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LIMITS AND COLIMITS IN THE CATEGORY OF SMALL CATEGORIES*

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Abstract. The aim of this note is to show some properties of the homotopy groups of limits and colimits in the category of small categories $\mathbb{C}at$ and to give a version of Milnor's theorem in this category.

Moreover, one proves that the homotopy limit (in the sense of Bousfield and Kan, see [1, ch XI]) of a diagram of nerves of categories is itself the nerve of a category. In fact, if $F : I \longrightarrow \mathbb{C}at$ is a functor and $\tilde{F}$ is its Eilenberg-Moore rectification (see [6]) then holim $NF = N(\lim \tilde{F})$.

For a similar result on the homotopy colimit see [7].

1. Preliminaries. Let $\mathbb{C}$ be a category. A cohomotopy system in $\mathbb{C}$ is a quadruple $(P; p_o, p_1, s)$, where $P : \mathbb{C} \longrightarrow \mathbb{C}$ is a functor, whereas $p_o, p_1 : P \longrightarrow id_{\mathbb{C}}$, $s: id_{\mathbb{C}} \longrightarrow P$ are such natural transformations that $p_o s = p_1 s = id_{\mathbb{C}}$.

Refering to Kamps (see [4]) we can define the Hurewicz fibration and cofibration. Moreover, we also have a homotopy relation in such a category.

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This quadruple determines a sequence of functors $P^n : \mathbb{C} \longrightarrow \mathbb{C}$, where $P^0 := id_{\mathbb{C}}$, $P^{n+1} := P(P^n)$, for $n \geq 0$ and natural transformations

$$d^{\delta}_{i,n} := P^{i-1} p_{\delta} P^{n-i} : P^n \longrightarrow P^{n-1},\ i = 1,\dots,n,\ \delta = 0,1$$

and $s_{i,n} := P^{i-1} s P^{n+1-i} : P^n \longrightarrow P^{n+1}$, $i = 1,\dots,n+1$.

Lemma 1.1. (see [4]). A sequence of functors and natural transformations $(P^n;\ d^{\delta}_{i,n}, s_{i,n})_{n \geq 0}$ defines a cubical object in the endofunctors category of $\mathbb{C}$.

Denote by $\mathbb{C}at$ ($\mathbb{C}at^*$) the category of (pointed) small categories and by $Set^{\square op}$ ($Set^{*\square op}$) the category of (pointed) cubical sets.

The cohomotopy system in $\mathbb{C}at$ ($\mathbb{C}at^*$) is defined in the following way. Let $\mathbb{Z}$ be the category given by:

$$\dots \longrightarrow -2 \longleftarrow -1 \longrightarrow 0 \longleftarrow 1 \longrightarrow 2 \longleftarrow \dots\dots$$

For $\mathbb{C} \in ob\mathbb{C}at$ a functor $\sigma : \mathbb{Z} \longrightarrow \mathbb{C}$ is called finite iff there exist $m_0, n_0 \in ob\mathbb{Z}$ such that $\sigma(m) = \sigma(m_0)$, $\sigma(n) = \sigma(n_0)$ and $\sigma(m \longrightarrow m') = id_{\sigma(m_0)}$, $\sigma(n \longrightarrow n') = id_{\sigma(n_0)}$ for $m, m' \leq m_0$ and $n, n' \geq n_0$. The above conditions will be written briefly as $\sigma(m_0) = \sigma(-\infty)$ and $\sigma(n_0) = \sigma(+\infty)$. The full subcategory given by finite functors of $\mathbb{C}at(\mathbb{Z},\mathbb{C})$ is denoted by $P(\mathbb{C})$.

Then $P : \mathbb{C}at \longrightarrow \mathbb{C}at$ is the functor and for $\mathbb{C} \in ob\mathbb{C}at$ there are functors $s(\mathbb{C}) : \mathbb{C} \longrightarrow P(\mathbb{C})$, and $p_0(\mathbb{C}), p_1(\mathbb{C}) : P\mathbb{C} \longrightarrow \mathbb{C}$ defined by $s(C)(C)(k) = C$ for $C \in ob\mathbb{C}$, $k \in ob\mathbb{Z}$ and $p_{\delta}(\mathbb{C})(\sigma) = \sigma((-1)^{\delta}\infty)$ for $\delta = 0,1$.

