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POLYNOMIAL RINGS OVER NON-COMMUTATIVE RINGS*

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Let R be an associative, not necessarily commutative ring with identity. It is shown that the polynomial ring $R[x_\alpha]$, where the x_α's constitute an arbitrary set of central indeterminates, has a semisimple (resp. right Artinian, Quasi-Frobenius) classical right quotient ring if and only if, R has the same property.

In [10], Lance Small showed that if an associative ring R with identity element is a right order in a semisimple ring $Q_{c\ell}(R)$, then so is the polynomial ring $R[x_1,x_2,...x_n]$ where the x's are central indeterminates. In [11] he showed that a similar conclusion holds if R is a Noetherian order in a right Artinian ring. Shock [9] extended these results to the case where the indeterminates form a countably infinite set. In continuing these investigations we prove in this paper that if I is any index set, a ring R is a right order in a P-ring if and only if $S = R[x_\alpha]_{\alpha\in I}$ is a right order in a P-ring where P is any one of P_1: semisimple, P_2: right Artinian, P_3: Quasi-Frobenius.

For the case $P = P_1$, a connection between the associated matrix of rings of $Q_{c\ell}(S)$ and $Q_{c\ell}(R)$ is established. (Theorem 2.5).

Sufficiency for the cases $P = P_2$ and $P = P_3$ exploit the fact that any polynomial ring over R has among its ring of quotients, a group ring RG where G is a free abelian group, the rank of which can be chosen to be the cardinality of I. In establishing necessity for

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P_3, it is first observed that if $Q_{max}(S)$ is self-injective, so is $Q_{max}(R)$.

The set of all polynomials which are not zero-divisors has not yet been successfully described in terms of the coefficient ring R, even when R is commutative or I finite. However, in each of the cases P_1, P_2, P_3 we exhibit a well described right Ore semigroup M(P) of S such that

$$Q_{c\ell}(S) = \{fg^{-1} | f \varepsilon S, \ g \varepsilon M(P)\}$$

§1. Preliminaries: Throughout this paper, a ring R is associative with identity element. Objects of mod-R will be denoted by M_R, and will be unitary. If S is a subset of R, the right and left annihilators of S in R will be denoted by $r_R(S)$ and $\ell_R(S)$ respectively. An element a of R is regular if and only if $r_R(a) = \ell_R(a) = 0$. The set of all regular elements of R/A, where A is a two sided ideal of R, will be denoted by $C_R(A)$. In particular, $C_R(0)$ is the set of all regular elements of R.

Defn. 1.1: Let $\{x_\alpha\}_{\alpha \varepsilon I}$ be any set indexed by a set I of arbitrary cardinality, and let < be a (fixed) well ordering on I. Let G be the (multiplicatively written) free abelian group on $\{x_\alpha\}$, and

$$RG = \{f: G \to R | f \text{ has finite support}\},$$

the associated group ring. Let

$$M = \{x_{\alpha_1}^{n_1} x_{\alpha_2}^{n_2} \ldots x_{\alpha_k}^{n_k} \ \varepsilon \ G | n_i \geq 0 \text{ for each } i\}$$

Then M is a submonoid of G and the subring $RM = \{f \varepsilon RG | f(g) = 0$ for every $g \notin M\}$ is called the polynomial ring in $\{x_\alpha\}_{\alpha \varepsilon I}$ over R. Henceforth, we shall denote RM by S or $R[x_\alpha]$.

If $\phi: R \to RG$ and $\psi: G \to RG$ are defined by

$$\phi(a)(g) = a \text{ if } g = 1, \text{ and } 0 \text{ otherwise}$$

$$\psi(g)(g') = 1 \text{ if } g' = g, \text{ and } 0 \text{ otherwise}$$

then ϕ (resp. ψ) is a ring (resp. monoid) monomorphism. For $a \in R$ and $g \in G$, we write ag to mean $\phi(a)\psi(g) \in RG$. Then each $f \in R[x_\alpha]$ has a decomposition

$$f = \sum_{i=1}^{n} f_i m_i$$

where $f_i \in R$, $m_i \in M$ for each i, and $f = 0$ if and only if $f_i = 0$ for every i. The set of <u>coefficients</u> of f will be denoted by $\{f_i\}_{i=1}^{n}$.

Our definition of a polynomial ring is easily seen to be consistent with the "usual" definition for a polynomial ring. In fact for any $n \geq 2$

$$R[x_1, x_2, \ldots, x_n] = R[x_1, x_2, \ldots, x_{n-1}][x_n]$$

canonically as rings.

