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ON THE LOGARITHMIC CONVERGENCE EXPONENT AND GEOMETRIC MEANS OF AN INTEGRAL FUNCTION OF ORDER ZERO

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1. For a non constant integral function of order zero, the logarithmic order ρ^* and the lower logarithmic order λ^* are given as [1],

(1.1) $$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \log M(r,f)}{\log \log r} = \begin{matrix}\rho^* \\ \lambda^*\end{matrix}$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

The geometric means of $|f(z)|$ for $0 < K < \infty$, are defined as

(1.2) $$G(r) = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log | f(re^{i\theta})| \, d\theta \right\}$$

and

(1.3) $$g_K(r) = \exp \left\{ \frac{(K+1)}{r^{K+1}} \int_0^r x^K \log G(x) dx \right\}.$$

Another geometric mean of $|f(z)|$ is defined as [2],

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$$(1.4)\quad g^*_K(r) = \exp\left\{\frac{(K+1)}{(\log r)^{K+1}} \int_1^r (\log x)^K \log G(x)\frac{dx}{x}\right\}.$$

The logarithmic convergence exponent ρ^*_1 and lower logarithmic convergence exponent λ^*_1 are given as ([3], p.58)

$$(1.5)\qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \text{ing}\end{matrix} \frac{\log n(r)}{\log\log r} = \begin{matrix}\rho^*_1\\ \\ \lambda^*_1\end{matrix},$$

where $(0 \leqslant \lambda^*_1 \leqslant \rho^*_1 \leqslant \infty)$.

Jain, P.K. and Chungh, V.D. ([2],[3],[4]) have discussed some properties of these geometric means. In this paper we have also studied few properties of $g^*_K(r)$ which are given in the form of the theorems.

2. Theorem 1: - Let $f(z)$ be an integral function of order zero. Then, for $0 < r_1 < r_2$, we have

$$(2.1)\qquad \left\{(\log r_2)^{K+1} - (\log r_1)^{K+1}\right\} \log G(r_1) \leqslant$$

$$\leqslant (\log r_2)^{K+1} \log g^*_K(r_2) - (\log r_1)^{K+1} \log g^*_1(r_1) \leqslant$$

$$\left\{(\log r_2)^{K+1} - (\log r_1)^{K+1}\right\} \log G(r_2),$$

where K is any positive number.

Proof. From (1.4), we have

$$(2.2)\quad (\log r_1)^{K+1} \log g^*_K(r_1) = (K+1)\int_1^{r_1} \log G(x)(\log x)^K \frac{dx}{x}.$$

Similarly,

$$(2.3)\quad (\log r_2)^{K+1} \log g_K^*(r_2) = (K+1)\int_1^{r_2} \log G(x)(\log x)^K \frac{dx}{x}.$$

From (2.2) and (2.3) we get,

$$(2.4)\quad (\log r_2)^{K+1} \log g_K^*(r_2) - (\log r_1)^{K+1} \log g_k^*(r_1)$$

$$= (K+1)\int_{r_1}^{r_2} \log G(x)(\log x)^K \frac{dx}{x}.$$

From (2.4), (2.1) follows since $G(x)$ is an increasing function of x.

3. Theorem 2: - Let $f(z)$ be an integral function of order zero and logarithmic convergence exponent ρ_1^* and lower logarithmic convergence exponent λ_1^*, then,

$$(3.1)\quad \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \log \left(\left\{\frac{G(r)}{g_K^*(r)}\right\}^{1/(\log r)}\right)}{\log \log r} = \begin{matrix}\rho_1^* \\ \lambda_1^*\end{matrix} .$$

In order to prove the above theorem we first prove the following lemma.

Lemma 1. $$\left\{\frac{g_K^*(r)}{G(r)}\right\}^{1/\log r} = \exp\left\{\frac{-1}{(\log r)^{K+2}}\int_1^r (\log x)^{K+1}\, n(x)\frac{dx}{x}\right\}.$$

Proof of Lemma 1 : -

$$\frac{-1}{(\log r)^{K+1}}\int_1^r (\log x)^{K+1}\,\frac{d}{dx}\,(\log G(x))dx$$

$$= \frac{-1}{(\log r)^{K+1}}\left\{(\log r)^{K+1}\log G(r) - \int_1^r (K+1)(\log x)^K \log G(x)\frac{dx}{x}\right\}$$

$$= -\log G(r) + \frac{(K+1)}{(\log r)^{K+1}}\int_1^r (\log x)^K \log G(x)\,\frac{dx}{x}$$

$$= -\log G(r) + \log g_K^*(r)$$

$$= \log\left(\frac{g_K^*(r)}{G(r)}\right).$$

Hence,

$$(3.2)\quad \exp\left\{\frac{-1}{(\log r)^{K+2}}\int_1^r (\log x)^{K+1}\frac{d}{dx}(\log G(x))dx\right\} = \left\{\frac{g_K^*(r)}{G(r)}\right\}^{1/\log r}$$

From (1.2) and using Jensen's formula we get $\frac{d}{dx}(\log G(x)) = \frac{n(x)}{x}$.

