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ON KÜNNETH RELATIONS

Marek Golasiński

Abstract. The aim of this note is to subsume a number of apprently quite distrinct results in one general theorem. For a left exact functor $T : R\text{-Mod} \longrightarrow Ab$ and a cochain complex C^* we give a long exact sequence including the canonical map $H^nT \longrightarrow TH^n$, where H^n is the n-th cohomology functor. Under the appropriate hypothesis the usual form of the Künneth relation (see [1], chap. VI) is a special case of our long exact sequence (Remark 1.2). Also the latest results of Coelho-Pezennec (see [2]) are contained in this long sequence (Proposition 2.3).

In particular, we obtain a simple proof of the following results of Osofsky on upper bounds of cohomological dimensions (see [7], [8]). If I is a directed set and the cardinal number of it is no greater than $\aleph_m$, then:

1. $l.pd_{colim_I R_i}\, colim_I M_i \leqslant sup_I l.pd_{R_i}\, M_i + m + 1$ in the category of modules,
2. $cd_R colim_I G_i \leqslant sup_I cd_R G_i + m + 1$ in the category of

groups.

1. Main results. Let R be a ring and let R-Mod and Ab denote the category of left R-modules and the category of abelian groups, respectively. For a left exact functor $T\colon R\text{-Mod} \longrightarrow Ab$ denote by T^n the right n-th derived functor of T and for a cochain complex $C^* = (C^n, d^n)$ of R-modules we put $Z^n = \operatorname{Ker} d^n$, $B^n = \operatorname{Im} d^{n-1}$ and $H^n = Z^n/B^n$, where n runs over the integers.

Theorem 1.1. (General theorem). If $T^k C^n = 0$ for $k \geqslant 1$ and all integers n, then there exists a long exact sequence of abelian groups

$$0 \longrightarrow T^1 Z^{n-1} \longrightarrow H^n T \longrightarrow TH^n \longrightarrow T^2 Z^{n-1} \longrightarrow$$
$$\longrightarrow T^1 Z^n \longrightarrow T^1 H^n \longrightarrow T^3 Z^{n-1} \longrightarrow \dots \longrightarrow$$
$$\longrightarrow T^k Z^n \longrightarrow T^k H^n \longrightarrow T^{k+2} Z^{n-1} \longrightarrow \dots .$$

Proof. The canonical short exact sequence of R-modules $0 \longrightarrow Z^n \xrightarrow{i^n} C^n \xrightarrow{d^n} B^{n+1} \longrightarrow 0$ yields a long exact sequence of abelian groups

$$0 \longrightarrow TZ^n \xrightarrow{Ti^n} TC^n \xrightarrow{Td^n} TB^{n+1} \xrightarrow{\alpha_0^n} T^1 Z^n \xrightarrow{T^1 i^n} T^1 C^n \xrightarrow{T^1 d^n}$$
$$\longrightarrow T^1 B^{n+1} \xrightarrow{\alpha_1^n} \dots \xrightarrow{\alpha_{k-1}^n} T^k Z^n \xrightarrow{T^k i^n} T^k C^n \xrightarrow{T^k d^n} T^k B^{n+1} \xrightarrow{\alpha_k^n} \dots .$$

where $\alpha_k^n : T^k B^{n+1} \longrightarrow T^{k+1} Z^n$ is the connecting map.

By assumption $T^k C^n = 0$ for $k \geqslant 1$ and all integers n. So we get a short exact sequence

$$\text{a)} \quad 0 \longrightarrow TZ^n \xrightarrow{Ti^n} TC^n \xrightarrow{Td^n} TB^{n+1} \xrightarrow{\alpha_0^n} T^1Z^n \longrightarrow 0$$

and a family of isomorphisms

$$\text{b)} \quad T^kB^{n+1} \underset{\approx}{\xrightarrow{\alpha_k^n}} T^{k+1}Z^n \quad \text{for } k \geqslant 1 \text{ and all integers } n.$$

The sequence a) yields an isomorphism

$$\text{a')} \qquad TB^{n+1}/\operatorname{Im} Td^n \underset{\approx}{\xrightarrow{\overline{\alpha_0^n}}} T^1Z^n.$$

