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A NOTE ON PROJECTIVE FOLIATIONS

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In [12], Nishikawa and Sato studied conformal and projective foliations defined as foliations whose second order transversal bundle is endowed with either a conformal or a projective projectable structure. (See, for instance [7] for such structures on manifolds.) Namely, they proved the existence of corresponding projectable normal Cartan connections from which they deduce that the same strong Bott vanishing phenomenon like for Riemannian foliations holds. Then, Nishikawa studied characteristic classes of projective foliations in [13].

Independently, I discussed conformal foliations in [16] using the classical definition of conformal structures by means of Riemannian metrics, and Montesinos [11] proved the strong Bott vanishing theorem for conformal foliations, by using this classical approach.

The aim of this Note is to present projective foliations by using the alternate approach to projective structures known as geometry of paths [6], and by constructing the normal connection with a vector bundle version of the original Cartan technique [4]. This approach will provide us not only with the Bott-Nishikawa-Sato vanishing theorem of [12, 13] but also with projectively invariant representative forms of the real Pontrjagin classes of manifolds and of transverse bundles of foliations. Furthermore, we shall obtain a cohomological obstruction to the existence of a transversal projective projectable (I am using the name foliate instead) structure.

Beyond all this, since two approaches to projective structures on manifolds are available, it seems natural to use them both in studying projective foliations as well.

1. Projective Structures on Manifolds. The definitions of this section are a reformulation of those given in [6]. Let V^n be a differentiable manifold (in this paper, we are always in the C^∞-category), and U an open subset of V endowed with a torsionless linear connection ∇. Then we call (U,∇) a c-chart. Two c-charts $(U,\nabla),(U',\nabla')$ are called projectively related if either $U \cap U' = \phi$ or $U \cap U' \neq \phi$, and over $U \cap U'$ one has

$$\nabla'_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X \,, \tag{1.1}$$

where X,Y are local vector fields and ψ is a well-defined 1-form. A family $F = \{(U_\alpha,\nabla_\alpha)\}$ of c-charts is a projective atlas on V if any two of its charts are projectively related, and $\{U_\alpha\}$ is a covering of V. A maximal projective atlas is a projective structure on V. Of course, any projective atlas yields a unique projective structure. A pair consisting of a manifold V and a projective structure on it is a projective manifold.

Following are some well-known facts concerning projective manifolds [6]:

i) A global torsionless linear connection on V, and in particular a Riemannian connection provide a projective structure. Conversely, if we glue up local c-charts by a partition of unity, we get a global c-chart of the same maximal atlas. Hence a projective structure can always be represented by a global connection, but not in a canonical manner.

ii) The unparametrized geodesics of the connections ∇_α of a projective structure are the same, and they yield the system of paths of the structure which, in fact, is the characteristic object of the projective manifold.

iii) A projective structure can always be defined by a Ricci symmetric atlas (i.e., one whose connections ∇_α have a symmetric Ricci curvature tensor). (Then, ψ of (1.1) is a closed form.) Moreover, we can even represent the structure by a single torsionless Ricci-symmetric linear connection.

iv) Let $n = \dim V > 1$, and set

$$W(X,Y)Z = R_\alpha(X,Y)Z + \frac{1}{n+1}\left[B_\alpha(X,Y) - B_\alpha(Y,X)\right]Z + \tag{1.2}$$

$$+ \frac{1}{n^2-1}\left\{\left[nB_\alpha(Z,Y) + B_\alpha(Y,Z)\right]X - \left[nB_\alpha(Z,X) + B_\alpha(X,Z)\right]Y\right\} ,$$

where R_α is the curvature of ∇_α, and B_α is the corresponding Ricci tensor. Then W does not depend on the index α, and it defines the Weyl projective curvature tensor. $W = 0$ for $n = 2$, and for $n \geqslant 3$, $W = 0$ iff V is a projectively Euclidean manifold, i.e., one which has a projective atlas consisting of flat connections. (E.g., the Riemannian manifolds of constant curvature belong to this class.)

v) [5] The Chern-Pontrjagin forms of a Riemannian manifold are invariant by projective transformations between Levi-Civita connections.

This latter fact can be extended as follows. Using the usual local components of the tensor (1.2), let us define the local 2-forms

$$W^i_j = \frac{1}{2} W^i{}_{jk\ell}\, dx^k \wedge dx^\ell , \tag{1.3}$$

and the global differential forms

$$P_j(V) = \frac{1}{(2\pi)^{2j}(2j)!} \sum_{h,k=1}^{n} \delta^{k_1 \ldots k_{2j}}_{h_1 \ldots h_{2j}}\, W^{h_1}_{k_1} \wedge \ldots \wedge W^{h_{2j}}_{k_{2j}} . \tag{1.4}$$

The forms (1.4) can be computed by using a global Ricci-symmetric torsionless linear connection as a projective atlas of V (see iii) above). In this case the computation of A. Avez [1] (originally done for conformal structures) applies, and $P_j(V)$ are seen to be equal to the usual Chern-Pontrjagin forms of the chosen connection.

