LIMITS AND COLIMITS IN THE CATEGORY OF SMALL CATEGORIES*  
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Abstract. The aim of this note is to show some properties of the homotopy groups of limits and colimits in the category of small categories Cat and to give a version of Milnor's theorem in this category.

Moreover, one proves that the homotopy limit (in the sense of Bousfield and Kan, see [1, ch XI]) of a diagram of nerves of categories is itself the nerve of a category. In fact, if $F : I \to \text{Cat}$ is a functor and $\tilde{F}$ is its Eilenberg-Moore rectification (see [6]) then $\text{holim } NF = N(\text{lim } \tilde{F})$.

For a similar result on the homotopy colimit see [7].

1. Preliminaries. Let $C$ be a category. A cohomotopy system in $C$ is a quadruple $(P; p_0, p_1, s)$, where $P : C \to C$ is a functor, whereas $p_0, p_1 : P \to \text{id}_C$, $s : \text{id}_C \to P$ are such natural transformations that $p_0 s = p_1 s = \text{id}_C$.

Refering to Kamps (see [4]) we can define the Hurewicz fibration and cofibration. Moreover, we also have a homotopy relation in such a category.

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This quadruple determines a sequence of functors
\[ P^n : \mathcal{C} \to \mathcal{C}, \text{ where } P^0 = \text{id}_\mathcal{C}, \text{ for } n \geq 0 \]
and natural transformations
\[ \delta_{i,n}^\delta = p_i^{-1} p_{n-i}^\delta : P^n \to P^{n-1}, \text{ for } n \geq 0, \delta = 0,1 \]
and \[ s_{i,n} = p_i^{-1} s p_{n+1-i} : P^n \to P^{n+1}, \text{ for } n \geq 0. \]

Lemma 1.1. (see [4]). A sequence of functors and natural transformations \( (P^n, \delta_{i,n}^\delta, s_{i,n}) \) defines a cubical object in the endofunctors category of \( \mathcal{C} \).

Denote by \( \text{Cat}(\text{Cat}^\text{op}) \) the category of (pointed) small categories and by \( \text{Set}^{\text{op}}(\text{Set}^\text{op}) \) the category of (pointed) cubical sets.

The cohomotopy system in \( \text{Cat}(\text{Cat}^\text{op}) \) is defined in the following way. Let \( \mathcal{Z} \) be the category given by:

\[ \ldots \to -2 \to -1 \to 0 \to 1 \to 2 \to \ldots \]

For \( C \in \text{obCat} \) a functor \( \sigma : \mathcal{Z} \to \mathcal{C} \) is called finite iff there exist \( m_0, n_0 \in \text{obZ} \) such that \( \sigma(m) = \sigma(m_0), \sigma(n) = \sigma(n_0) \) and \( \sigma(m \to m') = \text{id}_{\sigma(m_0)}, \sigma(n \to n') = \text{id}_{\sigma(n_0)} \) for \( m, m' \leq m_0 \) and \( n, n' \geq n_0 \). The above conditions will be written briefly as \( \sigma(m_0) = \sigma(-\infty) \) and \( \sigma(n_0) = \sigma(+\infty) \). The full subcategory given by finite functors of \( \text{Cat}(\mathcal{Z}, \mathcal{C}) \) is denoted by \( P(C) \).

Then \( P : \text{Cat} \to \text{Cat} \) is the functor and for \( C \in \text{obCat} \) there are functors \( s(C) : \mathcal{C} \to P(C), \) and \( p_0(C), p_1(C) : P(C) \to \mathcal{C} \) defined by \( s(C)(C)(k) = C \) for \( C \in \text{obC} \), \( k \in \text{obZ} \) and \( p_\delta(C)(\sigma) = \sigma((-1)\delta) \) for \( \delta = 0,1 \).

Hence we obtain the cohomotopy system \( (P; p_0, p_1, s) \) in \( \text{Cat} \) and the functor \( Q : \text{Cat} \times \text{Cat} \to \text{Set}^{\text{op}}(\text{Set}^\text{op}) \), where \( Q(C, C')_n = \).
\[ \text{Cat}(\mathcal{C}, P^n(\mathcal{C})) \text{ for } \mathcal{C}, \mathcal{C}' \in \text{obCat}. \text{ In particular, for } \mathcal{C} = \ast \text{ we have the functor } Q : \text{Cat} \rightarrow \text{Set}^{\text{op}}. \]

In \text{Cat} we can define the Serre fibration (see [2]). Moreover, in \text{Cat} it is easy to define a notion of the loop functor \( \Omega \), the homotopy fibre of a map etc.

For further considerations we shall need the following

**Theorem 1.2.** (see [2]). For a functor \( p : E \rightarrow B \) the following conditions are equivalent:

i) \( p : E \rightarrow B \) is the Serre fibration,

ii) \( Q(p) : Q(E) \rightarrow Q(B) \) is the Kan fibration.

**Corollary 1.3.** For any \( \mathcal{C} \in \text{obCat} \) the cubical set \( Q(\mathcal{C}) \) satisfies the Kan extension condition.