Moreover, the canonical short exact sequence of R-modules $0 \longrightarrow B^n \xrightarrow{j^n} Z^n \xrightarrow{\beta^n} H^n \longrightarrow 0$ yields a long exact sequence of abelian groups

$$\text{c)} \quad 0 \longrightarrow TB^n \xrightarrow{Tj^n} TZ^n \xrightarrow{T\beta^n} TH^n \xrightarrow{\gamma_0^n} T^1B^n \xrightarrow{T^1j^n} T^1Z^n \xrightarrow{T^1\beta^n} T^1H^n \xrightarrow{\gamma_1^n} \cdots \xrightarrow{\gamma_{k-1}^n} T^kB^n \xrightarrow{T^kj^n} T^kZ^n \xrightarrow{T^k\beta^n} T^kH^n \xrightarrow{\gamma_k^n} \cdots,$$

where $\gamma_k^n : T^kH^n \longrightarrow T^{k+1}B^n$ is the usual connecting map. Hence, by b) we obtain the following long exact sequence of abelian groups

$$0 \longrightarrow TZ^n/TB^n \xrightarrow{\overline{T\beta^n}} TH^n \xrightarrow{\alpha_1^{n-1} \circ \gamma_0^n} T^2Z^{n-1} \xrightarrow{T^1j^n(\alpha_1^{n-1})^{-1}} T^1Z^n \xrightarrow{T^1\beta^n} T^1H^n \xrightarrow{\alpha_2^{n-1} \circ \gamma_1^n} \cdots \xrightarrow{\alpha_k^{n-1} \circ \gamma_{k-1}^n} T^{k+1}Z^{n-1} \xrightarrow{T^kj^n(\alpha_k^{n-1})^{-1}} T^kZ^n \xrightarrow{T^k\beta^n} T^kH^n \xrightarrow{\alpha_{k+1}^{n-1}\ \gamma_k^n} \cdots .$$

The functor $T : R\text{-Mod} \longrightarrow Ab$ is left exact, hence $\operatorname{Ker} Td^n = TZ^n$ and the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & \operatorname{Im} Td^{n-1} & \xrightarrow{j'} & \operatorname{Ker} Td^n & \xrightarrow{\beta'} & H^nT & \longrightarrow & 0 \\ & & \Big\downarrow \varphi & & \Big\| & & \Big\downarrow \psi & & \\ 0 & \longrightarrow & TB^n & \xrightarrow{Tj^n} & TZ^n & \xrightarrow{\delta} & TZ^n/TB^n & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

yields $\operatorname{Ker}\psi = \operatorname{coKer}\varphi = TB^n/\operatorname{Im} Td^{n-1}$, by the Snake Lemma. Let $\eta : \operatorname{Ker}\psi \longrightarrow H^nT$ be the canonical inclusion. Then, finally, we obtain the long exact sequence

$$0 \to T^1Z^{n-1} \xrightarrow{\eta\cdot(\alpha_0^{\overline{n-1}})^{-1}} H^nT \xrightarrow{T\beta^n \circ \psi} TH^n \xrightarrow{\alpha_1^{n-1} \circ \gamma_0^n} T^2Z^{n-1} \xrightarrow{T^1 j^n \circ (\alpha_1^{n-1})^{-1}}$$

$$\to T^1Z^n \xrightarrow{T^1\beta^n} T^1H^n \xrightarrow{\alpha_2^{n-1} \circ \gamma_1^n} T^3Z^{n-1} \xrightarrow{T^2 j^n \circ \alpha_2^{n-1}} \cdots \longrightarrow$$

$$\xrightarrow{T^k j^n \circ (\alpha_k^{n-1})^{-1}} T^kZ^n \xrightarrow{T^k\beta^n} T^kH^n \xrightarrow{\alpha_{k+1}^n \circ \gamma_k^n} T^{k+2}Z^{n-1} \xrightarrow{T^{k+1} j_o^n (\alpha_{k+1}^{n-1})^{-1}} \cdots \blacksquare$$

As a corollary we get the usual form of the Künneth relation (see [1], chap. VI).

<u>Remark 1.2</u>. If $T^k = 0$ for $k \geqslant 2$, then $T^1Z^n \xrightarrow[\approx]{T^1\beta^n} T^1H^n$ and the short sequence

$$0 \longrightarrow T^1H^{n-1} \xrightarrow{\eta\cdot(\alpha_0^{n-1})^{-1} \circ (T^1\beta^{n-1})^{-1}} H^nT \xrightarrow{T\beta^n \circ \psi} TH^n \to 0$$

is exact.

