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ESTIMATES FOR SOME SQUARE FUNCTIONS OF LITTLEWOOD-PALEY TYPE

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1. Introduction

One of the classical results of the Littlewood-Paley theory states that the L^p-norm of a function is equivalent to the L^p-norm of the quadratic mean of its partial sums corresponding to the dyadic intervals, which form a decomposition of $\mathbb{R}^n$. The precise statement can be seen in [10], Ch. IV. More recently, other types of partitions have been considered and, in particular, that obtained from a fixed interval and its translates, i.e., in the one-dimensional case:

$$\Delta f(x) = \left(\sum_{-\infty}^{\infty} \left| \int_k^{k+1} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \right|^2 \right)^{1/2}$$

For the operator Δ, one does not obtain equivalence of norms (except for L^2), but only the inequality $\|\Delta f\|_p \leq C_p \|f\|_p$ which, moreover, is only valid in the range $2 \leq p < \infty$. A sketch of proof for this result appears in [1], where it is shown to be a basic ingredient of A. Córdoba's approach to the estimates for spherical summation multipliers. A more detailed proof is given in [2].

This paper grew out of conversations on this subject with A. Córdoba, to whom I am indebted for sharing his knowledge with me. The purpose is two-fold.

First, we give in section 2 another proof of the result just mentioned and some slight generalisations. This new a simpler approach is based on the unexpected pointwise inequality: $Gf(x)^2 \leq \text{Const.}\ M(|f|^2)(x)$ (where G is a smooth version of Δ), from which, weighted L^p inequalities for the operator Δ are also obtained almost immediately.

Secondly, we explore some of the analogues of Δ in order to get a deeper understanding of what is really going on with these quadratic operators. In particular, some continuous analogues of Δ considered in section 4 lead to a striking result on pointwise convergence of averages of ball multipliers. Further generalisation is gained in section 5, where we deal with Δf and its smooth version Gf in locally compact Abelian groups, a context which requires yet another different proof of the L^p-inequalities.

The notation used is fairly standard. We denote by $M = M_1$ the Hardy-Littlewood maximal operator in $\mathbb{R}^n$ and, more generally, we define

$$M_q f(x) = M(|f|^q)(x)^{1/q} = \sup_{x \in Q}(|Q|^{-1} \int_Q |f(y)|^q dy)^{1/q}$$

where $1 \leq q \leq \infty$ and Q stands for an arbitrary cube in $\mathbb{R}^n$. The classes of weights considered are: A_p (the usual Muckenhoupt's class) and A_p^*, which consists of all $w(x) > 0$ such that

$$\sup_I (|I|^{-1} \int_I w)(|I|^{-1} \int_I w^{1-p'})^{p-1} < \infty$$

where I is an arbitrary bounded n-dimensional interval, and the usual modification is considered for $p = 1$. Weights in A_p^* correspond to products of operators which are bounded with respect to $w \in A_p$, such as the strong maximal function or the double Hilbert transform (see [5]).

2. Quadratic Operators of Discrete Type

Given $m \in L^\infty(\mathbb{R}^n)$, we denote by T_m the multiplier operator: $(T_m f)\hat{} = \hat{f}.m$, which is well defined in $L^2(\mathbb{R}^n)$. When T_m can be extended to a bounded operator in L^p, we shall denote again this extension by T_m. In particular, if m is a Schwartz function, $m\in\mathcal{S}(\mathbb{R}^n)$, we know that T_m can be defined in L^p for all $1 \leq p \leq \infty$. The quadratic operator to be considered here is

$$(1) \qquad Gf(x) = \{\sum_{k\in Z^n} |T_{m(k+.)}\, f(x)|^2\}^{1/2}$$

where $m(k+.)$ means the translation of our multiplier: $m(k+.)(\xi) = m(k+\xi)$. Finally, if Q stands for the unit cube in $\mathbb{R}^n$, $Q = [-\frac{1}{2}, \frac{1}{2}]^n$, we define the functions:

$$q_j(x) = |2^jQ|^{-1}\, \chi_{2^jQ}\,(x) = 2^{-jn}\, q_o(2^{-j}x)$$

Theorem A: *If* $m\in\mathcal{S}(\mathbb{R}^n)$, *then the pointwise majorisation*

$$(2) \quad Gf(x) \leq \sum_{j=0}^{\infty} c_j\{q_j * |f|^2(x)\}^{1/2} \leq C\, M_2f(x)$$

holds for every $f\in L^1+L^\infty$, *where the constants* c_j *depend only on* m *and* $\sum_{j=0}^{\infty} c_j = C < \infty$. *As a consequence,* G *is a bounded operator in* $L^p(\mathbb{R}^n)$, $2 \leq p \leq \infty$, *and more precisely:*

$$(3) \quad \int Gf(x)^p\, w(x)\, dx \leq C_p(w) \int |f(x)|^p\, w(x)\, dx$$

for all $w\in A_{p/2}$, $2 \leq p < \infty$.