Hence, the Pontrjagin classes of a general projective manifold V can be represented by projectively invariant forms, and these forms are given by (1.4). Particularly, every projectively Euclidean manifold has vanishing Pontrjagin classes.

2. Projective Foliations. Now, we shall apply the schema of Section 1 to the transverse bundle of a foliation. Let M^n be a manifold, and F a foliation of codimension q on M (see, for instance [2] for generalities on foliations). Let E be the tangent bundle of F, and $Q = \mathrm{Tr}\, F = TM/E$ be its transverse bundle. Then, we have the natural projection $\pi : TM \to Q$, and we shall denote $\pi(X) = \overline{X}$, and $\tilde{X}$ any element of $\pi^{-1}(\overline{X})$. The dual bundle Q^* is a subbundle of T^*M. Our convention will be to attach the label foliate to everything which is constant on the leaves of F, and the label basic to everything which depends only on the "differentials in Q^* ". Particularly, a basic connection ∇ on Q is characterized by [2]

$$\nabla_X \overline{Z} = \overline{[X,\tilde{Z}]}. \tag{2.1}$$

Such a connection has the torsion

$$T(\overline{X},\overline{Y}) = \nabla_{\tilde{X}}\overline{Y} - \nabla_{\tilde{Y}}\overline{X} - \overline{[\tilde{X},\tilde{Y}]} \tag{2.2}$$

(which does not depend on the choice of $\tilde{X},\tilde{Y}$), and it is torsionless if $T = 0$. Moreover, Q is a foliate bundle, and the basic connection ∇ is foliate if for every foliate sections $\overline{Z},\overline{X}$, the section $\nabla_{\tilde{X}}\overline{Z}$ is also foliate. It is known that Q has always basic connections but may have no foliate connections [9]. Finally, two torsionless basic connections on Q will be called transversally projectively related if for any vector field X on M, and section $\overline{Z}$ of Q one has

(2.3) $$'\nabla'_X\overline{Z} = \nabla_X\overline{Z} + \lambda(X)\overline{Z} + \lambda(\tilde{Z})\overline{X} \quad ,$$

for some basic 1-form λ (i.e., $\lambda \in Q^*$), which implies that $\lambda(\tilde{Z})$ depends on $\overline{Z}$ alone).

Now, we shall refer to basic connections on Q, define like in Section 1 transversal c-charts and atlases, and get thereby the notion of a transversal projective structure of the foliation F. Furthermore, if all the connections ∇_α of a transversal projective atlas are foliate connections we shall say that this atlas defines a foliate transversal projective structure. A foliation F endowed with a foliate transversal projective structure is called a projective foliation. (In this case, the 1-forms λ of (2.3) are foliate forms.) It is obvious that the transversal projective structure of a projective foliation F of codimension q is locally the pull-back of a projective structure of R^q by the local submersions which define F [2], and the latter are related by projective diffeomorphisms. This proves that our definition of a projective foliation is equivalent to that of [12]. Moreover, one can get transversal paths which are the pull-backs of the paths of the projective structure of R^q mentioned above.

Like in Section 1, we see that a global torsionless basic Q-connection defines a transversal projective structure of F, and every such structure has atlases consisting of a single global chart. Particularly, the transversal part of the second connection of a Riemannian metric of M with respect to F [14,15] offers a transversal projective structure of F, which proves the existence of such structures for every F. But, generally, only local foliate transversal projective structures exist.

Following [14,15], we shall cover M by flat coordinate neighbourhoods, with local coordinates (x^a, x^u) $(a,b,\ldots = 1,\ldots,q\ ;\ u,v,\ldots = q+1,\ldots,n)$, such that x^a = const. define the leaves of F, and the changes of the coordinates are locally of the form

$$\tilde{x}^a = \tilde{x}^a(x^b),\ \tilde{x}^u = \tilde{x}^u(x^b,x^v) \ . \tag{2.4}$$

Then, we choose once and for ever an auxiliary Riemann metric g, we identify Q with the corresponding normal bundle of F, and take the local bases and cobases

$$X_a = \frac{\partial}{\partial x^a} - t^u_a \frac{\partial}{\partial x^u} \in Q(\perp E),\quad X_u = \frac{\partial}{\partial x^u} \in E \ , \tag{2.5}$$

$$dx^a \ ,\ \theta^u = dx^u + t^u_a\, dx^a \ . \tag{2.6}$$

All the following tensor components are with respect to (2.5), (2.6).