If $f \in R[x_\alpha]$, the <u>leading coefficient</u> $L(f) \in R$ of f is defined as follows. If $f = 0$, let $L(f) = 0$. Suppose $f \neq 0$. If $|I| = 1$, $L(f)$ is defined as usual. In general, $\exists$ a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of I such that $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ and $f \in R[x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}]$. Since $R[x_{\alpha_1}, x_{\alpha_2} \ldots, x_{\alpha_n}] \cong R[x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_{n-1}}][x_{\alpha_n}]$ we may regard f as a polynomial in x_{α_n} and proceed by induction to obtain $0 \neq L(f) \in R$.

The following lemma follows from the definitions.

<u>Lemma 1.2</u>: For any f, g in $R[x_\alpha]$,

$$L(f)L(g) \neq 0 \Rightarrow L(fg) = L(f)L(g) \Rightarrow L(fg) \neq 0$$

Hence if $fg = 0$, then $L(f)L(g) = 0$.

Next we examine some connections between the lattice of right ideals of a polynomial ring and that of its coefficient ring. Let $S = R[x_\alpha]$. By $L_r(R)$ (resp. $L_r(S)$ we shall mean the lattice of right ideals of R (resp.S). If $K \in L_r(R)$, let

$$K[x_\alpha] = \{f \in S \mid f_i \in K \text{ for all } i\}$$

Then $K[x_\alpha] = KS$ is the right ideal in S generated by K.

<u>Defn. 1.3</u>: Define $\omega : L_r(R) \to L_r(S)$ and

$$\Omega : L_r(S) \to L_r(R)$$

by $\omega(K) = K[x_\alpha]$ and

$\Omega(J) = J \cap R$

The next lemma requires a routine verification.

<u>Lemma 1.4</u>: (i) For all $K \in L_r(R)$, $(\Omega\omega)(K) = K$, so Ω is surjective.

(ii) ω is 1-1, preserves inclusions, arbitrary and direct sums, arbitrary intersections and finite products.

(iii) If A is a subset of R and

$$A' = \{f \in S \mid f_i \in A \text{ for every } i\}$$

then $\omega(r_R(A)) = r_S(A')$ so takes right annihilators into right annihilators. Moreover, if A is an ideal,

$$\omega(\ell_R(A)) = \ell_S(\omega(A))$$

(iv) Ω preserves inclusions and arbitrary intersections.

A ring is right Goldie finite dimensional (abbreviated right f.d.) if any direct sum of right ideals of R has at most finitely many non-zero summands. A right Goldie ring is a right f.d. ring which satisfies the maximum condition on annihilator right ideals.

Lemma 1.5: If R is right f.d., so is $S = R[x_\alpha]_{\alpha \in I}$. If S is right Goldie, so is R.

Proof: The first part is due to Shock (Theorem 2.6 of [9]) where the statement is proved for countable I. The same proof adapts readily to the general case.

Let S be right Goldie. If $\{K_\beta\}_{\beta \varepsilon B}$ is an independent family of right ideals in R with

$$K = \bigoplus_{\beta \varepsilon B} K_\beta$$

then by 1.4 (ii)

$$\omega(K) = \bigoplus_{\beta \varepsilon B} \omega(K_\beta)$$

so $\omega(K_\beta) = 0$ (hence $K_\beta = 0$) for all but finitely many β, proving R is right f.d.

Let $K_1 \subseteq K_2 \subseteq \dots \subseteq K_n \subseteq \dots$ (*)

be any ascending sequence of right annihilators in R. By 1.4,

$$\omega(K_1) \subseteq \omega(K_2) \subseteq \dots \subseteq \omega(K_n) \subseteq \dots \qquad (**)$$

is an ascending chain annihilator right ideals of S so that (**), hence (*), becomes eventually constant.

The prime radical rad R is the meet of all the prime ideals of R. R is semiprime if and only if rad R = 0. The right singular ideal of R is the ideal

$$Z_r(R) = \{a \varepsilon R \mid r_R(a) \text{ is an essential right ideal of } R\}.$$

Lemma 1.6: Let $S = R[x_\alpha]$. Then

(i) rad S = ω(rad R)

(ii) $Z_r(S) = \omega(Z_r(R))$

Proof: (i) is due to Amitsur [1] and (ii) to Shock [9].

Lemma 1.7: Let A be an ideal of R. Then

$$R[x_\alpha]/A[x_\alpha] = S/\omega(A) \cong R/A[x_\alpha].$$

Proof: Let π: $R \to R/A$ be the natural ring epimorphism and define

$$\bar{\pi} : S \to R/A[x_\alpha] \text{ by}$$

$$\bar{\pi}(f) = \sum_{i=1}^{n} \pi(f_i)m_i \quad \text{whenever } f = \sum_{i=1}^{n} f_i m_i \in S.$$

Then $\bar{\pi}$ is a ring epimorphism with kernel $A[x_\alpha] = \omega(A)$.

