = +\{\mu_1^1(T-I) - \phi_{U_1}(\mu\otimes\mu)\}d$$

$$= \mu_1^1(T-I)d - \phi_{U_1}(\mu\otimes\mu)d$$

$$= \mu_1^1 d(T-I) - \phi_{U_1}d(\mu\otimes\mu)$$

$$= -\phi_{U_1}d(\mu\otimes\mu)$$

$$= +\phi_U(I-T)(\mu\otimes\mu)$$

equals to zero on $(A\otimes A)^0$ we get

$$-\mu_2^2 d = \psi\hat{R}_2^1\hat{h}\phi d = \psi\hat{R}_2^1\hat{h}\hat{d}\phi =$$

$$= \psi\hat{R}_2^1(1-\hat{d}\hat{h})\phi = \psi\hat{R}_2^1\phi = R_2^1 ,$$

which is the formula

$$-\mu_2^2 d = \mu_1^1(T-I) - \phi_{U_1}(\mu \otimes \mu) ,$$

valid on $(A \otimes A)^1$.

Next we assume by induction on n that μ_2^ℓ for $\ell < n-1$ have been defined satisfying

$$d\mu_2^\ell - \mu_2^{\ell+1} d = \mu_1^\ell(T-I) - \phi_{U_1}(\mu \otimes \mu).$$

Set

$$V_2^{n-1} = R_2^{n-1} - d\mu_2^{n-1} = \mu_1^{n-1}(T-I) - \phi_{U_1}(\mu \otimes \mu) - d\mu_2^{n-1} .$$

Then

$$\begin{aligned} V_2^{n-1}d &= \mu_1^{n-1}(T-I)d - \phi_{U_1} d(\mu \otimes \mu) - d\mu_2^{n-1}d \\ &= \mu_1^{n-1}d(T-I) - \phi_{U_1} d(\mu \otimes \mu) - \\ &- d\{d\mu_2^{n-2} - \mu_1^{n-2}(T-I) + \phi_{U_1}(\mu \otimes \mu)\} \\ &= (d\mu_1^{n-2} + \mu_1^{n-1}d)(T-I) - \\ &- (\phi_{U_1} d + d\phi_{U_1})(\mu \otimes \mu) \end{aligned}$$

Using the identity (2) we get

$$V_2^{n-1}d = (d\mu_1^{n-2} + \mu_1^{n-1}d)(T-I) + \phi_U(T-I)(\mu \otimes \mu).$$

And from the identity (3) we can conclude that

$$V_2^{n-1}d = 0.$$

Now when we define μ_2^n by

$$-\mu_2^n = \psi\hat{V}_2^{n-1}\hat{h}\phi$$

we get the desired identity

$$-\mu_2^n d = V_2^{n-1}$$

Now we assume that we already have the formulas for $j < i$ and for the dimensions $\ell < n-1$, that means that we have

$$\mu_j^0 = \mu_j^1 = \ldots = \mu_j^{j-1} = 0$$

$$d\mu_j^\ell + (-1)^{j+1}\mu_j^{\ell+1}d = \mu_{j-1}^\ell(T+(-1)^{j+1}I) - \phi_{U_{j-1}}(\mu\otimes\mu).$$

If we denote the right hand side by R_j^ℓ then for $V_i^{n-1} = R_i^{n-1} - d\mu_i^{n-1}$ we get

$$\begin{aligned}
V_i^{n-1}d &= R_i^{n-1}d - d\mu_i^{n-1}d - \\
&= \mu_{i-1}^{n-1}d(T+(-1)^{i+1}I) - \phi_{U_{i-1}}d(\mu\otimes\mu) + \\
&+ (-1)^i d\mu_{i-1}^{n-2}(T+(-1)^{i+1}I) + (-1)^{i+1}d\phi_{U_{i-1}}(\mu\otimes\mu) \\
&= (-1)^i\{(d\mu_{i-1}^{n-2}+(-1)^i\mu_{i-1}^{n-1}d)(T+(-1)^{i+1}I) + \\
&+ (-1)^{i+1}(d\phi_{U_{i-1}}+(-1)^i\phi_{U_{i-1}}d)(\mu\otimes\mu)\} \\
&= (-1)^i\mu_{i-1}^{n-2}(T+(-1)^iI)(T+(-1)^{i+1}I) + \\
&+ (-1)^{i+1}\phi_{U_{i-2}}(\mu\otimes\mu)(T+(-1)^{i+1}I) + \\
&+ (-1)^{i+1}(d\phi_{U_{i-1}}+(-1)^i\phi_{U_{i-1}}d(\mu\otimes\mu) = 0.
\end{aligned}$$

Now we define μ_i^n by setting

$$(-1)^{i+1}\mu_i^n = \psi\hat{V}_i^{n-1}\hat{h}\phi.$$

Then

$$(-1)^{i+1}\mu_i^n d = V_i^{n-1} = R_i^{n-1} - d\mu_i^{n-1}$$

or equivalently

$$d\mu_i^{n-1} + (-1)^{i+1}\mu_i^n d = \mu_{i-1}^{n-1}(T+(-1)^{i+1}I) - \phi_{U_{i-1}}(\mu\otimes\mu).$$

This completes the proof of the Lemma 2.

Remarks.

1. Following the lines of Gugenheim, V.K.A.M. "On Chen's iterated integrals," Ill. J. Math. 21 (1977), 703-715; Theorem 1 can be used to generalize Chen's iterated integrals to the deRham complex of Cartan-Miller ([3], [8]).

2. Let $Z^{n,n}A$ be the cocycles in $A^{n,n} = A_{Z_2}^{n,n}$ and let $CZ^{n,n}A$ be the cone on $Z^{n,n}A$. Campbell [2] proved that the "algebraic" fibration

$$Z^{n,n}A \to CZ^{n,n}A \to Z^{n+1,n+1}A$$

is isomorphic (via the integration map) to the principal Kan fibration

$$K(Z_2,n) \to L(Z_2,n+1) \to K(Z_2,n+1).$$

Hence that the construction of Steenrod squares can be applied to the "algebraic" fibration.

3. While the non-commutativity of the cup product implies the existence of higher homotopies (Theorem 1) and the existence of the cohomology operations (Theorem 2) the

existence of the homotopies μ_1 and μ_2 has still another implication. Namely, if the commutative cochain problem over a commutative ring R (not necessarily with a unit) has a solution then for each element a in R the element a^2 in R is divisible by 2. This explains, for example, why the commutative cochain problem does not have a solution over the integers.

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