Proof: Let $g \in \mathcal{S}(\mathbb{R}^n)$ be the inverse Fourier transform of m. For each finite sequence $\lambda = (\lambda_k)$ of unit ℓ^2-norm, we define

$$G_\lambda f(x) = \sum_k \lambda_k T_{m(k+.)} f(x) =$$

$$= \sum_k \lambda_k \int e^{-2\pi i k.y} g(y) f(x-y)\, dy =$$

$$= \int g(y)\, h_\lambda(y)\, f(x-y)\, dy$$

where $h_\lambda(y)$ is periodic: $h_\lambda(y+k) = h_\lambda(y)$ $(k \in \mathbb{Z}^n)$ and has unit norm in $L^2(Q)$. Now, we define:

$$c_o = \sup \{|g(x)| : x \in Q\}$$

$$c_j = 2^{jn} \sup \{|g(x)| : x \in (2^jQ) \setminus (2^{j-1}Q)\} \qquad (j=1,2,\ldots)$$

so that $\sum c_j < \infty$ (because $g \in \mathcal{S}$), and

$$|G_\lambda f(x)| \le \sum_{j=0}^{\infty} c_j\, 2^{-jn} \int_{2^jQ} |h_\lambda(y)\, f(x-y)|\, dy$$

$$\le \sum_{j=0}^{\infty} c_j \left(2^{-jn} \int_{2^jQ} |f(x-y)|^2\, dy\right)^{1/2}$$

Since $G f(x) = \sup_{\lambda} |G_\lambda f(x)|$, the first inequality in (2) is proved, and the second one follows because $q_j * f \le Mf$ for every function f.

Since M_2 is a bounded operator in $L^\infty(\mathbb{R}^n)$ and in $L^p(w)$ when $p > 2$ and $w \in A_{p/2}$, only the case $p = 2$ of (3) remains to be proved. But this is a consequence of the first estimate in (2) together with the observation that

$$\int (|Q|^{-1} \chi_Q) * f(x)\ w(x)\ dx \le C \int |f(x)|\ w(x)\ dx$$

for every cube Q and every weight $w \in A_1$. ∎

The previous theorem is a smooth version of the Littlewood-Paley type inequalities we actually wish to obtain, which are concerned with the partial sum operators S_I, where I is an arbitrary n-dimensional interval and $(S_I f)^\wedge = \hat{f}\, \chi_I$. Now, there is a standard truncation argument to obtain the strong result from its smooth version, which is based on the inequality

$$(4) \quad \| (\sum_j |S_{I_j} f_j|^2)^{1/2} \|_p \le C_p \| (\sum_j |f_j|^2)^{1/2} \|_p \qquad (1 < p < \infty)$$

which holds for arbitrary intervals I_j and functions $f \in L^p(\mathbb{R}^n)$ (see Stein [10]).

The intervals $\{I_j\}$ are said to be almost congruent (with constant $C \geq 1$) if

$$\sup_j \ell_i(I_j) \leq C \inf_j \ell_i(I_j) \qquad (i = 1,2, \ldots, n)$$

where $\ell_i(\cdot)$ stands for the side length of an interval along the x_i-direction.

Theorem B: If $2 \leq p < \infty$ and $w \in A^*_{p/2}$, then, for every sequence $\{I_j\}$ of disjoint almost congruent intervals in $\mathbb{R}^n$, the inequality

$$(5) \quad \left\| \left(\sum_j |S_{I_j} f|^2\right)^{1/2} \right\|_{L^p(w)} \leq C_p(w) \, \|f\|_{L^p(w)}$$

holds for all $f \in L^p(w)$.

