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A CHARACTERIZATION OF THE RADON-NIKODYM PROPERTY

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§1. INTRODUCTION.

The aim of this paper is to give a new characterization of the Radon-Nikodym property in terms of martingales in X-valued Orlicz spaces.

Let X be a Banach space and put Σ, the Lebesgue measurable sets in $[0,1]$. It is well known (see [1]):

(1.1). X has the Radon-Nikodym property with respect to $[0,1]$ if and only if every bounded uniformly integrable martingale in $L^1_X[0,1]$, (f_n, B_n) where $\sigma(\bigcup B_n) = \Sigma$, is convergent in $L^1_X[0,1]$.

We are interested in a generalization of this fact.

In this paper we shall prove the following

Theorem (1.2).

Let ϕ be a Young function with the Δ_2-condition. X has the Radon-Nikodym property if and only if every bounded martingale in L^ϕ_X, (f_n, B_n) where $\sigma(\bigcup B_n) = \Sigma$, is convergent in L^ϕ_X.

The definitions and the main results relating to X-valued martingales and Orlicz spaces may be found in [1] and [2] respectively.

We are going to denote by ϕ a Young function, and $\tilde{L}^{\phi}_{X} = \{f:[0,1] \to X$ strongly measurable with respect Lebesgue measure s.t. $\rho(f,\phi) = \int_0^1 \phi(||f(x)||)dx < \infty\}$.

Let ψ be the complementary Young function of ϕ . We shall write $||f||_{\phi} = \sup\{ \int_0^1 ||f(x)||\,|g(x)|dx$ with $g \in \tilde{L}^{\psi}_{\mathbb{R}}, \rho(g,\psi) \leq 1\}$ and $L^{\phi}_{X} = \{f:[0,1] \longrightarrow X$ strongly measurable with $||f||_{\phi} < \infty\}$.

It is well known that L^{ϕ}_{X} is a vector space and $||f||_{\phi}$ is a norm on it.

Besides, $\tilde{L}^{\phi}_{X} = L^{\phi}_{X}$ if and only if ϕ verifies the Δ_2-condition. It is easy to prove that the convergence and the boundedness in $\tilde{L}^{\phi}_{X}$ and L^{ϕ}_{X} are equivalent using the following fact:

(1.3) Suppose ϕ verifies Δ_2-condition, i.e. there exists $K > 0$ and $T \geq 0$ such that $\phi(2t) \leq K\phi(t)$ for all $t \geq T$.

If there exists m belongs to $\mathbb{N}$ with $\rho(f,\phi) \leq 1/K^m$ then $||f||_{\phi} \leq \frac{\phi(T)+2}{2^m}$. See [2], page 158 for a proof.

§2. PREVIOUS LEMMAS .

Lemma 1.

If $(f_n , n \in N)$ is a bounded sequence in $\tilde{L}^{\phi}_{X}$, then $(f_n, n \in N)$ is a bounded uniformly integrable sequence in L^1_X .

Proof.

For a Young function we have

(2.1) $\frac{\phi(t)}{t} \longrightarrow \infty$ as $t \to \infty$., and by (2.1) we obtain $||f_n||_{L^1_X} \leq \rho(f_n,\phi) + A$, where A is a constant.
We have only to show that $\int_E ||f_n(x)||dx \longrightarrow 0$ as $m(E) \longrightarrow 0$. Given $\varepsilon > 0$, by (2.1) , there exists $\tau > 0$ such that

(2.2) $\frac{\phi(t)}{t} > \frac{2c}{\varepsilon}$ for $t > \tau$ where $\sup_n \rho(f_n,\phi) \leq C$.

Let $\delta = \varepsilon/2\tau$. If $m(E) < \delta$ and denoting
$A_n = \{x : ||f_n(x)|| \leq \tau\} \cap E$ and
$B_n = \{x : ||f_n(x)|| > \tau\} \cap E$ we obtain

$$\int_E ||f_n(x)|| = \int_{A_n} ||f_n(x)||dx +$$

$$+ \int_B ||f_n(x)||dx \underset{(2.2)}{<} \tau m(E) + \frac{\varepsilon}{2c} \int_0^1 \phi(||f_n(x)||)dx<\varepsilon$$

Lemma 2.