Hence, form (3.2), we get

$$\left(\frac{g_K^*(r)}{G(r)}\right)^{1/\log r} = \exp\left\{\frac{-1}{(\log r)^{K+2}}\int_1^r (\log x)^{K+1}\,n(x)\,\frac{dx}{x}\right\}.$$

Proof of theorem 2 : - From Lemma 1, we have

$$\left\{\frac{G(r)}{g_K^*(r)}\right\}^{1/\log r} = \exp\left\{\frac{1}{(\log r)^{K+2}}\int_1^r (\log x)^{K+1}\,n(x)\,\frac{dx}{x}\right\}$$

$$\leqslant \exp\left\{\frac{1}{(\log r)^{K+2}}\,n(r)\int_1^r (\log x)^{K+1}\,\frac{dx}{x}\right\}$$

$$= \exp\left\{\frac{n(r)}{(K+2)}\right\}.$$

Hence, using (1.5), we get

$$(3.3)\qquad \lim_{r\to\infty}\frac{\sup}{\inf}\ \frac{\log\log\left\{\frac{G(r)}{g_K^*(r)}\right\}^{1/\log r}}{\log\log r} \leqslant \lim_{r\to\infty}\frac{\sup}{\inf}\ \frac{\log n(r)}{\log\log r} = \frac{\rho_1^*}{\lambda_1^*}$$

Further,

$$\left\{\frac{G(r^2)}{g_K^*(r^2)}\right\}^{1/\log(r^2)} = \exp\left\{\frac{1}{(\log r^2)^{K+2}}\int_1^{r^2}(\log x)^{K+1}\, n(x)\,\frac{dx}{x}\right\}$$

$$> \exp\left\{\frac{1}{(\log r^2)^{K+2}}\int_r^{r^2}(\log x)^{K+1}\, n(x)\,\frac{dx}{x}\right\}$$

$$\geqslant \exp\left\{\frac{1}{(\log r^2)^{K+2}}\, n(r)\int_r^{r^2}(\log x)^{K+1}\,\frac{dx}{x}\right\}$$

$$= \exp\left[\frac{n(r)}{(\log r^2)^{K+2}}\left\{\frac{(\log r^2)^{K+2} - (\log r)^{K+2}}{(K+2)}\right\}\right]$$

$$= \exp\left\{\frac{n(r)}{K+2}\left(1 - \left(\tfrac{1}{2}\right)^{K+2}\right)\right\}.$$

Hence,

$$(3.4)\qquad \lim_{r\to\infty}\frac{\sup}{\inf}\ \frac{\log\log\left\{\frac{G(r^2)}{g_K^*(r^2)}\right\}^{1/\log r^2}}{\log\log(r^2)} \geqslant$$

$$\geqslant \lim_{r\to\infty}\frac{\sup}{\inf}\ \frac{\log n(r)}{\log\log r} = \frac{\rho_1^*}{\lambda_1^*}.$$

From (3.3) and (3.4), (3.1) follows.

4. Theorem 3:- Let us set

$$(4.1)\qquad \lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\left\{\frac{G(r)}{g_K^*(r)}\right\}^{1/\log r}}{(\log r)^{\rho_1^*}}=\begin{matrix}p\\ q\end{matrix},$$

and

$$(4.2)\qquad \lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{n(r)}{(\log r)^{\rho_1^*}}=\begin{matrix}c\\ d\end{matrix}.$$

Then we have,

$$\text{(i)}\quad \frac{d}{(K+\rho_1^*+2)}\leqslant q\leqslant p\leqslant\frac{c}{(K+\rho_1^*+2)},$$

$$\text{(ii)}\quad \left(\frac{c}{d}\right)^{\frac{K+2}{\rho_1^*}}q\leqslant\frac{d}{(K+\rho_1^*+2)}+d\left\{\frac{\left(\frac{c}{d}\right)^{\frac{K+2}{\rho_1^*}}-1}{K+2}\right\},$$

$$\text{(iii)}\quad \left\{\frac{(K+2)(c-d)+c\,\rho_1^*}{c\rho_1^*}\right\}\geqslant\frac{c}{(K+\rho_1^*+2)}.$$

Proof. From Lemma 1, for $h > 0$, we have

$$\log\left\{\frac{G(r^{1+h})}{g_K^*(r^{1+h})}\right\}=\frac{1}{\{\log(r^{1+h})\}^{K+1}}\int_1^{r^{1+h}}(\log x)^{K+1}\,n(x)\frac{dx}{x}$$

$$=\frac{1}{\{\log(r^{1+h})\}^{K+1}}\left[\left\{\int_1^{r_o}+\int_{r_o}^{r}+\int_r^{r^{1+h}}\right\}(\log x)^{K+1}n(x)\frac{dx}{x}\right].$$

$$<0\{(\log r)^{-K-1}\}+\frac{(c+\epsilon)}{\{\log(r^{1+h})\}^{K+1}}\int_{r_o}^{r}(\log x)^{K+\rho_1^*+1}\,\frac{dx}{x}$$

$$+ \frac{n(r^{1+h})}{\{\log(r^{1+h})\}^{K+1}} \int_{r}^{r^{1+h}} (\log x)^{K+1} \frac{dx}{x}$$