Now, a basic connection on Q has local equations

$$\nabla_{X_b} X_a = \gamma^c_{ab} X_c \ ,\ \nabla_{X_u} X_a = 0 \ , \tag{2.7}$$

and it has no torsion iff $\gamma^c_{ab} = \gamma^c_{ba}$. A projective transformation (2.3) takes the form

$$\gamma'^c_{ab} = \gamma^c_{ab} + \delta^c_a\lambda_b + \delta^c_b\lambda_a \ \ , \tag{2.8}$$

where $\lambda = \lambda_a dx^a$ is the 1-form of (2.3). The curvature R of ∇ is given by

$$R(X_a,X_b)X_c = R^e{}_{cab}X_e \ ,\ R(X_a,X_u)X_c = R^e{}_{cau}X_e \ ,\ R(X_u,X_v)X_c = 0, \tag{2.9}$$

where

$$R^e{}_{cab} = X_a\gamma^e_{cb} - X_b\gamma^e_{ca} + \gamma^h_{cb}\gamma^e_{ha} - \gamma^h_{ca}\gamma^e_{hb} \ ;\ R^e{}_{cau} = -X_u\gamma^e_{ca} \ , \tag{2.10}$$

$$R^e{}_{cab} + R^e{}_{abc} + R^e{}_{bca} = 0,\quad R^e{}_{cau} = R^e{}_{acu} \ . \tag{2.11}$$

Let us also recall that the decomposition $TM = E \oplus Q$ $(Q \perp E)$ induces a decomposition of differential forms into components of type (p,r) (which contain p forms dx^a, and r forms θ^u in their local expression), and a decomposition $d = d' + d'' + \partial$ of the exterior differentiation d into components of the respective type (1,0), (0,1), (2,-1) [14,15].

A basic connection ∇ has the following important associated 2-form

$$\beta(X,Y) = \sum_{c=1}^{q} dx^c\big(R(X,Y)\overline{X}_c\big) = \sum_{c=1}^{q} dx^c\big(R(X,Y)X_c\big) , \tag{2.12}$$

which, obviously, does not depend on the choice of g. One has

Proposition 2.1. *The form β is an exact 2-form.*

Proof. From (2.12) and (2.10), we get

$$\beta = d(\gamma^c_{ca}dx^a) , \tag{2.13}$$

but we are not yet done since $\tilde{\omega} = \gamma^c_{ca}dx^a$ is only a local 1-form. But denoting $h = g/Q$, using the computations of [18], and applying (2.13) we shall find that the β-form of the connection Γ^a_{bc} induced on Q by the second connection of g (already mentioned earlier) is

$$\tilde{\beta} = d\big(\Gamma^c_{ca}dx^a\big) = d\Big[\big(X_c \ \ln\sqrt{\det h}\big)dx^c\Big] = dd' \ln\sqrt{\det h} = \tag{2.14}$$

$$= d(d-d'') \ln\sqrt{\det h} = -dd'' \ln\sqrt{\det h} \ .$$

Here, in view of formulas (2.4), we can see that $d'' \ln\sqrt{\det h}$ is a well defined global 1-form. Furthermore, $t^c_{ab} = \gamma^c_{ab} - \Gamma^c_{ab}$ is a "tensor", whence $t^c_{ca}dx^a$ is a global 1-form. This yields

$$\beta = d\big(t^c_{ca}dx^a - d'' \ln\sqrt{\det h}\big) , \tag{2.15}$$

and proves the proposition.

The idea of the above proof is the one used in [6] to get a projectively related connection with symmetric Ricci curvature on a projective manifold (see iii) of Section 1). Indeed, on a manifold, the symmetry of the Ricci tensor is equivalent to $\beta = 0$. However, in our case we cannot get $\beta = 0$ (globally) but, if we apply the same proof as in [6, p.88], we can obtain a projectively related connection on Q such that $\beta = d\alpha$ with α of type (0,1).

Now, let us consider again a transversal projective structure of the foliation F, defined by transversal c-atlas with basic connections ∇_α, and let us define

$$(2.16) \qquad W(X,Y)\bar{Z} = R_\alpha(X,Y)\bar{Z} - \frac{1}{q+1}\left[\beta_\alpha(X,Y)\bar{Z} + \beta_\alpha(\tilde{Z},Y)\bar{X}\right] ,$$

where the field Y is tangent to F. A straightforward checking shows that W does not depend on the choice of $\tilde{Z}$ corresponding to $\bar{Z}$, and it is invariant by (2.3). (The condition $Y \in E$ is essential.) This checking is easy by using the local components

$$(2.17) \qquad W^e{}_{cau} = \underset{(\alpha)}{R}{}^e{}_{cau} - \frac{1}{q+1}\left(\delta^e_c\beta^{(\alpha)}_{au} + \delta^e_a\beta^{(\alpha)}_{cu}\right),$$

where $\beta^{(\alpha)}_{au} = \underset{(\alpha)}{R}{}^c{}_{cau}$, and the formulas (2.10), (2.8).