§2. ORDERS IN SEMISIMPLE RINGS: The maximal ring of right quotients of R is denoted by $Q_{max}(R)$. Asano [2] has shown that if N is a multiplicatively closed subset of $C_R(0)$ and R satisfies the right Ore condition with respect to N i.e. (for all $a\varepsilon R$, $n\varepsilon N$, $\exists\ a_1\varepsilon R$, $n_1\varepsilon N$ such that $an_1 = na_1$) then there is a ring R_N containing R, unique up to an isomorphism that extends the identity map of R satisfying

(i) if $n\varepsilon N$, n is a unit in R_N

(ii) if $x\varepsilon R_N$, $a\varepsilon R$, $n\varepsilon N$ such that $x = an^{-1}$.

Such an N is called a right Ore semigroup and R_N is called a partial ring of quotients of R with respect to N. R_N is a ring of quotients of R in the sense of Utumi [13] wo we may suppose $R_N \subseteq Q_{max}(R)$. If $N = C_R(0)$ we say R has a classical right quotient ring and write $R_N = Q_{c\ell}(R)$ in this case. If $Q_{c\ell}(R)$ exists, we say that R is a right order in $Q_{c\ell}(R)$.

It is well known that the following are equivalent (see for example 3):

(a) $Q_{c\ell}(R)$ exists and is semisimple

(b) R is semiprime right Goldie

(c) R is semiprime, right finite dimensional and $Z_r(R) = 0$.

Then $Q_{c\ell}(R) = Q_{max}(R)$.

The next theorem is a direct consequence of 1.5 and 1.6 and the equivalence (a) $\Leftrightarrow$ (c) above.

Theorem 2.1: Let $S = R[x_\alpha]$. Then R is a right order in a semisimple ring if and only if S is such an order.

The Artin-Wedderburn Theorems now guarantee that under the hypotheses of 2.1, $Q_{c\ell}(R)$ and $Q_{c\ell}(S)$ are finite direct sums of matrix rings over division rings. We proceed to investigate the connections between these matrix rings.

Lemma 2.2: Let D be a division ring. Then

$$Q_{max}(D[x_\alpha]) = Q_{c\ell}(D[x_\alpha]) \text{ is a division ring.}$$

Proof: By Lemma 1.2, $S = D[x_\alpha]$ is a domain, and is right Ore with respect to $C_S(0)$, by 2.1. The assertion now follows.

Lemma 2.3: (i) If R and T are rings with $R \cong T$, then $R[x_\alpha] \cong T[x_\alpha]$.

(ii) If $\{R_i\}$ $1 \leq i \leq n$, is a family of rings

(a) $\left(\bigoplus_{i=1}^{n} R_i\right)[x_\alpha] \cong \bigoplus_{i=1}^{n} (R_i[x_\alpha])$ and

(b) $Q_{max}(\bigoplus_{i=1}^{n} R_i) = \bigoplus_{i=1}^{n} Q_{max}(R_i)$

(iii) If R_n is the nxn matrix ring over R,

(a) $R_n[x_\alpha] \cong (R[x_\alpha])_n$

(b) $Q_{max}(R_n) = (Q_{max}(R))_n$

Proof: The isomorphisms in (i), (ii) and (iii) are all natural. (ii)(b) and (iii)(b) are due to Utumi (Theorems 2.1 and 2.3 of [13]).

Lemma 2.4: Let N be a right Ore semigroup of R. Then N is a right Ore semigroup of S and

$$S_N = R_N[x_\alpha]$$

Proof: Since $C_R(0) \subseteq C_S(0)$, N is a multiplicatively closed subset of S. Each $n \in N$ is a unit in $R_N \subseteq R_N[x_\alpha]$.

If $\tilde{f} \in R_N[x_\alpha]$, then there exists $f \in S$ and $n \in N$ such that $\tilde{f} = fn^{-1}$. To see this let $\tilde{f} = \sum_{i=1}^{n} \tilde{f}_i m_i$ where $\tilde{f}_i \in R_N$ for each i. For $1 \leq i \leq n$, $\exists$ $f_i \in R$, $n_i \in N$ such that $\tilde{f}_i = f_i n_i^{-1}$. By the common multiple property (see for example, Lemma 1.4 of [10]), $\exists$ $n \in N$ such that $n_i^{-1} n \in R$ for every i. So $\tilde{f}n = f \in S$ and $\tilde{f} = fn^{-1}$ where $f \in S$ and $n \in N$. It readily follows that N is a right Ore semigroup of S and $S_N = R_N[x_\alpha]$.