Proof: Suppose first that all I_j are bounded. Since everything is invariant under dilations in each coordinate, we can assume that

$$1 \leq \ell_i(I_j) \leq C \qquad (1 \leq i \leq n)$$

for every interval I_j in our sequence. Then, each interval contains at least one lattice point: $k_j \in I_j \cap \mathbb{Z}^n$.

Take a Schwartz function m such that $m(\xi) = 1$ when $\xi \in I = [-C, C]^n$, so that

$$S_{I_j} f = S_{I_j} (T_{m(-k_j+\cdot)} f) \qquad (f \in L^p)$$

and an application of inequality (4) together with Theorem A yields:

$$\| (\sum_j |S_{I_j} f|^2)^{1/2} \|_p \leq C_p \| G f \|_p$$

$$\leq C_p' \| f \|_p \qquad (2 \leq p < \infty)$$

This proves the case $w(x) \equiv 1$. For a general $w \in A^*_{p/2}$ the argument is exactly the same, but we must use the weighted version of (4), namely, that such inequality holds not only in $L^p(\mathbb{R}^n)$, but also in $L^p(w)$ if $w \in A^*_p$ and $1 < p < \infty$. (This actually a rather straightforward consequence of a general theorem of Marcinkiewicz and Zygmund [6] together with the boundedness in $L^p(w)$ of the multiple Hilbert transform; see also Kurtz [5]).

Finally, it may be the case that, for some $i = 1,2, \ldots, n$, we have $\ell_i(I_j) = \infty$ for all I_j. The necessary modifications to deal with this case are rather trivial, are are left to the reader. ∎

Given a sequence of disjoint consecutive intervals in $\mathbb{R}$, if they all have the same length we have just proved that inequality (5) holds true, while in the case of lengths increasing at an exponential rate, the same inequality is obtained (and not only for $p \geq 2$, but for all $1 < p < \infty$) by classical Littlewood-Paley theory. It seems natural to expect the same kind of result when the lengths increase arbitrarily. A partial positive answer is contained in our next result which, for the sake of simplicity, will be stated in its one-dimensional version.

Theorem C: Let $\{a_j\}_{j \in \mathbb{Z}}$ be an odd sequence of real numbers (i.e. $a_{-j} = -a_j$), and assume that its positive part is convex and slowly increasing, i.e.

$$a_{j-1} \leq a_j \leq \frac{1}{2} (a_{j-1} + a_{j+1}) \qquad (j \geq 1)$$

$$a_{2j} \leq C\, a_j \qquad (j \geq 1)$$

Then, the quadratic expression

$$\Delta f(x) = \left(\sum_{-\infty}^{\infty} \left| \int_{a_{j-1}}^{a_j} \hat{f}(\xi)\, e^{2\pi i x \xi} d\xi \right|^2 \right)^{1/2}$$

defines a bounded operator in $L^p(\mathbb{R})$, $2 \leq p < \infty$.

Proof: It suffices to consider the operator Δ_+ corresponding to the sum $\sum_{j=1}^{\infty}$. Define the intervals

$$I_j = [a_{j-1}, a_j) \qquad (j \geq 1)$$

$$J_k = [a_{2^{k-1}}, a_{2^k}) = \bigcup_{2^{k-1}<j\leq 2^k} I_j \qquad (k \geq 1)$$

If $2^{k-1} < j \leq h \leq 2^k$, we have $|I_j| \leq |I_k|$ by the convexity of the sequence; but we also have

$$|I_h| \leq 2^{-k}(a_{2^{k+1}} - a_{2^k}) \leq 2^{-k}(C - 1)\, a_{2^k}$$

$$|I_j| \geq 2^{-(k-2)}(a_{2^{k-1}} - a_{2^{k-2}}) \geq 2^{1-k}\, a_{2^{k-1}}$$

Thus, $|I_j| \leq |I_h| \leq \frac{1}{2} C(C-1)\, |I_j|$, which shows that the intervals in each dyadic block $\{I_j\}_{2^{k-1}<j\leq 2^k}$ are almost congruent with a constant independent of k. Now, it is important to point out that in Theorem B, $C_p(w)$ depends only on p, w and on the congruence constant of the sequence of intervals. Thus, the operators

$$\Delta_k f = \{\sum_{2^{k-1}<j\leq 2^k} |S_{I_j} f(x)|^2\}^{1/2} \qquad (k \in N)$$

are uniformly bounded in $L^2(w)$ for every $w \in A_1$, and it is a well known fact (see [7]) that this implies the vector valued inequality