The operator W yields a well-defined 2-form of type (1,1) on M, with values in the foliate vector bundle $Hom(Q,Q)$, which has the local components (2.17). We shall denote this form by w, and call it the <u>auxiliary Weyl form</u>. It provides us with a cohomological obstruction to the existence of a foliate transverse projective structure since we have

<u>Theorem 2.2. The auxiliary Weyl form w is d"-closed, and it defines a d"-cohomology class which is independent on the transverse projective structure of F. The foliation F admits a foliate transverse projective structure iff w is also d"-exact.</u>

Proof. Let us start with a transverse projective structure of F, and the corresponding form w. Let ∇_α be one of the local connections of this structure, and β_α be the corresponding form (2.12). Then, if we consider a transformation (2.3) (or (2.8)) where λ is a basic form, such that $\beta + (q+1)d\lambda = 0$, we get another connection of the same projective structure whose β-form is zero. By (2.13) such a form λ exists locally. Hence, we can always choose a projectively equivalent atlas whose connections ∇_α have vanishing forms β_α. (But, generally, this new atlas has more than one chart.) Now, since W is projectively invariant, we can express it with these connections ∇_α, and (2.16) yields

$$W(X,Y)\overline{Z} = R_\alpha(X,Y)\overline{Z} \quad . \tag{2.18}$$

It follows that w is precisely the (1,1)-type part of the curvature of a basic connection, and it is known from [9] that the latter is d''-closed.

Now, let us note that the d''-exactness of w means that some "tensor" of local components t^e_{ca} exists such that

$$W^e_{cau} = X_u t^e_{ca} \quad . \tag{2.19}$$

But then, it follows from (2.10) and (2.18) that the d''-cohomology class of w is well defined, and it does not depend on the transverse projective structure used for F. It is known [15] that this class represents an element

$$[w] \in H^1\left(M,\Phi^1\left(\mathrm{Hom}(Q,Q)\right)\right) \quad ,$$

where the second argument denotes the sheaf of germs of foliate (1,0)-forms with values in Hom(Q,Q). We shall say that $[w]$ is the projective Molino-Atiyah class of F [9]. Particularly, if a foliate transverse projective structure exists, then (2.10) and (2.18) yield that its auxiliary Weyl form is $w=0$, whence $[w]=0$, and w of any other transverse projective structure is d''-exact.

Conversely, if w is d''-exact, we have (2.19), and we can go over from everyone of the connections γ^a_{bc} of the β-vanishing atlas to the connection $\tilde{\gamma}^a_{bc} = \gamma^a_{bc} - t^a_{bc}$, which will obviously be (local) foliate connections. But, any two of the connections $\tilde{\gamma}$ have the same difference as the corresponding connections γ, hence they are projectively related. Therefore, we obtained a foliate transverse projective structure for F, as desired. Q.E.D.

Hence, the projective Molino-Atiyah class is the obstruction to the existence of a foliate transverse projective structure. (The original Molino-Atiyah class of [9] was the obstruction to the existence of a foliate connection of F.)

We shall end this section by constructing projectively invariant representatives of the real Pontrjagin classes of Q analogous to the forms (1.4).

Formally, the formulas (2.8), (2.10), and (2.11) are the same as in the case of a projective structure on a manifold (where $X_a = \partial/\partial x^a$), whence we conclude that the computations of [6, p. 87-89] hold good, and, if $q \geqslant 2$, the "tensor"

$$W^e{}_{cab} = R^e{}_{cab} + \frac{1}{q+1}\delta^e_c(B_{ab} - B_{ba}) + \tag{2.20}$$

$$+ \frac{1}{q^2-1}\left[\delta^e_a(qB_{cb} + B_{bc}) - \delta^e_b(qB_{ca} + B_{ac})\right],$$

where $B_{ab} = R^c{}_{abc}$ (and, of course, all the components are with respect to (2.5)) is projectively invariant. Generally (2.20) depends on the choice of the auxiliary metric g, but if F is a projective foliation, (2.20) is the pull-back of the projective curvature tensor of the "local bases" of F, and it is an invariant of the foliate projective structure of Q. We call the W of (2.20) the projective curvature tensor of the basic connection ∇ of Q; it vanishes identically for $q=2$.

Now, with (2.17) and (2.20), we can define the local differential forms

$$\omega^e_c = \frac{1}{2} W^e{}_{cab} dx^a \wedge dx^b + W^e{}_{cau} dx^a \wedge \theta^u , \tag{2.21}$$

where θ^u is defined in (2.6). We get

$$\omega^e_c = \Omega^e_c + dx^e \wedge \zeta_c - \frac{1}{q+1} \delta^e_c \beta , \tag{2.22}$$

where Ω^e_c are the usual curvature forms, and

$$\zeta_c = \frac{1}{q^2-1} \left(q B_{cb} + B_{bc}\right) dx^b - \frac{1}{q+1} \beta_{cu} dx^u . \tag{2.23}$$

The real Pontrjagin classes of Q are represented by the forms

$$P_{2j}(Q) = \frac{1}{(2\pi)^{2j}(2j)!} \sum_{a,b=1}^{q} \delta^{b_1 \ldots b_{2j}}_{a_1 \ldots a_{2j}} \Omega^{a_1}_{b_1} \wedge \ldots \wedge \Omega^{a_{2j}}_{b_{2j}} , \tag{2.24}$$

and these forms belong to the real exterior algebra generated by

$$\Pi_h = \Omega^{a_1}_{a_2} \wedge \Omega^{a_2}_{a_3} \wedge \ldots \wedge \Omega^{a_h}_{a_1} \qquad (h = 1,2,\ldots) . \tag{2.25}$$