Hence we obtain the cohomotopy system $(P; p_0, p_1, s)$ in $\mathbb{C}at$ and the functor $Q : \mathbb{C}at \times \mathbb{C}at \longrightarrow Set^{\square op}$, where $Q(\mathbb{C},\mathbb{C}')_n =$

$= \mathbb{C}at(\mathbb{C},P^n(\mathbb{C}))$ for $\mathbb{C},\mathbb{C}' \in ob\mathbb{C}at$. In particular, for $\mathbb{C} = *$ we have the functor $Q : \mathbb{C}at \longrightarrow Set^{oop}$.

In $\mathbb{C}at$ we can define the Serre fibration (see [2]). Moreover, in $\mathbb{C}at$ it is easy to define a notion of the loop functor Ω, the homotopy fibre of a map etc.

For further considerations we shall need the following

Theorem 1.2. (see [2]). For a functor $p : \mathbb{E} \longrightarrow \mathbb{B}$ the following conditions are equivalent:

i) $p : \mathbb{E} \longrightarrow \mathbb{B}$ is the Serre fibration,

ii) $Q(p) : Q(\mathbb{E}) \longrightarrow Q(\mathbb{B})$ is the Kan fibration.

Corollary 1.3. For any $\mathbb{C} \in ob\mathbb{C}at$ the cubical set $Q(\mathbb{C})$ satisfies the Kan extension condition.

One can prove that for any $\mathbb{C},\mathbb{C}' \in ob\mathbb{C}at$ the cubical set $Q(\mathbb{C},\mathbb{C}')$ also satisfies the Kan extension condition.

2. The homotopy groups and Milnor's theorem. Following the paper [2] for $\mathbb{C} \in ob\mathbb{C}at^*$ we put $\pi_n(\mathbb{C},*) := \pi_n(Q(\mathbb{C}),*)$, where $\pi_n(Q(\mathbb{C}),*)$ is the n-th homotopy group of the cubical set $Q(\mathbb{C})$ (see [3]).

Theorem 2.1. (see [2]) i) A map $f : \mathbb{C} \longrightarrow \mathbb{D}$ induces the long exact sequence

$$\dots \longrightarrow \pi_n(\mathbb{C},*) \longrightarrow \pi_n(\mathbb{D},*) \longrightarrow \pi_{n-1}(f_h^{-1}(*),*) \longrightarrow \dots,$$

ii) if $f: \mathbb{C} \longrightarrow \mathbb{D}$ is the Serre fibration then $f_n^{-1}(*) \underset{W}{\simeq} f^{-1}(*)$ where $\underset{W}{\simeq}$ is the weak homotopy equivalence.

Let $\mathbb{I}$ be a small category. Thomason (see [7]). proved that for a functor $F : \mathbb{I} \longrightarrow \mathbb{C}at$, the classifying space of the Grothendieck construction $B(\mathbb{I} \int F)$, is homotopy equivalent to the realization of the Bousfield-Kan homotopy colimit $|\text{hocolim } NF|$. There also exists a relation between the homotopy groups of hocolim F and colim F.

Let $F : \mathbb{I} \longrightarrow \mathbb{C}at$ be a functor. The Grothendieck construction on F, $\mathbb{I} \int F$, is the category with objects: the pairs (i,X) with i an object of $\mathbb{I}$ and X an object of $F(i)$, and with morphisms $(\alpha,x):(i_1,X_1) \longrightarrow (i_0,X_0)$ given by a morphism $\alpha: i_1 \longrightarrow i_0$ in $\mathbb{I}$ and a $x : F(\alpha)(X_1) \longrightarrow X_0$ in $F(i_0)$.. The composition is defined by $(\alpha,x)(\alpha',x') =$ $= (\alpha\alpha', xF(\alpha)(x'))$.

For $F : \mathbb{I} \longrightarrow \mathbb{C}at^*$ let $p : \mathbb{I} \int F \longrightarrow \text{colim}^* F$ be the functor given by $p(i,X) = X$, then $p^{-1}(*) = \mathbb{I}$. Moreover, for any $C \in \text{ob colim}^* F$ we have the pair of functors $p^{-1}(C) \longrightarrow p/_C$ and $p^{-1}(C) \longrightarrow C/_p$ given in the obvious way, where $p/_C$ and $C/_p$ are comma categories. It is not difficult to see that $p^{-1}(C)^p \longrightarrow p/_C$ has a left adjoint and $p^{-1}(C) \longrightarrow C/_p$ has a right adjoint. Hence p is the Serre fibration (see [5]).