Theorem 2.5: Let R be a semiprime right Goldie ring. If

$$Q_{c\ell}(R) = \bigoplus_{i=1}^{k} D_{n_i}^{(i)}$$

where $D^{(i)}$ is a division ring for all i, then

$$Q_{c\ell}(S) \approx \bigoplus_{i=1}^{k} (Q_{c\ell}(D^{(i)}[x_\alpha]))_{n_i} .$$

<u>Proof</u>: Let $T^{(i)} = Q_{c\ell}(D^{(i)}[x_\alpha]) = Q_{max}(D^{(i)}[x_\alpha])$, $1 \leq i \leq k$. By 2.2, $T^{(i)}$ is a division ring, for each i.

Let $N = C_R(0)$. Then

$$R_N = Q_{c\ell}(R) = \bigoplus_{i=1}^{k} D^{(i)}_{n_i} \text{ exists so by 2.4}$$

$$S_N = R_N[x_\alpha] = (\bigoplus_{i=1}^{k} D^{(i)}_{n_i})[x_\alpha].$$

By 2.3, $S_N = \bigoplus_{i=1}^{k} (D^{(i)}[x_\alpha])_{n_i}$.

Again by 2.3

$$Q_{max}(S_N) = \bigoplus_{i=1}^{k} (Q_{max}(D^{(i)}[x_\alpha]))_{n_i} = \bigoplus_{i=1}^{k} T^{(i)}_{n_i}$$

Since S_N is a ring of quotients of S,

$$Q_{c\ell}(S) = Q_{max}(S) = Q_{max}(S_N) \approx \bigoplus_{i=1}^{k} T^{(i)}_{n_i}$$

as asserted.

Next we determine whether

$$Q_{c\ell}(S) = \{ fg^{-1} | f \in S,\ g \in C_S(0)\}$$

can be described completely in terms of R. Evidently this can be done if all the regular elements of $S = R[x_\alpha]$ can be found, given R, and we have not been able to do so in general. However, $Q_{c\ell}(S)$ can still be obtained by localizing at a known set of regular elements.

Let $M = \{g \in S \mid L(g) \in C_R(0)\}$.

By 1.2, M is a multiplicatively closed subset of $C_S(0)$.

<u>Lemma 2.6</u>: If R is a semiprime right Goldie ring and $g \in C_S(0)$, then $\exists\ g' \in S$ such that $gg' \in M$.

<u>Proof</u>: For any right ideal B' of S, let

$$B = \{L(f) \mid f \in B'\}$$

Then B is a right ideal of R. If B' is an essential right ideal of S, then B is an essential right ideal of R. To see this, let $0 \neq A \in L_r(R)$. Then $0 \neq \omega(A) \in L_r(S)$ so $\omega(A) \cap B' \neq 0$. If $0 \neq f \in \omega(A) \cap B'$ we have $0 \neq L(f) \in A \cap B$.

Now let $B' = gS$ where $g \in C_S(0)$. Since S is right f.d., gS is an essential right ideal of S so that

$$B = \{L(gg') \mid g' \in S\}$$

is an essential right ideal of R. Since R is semiprime right Goldie, B contains a regular element. Hence $\exists\ g' \in S$ such that $gg' \in M$.

<u>Theorem 2.7</u>: Let R be a semiprime right Goldie ring. Then

$$Q_{c\ell}(S) = \{fg^{-1} \mid f \in S,\ g \in M\} = S_M.$$

<u>Proof</u>: Let $q \in Q_{c\ell}(S)$. Then $\exists\ g_1 \in C_S(0)$, $f_1 \in S$ such that $qg_1 = f_1 \in S$. By 2.6, $\exists\ g'_1 \in S$ such that $g_1 g'_1 \in M$. Let $g = g_1 g'_1$, $f = f_1 g'_1$. Then $q = fg^{-1}$ where $f \in S$, $g \in M$.

§3. <u>ORDERS IN ARTINIAN RINGS</u>: In this section, we prove the analog of Theorem 2.1 for orfers in Artinian rings (Theorem 3.6). Again we show that $Q_{c\ell}(S)$ can be obtained by localizing at a well described set of regular elements of S.

The rather cumbersome definition of a polynomial ring in §1 will be put to use in the next lemma, where it is observed that a polynomial ring has, among its rings of quotients, a group ring.

<u>Lemma 3.1</u>: If G and M are as in Defn. 1.1, then M is a right Ore semigroup of $S = R[x_\alpha]$. Moreover, $S_M = RG$, where S_M is the partial ring of quotients of S with respect to M.