Now, let us construct the analogous forms

$$\tilde{\Pi}_h = \omega^{a_1}_{a_2} \wedge \omega^{a_2}_{a_3} \wedge \ldots \wedge \omega^{a_h}_{a_1} \qquad (h = 1,2,\ldots). \tag{2.26}$$

Then, since in view of (2.11), we have

$$\zeta_c \wedge dx^c = - \frac{1}{q+1} \beta , \tag{2.27}$$

it follows that $\tilde{\Pi}_h$ belongs to the real exterior algebra generated by Π_k $(k \leq h)$ and β. For instance, we have

$$\tilde{\Pi}_1 = \Pi_1 - \beta, \; \tilde{\Pi}_2 = \Pi_2 - \frac{2}{q+1} \Pi_1 \wedge \beta + \frac{1}{q+1} \beta \wedge \beta , \text{ etc.} \tag{2.28}$$

Since Π_h are closed forms, and β is exact, it follows that the forms $\tilde{\Pi}_h$ are cohomologous to Π_h. Moreover, we get in fact

Theorem 2.3. The real Pontrjagin classes of the transverse bundle Q of a foliation $\mathcal{F}$ of codimension ≥ 2 are representable via the de Rham isomorphism by the differential forms

$$(2.29) \qquad \tilde{P}_{2j}(Q) = \frac{1}{(2\pi)^{2j}(2j)!} \sum_{a,b=1}^{q} \delta^{b_1 \ldots b_{2j}}_{a_1 \ldots a_{2j}} \, \omega^{a_1}_{b_1} \wedge \ldots \wedge \omega^{a_{2j}}_{b_{2j}} \ .$$

Each form $\tilde{P}_{2j}(Q)$ is cohomologous to $P_{2j}(Q)$, and it provides a projectively invariant representative form of the j-th Pontrjagin class of Q.

If $\mathcal{F}$ is a projective foliation, we can use the local foliate connections of the projective structure, for which $W^{e}{}_{cau} = 0$. Hence the forms (2.21) are of the type (2,0), and we refind the result of [12]

Corollary 2.4. If $\mathcal{F}$ is a projective foliation of codimension $q \geq 2$, one has necessarily $\mathrm{Pont}^k Q = 0$ for $k > q$.

3. The Cartan Normal Connection. Let $(M,\mathcal{F})$ be a foliate manifold with transversal projective structure defined by a global basic torsionless connection ∇ of Q. (We shall use again the notations of Section 2). Inspired by Cartan [4], let us consider the canonical line bundle $K(\mathcal{F}) = \Lambda^q Q^*$ of Q, then define

$$(3.1) \qquad \tau(\mathcal{F}) = K(\mathcal{F}) \oplus \big(K(\mathcal{F}) \otimes Q\big) \ ,$$

which is a foliate vector bundle of rank $q+1$, with the local bases

$$(3.2) \qquad e = dx^1 \wedge \ldots \wedge dx^q \quad , \quad e_a = e \otimes \frac{\partial}{\partial x^a} \quad ,$$

and the transition cocycle

$$(3.3) \qquad \big(p^{\beta}_{\alpha}\big) = \begin{pmatrix} J & 0 \\ 0 & J \dfrac{\partial x^b}{\partial \tilde{x}^a} \end{pmatrix} \quad ,$$

where $\alpha,\beta,\ldots = 0,1,\ldots,q$; $J = \det(\partial\tilde{x}^a/\partial x^b)$. The elements of the inverse matrix of (3.3) will be denoted by q^β_α , i.e., $p^\gamma_\alpha q^\beta_\gamma = \delta^\beta_\alpha$.

Now, with respect to (3.2), a connection on $\tau(F)$ is given locally by a matrix (π^β_α) of connection forms with the transition law

$$\tilde{\pi}^\beta_\alpha = p^\lambda_\alpha q^\beta_\mu \pi^\mu_\lambda + q^\beta_\lambda dp^\lambda_\alpha \quad , \tag{3.4}$$

or, after using (3.3)

$$\begin{gathered} \tilde{\pi}^0_0 = \pi^0_0 + J^{-1}dJ \quad , \quad \tilde{\pi}^a_0 = \frac{\partial\tilde{x}^a}{\partial x^b}\pi^b_0 \quad , \quad \tilde{\pi}^0_a = \frac{\partial x^b}{\partial\tilde{x}^a}\pi^0_b \quad , \\ \tilde{\pi}^b_a = \frac{\partial x^c}{\partial\tilde{x}^a}\frac{\partial\tilde{x}^b}{\partial x^e}\pi^e_c + \frac{\partial\tilde{x}^b}{\partial x^c} d\left(\frac{\partial x^c}{\partial\tilde{x}^a}\right) + \delta^b_a J^{-1}dJ \quad , \quad \tilde{\pi}^a_a = \pi^a_a + (q-1) J^{-1}dJ \ . \end{gathered} \tag{3.5}$$