Moreover, the above Theorem yields the following result.

Corollary 1.3. If the maps $T^kZ^n \xrightarrow{T^k\beta^n} T^kH^n$ induced by the canonical map $Z^n \xrightarrow{\beta^n} H^n$ are left split (i.e. there exists a map $\rho^n : T^kH^n \longrightarrow T^kZ^n$ such that $\rho^n \circ T^k\beta^n = id_{T^kZ^n}$), then there exists a long exact sequence of abelian groups

$$\ldots \to T^{2k+1}H^{n-k-1} \to \ldots \longrightarrow T^3H^{n-2} \longrightarrow T^1H^{n-1} \longrightarrow H^nT \longrightarrow TH^n \longrightarrow$$

$$\longrightarrow T^2H^{n-1} \longrightarrow T^4H^{n-2} \longrightarrow \ldots \longrightarrow T^{2k}H^{n-k} \longrightarrow \ldots \quad .$$

Proof. In virtue of assumption the sequence c) from the proof of Theorem 1.1 determines the short exact sequence

$$0 \longrightarrow TZ^n/_{TB^n} \xrightarrow{\overline{T\beta^n}} TH^n \xrightarrow{\gamma_0^n} T^1B^n \longrightarrow 0$$

and the split short exact sequences

$$0 \longrightarrow T^kZ^n \underset{\rho n}{\overset{T^k\beta^n}{\rightleftarrows}} T^kH^n \underset{\delta n}{\overset{\gamma_k^n}{\rightleftarrows}} T^{k+1}B^n \longrightarrow 0$$

for $k \geqslant 1$ and all integers n.

Hence, using the isomorphisms b) from Theorem 1.1 we obtain the following diagram

$$\vdots$$

$$0\to T^{2k+3}Z^{n-k-2}\xrightarrow{\delta_0^{n-k-1}(\alpha_{2k+2}^{n-k-2})^{-1}} T^{2k+1}H^{n-k-1}\xrightarrow{\rho^{n-k-1}} T^{2k+1}Z^{n-k-1}\longrightarrow 0$$

$$\vdots$$

$$0\to T^{3}Z^{n-2}\xrightarrow{\delta_0^{n-1}(\alpha_{2}^{n-2})^{-1}} T^{1}H^{n-1}\xrightarrow{\rho^{n-1}} T^{1}Z^{n-1}\longrightarrow 0$$

$$0\to T^{1}Z^{n-1}\xrightarrow{\eta\cdot(\alpha_{0}^{n-1})^{-1}} H^{n}T\xrightarrow{\psi} TZ^{n}/_{TB^{n}}\longrightarrow 0$$

$$0\to TZ^{n}/_{TB^{n}}\xrightarrow{T\beta^{n}} TH^{n}\xrightarrow{\alpha_1^{n-1}\circ\gamma_0^{n}} T^{2}Z^{n-1}\longrightarrow 0$$

$$0\to T^{2}Z^{n-1}\xrightarrow{T^{2}\beta^{n-1}} T^{2}H^{n-1}\xrightarrow{\alpha_3^{n-2}\cdot\gamma_2^{n-1}} T^{4}Z^{n-2}\longrightarrow 0$$

$$0\to T^{4}Z^{n-2}\xrightarrow{T^{4}\beta^{n-2}} T^{4}H^{n-2}\xrightarrow{\alpha_5^{n-3}\circ\gamma_4^{n-2}} T^{6}Z^{n-3}\longrightarrow 0$$

$$\vdots$$

$$0\to T^{2k}Z^{n-k}\xrightarrow{T^{2k}\beta^{n-k}} T^{2k}H^{n-k}\xrightarrow{\alpha_{2k+1}^{n-k}\circ\gamma_{2k}^{n-k}} T^{2k+2}Z^{n-k-1}\longrightarrow 0$$

$$\vdots$$

Composing the above short exact sequences we obtain the announced long exact sequence of abelian groups.□

2. Applications. Let I be a directed set. It is well known that the functor colim_I is exact. Moreover, if the cardinal number of I is no greater than $\aleph_m$, then $\lim_I^{m+k} = 0$ for $k \geq 2$ (see [4]).