Therefore, following the Thomason's result we have

<u>Corollary 2.2.</u> For a functor $F : \mathbb{I} \longrightarrow \mathbb{C}at$ there is the long exact sequence

$$\ldots \longrightarrow \pi_n(\mathbb{I},*) \longrightarrow \pi_n(\text{holim } F,*) \longrightarrow \pi_n\ (\text{colim}^* F,*) \longrightarrow \ldots.$$

In particular, if $\mathbb{I}$ is a contractible category (for instance, a left of right filtering category) then

$$\pi_n\ (\text{hocolim } F,*) \simeq \pi_n\ (\text{colim}^* F,*) \quad \text{for } n \geqslant 0.$$

On the base of the proof of Theorem 3.1 from [1,ch.IX] and with references to the fact that the cubical set $Q(\mathbb{C},\mathbb{C}')$ satisfies the Kan extension condition, we obtain

Theorem 2.3. (Milnor's theorem). Let $\mathbb{I}$ be a countable small right filtering category. If $F : \mathbb{I}^{op} \longrightarrow \mathbb{C}at^*$ and $F' : \mathbb{I} \longrightarrow \mathbb{C}at^*$ are such functors that for any map $\alpha : i \longrightarrow i'$ in $\mathbb{I}$ $F(\alpha) : F(i') \longrightarrow F(i)$ is the Hurewicz fibration and $F'(\alpha) : F'(i) \longrightarrow F'(i')$ is the Hurewicz cofibration then there is the short exact sequence of pointed sets

$$* \longrightarrow \varprojlim^1 [F'(i),\Omega F(i)] \longrightarrow [\varinjlim F'(i), \varprojlim F(i)] \longrightarrow \varprojlim [F'(i),F(i)] \longrightarrow *$$

where $[\ ,\ ]$ denotes the set of homotopy classes of maps and $\varprojlim^1$ - the 1-th derived functor of $\varprojlim$.

Corollary 2.4. i) If $F(i) = F$ for any $i \in ob\, \mathbb{I}$ then

$$* \longrightarrow \varprojlim^1 [F'(i), \Omega F] \longrightarrow [\varinjlim F'(i),F] \longrightarrow \varprojlim [F'(i),F] \longrightarrow *$$

is the Milnor's sequence.

ii) If $F'(i) = F'$ for any $i \in ob\, \mathbb{I}$ then

$$* \longrightarrow \varprojlim^1 [F', \Omega F(i)] \longrightarrow [F', \varprojlim F(i)] \longrightarrow \varprojlim [F',F(i)] \longrightarrow *$$

is the Vogt-Cohen's sequence.

Remark that the following diagram

$$\begin{array}{ccccccc}
* \longrightarrow \varprojlim^1 [F'(i),\Omega \varprojlim F(i)] & & & & & & \varprojlim [\varinjlim F'(i),F(i)] \longrightarrow * \\
\downarrow \varphi & \searrow & & & & \nearrow & \downarrow \psi \\
* \longrightarrow \varprojlim^1 [F'(i),\Omega F(i)] & \longrightarrow & & [\varinjlim F'(i),\varprojlim F(i)] & & \longrightarrow & \varprojlim [F'(i),F(i)] \longrightarrow * \\
\uparrow \varphi' & \nearrow & & & & \searrow & \uparrow \psi' \\
* \longrightarrow \varprojlim^1 [\varinjlim F'(i),\Omega F(i)] & & & & & & \varprojlim [F'(i),\varprojlim F(i)] \longrightarrow *
\end{array}$$

is commutative. From the "Snake Lemma" we have that $\operatorname{coker} \varphi = \ker \psi$ and $\operatorname{coker} \varphi' = \ker \psi'$.