Whence, let us note that $\tau(F)$ admits connections such that π^0_0 is an arbitrary basic connection on $K(F)$, and

$$\pi^a_0 = dx^a \quad , \quad \pi^a_a - (q-1)\pi^0_0 = 0 \ . \tag{3.6}$$

Particularly, let us look at the connections ∇ (ω^b_a) and ∇' (ω'^b_a) of the projective structure of Q, with coefficients γ^b_{ac} , γ'^b_{ac} related by (2.8), then take in (2.8)

$$\lambda_a = -\frac{1}{q+1}\left(\gamma^b_{ba} + \pi^0_{0a}\right) \quad , \tag{3.7}$$

where $\pi^0_0 = \pi^0_{0a}dx^a$ is a chosen basic connection on $K(F)$. This yields the unique global connection of the projective structure of Q for which $\omega'^a_a + \pi^0_0 = 0$. Then we see by (3.5) that

$$\pi^b_a = \omega'^b_a + \delta^b_a\pi^0_0 \tag{3.8}$$

fits into the definition of a connection of $\tau(F)$ satisfying (3.6). From (2.8), (3.7), (3.8), we also get

(3.9) $$\pi^b_a = \theta^b_a + \frac{q}{q+1}\,\delta^b_a\pi^0_0 - \frac{1}{q+1}\,\pi^0_{0a}dx^b \quad ,$$

where $\theta^b_a = \theta^b_{ac}dx^c$ and

$$\theta^b_{ac} = \gamma^b_{ac} - \frac{1}{q+1}\left(\delta^b_a\gamma^e_{ec} + \delta^b_c\gamma^e_{ea}\right) \quad ,$$

which correspond to the <u>coefficients of the projective connection</u> of [6, p.98].

Hereafter, on $\tau(F)$ we are refering to the connections (3.6), (3.9) only. Now, let us consider the following Cartan change of the local bases (3.2) [4]

(3.10) $$e = e, \quad u_a = e_a + \xi_a e \quad ,$$

and choose $\xi_a = [1/(q+1)]\pi^0_{0a}$. Then we get the following new connection forms of our connections on $\tau(F)$

(3.11) $$\kappa^0_0 = \frac{q}{q+1}\,\pi^0_0 \quad , \quad \kappa^a_0 = dx^a \; , \; \kappa^b_a = \theta^b_a + \frac{q}{q+1}\,\delta^b_a\pi^0_0 \; .$$

The transition cocycle of the bases (3.10) is

(3.12) $$\begin{pmatrix} J & 0 \\ \frac{1}{q+1}\frac{\partial J}{\partial \tilde{x}^a} & J\frac{\partial x^b}{\partial \tilde{x}^a} \end{pmatrix} = \begin{pmatrix} J & 0 \\ \frac{1}{q+1}\,J\,\frac{\partial x^b}{\partial x^e}\frac{\partial^2 \tilde{x}^e}{\partial x^b \partial x^c}\frac{\partial x^c}{\partial \tilde{x}^a} & J\frac{\partial x^b}{\partial \tilde{x}^a} \end{pmatrix} \quad ,$$

and it is differentially but not necessarily foliately cohomologous with (3.3). Hence (3.3) and (3.12) define different foliate vector bundles. <u>Hereafter, we shall always refer to $\tau(F)$ as the foliate bundle (3.12), and to the connections (3.11).</u>

Now, let us denote by $K^\beta_\alpha = d\kappa^\beta_\alpha - \kappa^\gamma_\alpha \wedge \kappa^\beta_\gamma$ the curvature forms of our connections, whence

$$(3.13)\qquad K^a_0 = 0,\quad K^0_0 = d\kappa^0_0 - dx^a \wedge \kappa^0_a\ ,\quad K^b_a = \Theta^b_a - \kappa^0_a \wedge dx^b + \delta^b_a d\kappa^0_0\ ,$$

where

$$(3.14)\qquad \Theta^b_a = d\theta^b_a - \theta^c_a \wedge \theta^b_c\ ,$$

and these are projectively invariant forms.

Furthermore, let us note that (3.6), (3.9) and (3.11) imply

$$(3.15)\qquad \kappa^a_a - q\kappa^0_0 = 0\ .$$

This suggests us to impose as a further condition

$$(3.16)\qquad K^a_a - qK^0_0 = 0$$

(which has an invariant meaning by (3.12)). After explicitation, (3.16) turns out to be equivalent to

$$(3.17)\qquad dx^a \wedge \kappa^0_a = 0\ ,$$

and, if we denote $\kappa^0_a = \kappa_{ab}dx^b$, (3.17) becomes $\kappa_{ab} = \kappa_{ba}$.