Let $\{R_i, \varphi_{ij}\}_{i,j\in I}$ and $\{M_i, \psi_{ij}\}_{i,j\in I}$ be directed systems of rings and abelian groups respectively, such that each M_i is a left R_i-module and $\psi_{ij}(r_i m_j) = \varphi_{ij}(r_i)\psi_{ij}(m_i)$ for $r_i \in R_i$, $m_i \in M_i$ and $i < j$.

(Such systems will be called *consistent*).

Then, $M = \mathrm{colim}_I M_i$ is a left $R = \mathrm{colim}_I R_i$-module and $M \approx \mathrm{colim}_I R \otimes_{R_i} M_i$ in the category of R-Mod.

For further purposes the following lemmas will be useful.

Lemma 2.1. If each M_i is a (pure) projective R_i-module for all $i \in I$, then

$$\lim_I^n \mathrm{Hom}_{R_i}(M_i N) \approx \mathrm{Ext}_R^n(\mathrm{colim}_I M_i N) (\approx \mathrm{Pext}_R^n(\mathrm{colim}_I M_i, N))$$

for any R-module N.

Proof. A directed system $\{M_i, \psi_{ij}\}_{i,j \in I}$ yields an exact sequence of R-modules (see [3], Appendix I)

$$\ldots \longrightarrow \bigoplus_{i_0 < \ldots < i_n} R \otimes_{R_{i_0}} M_{i_0} \longrightarrow \ldots \longrightarrow \bigoplus_{i_0 < i_1} R \otimes_{R_{i_0}} M_{i_0} \longrightarrow \bigoplus_{i_0 \in I} R \otimes_{R_{i_0}} M_{i_0} \longrightarrow$$

$$\longrightarrow \mathrm{colim}_I R \otimes_{R_i} M_i \approx \mathrm{colim}_I M_i .$$

If each M_i is a (pure) projective R_i-module for all $i \in I$ then the above sequence is an R-(pure) projective resolution of $\mathrm{colim}_I M_i$.

Applying the functor $\mathrm{Hom}_R(-,N)$ we obtain the following chain complexes:

$$0 \longrightarrow \mathrm{Hom}_R(\bigoplus_{i\in I} R \otimes_{R_i} M_i, N) \longrightarrow \mathrm{Hom}_R(\bigoplus_{i_0<i_1} R \otimes_{R_{i_0}} M_{i_0} N) \longrightarrow \cdots$$

$$\wr\wr \qquad\qquad\qquad \wr\wr$$

$$0 \longrightarrow \prod_{i\in I} \mathrm{Hom}_{R_i}(M_i, N) \longrightarrow \prod_{i_0<i_1} \mathrm{Hom}_{R_{i_0}}(M_{i_0}, N) \longrightarrow \cdots .$$

Consequently, $\lim_I^n \mathrm{Hom}_{R_i}(M_i,N) \approx \mathrm{Ext}_R^n(\mathrm{colim}_I M_i, N)$

$(\approx \mathrm{Pext}_R^n(\mathrm{colim}_I M_i, N))$.

Let F_{M_i} denotes the free R_i-module generated by the elements of M_i, then $F_{\mathrm{colim}_I M} \approx \mathrm{colim}_I F_{M_i}$. Hence, we obtain the following generalization of Lemma 9.5 from [1].

Lemma 2.2. There exist R_i-(pure) projective resolutions $\mathbb{P}_i$ of M_i forming a consistent directed system $\{\mathbb{P}_i, \overline{\psi}_{ij}\}$ $i,j\in I$ such that $\mathbb{P} = \mathrm{colim}_I \mathbb{P}_i$ is an R-(pure) projective resolution of $\mathrm{colim}_I M_i$.

The two lemmas stated above will be used in the sequel.

Let $\{C_*^i, \psi_{ij}\}_{i,j\in I}$ be a consistent directed system of chain complexes such that C_n^i are R_i-modules for all $i \in I$. Put $C_* = \mathrm{colim}_I C_*^i$ and $Z_n'^i = \mathrm{coKer}\ d_n^i$.

Then the following generalization of the Coelho-Pezennec result is a simple consequence of Theorem 1.1 and Lemma 2.1.