(6) $\quad \| (\sum_k |\Delta_k f_k|^2)^{1/2} \|_p \leq C_p \| (\sum_k |f_k|^2)^{1/2} \|_p$

$(2 \leq p < \infty)$

On the other hand, we have

$$|J_{k+1}| \geq 2^k |I_{2^k}| \geq 2 |J_k| \qquad (k \geq 1)$$

so that, by classical Littlewood-Paley theory (see [10])

(7) $\quad \| (\sum_k |S_{J_k} f|^2)^{1/2} \|_p \leq C_p \| f \|_p \qquad (1 < p < \infty)$

and we only have to combine (6) and (7) taking into account that

$$|\Delta_+ f(x)|^2 = |S_{I_1} f(x)|^2 + \sum_{k \geq 1} |\Delta_k (S_{J_k} f)(x)|^2 \qquad \blacksquare$$

3: Further Results for the Discrete Case

a) Together with the operator G in Theorem A, one can consider its dilations:

$$G_\delta f(x) = G(f^\delta)(\tfrac{x}{\delta}) = (\sum_{k \in \mathbb{Z}^n} |T_{m(k+\delta \cdot)} f(x)|^2)^{1/2}$$

where $f^{\delta}(x) = f(\delta x)$ and $m(k+\delta\cdot)(\xi) = m(k+\delta\xi)$. Since M is dilation invariant, one still has the pointwise majorization for the maximal operator

$$G^*f(x) = \sup_{\delta>0} G_\delta f(x) \leq C\, M_2\, f(x)$$

from which, pointwise convergence results (when $\delta \to 0$) can be derived. For instance, we have

Corollary 1: If $m \in \mathcal{S}(\mathbb{R}^n)$, *then*

$$\lim_{\delta\to 0} \sum_k \left| T_{m(k+\delta\cdot)} f(x) - m(k)\, f(x) \right|^2 = 0$$

for every $f \in L^p(\mathbb{R}^n)$, $2 \leq p < \infty$.

(see Section 4 below for details)

b) The method of Theorem A can be applied more generally to an operator of the form

$$(8) \quad G\,f(x) = \Big(\sum_j \Big| \int_{\mathbb{R}^n} r_j(y)\, K(y)\, f(x-y)\, dy \Big|^2\Big)^{1/2}$$

where $K(y)$ is such that its least decreasing radial majorant is bounded and integrable, and $\{r_j\}$ is an orthonormal system in $L^2(I)$ for some bounded interval $I \subset \mathbb{R}^n$, each r_j being extended by periodicity (or a system of homogeneous functions of degree zero which are orthonormal on the unit sphere). Moreover, in this case, we still have the majorization $G^*f(x) \leq C\, M_2\, f(x)$ and the corresponding result of pointwise convergence.

In the particular case $I = [0,1]^n$, $\{r_j\}$ = Rademacher functions of n variables, we have a stronger condition than orthogonality:

$$\| \sum_j \lambda_j r_j \|_{L^q(I)} \sim (\sum_j |\lambda_j|^2)^{1/2} \qquad (q < \infty)$$

which permits pushing down the exponent 2 in our G-function:

Corollary 2: *If* Gf *is defined by* (8) *and* $\{r_j\}$ *are the n-dimensional Rademacher functions, then, for every* $\varepsilon > 0$, *we have the pointwise estimate*

$$G\, f(x) \leq C_\varepsilon\, M_{1+\varepsilon}\, f(x) \qquad (f \in L^1 + L^\infty)$$

c) In the unweighted case (i.e. $w(x) \equiv 1$) of Theorem B, we also obtain by the usual argument the dual inequalities:

$$\| f\|_q \leq C_q \, \| (\sum_j |S_{I_j} f|^2)^{1/2}\|_q \qquad (1 < q \leq 2)$$

provided that the intervals I_j cover almost all $\mathbb{R}^n$. The same can be said in Theorem C.