If ϕ verifies the Δ_2-condition then the simple functions are dense in $\tilde{L}^\phi_X$.

Proof:

Given $f \in \tilde{L}^\phi_X$, since f is strongly measurable, there exists a sequence $(f_n , n \in N)$ of countably valued functions such that

(2.3) $\|f_n(x)-f(x)\| < \frac{1}{n}$ for almost all $x \in [0,1]$ and for all $n \in N$. Suppose $f_n = \sum_{m=o}^{\infty} x_{n,m} \chi_{E_{n,m}}$ where $x_{n,m} \in X$ and $\chi_{E_{n,m}}$ are the characteristic functions of disjoint measurables sets.

Since $2\|f_n(x)\| < 2\|f(x)\| + \frac{2}{n}$ a.e. and ϕ is a convex function, we have $2f_n \in \widetilde{L}_X^{\phi}$. Therefore, there is a number $p_n \in N$ such that

$$(2.4) \qquad \int_{\bigcup_{m=p_n}^{\infty} E_{n,m}} \phi(2\|f_n(x)\|)\,dx < \frac{1}{n}$$

We consider the simple function $g_n = \sum_{m=o}^{p_n} x_{n,m} \chi_{E_{n,m}}$.
By (2.3) and (2.4)

$$\int_o^1 \phi(\|f(x)-g_n(x)\|)\,dx \leq \frac{1}{2} \int_o^1 \phi(2\|f(x)-f_n(x)\|\,dx$$

$$+ \frac{1}{2} \int_o^1 \phi(2\|f_n(x)-g_n(x)\|)\,dx \leq \frac{1}{2}\phi(\tfrac{2}{n}) + \frac{1}{n}$$

Since $\phi(t) \to 0$ as $t \to 0^+$ the proof is finished.

Lemma 3

Let $(B_\tau, \tau \in I)$ be a family of sub-σ-fields of $\sum$. Suppose ϕ with Δ_2-condition.

If f_n convergs to f in L_X^{ϕ} then $E(f_n/B_\tau)$ convergs to $E(f/B_\tau)$ uniformly in B_τ, where $E(./B_\tau)$ denotes the conditional expectation relative to B_τ.

Proof

It may be proved with a slight, modification in the argument in [1], page 122 that if B is a sub-σ-field of Σ, then $\rho(E(G/B),\phi) \leq \rho(g,\phi)$ for all $g \in \tilde{L}_X^\phi$.

Now, given $\varepsilon > 0$, let m_o be a number such that $\max\left(\frac{\phi(T)+2}{2^{m_o}}, \frac{1}{K^{m_o}}\right) < \varepsilon$ where K,T are the constants in the Δ_2-condition.

Since $||f_n - f||_\phi \to 0$ $(n \to \infty)$ then $\rho(f_n - f,\phi) \to 0$ $(n \to \infty)$ so there is a number n_o such that if $n > n_o$ we have $\rho(f_n - f,\phi) < \frac{1}{K^{m_o}}$. This implies, by (1.3) and the first result in the proof, that $||E(f_n - f/B_\tau)||_\phi < \varepsilon$ for $n \geq n_o$ and it is true for all $\tau \in I$.

§3. PROOF OF THE THEOREM (1.2).

Suppose X has the Radon-Nikodym property and let (f_n, B_n) be a bounded martingale in L_X^ϕ with $\sigma(\bigcup B_n) = \Sigma$. By lemma 1 and (1.1), there is a function f in L_X^1 such that $f_n \longrightarrow f$ in L_X^1 and $f_n = E(f/B_n)$ as it may be seen in [1]. Since the convergence of martingales in L_X^1 implies the convergence almost everywhere, we obtain, using the continuity of ϕ that $\phi(||f_n(x)||) \longrightarrow \phi(||f(x)||)$ a.e. and by Fatou's Lemma

$$\int \phi(||f(x)||)\,dx \leq \lim\ \inf \int \phi(||f_n(x)||\,dx \leq M$$

Therefore f belongs to $\tilde{L}_X^\phi$, which coincides with L_X^ϕ.