<u>Proof</u>: For any $f \in M$, $L(f) = 1 \in C_R(0)$ so $f \in C_S(0)$ and $M \subseteq C_S(0)$. Clearly M is multiplicatively closed. Each $m \in M$ is a unit in G, hence in RG. Next we show that

$$\tilde{f} \in RG \Rightarrow \exists \; f \in S, \; m \in M \text{ such that } \bar{f} = fm^{-1}.$$

For each $\alpha \in I$, define $\tilde{\alpha}: G \to Z$ (the integers) by $\tilde{\alpha}(g) = n \Leftrightarrow g = x_\alpha^n x_{\alpha_1}^{n_1}, \ldots, x_{\alpha_k}^{n_k}$. Let $\bar{f} \in RG$ be any element.

Then $\exists \; f_i \in R, \; g_i \in G, \; 1 \leq i \leq n$ such that

$$\tilde{f} = \sum_{i=1}^{n} f_i g_i$$

For each $\alpha \in I$ let

$$n_\alpha = \max\{|\alpha(g_i)| \mid 1 \leq i \leq n\}.$$

Then $n_\alpha \geq 0$ for every α and $n_\alpha = 0$ for all but finitely many α.

Let

$$m = \prod_{\alpha \in I} x_\alpha^{n_\alpha}.$$

Then $m \in M$ and $g_i m \in M$ for every i, $1 \leq i \leq n$. Hence

$$\tilde{f}m = \sum_{i=1}^{n} f_i g_i m = f \in S$$

and $\tilde{f} = fm^{-1}$, as required. It now follows that M is a right Ore semigroup of S and $RG = S_M$.

The next Lemma is due to Small (Lemma 1.7 of [9]).

Lemma 3.2: If N is a right Ore semigroup of a ring T and T_N is a right order in a right Artinian ring, then T is a right order in a right Artinian ring. In fact $Q_{c\ell}(T) = Q_{c\ell}(T_N)$.

A Quasi-Forbenius (QF) ring is a right Artinian right self-injective ring. For other characterizations of QF-rings, see [4].

Defn. 3.3: For the sake of brevity, say a ring T has P_2 (resp. has P_3) if T is a right order in a right Artinian ring (resp. QF-ring).

Lemma 3.4: Let G be a group which has a collection of subgroups $\{G_\beta | \beta$ is an ordinal$\}$ such that for all β

(a) G_β is a normal subgroup of $G_{\beta+1}$

(b) $G_{\beta+1}/G_\beta$ is either finite or cyclic, of which at most finitely many are finite, and

(c) $G_\eta = G$ for some ordinal η.

Then RG has P_2 (resp. P_3) if R has P_2 (resp. P_3).

Proof: For P_2 see Hughes [7] and for P_3, see Horn [6].

Theorem 3.5: If R has P_2 (resp. P_3), then $S = R[x_\alpha]$ has P_2 (resp. P_3).

Proof: Evidently, any free abelian group satisfies the conditions of Lemma 3.4. Hence if G is as in definition 1.1, then by 3.1, 3.2 and 3.4,

R has P_2 (resp. P_3) $\Rightarrow$ RG has P_2 (resp. P_3)

$\Rightarrow$ S_M has P_2 (resp. P_3)

$\Rightarrow$ S has P_2 (resp. P_3)

An ideal K of a ring T is said to be locally nilpotent if for every finite subset F of K, there is an integer k = k(F) such that every product involving k elements from F, vanishes. The Levitzki radical of T, L(T), is the sum of all the locally nilpotent ideals of T, and is itself a locally nilpotent ideal.

For the moment, let S be any ring, and let rad S = N(S). Small [11] has shown that S has P_2 if, and only if,

(i) L(S) is nilpotent

(ii) For each integer $k \geq 0$, $S_k = S/_{J_k}$ is a right Goldie ring, where

$$J_k = \ell_S(L(S)^k) \cap L(S) \text{ and } L(S)^0 = S.$$

(iii) a is regular in S whenever a+N(S) is regular in $S/_{N(S)}$.

Theorem 3.6: A ring R is a right order in the right Artinian ring if, and only if, $S = R[x_\alpha]_{\alpha\in I}$ is a right order in a right Artinian ring, for any index set I.

Proof: Sufficienty has been shown in 3.5. Suppose then that S has P_2. We show that R satisfies conditions (i), (ii) and (iii) above.