On the other hand, the following weighted version of Theorem C can be obtained: "The operator Δ is bounded in $L^p(w)$ if $2 \leq p < \infty$ and $w \in A_{p/2}$". This is proved by using the known weighted L^2-inequalities for Littlewood-Paley operators (see [5]) together with the extrapolation theorem of [4] and [8].

d) The fact that Theorems A, B and C only contain results for L^p if $p \geq 2$, deserves a word of explanation. Consider the simplest case: One-dimensional intervals of the same length and $w(x) \equiv 1$. The inequality

$$\| (\sum_k |S_{[k,k+1]} f|^2)^{1/2} \|_p \leq C \| f \|_p \qquad (f \in L^p(\mathbb{R}))$$

cannot hold if $p < 2$ because, if we define $f_k \in L^p(\mathbb{R})$ by $\hat{f}_k = \chi_{[k,k+1]}$ and apply this inequality to

$$f(x) = \sum_{k=0}^{N-1} f_k(x) = N f_o (Nx)$$

we are led to: $\| N^{1/2} f_o \|_p \leq C N^{1-1/p} \| f_o \|_p$, which, when $N \to \infty$, forces $2 \leq p$.

Given a multiplier $m \in L^\infty(\mathbb{R}^n)$, the continuous analogue of the square function considered in section 2 is

$$(9) \quad G\, f(x) = \{\int_{\mathbb{R}^n} |T_{m(u+\cdot)}\, f(x)|^2\, du\}^{1/2}$$

The L^p estimates are even simpler in this case:

Theorem D: Suppose that $m \in L^2 \cap L^\infty(\mathbb{R}^n)$, *and let* $K(x) = |\hat{m}(-x)|^2$. *Then:*

i) $G\, f(x) \le \{K * |f|^2\, (x)\}^{1/2} \quad (f \in L^2(\mathbb{R}^n))$

and G *extends to a bounded operator from* $L^p(\mathbb{R}^n)$ *to itself,* $2 \le p \le \infty$.

ii) If we assume moreover that

$|\hat{m}(x)| \le C(1+|x|)^{-n/2-\varepsilon}$ *for some* $\varepsilon > 0$, *then*

$G\, f(x) \le C\, M_2\, f(x) \qquad (f \in L^p;\ 2 \le p \le \infty)$

and, for every function $f \in L^p(\mathbb{R}^n)$, $2 \le p < \infty$, *we have*

$$\lim_{\delta\to 0} \int_{\mathbb{R}^n} |T_{m(u+\delta\cdot)}\, f(x) - m(u)\, f(x)|^2\, du = 0 \quad \underline{a.e.}$$

Proof: A priori, we only know that G is well defined in L^2. Let $g \in L^2$ be the inverse Fourier transform of m, so that $G\, f(x) = \lim_{N\to\infty} G_N\, f(x)$, where

$$G_N f(x) = \{ \int_{|\xi|\le N} | \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot y} g(y) f(x-y) dy |^2 d\xi \}^{1/2}$$

When $f \in L^2$, the inner integral is absolutely convergent, and, if we fix $x \in \mathbb{R}^n$ and $N > 0$, there exists a function $\lambda(\xi)$ (depending on both x and N) of unit norm in $L^2(\mathbb{R}^n)$ such that

$$G_N f(x) = \int_{|\xi|\le N} \int_{\mathbb{R}^n} \lambda(\xi) e^{-2\pi i\xi\cdot y} g(y) f(x-y) dy d\xi$$

$$= \int h(y) g(y) f(x-y) dy$$

where $h \in L^2$ and $\| h \|_2 = 1$. Since $K = |g|^2 \in L^1$, we apply Scharz inequality to obtain

$$G_N f(x) \le \{ \int K(y) |f(x-y)|^2 dy \}^{1/2}$$

and part (i) is proved by letting $N \to \infty$.

Now, under the hypothesis of (ii), $|K(x)|$ is dominated by a decreasing, radial, integrable function, and thus:

$$(K * |f|^2)^{1/2} \le C M(|f|^2)^{1/2} = C M_2 f$$

In particular, the maximal operator:

$$\sup_{\delta\to 0} \{ \int_{\mathbb{R}^n} |T_{m(u+\delta\cdot)} f - m(u) f|^2 du\}^{1/2}$$

is bounded in $L^p(\mathbb{R}^n)$, $2 < p \leq \infty$ and of weak type (2,2). By a standard technique, the pointwise convergence result will be proved for every $f \in L^p$, $2 \leq p < \infty$, if we establish this result for every Schwartz function. But, if $f \in \mathcal{S}(\mathbb{R}^n)$:

$$\{\int |T_{m(u+\delta\cdot)} f(x) - m(u) f)(x)|^2 du\}^{1/2} \leq$$

$$\leq \{\int \{\int |m(u+\delta\xi) - m(u)| \; |\hat{f}(\xi)| \; d\xi\}^2 du\}^{1/2} \leq$$

$$\leq \int |\hat{f}(\xi)| \{ \int |m(u+\delta\xi) - m(u)|^2 du\}^{1/2} d\xi$$

Since $m \in L^2$, the inner integral in the last expression is bounded independently of δ, ξ, and it vanishes when $\delta \to 0$, so that an application of Lebergue's dominated convergence theorem ends the proof. ∎