One can prove that for any \( \mathcal{C}, \mathcal{C}' \in \text{obCat} \) the cubical set \( Q(\mathcal{C}, \mathcal{C}') \) also satisfies the Kan extension condition.

**2. The homotopy groups and Milnor's theorem.** Following the paper [2] for \( \mathcal{C} \in \text{obCat}^{\ast} \) we put \( \pi_n(\mathcal{C}, \ast) = \pi_n(Q(\mathcal{C}), \ast) \), where \( \pi_n(Q(\mathcal{C}), \ast) \) is the \( n \)-th homotopy group of the cubical set \( Q(\mathcal{C}) \) (see [3]).

**Theorem 2.1.** (see [2]) i) A map \( f : \mathcal{C} \rightarrow \mathcal{D} \) induces the long exact sequence

\[ \ldots \rightarrow \pi_n(\mathcal{C}, \ast) \rightarrow \pi_n(\mathcal{D}, \ast) \rightarrow \pi_{n-1}(f^{-1}(\ast), \ast) \rightarrow \ldots. \]

ii) If \( f : \mathcal{C} \rightarrow \mathcal{D} \) is the Serre fibration then \( f^{-1}(\ast) \sim w f^{-1}(\ast) \) where \( \sim w \) is the weak homotopy equivalence.

Let \( \mathcal{I} \) be a small category. Thomason (see [7]). proved that for a functor \( F : \mathcal{I} \rightarrow \text{Cat} \), the classifying space of the Grothendieck construction \( B(\mathcal{I} / F) \), is homotopy equivalent to
the realization of the Bousfield-Kan homotopy colimit
\( \text{hocolim} \ NFI \). There also exists a relation between the homotopy groups of \( \text{hocolim} \ F \) and \( \text{colim} \ F \).

Let \( F : \mathcal{I} \to \text{Cat} \) be a functor. The Grothendieck construction on \( F, \mathcal{I}/F, \) is the category with objects: the pairs \((i, X)\) with \( i \) an object of \( \mathcal{I} \) and \( X \) an object of \( F(i) \), and with morphisms \((\alpha, x) : (i_1, X_1) \to (i_0, X_0)\) given by a morphism \( \alpha : i_1 \to i_0 \) in \( \mathcal{I} \) and a \( x : F(\alpha)(X_1) \to X_0 \) in \( F(i_0) \). The composition is defined by \((\alpha, x)(\alpha', x') = (\alpha \alpha', xF(\alpha)(x'))\).

For \( F : \mathcal{I} \to \text{Cat} \) let \( p : \mathcal{I}/F \to \text{colim}^* F \) be the functor given by \( p(i, X) = X \), then \( p^{-1}(*) = \mathcal{I} \). Moreover, for any \( C \in \text{ob} \text{ colim}^* F \) we have the pair of functors \( p^{-1}(C) \to p/C \) and \( p^{-1}(C) \to C/p \) given in the obvious way, where \( p/C \) and \( C/p \) are comma categories. It is not difficult to see that \( p^{-1}(C)p \to p/C \) has a left adjoint and \( p^{-1}(C) \to C/p \) has a right adjoint. Hence \( p \) is the Serre fibration (see [5]).

Therefore, following the Thomason's result we have

**Corollary 2.2.** For a functor \( F : \mathcal{I} \to \text{Cat} \) there is the long exact sequence

\[
\cdots \to \pi_n(\mathcal{I}, \ast) \to \pi_n(\text{hocolim} F, \ast) \to \pi_n(\text{colim}^* F, \ast) \to \cdots
\]

In particular, if \( \mathcal{I} \) is a contractible category (for instance, a left or right filtering category) then

\[
\pi_n(\text{hocolim} F, \ast) \cong \pi_n(\text{colim}^* F, \ast) \quad \text{for} \ n \geq 0.
\]

On the base of the proof of Theorem 3.1 from [1,ch.XIX] and with references to the fact that the cubical set \( Q(\mathcal{I}, C') \) satisfies the Kan extension condition, we obtain
**Theorem 2.3. (Milnor's theorem).** Let $I$ be a countable small right filtering category. If $F : I^{op} \to \text{Cat}^*$ and $F' : I \to \text{Cat}^*$ are such functors that for any map $\alpha : i \to i'$ in $I$ $F(\alpha) : F(i') \to F(i)$ is the Hurewicz fibration and $F'(\alpha) : F'(i) \to F'(i')$ is the Hurewicz cofibration then there is the short exact sequence of pointed sets

$$* \to \lim^1 \{F'(i), \Omega F(i)\} \to \lim F'(i), \lim F(i) \to \lim [F'(i), F(i)] \to *$$

where $\{,\}$ denotes the set of homotopy classes of maps and $\lim^1$ - the 1-th derived functor of $\lim$.

**Corollary 2.4.** i) If $F(i) = F$ for any $i \in \text{ob} I$ then

$$* \to \lim^1 \{F'(i), \Omega F\} \to \lim F'(i), F \to \lim [F'(i), F] \to *$$

is the Milnor's sequence.

ii) If $F'(i) = F'$ for any $i \in \text{ob} I$ then

$$* \to \lim^1 \{F', \Omega F(i)\} \to \{F', \lim F(i)\} \to \lim [F', F(i)] \to *$$

is the Vogt-Cohen's sequence.