(i) L(R) is nilpotent: We first observe that

$$\omega(L(R)) = L(R)[x_\alpha]$$

is a locally nilpotent ideal of S. For let

$$F = \{f^{(1)}, f^{(2)}, \ldots, f^{(n)}\} \subseteq \omega(L(R))$$

For each i, $1 \leq i \leq n$, let

$$f^{(i)} = \sum_{j=1}^{n(i)} f_j^{(i)} m_{ij}$$

Then $F' = \{f_j^{(i)}\}_{i,j}$ is a finite subset of L(R), which is locally nilpotent. Hence there is an integer k such that any product of k elements from F' vanishes. So any product of k elements from F vanishes, proving that $\omega(L(R)) \subseteq L(S)$. Let n be the index of nilpotency of L(S). Then by 1.4,

$$0 = (\omega(L(R)))^n = \omega((L(R)^n) \text{ so } (L(R))^n = 0$$

(ii) $J_0 = \ell_S(L(S)^0) \cap L(S) = 0$ so $S_0 = S$ is a right Goldie ring. Since S satisfies the maximum condition on annihilator right ideals, we have by a result of Herstein and Small [5] that L(S) contails all nil ideals. In particular $L(S) \supseteq N(S)$. Since L(S) is nilpotent L(S) = N(S). Since S is right Goldie, so is R by 1.5. By the same reasoning, L(R) = N(R).

Let $I_K = \ell_R(N(R)^k) \cap N(R)$, $k \geq 0$.

Since $\omega(N(R)) = \omega(\text{rad } R) = \text{rad } S = N(S)$, by 1.6, we obtain by repeated use of 1.4 that

$$\omega(I_k) = J_k \text{ for every } k \geq 0$$

Let $R_k = R/I_k$. Then by 1.7,

$$R_k[x_\alpha] \cong S/J_k = S_k$$

Hence by 1.6, R_k is right Goldie for each $k \geq 0$-

(iii) Let a+N(R) be regular in $R/N(R)$. Then a+N(R) is regular $R/N(R)[x_\alpha]$. The map

$$\phi : S \rightarrow R/N(R)[x_\alpha]$$

defined by $\phi(\sum_{i=1}^{n} f_i m_i) = \sum_{i=1}^{n} (f_i+N(R))m_i$

induces an isomorphism between $S/_{N(S)}$ and $R/_{N(R)}[x_\alpha]$ in which a+N(R) corresponds to a+N(S). Hence a+N(S) is regular in $S/_{N(S)}$. Since S satisfies condition (iii), a is regular in S. In particular $a \in C_R(0)$.

This proves the theorem.

<u>Theorem 3.7</u>: Let R be a right order in a right Artinian ring, and let $S = R[x_\alpha]$. If

$$M = \bigcup_{h \in S} \{h+N(S) \mid L(h) \in C_R(0)\}$$

then M is a right Ore semigroup of S and $Q_{c\ell}(S) = S_M$.

<u>Proof</u>: It is well known that if a ring T is a right order in a right Artinian (indeed semiprimary) ring, then $T/_{N(T)}$ is semiprime Goldie and $a \in C_T(0)$ if and only if a+N(T) is regular in $T/_{N(T)}$ (See for example, Prop. 3.1, p. 286 of [12]).

(i) <u>$M \subseteq C_S(0)$:</u> Let $f \in M$. Then $\exists$ $h \in C_S(0)$ such that f+N(S) = h+N(S). Since S is a right order in a right Artinian ring, $f \in C_S(0)$.

Clearly, M is multiplicatively closed.

(ii) <u>$f \in C_S(0) \Rightarrow f' \in S$ such that $ff' \in M$</u>: Let $\tilde{\pi}: S \to R/_{N(R)}[x_\alpha]$ be as in Lemma 1.7 and let $\bar{\pi}(g) = \bar{g}$ for each $g \in S$.

Let $f \in C_S(0)$. Then f+N(S) is regular in $S/_{N(S)}$. In the induced isomorphism $S/_{N(S)} \cong R/_{N(R)}[x_\alpha]$ the image $\bar{f}$ of f+n(S) is regular in $R/_{n(R)}[x_\alpha]$. By 2.6 and the fact that $R/_{N(R)}$ is a semiprime right Goldie ring, $\exists$ $f' \in S$ such that $L(\bar{f}\bar{f}')$ is regular in $R/_{N(R)}$.

Let $\bar{f}\bar{f}' = \overline{ff'} = \sum_{i=1}^{n} (a_i+N(R))m_i$

where $a_n + N(R) = L(\overline{ff'})$.

Let $h = \sum_{i=1}^{n} a_i m_i \ \varepsilon\, S$.

Since $a_n \neq 0$, $L(h) = a_n$. But $a_n \ \varepsilon\ C_R(0)$ since $a_n+N(R)$ is regular in $R/_{N(R)}$. By defn. of M, $h+N(S) \subseteq M$. But $\bar{h} = \overline{ff'}$ so

$ff' \ \varepsilon\ h + N(R)[x_\alpha] = h + N(S)$, proving $ff' \varepsilon M$ for some $f' \varepsilon S$.

We proceed as in 2.7 to complete the proof.