The following particular case of Theorem D is worth mentioning: Let $m = \chi_B$, where B is the unit ball in $\mathbb{R}^n$. Then

$$|\hat{m}(x)| = |x|^{-n/2} |J_{n/2} (2\pi|x|)| \leq C(1+|x|)^{-n/2-1/2}$$

and both parts of the theorem can be applied. If we write S_E for the partial sum operator corresponding to the multiplier χ_E, then the first part shows that it makes sense to define the operator:

$$f \to \{\int_{\mathbb{R}^n} |S_{u+B}\, f|^2\, du\}^{1/2}$$

for every $f \in L^p(\mathbb{R}^n)$, $2 \leq p \leq \infty$, even though, by Fefferman's theorem ([3]), each one of the operators S_{u+B} can be defined only in L^2. Moreover, the second part gives:

Corollary 3: *If* $f \in L^p(\mathbb{R}^n)$, $2 \leq p < \infty$, *then*

$$S^*f(x) = \sup_{0<r<\infty} \{\int_{\mathbb{R}^n} |S_{r(u+B)}\, f(x)|^2\, du\}^{1/2} \leq C\, M_2\, f(x)$$

and

$$\lim_{r\to\infty} \int_{|u|<1} |S_{r(u+B)}\, f(x) - f(x)|^2\, du = 0 \qquad \textit{a.e.}$$

Given an open ball V in $\mathbb{R}^n$ containing the origin, it is not known whether

$$(10) \quad \lim_{r\to\infty} S_{rV}\, f(x) = f(x) \qquad \text{a.e.}$$

is true for every $f \in L^2$. What Corollary 3 shows is that a certain average of the statements (10) for all balls containing the origin is true, but this is only a poor substitute which is far away from (10) itself.

5: The Case of Locally Compact Groups

An essential part of the results in sections 2 and 4 can be formulated in the context of locally compact Abelian (ℓ.c.a.) groups, providing some sort of unified version of the discrete and continuous cases. Since there is no Hardy-Littlewood operator in this general context, we only obtain the L^p inequalities, but not the pointwise estimates, and a different, slightly longer approach must be followed.

Let X be a ℓ.c.a. group with dual group $\hat{X}$. A general element of X (resp. $\hat{X}$) will be denoted by x, y, etc. (resp. ξ, α, etc.), and ds, $d\xi$ will be the Haar measures on X and $\hat{X}$, which we shall assume to be suitably normalised with respect to each other. The letters H and Γ will be used to represent closed subgroups of X and $\hat{X}$, their Haar measures being m_H and m_Γ, respectively.

The operator to be considered here is

$$G\,f(x) = \{ \int_\Gamma |(\alpha g) * f(x)|^2 \, dm_\Gamma(\alpha)\}^{1/2}$$

$$= \{\int_\Gamma | \int <y, \alpha> g(y)\, f(x-y)\, dy|^2 \, dm_\Gamma(\alpha)\}^{1/2}$$

When $X = \hat{X} = \mathbb{R}^n$, we can take $\Gamma = \mathbb{Z}^n$, obtaining the operator defined in (1), or $\Gamma = \hat{X} = \mathbb{R}^n$, in which case we get the operator G of (9) (with $\hat{g} = m$ in both cases).