Hence we obtain

Corollary 2.5. There are the following exact sequences:

$$\text{i)}\ * \longrightarrow \varprojlim{}^1 [F'(i), \Omega \varprojlim F(i)] \longrightarrow \varprojlim{}^1 [F'(i), \Omega F(i)] \longrightarrow$$
$$\longrightarrow \varprojlim [F'(i), \varprojlim F(i)] \longrightarrow \varprojlim [F'(i), F(i)] \longrightarrow * ,$$
$$\text{ii)}\ * \longrightarrow \varprojlim{}^1 [\varinjlim F'(i), \Omega F(i)] \longrightarrow \varprojlim{}^1 [F'(i), \Omega F(i)] \longrightarrow$$
$$\longrightarrow \varprojlim [\varinjlim F'(i), F(i)] \longrightarrow \varprojlim [F'(i), F(i)] \longrightarrow * .$$

3. Homotopy limit in Cat. Let $\mathbb{I}$ be a small category and $Set^{\Delta^{op}}$ - the category of simplicial sets. A.K. Bousfield and D.M. Kan defined for $F : \mathbb{I} \longrightarrow Set^{\Delta^{op}}$ the homotopy limit - holim F. We will prove that the homotopy limit of a diagram of nerves of categories is itself the nerve of a category.

For $F : \mathbb{I} \longrightarrow \mathbb{C}at$ we define the functor $\widetilde{F} : \mathbb{I} \longrightarrow \mathbb{C}at$ (the Eilenberg-Moore rectification or Street "second construction", see [6]). For $i \in ob\ \mathbb{I}$ $\widetilde{F}(i)$ is the category whose objects are pairs (ψ, φ), where ψ is a function assigning each $\alpha : i \longrightarrow i'$ in $\mathbb{I}$ with source i an object $\psi(\alpha)$ of $F(i')$; and φ assigns each string $i \xrightarrow{\alpha} i' \xrightarrow{\beta} i''$ a map $\varphi_{\beta,\alpha} : \psi(\beta\alpha) \longrightarrow F(\beta)\psi(\alpha)$ in $F(i'')$, subject to

$$\begin{array}{ccc}
\psi(\gamma\beta\alpha) & \xrightarrow{\ \varphi_{\gamma\beta,\alpha}\ } & F(\gamma\beta)\psi(\alpha) \\
{\scriptstyle \varphi_{\gamma,\beta\alpha}}\downarrow & & \Vert \\
F(\gamma)\psi(\beta\alpha) & \xrightarrow{\ F(\gamma)(\varphi_{\beta,\alpha})\ } & F(\gamma)F(\beta)\psi(\alpha)
\end{array}$$

commute. A map $a : (\psi,\varphi) \longrightarrow (\psi',\varphi')$ is a function which assigns to each $\alpha : i \longrightarrow i'$ in $\mathbb{I}$ a map of $F(i')$, $a(\alpha) : \psi(\alpha) \longrightarrow \psi'(\alpha)$; subject to, for $i \xrightarrow{\alpha} i' \xrightarrow{\beta} i''$, that

$$\begin{array}{ccc}
F(\beta)\psi(\alpha) & \xrightarrow{\ F(\beta)(a(\alpha))\ } & F(\beta)\psi'(\alpha) \\
{\scriptstyle \varphi_{\beta,\alpha}}\uparrow & & \downarrow{\scriptstyle \varphi'_{\beta,\alpha}} \\
\psi(\beta\alpha) & \xrightarrow{\ a(\beta\alpha)\ } & \psi'(\beta\alpha)
\end{array}$$

commutes. The composition is given by $\overline{a} \cdot a(\alpha) = \overline{a}(\alpha)a(\alpha)$. For $\delta : i \longrightarrow \overline{i}$, $\widetilde{F}(\delta) : \widetilde{F}(i) \longrightarrow \widetilde{F}(\overline{i})$ is given on objects by $\widetilde{F}(\delta)(\psi,\varphi) = (\psi^{\delta},\varphi^{\delta})$, where $\psi^{\delta}(\alpha) = \psi(\alpha\delta)$, $\varphi^{\delta}_{\beta,\alpha} = \varphi_{\beta,\alpha\delta}$; and on morphisms by $\widetilde{F}(\delta)(a) = a^{\delta}$, $a^{\delta}(\alpha) = a(\alpha\delta)$.

Then we have the following

Theorem 3.1. For a functor $F : \mathbb{I} \longrightarrow \mathbf{Cat}$ there is a natural isomorphism $\operatorname{ho}\varprojlim NF \simeq N \varprojlim \widetilde{F}$, where N is the nerve functor.

The proof is straightforward.

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