<u>Corollary 3.8</u>: If R is a right order in a right Artinian ring and $f \varepsilon C_S(0)$, then $\exists$ $f' \varepsilon S$ such that some coefficient of ff' is regular in R.

<u>Remark</u>: Theorem 3.5 cannot be extended to orders in Noetherian rings, even when R is commutative. The following example was kindly provided by Prof. Vasconcelos:

Let k be a field and let k[[x,y,z]] be the power series ring in the indeterminates x,y,and z. Let $R = k[[x,y,z]]/_{(x^2,xy,xz)}$

Then R is a commutative Noetherian ring so that its classical quotient ring is Noetherian (Theorem 1.9 of [2]). Let $Q = Q_{c\ell}(R[T_1,T_2,....])$. If f goes to $\bar{f}$ in the natural map

$$k[[x,y,z]] \to k[[x,y,z]]/_{(x^2,xy,xz)},$$

the ideal in Q generated by

$$\{\bar{x}, \bar{y}T_1 - \bar{x}T_2, \bar{y}T_2 - \bar{z}T_3, ...\}$$

is not finitely generated, so Q is not Noetherian.

§4. ORDERS IN QUASI-FROBENIUS RINGS: Let M_R be an object of mod-R. If N_R is a submodule of M_R, we write $N_R \lhd M_R$ (resp. $N_R \blacktriangleleft M_R$) whenever M_R is an essential (resp. rational) extension of N_R.

Thus

$$N_R \lhd M_R \Leftrightarrow \text{whenever } 0 \neq m \in M, \ \exists \ r \in R \text{ such that } 0 \neq mr \in N \text{ and}$$

$$N_R \blacktriangleleft M_R \Leftrightarrow \text{whenever } 0 \neq m_1 \in M, \ m_2 \in M, \ \exists \ r \in R \text{ such that}$$

$$0 \neq m_1 r, \ m_2 r \in N.$$

The injective hull of M_R is designated by $E(M_R)$. If R is a subring of a ring T, then T is a ring of right quotients of R provided $R_R \blacktriangleleft T_R$. The maximal ring of right quotients $Q_{max}(R)$ of R is the biendomorphism ring of $E(R_R)$.

A characterization of $Q_{max}(R)$ we shall often use is

$$Q_{max}(R) = \{x \in E(R_R) \mid x^{-1}R \blacktriangleleft R_R\}$$

where $x^{-1}R$ is the right ideal

$$\{r \in R \mid xr \in R\}.$$

$Q_{max}(R)$ is a ring of right quotients of R containing every ring T of right quotients of R satisfying

$$R \subseteq T \subseteq E(R_R).$$

For details, refer to either [3] or [8].

In this section we show that a ring R is a right order in a QF-ring if and only if $S = R[x_\alpha]_{\alpha \in I}$ is such an order, for any index set I. (Theorem 4.5). To this end, we observe that if $Q_{max}(R[x_\alpha])$ is right self-injective, so is $Q_{max}(R)$ (Theorem 4.3).

Any object of mod-S is an object of mod-R via the inclusion map $R \to S$.

Lemma 4.1:

$$K_R \lhd R_R \Leftrightarrow (\omega(K))_R \lhd S_R \Leftrightarrow (\omega(K))_S \lhd S_S.$$

Proof: Suppose $K_R \lhd R_R$. If $0 \neq f \in S$, $g \in S$, then $\exists\ 0 \neq f_i \in R$ for some coefficient f_i of f. Let $g = \sum_{j=1}^{n} g_j m_j$. There is an $r \in R$ such that $0 \neq f_i r$ and $g_j r \in K$ for every j. Hence $0 \neq fr$ and $gr \in \omega(K)$, proving $(\omega(K))_R \lhd S_R$. Since $R \subset S$, it follows that $(\omega(K))_S \lhd S_S$. The implication $(\omega(K))_S \lhd S_S \Rightarrow K_R \lhd R_R$ is trivial.

Lemma 4.2: Let A be an object of mod-S. If A_S is injective, then so is A_R.

Proof: Let A_S be injective and let K be any right ideal of R. Let $\phi \in Hom_R(K_R, A_R)$. Define $\phi^*: \omega(K) \to A_S$ by

$$\phi^*\left(\sum_{i=1}^{n} f_i m_i\right) = \sum_{i=1}^{n} \phi(f_i) m_i.$$

Then $\phi^* \in Hom_S(\omega(K)_S, A_S)$.

Since A_S is injective, (by Baer's criterion for injective modules) $\exists$ an $a \in A$ such that $\phi^*(f) = af$ for every $f \in \omega(K)$. Hence for $k \in K$, $\phi(k) = \phi^*(k) = ak$, proving that A_R is injective.

Theorem 4.3: If $Q_{max}(S)$ is right self-injective, so is $Q_{max}(R)$.