Theorem E: *Let* $g \in L^1(X)$ *be a positive function such that*

$$(11) \quad \{ \int_\Gamma |\hat{g}(\alpha)|^2 \, dm_\Gamma(\alpha)\}^{1/2} = M < \infty$$

Then, the operator G *defined above is bounded in* $L^p(X)$ *when* $2 \leq p \leq \infty$, *and more precisely:*

$$\| Gf \|_p \leq M \| f \|_p \qquad (f \in L^p;\ 2 \leq p \leq \infty)$$

Proof: We define the closed subgroup of X

$$H = \Gamma^\perp = \{x \in X : <x,\alpha> = 1 \text{ for all } \alpha\in\Gamma\}$$

so that $\Gamma = (X/H)\hat{}$ The Haar measure $\tilde{m}$ in X/H is defined as the dual of m_Γ. Then, a unique Haar measure m_H can be fixed in H so that, defining

$$\tilde{f}(\bar{x}) = \int_H f(x+y)\, dm_H(y) \qquad\qquad (f \in L^1(X))$$

(where $\bar{x} = (x + H) \in X/H$), the following identity always holds

$$(12) \qquad \int_X f(x)\, dx = \int_{X/H} \tilde{f}(\bar{x})\, d\tilde{m}(\bar{x})$$

A series of simple lemmas will be needed in the proof:

Lemma 1: If $g \in L^1(X)$ is positive, then, for every $\xi \in \hat{X}$

$$\int_\Gamma |\hat{g}(\alpha-\xi)|^2\, dm_\Gamma(\alpha) \le \int_\Gamma |\hat{g}(\alpha)|^2\, dm_\Gamma(\alpha) = M^2$$

Proof: Since $(\tilde{f})^\wedge(\alpha) = \hat{f}(\alpha)$ for every $\alpha \in \Gamma$, Plancherel's theorem implies

$$\int_\Gamma |\hat{g}(\alpha-\xi)|^2\, dm_\Gamma(\alpha) = \int_{X/H} |(\xi g)^\sim(\bar{x})|^2\, d\tilde{m}(\bar{x})$$

$$\int_\Gamma |\hat{g}(\alpha)|^2\, dm_\Gamma(\alpha) = \int_{X/H} g(\bar{x})^2\, dm(\bar{x})$$

But $|\tilde{f}(\bar{x})| \le |f|^\sim(\bar{x})$ for every f, and in our particular case, we have $|(\xi g)^\sim(\bar{x})| \le \tilde{g}(\bar{x})$, which concludes the proof of the lemma.

In our next result, we shall denote by B a Banach space (with dual B^*) and by $L^1_B(X)$ the Bochner-Lebesgue space consisting of all (strongly) measurable B-valued functions $F(x)$ defined on X and such that $\|F(x)\|_B$ is integrable.

Lemma 2: Let $K(x)$ *be a measurable* B^**-valued function such that*

$$(13) \quad \int |K(x)\cdot b|\, dx \leq C\|b\|_B \qquad\qquad b \in B$$

Then, the formula

$$(14) \quad T\,F(x) = \int K(x-y)\cdot F(y)\, dy$$

defines a bounded operator from $L^1_B(X)$ *to* $L^1(X)$ *with norm* $\leq C$.

Proof: It suffices to verify that $\|TF\|_1 \leq C\|F\|_{L^1_B}$ for all simple integrable functions F, since these are dense in L^1_B. For each such function F, (13) implies that $TF(x)$ is well defined, and

$$\int |TF(x)|\, dx \leq \int\int |K(x-y)\cdot F(y)|\, dy\, dx =$$

$$= \int \{ \int |K(x)\cdot F(y)|\, dx\}\, dy \leq C \int \|F(y)\|_B\, dy.$$

In the next lemma, we take as our Banach space $B = B^* = L^2(\Gamma_o) = L^2(\Gamma_o; m_\Gamma)$, where Γ_o is a fixed compact subset of Γ. Functions $F \in L^2_B(X)$ are then isometrically identified with functions $f(x, \alpha) = F(x)(\alpha)$ in the product space $L^2(X \times \Gamma_o)$.

Lemma 3: Let $U : L^2 \to L^2_B$ be defined by $Uf(x, \alpha) = (\alpha g) * f(x)$. Then U is a bounded operator with norm $\leq M$, and its adjoint $T = U^* : L^2_B \to L^2$ is given by (14) where the kernel is: $K(x) = k(x, \alpha) = g(-x)\, \overline{\langle x, \alpha \rangle}$.

Proof: The first assertion follows from Plancherel's theorem and Lemma 1:

$$\int_X \|U f(x)\|_B^2\, dx = \int_{\Gamma_o} \int_{\hat{X}} |\hat{g}(\xi-\alpha)\ \hat{f}(\xi)|^2\, d\xi\, dm_\Gamma(\alpha)$$

$$\leq M^2 \int_{\hat{X}} |\hat{f}(\xi)|^2\, d\xi = M^2 \int_X |f(x)|^2\, dx$$

The kernel of U^* is obtained by a simple computation which is left to the reader (the fact that we take a compact subset $\Gamma_o \subset \Gamma$ makes all the integrals absolutely convergent).