$$= 0\left\{(\log r)^{-K-1}\right\} + \frac{(c+\epsilon)(\log r)^{K+\rho_1^*+2} - (\log r_o)^{K+\rho_1^*+2}}{(1+h)^{K+1}(K+\rho_1^*+2)(\log r)^{K+1}}$$

$$+ \frac{n(r^{1+h})[(1+h)^{K+2} - 1]\log r}{(K+2)(1+h)^{K+1}} .$$

Hence,

$$\frac{\log\left\{\frac{G(r^{1+h})}{g_K^*(r^{1+h})}\right\}^{1/\log r^{1+h}}}{\{\log(r^{1+h})\}^{\rho_1^*}} <$$

$$< \frac{(c+\epsilon)}{(1+h)^{\rho_1^*+K+2}(K+\rho_1^*+2)} + \frac{n(r^{1+h})[(1+h)^{K+2}-1]}{(K+2)(1+h)^{\rho_1^*+K+2}(\log r)^{\rho_1^*}} .$$

Taking limits of both sides and using (4.1) and (4.2) we get

$$(4.3)\quad (1+h)^{K+\rho_1^*+2}\, p \leqslant \frac{c}{(K+\rho_1^*+2)} + c(1+h)^{\rho_1^*}\frac{(1+h)^{K+2}-1}{K+2}$$

and

$$(4.4)\quad (1+h)^{K+\rho_1^*+2}\, q \leqslant \frac{c}{(K+\rho_1^*+2)} + d(1+h)^{\rho_1^*}\left\{\frac{(1+h)^{K+2}-1}{K+2}\right\}$$

Similarly, we obtain

$$(4.5)\qquad (1+h)^{K+\rho_1^*+2}\, p \geqslant \frac{d}{(K+\rho_1^*+2)} + c\left\{\frac{(1+h)^{K+2}-1}{K+2}\right\}$$

and

$$(4.6) \qquad (1+h)^{K+\rho_1^*+2} q \geq \frac{d}{(K+\rho_1^*+2)} + d\left\{\frac{(1+h)^{K+2}-1}{K+2}\right\}.$$

It can be seen that minima of right hand expansion of (4.3) and (4.4) occurs at $h = 0$ and $(1+h)^{\rho_1^*} = c/d$. Substituting $h = 0$ in (4.3) and $(1+h)^{\rho_1^*} = c/d$ in (4.4), we get second parts of (i) and (ii) respectively. Taking

$$(1+h)^{K+2} = \frac{(K+2)(c-d) + c^{\rho_1^*}}{c^{\rho_1^*}}$$

in (4.5) and $h = 0$ in (4.6) we get (iii) and first part of (1) respectively.

5. <u>Theorem</u> 4 : - If $f(z)$ is an integral function of order zero and logarithmic convergent exponent ρ_1^* and lower logarithmic convergent exponent λ_1^*, then

$$(5.1) \qquad \lim_{\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log\{\frac{G(r)}{g_K(r)}\}}{\log\log r} = \begin{matrix}\rho_1^*\\ \\ \lambda_1^*\end{matrix}.$$

<u>Proof</u>. From (1.2) and (1.3) and as $\frac{d}{dx}(\log G(x)) = \frac{n(x)}{x}$, we get

$$\frac{G(r)}{g_K(r)} = \exp\left[\frac{1}{r^{K+1}}\int_0^r x^{K+1}\frac{d}{dx}\left\{\log G(x)\right\}dx\right]$$

$$= \exp\left[\frac{1}{r^{K+1}}\int_0^r x^{K+1}\, n(x)dx\right]$$

$$\leqslant \exp_{r\to\infty}\left\{\frac{1}{r^{K+1}}\, n(r) \int_{o}^{r} x^{K+1}\, dx\right\}$$

$$= \exp\left\{\frac{n(r)}{(K+1)}\right\} .$$

Hence,

(5.2)
$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log\{\frac{G(r)}{g_K(r)}\}}{\log\log r} \leqslant$$

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log n(r)}{\log\log r} = \begin{matrix}\rho_1^*\\ \lambda_1^*\end{matrix}$$

On the other hand

$$\frac{G(2r)}{g_K(2r)} = \exp\left\{\frac{1}{(2r)^{K+1}} \int_{o}^{2r} x^K n(x)\, dx\right\}$$

$$> \exp\left\{\frac{1}{(2r)^{K+1}} \int_{r}^{2r} x^K n(x)\, dx\right\}$$

$$\geqslant \exp\left\{\frac{1}{(2r)^{K+1}}\, n(r) \int_{r}^{2r} x^K dx\right\}$$

$$= \exp\left\{\frac{n(r)}{2^{K+1}} \quad \frac{2^{K+1}-1}{K+1}\right\}$$

Hence,

(5.3)
$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log\{\frac{G(2r)}{g_K(2r)}\}}{\log\log(2r)} \geqslant$$

$$\geqslant \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log n(r)}{\log \log r} = \begin{matrix}\rho_1^* \\ \\ \lambda_1^*\end{matrix} .$$

Therefore from (5.2) and (5.3), (5.1) follows.

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