Proposition 2.3. (see [2]). If C_n^i are (pure) projective R_i-modules for all integers n, then the following long sequence

$$0 \longrightarrow \lim^1_I \operatorname{Hom}_{R_i}(Z'^{i}_{n-1},N) \longrightarrow H^n(C_*,N) \longrightarrow \lim_I H^n(C^i_*, N) \longrightarrow$$

$$\rightarrow \lim^2_I \operatorname{Hom}_{R_i}(Z'^{i}_{n-1},N) \rightarrow \lim^1_I \operatorname{Hom}_{R_i}(Z'^{i}_{n},N) \rightarrow \lim^1_I H^n(C^i_*,N) \longrightarrow$$

$$\longrightarrow \lim^3_I \operatorname{Hom}_{R_i}(Z'^{i}_{n-1},N) \longrightarrow \dots \longrightarrow$$

$$\rightarrow \lim^k_I \operatorname{Hom}_{R_i}(Z'^{i}_{n},N) \longrightarrow \lim^k_I H^n(C^i_*,N) \longrightarrow \lim^{k+2}_I \operatorname{Hom}_{R_i}(Z'^{i}_{n-1},N) \longrightarrow$$

$\longrightarrow \dots$ is exact.

Moreover, as direct consequences of this Proposition we obtain the results of Osofsky (see [7] abd [8]) and Kiełpiński-Simson (see [6]).

Let $1.\mathrm{pd}_R M$ $(1.\mathrm{P.pd}_R M)$ denote the left (pure) projective dimension of an R-module M and let 1.gl dimR (1.P.gl dimR) denote the left (pure) global dimension of a ring R.

<u>Corollary 2.4</u>. i) $1.\mathrm{pd}_{\mathrm{colim}_I R_i} \mathrm{colim}_I M_i \leqslant$

$$\leqslant \sup_I 1.\mathrm{pd}_{R_i} M_i + m + 1$$

$$(1.\mathrm{P.pd}_{\mathrm{colim}_I R_i} \mathrm{colim}_I M_i \leqslant \sup_I 1.\mathrm{P.pd}_{R_i} M_i + m + 1)$$

and

ii) $1.\mathrm{gl\,dim\,colim}_I R_i \leqslant \sup_I 1.\mathrm{gl\,dim} R_i + m + 1$

$(l.P.gl\ dim\ colim_I R_i \leq sup_I l.gl\ dim R_i + m + 1)$.

Proof. Applying Proposition 2.3 to the directed sistem $\{\mathbb{P}_i, \psi_{ij}\}_{i,j\in I}$ of projective resolutions of $\{M_i, \psi_{ij}\}_{i,j\in I}$ given by Lemma 2.2 we obtain the exact sequence

$$0 \longrightarrow \lim_I^1 Hom_{R_i}(Z'^{i}_{n-1}, N) \longrightarrow Ext_R^n(colim_I M_i, N) \longrightarrow \lim_I Ext^n_{R_i}(M_i, N) \longrightarrow$$

$$\longrightarrow \lim_I^2 Hom_{R_i}(Z'^{i}_{n-1}, N) \longrightarrow \lim_I^1 Hom_{R_i}(Z'^{i}_{n}, N) \longrightarrow \lim_I^1 Ext^n_{R_i}(M_i, N) \longrightarrow$$

$$\longrightarrow \lim_I^3 Hom_{R_i}(Z'^{i}_{n-1}, N) \longrightarrow \ \dots \ \longrightarrow$$

$$\longrightarrow \lim_I^k Hom_{R_i}(Z'^{i}_{n}, N) \longrightarrow \lim_I^k Ext^n_{R_i}(M_i, N) \longrightarrow \lim_I^{k+2} Hom_{R_i}(Z'^{i}_{n-1}, N) \longrightarrow$$

$\longrightarrow \dots$, where $R = colim_I R_i$.