We are now in a position to complete the proof of Theorem E. Let $G_o f$ denote the operator defined as Gf after replacing Γ by Γ_o. If it is proved that $\| G_o f \|_p \leq M \| f \|_p$ $(2 \leq p \leq \infty)$ for an arbitrary compact subset Γ_o of Γ, then, letting Γ_o increase to Γ, we get the desired result for Gf. But $|G_o f(x)| = \| U f(x) \|_B$, so that the inequalities to be obtained are

$$(15) \quad \| U f \|_{L^p_B} \leq M \| f \|_p \qquad (2 \leq p \leq \infty)$$

For $p = 2$, this was proved in Lemma 3. For $p = \infty$, (15) is equivalent, by duality, to

$$\| T F \|_{L^1} \leq M \| F \|_{L^1_B}$$

and this can be proved by verifying that the kernel $K(x)$ defined in Lemma 3 satisfies (14) with constant $C = M$. In fact, given $b = b(\alpha) \in B = L^2(\Gamma_o)$:

$$|K(-x) \cdot b| = \left| \int_{\Gamma_o} g(x) <x, \alpha> b(\alpha) \, dm_\Gamma (\alpha) \right|$$

$$= |h(\bar{x})| \; g(x)$$

with $h \in L^2(X/H)$ and $\| h \|_2 = \| b \|_B$, so that we can use the identity (12) to obtain

$$\int_X |K(x)\cdot b|\,dx = \int_{X/H} |h(\bar{x})|\,\tilde{g}(\bar{x})\,d\tilde{m}(\bar{x}) \le$$

$$\le \|h\|_2\,\|\tilde{g}\|_2 = \|b\|_B\,M$$

Finally, the case $2 < p < \infty$ of (15) follows by interpolation, and the proof is completed. ■

When $\Gamma = X$, (11) simply means that $g \in L^2(X)$. Moreover, in this case, the assumption $g \ge 0$ is not necessary, and we have a statement completely analogous to Theorem D(i). On the other hand, if Γ is discrete and $\hat{g}$ has compact support, then (11) is trivially verified. The truncation argument used in Theorem B can also be applied to the smooth G-function considered here: Given $m \in L^\infty(\hat{X})$, we say that it is an L^p-multiplier if the operator T_m defined (and bounded) in $L^2(X)$ by $(T_m f)\hat{} = \hat{f}\,m$ admits a bounded extension to $L^p(X)$.

__Theorem F__: __Let__ $m \in L^\infty(\hat{X})$ __be a compactly supported__ L^p-__multiplier, with__ $2 \le p < \infty$. __If__ Γ __is a discrete subgroup of__ X, __then, the operator__

$$\Delta f(x) = \{ \sum_{\alpha\in\Gamma} |T_{m(\alpha+\cdot)}\, f(x)|^2 \}^{1/2}$$

is bounded in $L^p(X)$.

Proof: First of all, we need the analogue of (4):

$$(16) \quad \left\| \left(\sum_{\alpha\in\Gamma} |T_{m(\alpha+\cdot)} f_\alpha|^2 \right)^{1/2} \right\|_p \leq$$

$$\leq C \left\| \left(\sum_{\alpha\in\Gamma} |f_\alpha|^2 \right)^{1/2} \right\|_p$$

which is a consequence of the theorem of Marcinkiewicz and Zygmund [6] together with the identity

$$T_{m(\alpha+\cdot)} f(x) = \langle \overline{x, \alpha} \rangle T_m(\alpha f)(x)$$

Then, we take a function $q(\xi)$ of positive type in $\hat{X}$ such that q has compact support and $q(\xi) = 1$ for all $\xi \in \mathrm{supp}(m)$. Since $q(\xi) = \hat{g}(\xi)$ for some positive $g \in L^1(X)$, we can form with this function g the above G-function, which can also be written now as

$$G f(x) = \left(\sum_{\alpha\in\Gamma} |T_{q(\alpha+\cdot)} f(x)|^2 \right)^{1/2}$$

Since G is a bounded operator in $L^p(X)$, we only have to apply (16) with $f_\alpha = T_{q(\alpha+\cdot)} f$. ∎

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