Proof: Let $Q' = Q_{max}(S)$. Now Q' is right self-injective if, and only if, Q'_S is injective (Prop. 4.3.3 of [8]). By Lemma 4.2, Q'_R is injective. Since $R_R \subset S_R \subseteq Q'_R$, we may assume that $R_R \subseteq E(R_R) \subseteq Q'_R$. The assertion will follow if we can show that $Q_{max}(R) = E(R_R)$.

We first claim that:

"If $a_1, a_2, \ldots, a_n$ in $E(R_R)$ and $m_1, m_2, \ldots m_n$ in M satisfy $\sum_{i=1}^{n} a_i m_i = 0$, then $a_i = 0$ for every i." (*)

Let $\sum_{i=1}^{n} a_i m_i = 0$. Note that $E(R_R)$ is a subset of Q', which is an object of mod-S, so that $a_i m_i$ is defined for each $i, 1 \leq i \leq n$. Suppose that some $a_i \neq 0$. Since $E(R_R)$ is an essential extension of R_R, $\exists$ $r \in R$ such that $f_i = a_i r \in R$ for each i, not all $f_i = 0$. But then $f = \sum_{i=1}^{n} f_i m_i = (\sum_{i=1}^{n} a_i m_i) r = 0 \in S$ so that $f_i = 0$ for every i, a contradiction.

Let $a \in E(R_R) \subseteq Q_{max}(S)$ Then

$$\begin{aligned} f = \sum_{i=1}^{n} f_i m_i \in \omega(a^{-1}R) &\Rightarrow f_i \in a^{-1}R \text{ for every } i \\ &\Leftrightarrow af_i \in R \text{ for every } i \\ &\Leftrightarrow af = \sum_{i=1}^{n} (af_i) m_i \in S \\ &\Leftrightarrow af \in a^{-1}S. \end{aligned}$$

Further if $af = \sum_{i=1}^{n} (af_i) m_i \in S$ we conclude from (*) and the fact that $af_i \in E(R_R)$ for each i, that $af_i \in R$ for each i. Hence $a^{-1}S = \omega(a^{-1}R)$. Since $a \in Q_{max}(S)$, $a^{-1}S \blacktriangleleft S_S$. By Lemma 4.1, $a^{-1}R \blacktriangleleft R_R$ proving $a \in Q_{max}(R)$. Hence $Q_{max}(R) = E(R_R)$ and $Q_{max}(R)$ is right self-injective.

Lemma 4.4: If a ring T of right quotients of R is right self-injective, that $T = Q_{max}(R)$ so that $Q_{max}(R)$ is itself right self-injective.

Proof: Let $E(R_R)$ be the injective hull of R_R. We may assume

$$R \subseteq T \subseteq U = Q_{max}(R) \subseteq E(R_R).$$

Since T_T is injective, there exists a submodule K_T of U_T such that

$$U_T = T_T \oplus K_T.$$

Since $R \subseteq T$, K is an R-submodule of $E(R_R)$ and $K \cap R \subseteq K \cap T = 0$. But $R_R \triangleleft E(R_R)$. Hence $K = 0$ and $T = U = Q_{max}(R)$.

Theorem 4.5: A ring R is a right order in QF-ring $\leftrightarrow S = R[x_\alpha]$ is such an order.

Proof: Sufficienty has been proved in 3.5. Let S be a right order in a QF-ring. Then $Q_{c\ell}(S)$ is a right self-injective right Artinian ring. Since $Q_{c\ell}(S)$ is a ring of right quotients of S, it follows from 4.4 that $Q_{c\ell}(S) = Q_{max}(S)$ is right self-injective. By Theorem 4.3 and its proof we may assume that

$$R \subseteq Q_{max}(R) = E(R_R) \subset Q_{max}(S) = Q_{c\ell}(S).$$

By 3.6, $Q_{c\ell}(R)$ exists and is right Artinian. We need only show that $Q_{c\ell}(R) = Q_{max}(R)$ to complete the proof. Let then $a \in Q_{max}(R)$. Since $a \in Q_{c\ell}(S)$, $\exists\ g \in C_S(0)$ such that $ag = f \in S$. By Corollary 3.8, $\exists\ g' \in S$ such that some coefficient d of gg' is in $C_R(0)$. But $g \in a^{-1}S$ so that $gg' \in a^{-1}S = \omega(a^{-1}R)$. Hence $d \in a^{-1}R \cap C_R(0)$ and $ad = r \in R$. Since $d^{-1} \in Q_{c\ell}(R)$ exists, $a = rd^{-1} \in Q_{c\ell}(R)$, proving that $Q_{c\ell}(R) = Q_{max}(R)$ is a QF-ring.

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