Hence, for $n > sup_I l.pd_{R_i} M_i$ we have the following isomorphisms

$$\lim_I^2 Hom_{R_i}(Z'^{i}_{n-1}, N) \approx \lim_I^1 Hom_{R_i}(Z'^{i}_{n}, N)$$

....

$$\lim_I^k Hom_{R_i}(Z'^{i}_{n-1}, N) \approx \lim_I^{k-1} Hom_{R_i}(Z'^{i}_{n}, N).$$

Therefore, for $n-k > sup_I l.pd_{R_i} M_i$

$$\lim_I^1 Hom_{R_i}(Z'^{i}_{n-1}, N) \approx \dots \approx \lim_I^k Hom_{R_i}(Z'^{i}_{n-k}, N).$$

But $\lim_I^k = 0$ for $k > m + 1$. Consequently,

$$\lim_I^1 \operatorname{Hom}_{R_i}(Z'^{\,i}_{n-1}, N) = 0 \quad \text{and} \quad \operatorname{Ext}_R^n(\operatorname{colim}_I M_i, N) = 0 \quad \text{for}$$

$n > \sup_I \text{l.pd}_{R_i} M_i + m + 1$. Hence,

$$\text{l.pd}_{\operatorname{colim}_I R_i} \operatorname{colim}_I M_i \leqslant \sup_I \text{l.pd}_{R_i} M_i + m + 1.$$

ii) For any R-module M we have $M = \operatorname{colim}_I M_i$, where $M_i = M$ are R_i-modules for all $i \in I$. Therefore, by i)

$$\text{l.pd}_{\operatorname{colim}_I R_i} M \leqslant \sup_I \text{l.pd}_{R_i} M + m + 1 \leqslant \sup_I \text{l.gl dim} R_i + m + 1$$

and hence $\text{l.gl dim colim}_I R_i \leqslant \sup_I \text{l.gl dim} R_i + m + 1$.

The analogous results for the left (pure) projective and global dimension are obtained by the same methods.■

In particular, if $\{G_i, \varphi_{ij}\}_{i,j \in I}$ is a directed system of groups, then for group-rings over a ring R we have $R[\operatorname{colim}_I G_i] \approx \operatorname{colim}_I R[G_i]$.

So, by the above Corollary $\text{pd}_{\operatorname{colim}_I R[G_i]} \Delta R \leqslant$ $\leqslant \sup_I \text{pd}_{R[G_i]} \Delta R + m + 1$, where ΔR denotes the trivial module over the appropriate group-ring.

Therefore, we get another result due to Osofsky (see [8])

$$\mathrm{cd}_R\ \mathrm{colim}_I G_i \leqslant \sup_I \mathrm{cd}_R G_i + m + 1,$$ where

cd_R denotes the R-cohomological dimension.

More generally, if $\{\mathbb{C}_i\ \varphi_{ij}\}_{i,j\in I}$ is a directed system of small categories, thèn using methods similar to those above, we obtain

$$\mathrm{cd}_R\ \mathrm{colim}_I \mathbb{C}_i \leqslant \sup_I \mathrm{cd}_R\ \mathbb{C}_i + m + 1.$$

Remark 2.5. By results from [5] and [9] we can replace the directed set I by any small category such that the functor colim_I is exact.

Now let R be a hereditary ring and let $\{C_*^i, \psi_{ij}\}_{i,j\in I}$ be a directed system of chain complexes over the category R-mod.

Proposition 2.6. If C_*^i and $C_* = \mathrm{colim}_I C_*^i$ are chain complexes of projective R-modules for all $i \in I$, then the following long sequence

$$\ldots \longrightarrow \lim_I^{2k+1} H^{n-k-1}(C_*^i,N) \longrightarrow \ldots \longrightarrow \lim_I^3 H^{n-2}(C_*^i,N) \longrightarrow$$

$$\longrightarrow \lim_I^1 H^{n-1}(C_*^1,N) \longrightarrow H^n(C_*,N) \longrightarrow \lim_I H^n(C_*^i,N) \longrightarrow$$

$$\longrightarrow \lim_I^2 H^{n-1}(C_*^i,N) \longrightarrow \lim_I^4 H^{n-2}(C_*^i,N) \to \ldots \to \lim_I^{2k} H^{n-k}(C_*^i,N) -$$

$\longrightarrow$ ... is exact for all integers n and any R-module M.