Remark that the following diagram

$$
\begin{array}{ccc}
* & \to & \lim^1 \{F'(i), \Omega \lim F(i)\} \\
\downarrow \phi & & \downarrow \phi' \\
* & \to & \lim^1 \{F'(i), \Omega F(i)\} \to \lim F'(i), \lim F(i) \to \lim [F'(i), F(i)] \to *
\end{array}
$$

is commutative. From the "Snake Lemma" we have that $\text{coker} \, \phi = \text{ker} \, \psi$ and $\text{coker} \, \phi' = \text{ker} \, \psi'$.

Hence we obtain
Corollary 2.5. There are the following exact sequences:

\[ i \circ \lim^1 [F'(i), \Omega \lim F(i)] \rightarrow \lim^1 [F'(i), \Omega F(i)] \rightarrow \lim [F'(i), \lim F(i)] \rightarrow \lim [F'(i), F(i)] \rightarrow 0, \]

\[ ii) \circ \lim^1 [\lim F'(i), \Omega F(i)] \rightarrow \lim^1 [F'(i), \Omega F(i)] \rightarrow \lim [\lim F'(i), F(i)] \rightarrow \lim [F'(i), F(i)] \rightarrow 0. \]

3. Homotopy limit in \textit{Cat}. Let \( I \) be a small category and \( \text{Set}^{\text{op}} \) - the category of simplicial sets. A.K. Bousfield and D.M. Kan defined for \( F : I \rightarrow \text{Set}^{\text{op}} \) the homotopy limit - \( \text{holim} \) \( F \). We will prove that the homotopy limit of a diagram of nerves of categories is itself the nerve of a category.

For \( F : I \rightarrow \text{Cat} \) we define the functor \( \tilde{F} : I \rightarrow \text{Cat} \) (the Eilenberg-Moore rectification or Street "second construction", see [6]). For \( i \in \text{ob} I \) \( \tilde{F}(i) \) is the category whose objects are pairs \((\psi, \varphi)\), where \( \psi \) is a function assigning each \( \alpha : i \rightarrow i' \) in \( I \) with source \( i \) an object \( \psi(\alpha) \) of \( F(i') \); and \( \varphi \) assigns each string \( i \xrightarrow{\alpha} i' \xrightarrow{\beta} i'' \) a map \( \varphi_{\beta, \alpha} : \psi(\alpha) \rightarrow F(\beta) \psi(\alpha) \) in \( F(i'') \), subject to

\[
\begin{array}{ccc}
\psi(\gamma \beta \alpha) & \xrightarrow{\psi(\gamma \beta, \alpha)} & F(\gamma \beta) \psi(\alpha) \\
\varphi_{\gamma, \beta \alpha} & & \\
\downarrow \varphi_{\gamma, \beta \alpha} & & \\
F(\gamma) \psi(\beta \alpha) & \xrightarrow{F(\gamma) \varphi_{\beta, \alpha}} & F(\gamma) F(\beta) \psi(\alpha)
\end{array}
\]

commute. A map \( a : (\psi, \varphi) \rightarrow (\psi', \varphi') \) is a function which assigns to each \( \alpha : i \rightarrow i' \) in \( I \) a map of \( F(i') \), \( a(\alpha) : \psi(\alpha) \rightarrow \psi'(\alpha) \); subject to, for \( i \xrightarrow{\alpha} i' \xrightarrow{\beta} i'' \), that

\[
\begin{array}{ccc}
F(\beta) \psi(\alpha) & \xrightarrow{F(\beta) a(\alpha)} & F(\beta) \psi'(\alpha) \\
\varphi_{\beta, \alpha} & & \\
\downarrow \varphi_{\beta, \alpha} & & \\
\psi(\beta \alpha) & \xrightarrow{a(\beta \alpha)} & \psi'(\beta \alpha)
\end{array}
\]

138
commutes. The composition is given by $\tilde{a} \cdot a(\alpha) = \tilde{a}(\alpha)a(\alpha)$. For $\delta : \mathbb{I} \longrightarrow \mathbb{I}$, $\tilde{F}(\delta) : \tilde{F}(\mathbb{I}) \longrightarrow \tilde{F}(\mathbb{I})$ is given on objects by $\tilde{F}(\delta)(\psi, \phi) = (\psi^\delta, \phi^\delta)$, where $\psi^\delta(\alpha) = \psi(\alpha\delta)$, $\phi^\delta_{\beta, \alpha} = \phi_{\beta, \alpha\delta}$; and on morphisms by $\tilde{F}(\delta)(a) = a^\delta$, $a^\delta(\alpha) = a(\alpha\delta)$.

Then we have the following

**Theorem 3.1.** For a functor $F : \mathbb{I} \longrightarrow \text{Cat}$ there is a natural isomorphism $\text{holim} \, NF \cong N \text{ lim} \, \tilde{F}$, where $N$ is the nerve functor.

The proof is straightforward.
REFERENCES


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