ADDENDUM TO:

A NOTE ON PRIMITIVE GROUPS WITH SMALL MAXIMAL SUBGROUPS

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(1) J.Saxl has provided us with a proof of the conjecture made in the last paragraph of our paper:

REMARK. If E is a non-abelian finite simple group which is not an alternating group, and if F is a maximal subgroup of E with least $n = |E:F|$, then $E < H = A_n$ (the alternating group on the set $E:F$ of cosets of F in E) is such that the pair E,H satisfies the hypothesis of our Proposition.

(The case where $E = A_n$, $n \geq 5$, has been dealt with in our paper.)

P r o o f. (Since Saxl's proof involves an application of the O'Nan-Scott Theorem*, we shall present a more elementary approach.) $\mathrm{Core}_{A_n}(E) = 1$ is immediate from simplicity of $A_n > E$. Let $1 \neq K \leq A_n$ be normalised by E. Aiming at a contradiction we assume that $E \nleq K$; in addition K may be choosen as a counterexample of least order. As E is simple, $E \cap K = 1$, and so $G = EK$ splits. From our choice of K we get that K is minimal normal in G. Since E is already primitive on the set $E:F$, so is G.

First assume that K is non-abelian. Let $K = S_1 \times \ldots \times S_m$ be the decomposition of K into simple components. If $m > 1$, then E permutes $\{S_1, \ldots, S_m\}$ transitively, and so from our choice of n we infer that $m \geq n$. Now

$$|S_1|^n \mid |S_1|^m = |S_1 \times \ldots \times S_m| \mid |G| \mid |A_n| = \tfrac{1}{2}(n!),$$

which is impossible: the 2-part of $n!$ does not exceed 2^{n-1}. (The same argument works with any odd prime as well, so we do not need to invoke the Feit-Thompson-Theorem.) Hence $m = 1$, in which case K is simple and $G \leq \mathrm{Aut}(K)$ (to within isomorphism). Appealing to the Schreier conjecture, we obtain that $E \cong G/K$, a subgroup of $\mathrm{Out}(K)$, is soluble, which contradicts the hypothesis.

Therefore K is abelian. In this case primitivity of G on $E:F$ together with minimality of K as a normal subgroup of G yields that $|K| = |E:F| = n$. But $E \leq G$ acts non-trivially on $K \setminus \{1\}$ (via conjugation), a contradiction against the choice of n. ◻

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*) See [1], Appendix, for a correct version.

Thus every non-abelian finite simple group occurs as component of the base group of one of the wreath products discussed in our paper. A corresponding conclusion may be deduced from the first part of the Theorem stated below (avoiding the necessity to rely on the Schreier conjecture).

(2) L.G.Kovács has observed that the construction dealt with in our Proposition can be generalised, and then yields the most general example:

THEOREM. *Let* H *be a finite group with subgroups* X *and* Y *such that* $X \trianglerighteq Y$, X/Y *is non-abelian simple,* $\mathrm{Core}_H(X) = 1$ *and* $X \not\leq K$ *whenever* $K > Y$ *is a subgroup of* H *normalised by* X.

Put $N = N_H(X) \cap N_H(Y)$ *and define a group*

$$G = E \wr_N H, \text{ where } E = X/Y,$$

to be the twisted wreath product with respect to the action of N *on* E *given by viewing* N/Y *a subgroup of* $\mathrm{Aut}(E)$ *in the obvious way.*

Then G *is a primitive group; the base group* E^* *is non-abelian, minimal normal in* G, *coincides with the socle* $S(G)$ *of* G, *and is complemented in* G *by a corefree maximal subgroup (namely, by the distinguished copy of* H *contained in* G).

Conversely, if G *is a primitive group with non-abelian minimal normal socle* $S(G)$ *complemented by a maximal subgroup* H, *then*

$$G \cong E \wr_N H,$$

where E *is the simple component of* $S(G) = E_1 \times \ldots \times E_n$ *and* $N = N_H(E_1)$ *has subgroups* $Y = C_H(E_1)$ *and* $X = S(N \bmod Y)$ *with the properties listed above.*

P r o o f. The first part of this result can be obtained by modifying the proof of our Proposition appropriately. (Observe that the hypothesis ensures that $Y = C_N(X/Y)$.)

As for the converse, define N,Y,X as in the last part of the Theorem and note that N/Y is necessarily a maximal subgroup of the primitive group NE_1/Y: indeed, if $N < N_0 < NE_1$, then (to within isomorphism) $H < (E \cap N_0) \wr_N H < G = E \wr_N H$; cf. [6]. Then apply [2],1.1, in conjunction with the Schreier conjecture: since $\mathrm{Out}(E)$ is soluble, a group as in the last part of the Theorem cannot have a simple socle (see [1],6.3). The remainder of the proof is left to the reader (cf. [4]). □

For a slightly different formulation of the above Theorem (and a different proof, relying on the methods developed in the paper by Gross,Kovács [3]) the reader is referred to the forthcoming paper of Kovács [4]; see also the papers of Kovács,Praeger,Saxl [5], Aschbacher,Scott [1] (where the example $A_5 \wr_{A_5} A_6 \cong A_5 \wr A_6$ — the latter indicating the non-twisted wreath product with respect to the natural permutation representation of A_6 — is already mentioned) and of Gross,Kovács [3] for related and more general results.

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