Proof. Because $\{Z^n \mathrm{Hom}_R(C^i_*,N)\}_{i\in I} = \{\mathrm{Hom}_R(C^i_n/B^i_n,N)\}_{i\in I}$ and the sequence

$$0 \longrightarrow H_n C^i_* \longrightarrow C^i_n/B^i_n \longrightarrow C^i_n/Z^i_n \longrightarrow 0 \quad \text{splits,}$$

therefore

$$\{\mathrm{Hom}_R(C^i_n/B^i_n,N)\}_{i\in I} = \{\mathrm{Hom}_R(H_n C^i_*,N)\}_{i\in I} \oplus \{\mathrm{Hom}_R(C^i_n/Z^i_n,N)\}_{i\in I}$$

and

$$\lim^k_I Z^n \mathrm{Hom}_R(C^i_*,N) = \lim^k_I \mathrm{Hom}_R\, H_n(C^i_*,N) \oplus \lim^k_I \mathrm{Hom}_R(C^i_n/Z^i_n,N).$$

But C^i_n/Z^i_n are projective R-modules and $\mathrm{colim}_I\, C^i_n/Z^i_n = C_n/Z_n$ is a projective R-module, so by Lemma 2.1

$$\lim^k_I \mathrm{Hom}_R(C^i_n/Z^i_n,N) = \mathrm{Ext}^k_R(C_n/Z_n,N) = 0 \quad \text{for} \quad k \geqslant 1.$$

Moreover, by Universal Coefficient Theorem (see [1] chap. VI) we have natural epimorphisms $H^n(C^i_*,N) \longrightarrow \mathrm{Hom}_R(H_n C^i_*,N)$ for all $i \in I$. Consequently, the map

$$\lim^k_I Z^n \mathrm{Hom}_R(C^i_*,N) = \lim^k_I \mathrm{Hom}_R(H_n C^i_*,N) \longrightarrow \lim^k_I H^n(C^i_*,N)$$

splits and an appropriate long exact sequence is determined by Corollary 1.2. ■

Corollary 2.7. If $\{X_i,\varphi_{ij}\}_{i,j\in I}$ is a directed system of compact topological spaces, then the cochain functor commutes with limits. Thus, the following sequence of singular cohomology groups

$$0 \longrightarrow \lim_I^{2n-1} H^0(X_i,A) \longrightarrow \dots \longrightarrow \lim_I^3 H^{n-2}(X_i,A) \longrightarrow$$

$$\longrightarrow \lim_I^1 H^{n-1}(X_i,A) \longrightarrow H^n(X,A) \longrightarrow \lim_I H^n(X_i,A) \longrightarrow$$

$$\rightarrow \lim_I^2 H^{n-1}(X_i,A) \rightarrow \lim_I^4 H^{n-2}(X_i,A) \rightarrow \dots \rightarrow \lim_I^{2n} H^0(X_i,A) \rightarrow 0$$

is exact for any abelian group A, where $X = \operatorname{colim}_I X_i$.

Similary, if $\{G_i,\varphi_{ij}\}_{i,j \in I}$ and $\{C_i,\varphi_{ij}\}_{i,j \in I}$ are directed systems of groups and small categories respectively, then we obtain the appropriate long exact sequence as above.

REFERENCES

[1] Cartan H., Eilenberg S., "Homological algebra", Princeton, New Jersey (1956).

[2] Coelho M., Pezennec J., "Cohomologie d'une limite inductive de complexes. Applications aux fantômes de finitude en cohomologie tordue", C.R.Acad.Sc.Paris t.296. (1983) p. 101-104.

[3] Gabriel P., Zisman M., "Calculus fractions and homotopy Theory", Springer-Verlag, Berlin (1967).

[4] Goblot R., "Sur les derives de certaines limites projectives, applications aux modules", Bull. Sci.Math. (2) 94 (1970) p.251-255.

[5] Isbell J., Mitchell B., "Exact colimits", Bull. Amer. Math.

[6] Kiełpiński R., Simson D., "On pure homological dimension", Bull. Acad. Pol. Sc. XXIII, no.1, (1975) p.1-6.

[7] Osofsky B., "Upper bounds on homological dimension", Nagoya Math.J. 32(1968) p.315-322.

[8] __________, "Homological dimension and the continuum hypothesis", Trans.Amer.Math.Soc. 132 (1968) p.217-230.

[9] Stauffer H.,B., "A relationship between left exact and representable functors", Can.J.Math. XXIII, nº.2 (1971).

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Institute of Mathematics
Nicholas Copernicus University
Chopina 12/18, 87-100 Torun,
POLAND