Finally, let us put

$$(3.18)\qquad K^\beta_\alpha = \frac{1}{2} K^\beta_{\alpha ce} dx^c \wedge dx^e + K^\beta_{\alpha cu} dx^c \wedge \theta^u$$

(there will be no part of type (0,2) in view of (3.13)), and define also similar components for Θ^b_a. Then, consider

$$(3.19)\qquad K^b_a - \delta^b_a K^0_0 = \Theta^b_a - \kappa^0_a \wedge dx^b\ ,$$

and the corresponding coefficients

$$(3.20)\qquad \Xi^b_{ace} = K^b_{ace} - \delta^b_a K^0_{0ce} = \Theta^b_{ace} - \left(\kappa_{ac}\delta^b_e - \kappa_{ae}\delta^b_c\right)\ .$$

These coefficients yield the "tensor"

(3.21) $$T_{ac} = \Xi^b_{acb} = \Theta^b_{acb} - (q-1)\kappa_{ac} \quad ,$$

and it has an invariant meaning to ask T to be skew-symmetric, which gives (for $q \geqslant 2$)

(3.22) $$\kappa_{ac} = \frac{1}{2(q-1)} \left(\Theta^b_{acb} + \Theta^b_{cab}\right) .$$

This means that we are able to determine a canonical connection κ, if π^0_0 is chosen, and we proved

Theorem 3.1. Let F be a foliation of M of codimension $q \geqslant 2$, endowed with a transverse projective structure. Let us choose the auxiliary Riemannian metric g, and a basic connection π^0_0 on $K(F)$. Then, there is a unique connection on $\tau(F)$, which satisfies the conditions (3.11) and (3.22).

The connection of Theorem 3.1 will be called the normal Cartan connection (compare with [4] and [16]). If the projective structure of F is foliate, we may use in the above computations of κ^β_α local foliate connections ∇, and we see that the normal connection of $\tau(F)$ is "equal up to the choice of π^0_0" to the lifts of the normal connections of the "local bases" of F.

Now, in order to escape from the arbitrary connection form π^0_0 we have to go over to the projectivization of $\tau(F)$, and it is nice to do this in the language of principal bundles.

Let us consider the principal bundle $\mathcal{B}_\tau$ of the bases of $\tau(F)$, factorize it by the relation of proportionality, and get the bundle P_τ of the projective frames of the fibres of $\tau(F)$, whose structure group is the q-dimensional projective group. Then, let us take the principal subbundle $\mathcal{B}^0_\tau$ of $\mathcal{B}_\tau$ consisting of bases with the first vector proportional to e of (3.2), and perform the same factorization to get a subbundle P^0_τ of P_τ for which the structure group is the central-projective group (i.e., the group of the projective transformations with a given fixed point). Following [4], it is P^0_τ which plays the main role; we consider it as a foliate principal bundle with the transition cocycle (3.12).

It is known that the general projective group $P(q,R)$ is $G\ell(q+1,R)$/centre, whence the corresponding Lie algebra $p(q,R)$ is $g\ell(q+1,R)/\{\rho I\}$ (I is the unit matrix). Hence, (a'^β_α) and (a^β_α) of $g\ell(q+1,R)$ define the same element of $p(q,R)$ iff

$$a'^\beta_\alpha - \delta^\beta_\alpha a'^0_0 = a^\beta_\alpha - \delta^\beta_\alpha a^0_0 \quad , \tag{3.23}$$

and we can always take $a^\beta_\alpha - \delta^\beta_\alpha a^0_0$ as the representative of the corresponding element of $p(q,R)$. The central projective group $P_0(q,R)$ and the corresponding Lie algebra $p_0(q,R)$ are defined similarly but using only matrices (a^β_α) with $a^b_0 = 0$.

Now, the normal connection of $\tau(F)$ yields a connection on B_τ with the $g\ell(q+1,R)$-valued local connection forms (κ^β_α), and this induces a connection on P_τ. Because of (3.12), the matrices obtained from (κ^β_α) by replacing κ^a_0 with 0 will yield a connection on B^0_τ, and this induces a connection on P^0_τ, whose $p_0(q,R)$-valued local forms are represented by $(\kappa^\beta_\alpha - \delta^\beta_\alpha \kappa^0_0)$. As shown by (3.11) and (3.22), the latter matrices do not depend on π^0_0 any more.

Finally, let us also note another important property of P^0_τ. We start by introducing in the manifold B_τ the local coordinates $(x^a, x^u, \xi^\beta_\alpha)$, where ξ^β_α are the components of the vectors of a frame of B_τ with respect to the bases (3.10). Then ξ^β_α are "homogeneous coordinates" in P^0_τ, and, in view of (3.12), the local equations x^a = const., quotients of ξ^β_α = const. define on P^0_τ a foliation F^0_τ whose leaves cover the leaves of F (like in the Riemannian case [10]). The mentioned property (which is a reason of refering to P^0_τ) is that F^0_τ admits a transverse parallelization. (This is known for $q=n$ [7].)

Indeed, let (η^β_α) be the inverse matrix of (ξ^β_α). Then, the global $g\ell(q+1,R)$-

valued connection form of the normal connection on $\mathcal{B}_\tau$ is the matrix [8]

$$\Xi^\beta_\alpha = \eta^\beta_\lambda d\xi^\lambda_\alpha + \eta^\beta_\lambda \xi^\gamma_\alpha \kappa^\lambda_\gamma \quad , \tag{3.24}$$

and the induced $\mathcal{P}(q,\mathbb{R})$-form on P_τ is given by

$$\Xi^\beta_\alpha - \delta^\beta_\alpha \Xi^0_0 = \eta^\beta_\lambda d\xi^\lambda_\alpha - \delta^\beta_\alpha \eta^0_\lambda d\xi^\lambda_0 + \left(\eta^\beta_\lambda \xi^\gamma_\alpha - \delta^\beta_\alpha \eta^0_\lambda \xi^\gamma_0\right)\left(\kappa^\lambda_\gamma - \delta^\lambda_\gamma \kappa^0_0\right) \quad , \tag{3.25}$$

which are q^2+2q linearly independent 1-forms on P_τ. Furthermore, the restriction of the forms (3.25), Ξ^a_0 excepted, to P^0_τ define the normal connection on P^0_τ, whence they provide q^2+q independent 1-forms on P^0_τ. But, it is easy to see that $\Xi^a_0/P^0_\tau = \eta^a_b \xi^0_0 dx^b$, and if we add them we obtain in all q^2+2q independent 1-forms on the manifold P^0_τ, which constitute a global field of transverse coframes of the foliation F^0_τ. Clearly, these coframes depend only on the transverse projective structure of F, and on the auxiliary Riemann metric g of M. Moreover, if F is a projective foliation these coframes do not depend on g, and they are foliate with respect to F^0_τ. By going over to the corresponding dual frames, we see that we have obtained

Theorem 3.2. Let F be a foliation of codimension $q \geq 2$ on M, and g be an auxiliary Riemannian metric. Then, for every transverse projective structure of F, there is a uniquely defined global transverse parallelism (the "normal parallelism") of the foliation F^0_τ on P^0_τ. If the given projective structure is foliate, this parallelism is independent of g, and is foliate as well.

Therefore, for projective foliations we have a situation which is similar to the one encountered in the case of the Riemannian foliations [10], and one might try to use the methods of [10] in the study of the projective foliations.

Remark. Cartan's original method [3] could be used similarly in order to write down the normal connection of a conformal foliation. Namely, if F is a conformal foliation, and if $\{h_\alpha\}$ is a set of foliate metrics of Q/U_α (where $\{U_\alpha\}$ is a flat open covering of M), which defines the conformal structure [16], then one has some relations $h_\alpha = \phi_{\alpha\beta} h_\beta$, where $\phi_{\alpha\beta}$ are positive foliate real functions on $U_\alpha \cap U_\beta$, and define a 1-cocycle of the covering $\{U_\alpha\}$. This provides us with a foliate line bundle θ on M having $\phi_{\alpha\beta}$ as its transition cocycle, and one can see that the normal Cartan connection [3] can be obtained on $\theta \oplus (\theta^{\frac{1}{2}} \otimes Q) \oplus \theta^{-1}$.

REFERENCES

1. A. Avez, Characteristic Classes and Weyl Tensor: Applications to General Relativity, Proc. Nat. Acad. Sci., U.S.A., 66 (1970), 265-268.
2. R. Bott, Lectures on Characteristic Classes and Foliations, Lect. Notes in Math., 279, Springer-Verlag, Berlin, 1972, 1-94.
3. E. Cartan, Les espaces à connexion conforme, Ann. Soc. Polon. Math., 2 (1923), 171-221 (Oeuvres Complètes, III 1, Gauthier-Villars, Paris, 1955, 747-798.)
4. E. Cartan, Sur les variétés à connexion projective, Bull. Soc. Math. France, 52 (1924), 205-241. (Oeuvres Complètes, III 1, Gauthier-Villars, Paris, 1955, 825-862.)
5. S.S. Chern, Geometry of Characteristic Classes, Proc. 13th Biannual Sem. Canad. Math. Congress, Halifax 1971, 1-40.
6. L.P. Eisenhart, Non-Riemannian Geometry, American Math. Soc. Colloquium Publ. VIII, New Printing: American Math. Soc. Providence, R.I., 1972.
7. S. Kobayashi, Transformation Groups in Differential Geometry, Springer-Verlag, Berlin, New-York, 1972.

8. S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, I. II, John Wiley and Sons, New York, 1963, 1969.

9. P. Molino, Propriétés cohomologiques et propriétés topologiques des feuilletages à connexion transverse projetable, Topology 12 (1973), 317-325.

10. P. Molino, Géométrie globale des feuilletages riemanniens, Proc. Koninklijke Nederlandse Akad., Series A, 85 (1982), 45-76.

11. A. Montesinos, Conformal Curvature for the Normal Bundle of a Conformal Foliation, Ann. Inst. Fourier, Grenoble 32 (1982), 261-274.

12. S. Nishikawa and H. Sato. On Characteristic Classes of Riemannian, Conformal, and Projective Foliations, J. Math. Soc. Japan 28 (1976), 223-241.

13. S. Nishikawa, Residues and Characteristic Classes for Projective Foliations, Japanese J. of Math. 7 (1981), 45-108.

14. I. Vaisman, Variétés Riemanniennes Feuilletées, Czechosl. Math. J. 21 (1971), 46-75.

15. I. Vaisman, Cohomology and Differential Forms, M. Dekker, Inc. New York, 1973.

16. I. Vaisman, Conformal Foliations, Kodai Math. J. 2 (1979), 26-37.

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