We shall prove that $f_n \longrightarrow f$ in L_X^ϕ. From Lemma 2, we see that given $\varepsilon > 0$, there exists a number m_o and a sequence of simple functions such that

(3.1) $||s_m - f||_\phi < \varepsilon/2$ for $m \geq m_o$

and using Lemma 3 with B_n, there is m_1 in $\mathbb{N}$ such that

(3.2) $||E(f - s_m/B_n)||_\phi < \varepsilon/2$ for $m \geq m_1$ and for all n.

Since $\sigma(\bigcup B_n) = \Sigma$, we can take the functions s_m on measurable sets from $\bigcup B_n$.

Let m be a fixed number such that $m > \max(m_o, m_1)$. If $s_m = \sum_{i=1}^{p} x_{m,i} \chi_{E_{m,i}}$, let n_o be a number such that $E_{m,i} \subset B_{n_o}$ for $i = 1,\dots,p$ and in this case

$E(s_m/B_n) = s_m$ for $n \geq n_o$.

Therefore if $n \geq n_o$, by (3.1) and (3.2)

$$||f - f_n||_\phi \leq ||f - s_m||_\phi + ||s_m - f_n||_\phi =$$

$$= ||f - s_m||_\phi + ||E(s_m - f/B_n)|| < \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

To prove the converse we are going to use the characterization of the Radon-Nikodym property in terms of operators: For every $T: L^1[0,1] \longrightarrow X$ there exists a function f in L^∞_X such that $T(\varphi) = \int_0^1 \varphi(x) f(x)\,dx$ for $\varphi \in L^1[0,1]$ (see [1], page 63).

Let $T: L^1[0,1] \to X$ be a bounded operator. We consider B_n the σ-field generated by the dyadic intervals of length $\frac{1}{2^n}$, i.e. $B_n = \sigma(I_{n,i}\ ;\ i = 0,\dots,2^n - 1)$ where

$I_{n,i} = [\frac{i}{2^n}, \frac{i+1}{2^n})$.

Let $f_n = \sum_{i=o}^{2^n-1} 2^n T(\chi_{I_{n,i}}) \chi_{I_{n,i}}$. It is easy to prove that $E(f_{n+1}/B_n) = f_n$ and obviously $\sigma(\cup B_n) = \Sigma$. Since $||f_n|| = \sum_{i=o}^{2^n-1} 2^n ||T(\chi_{I_{n,i}})|| \chi_{I_{n,i}}$, it is clear that $||f_n(x)|| \leq ||T||$ for all $x \in [0,1]$. Then $\rho(f_n,\phi) \leq$ $\leq \phi(||T||)$ for all n, and we can find a function f in L_X^ϕ such that $f_n \longrightarrow f$ in L_X^ϕ. This is equivalent to $\phi(||f_n||) \longrightarrow \phi(||f||)$ in L^1 and therefore there is a subsequence $\phi(||f_{n_k}(x)||) \longrightarrow \phi(||f(x)||)$ a.e. Hence $\phi(||f(x)||) \leq \phi(||T||)$ a.e. and f belongs to L_v^∞.

To conclude the proof, we must only prove that

(3.3) $T(s) = \int_0^1 s(x) f(x)\, dx$ for all simple function on $\cup B_n$-measurable sets. First, we shall prove that

(3.4) $f_n = E(f/B_n)$.

If E is a B_n-measurable set, $\int_E f_n(x)\, dx = \int_E f_{n+k}(x)\, dx$ for $k \geq 1$ and then it is sufficent to prove that

$\int_E f_n(x)\, dx \to \int_E f(x)\, dx$ as $n \to \infty$. It is clear from the Holder inequality $\int_E \| f_n(x) - f(x) \|\, dx \leq \|f_n - f\|_\phi \|\chi E\|_\psi$

From (3.4), $\int_{I_{n,i}} f(x)\, dx = \int_{I_{n,i}} f_n(x)\, dx = T(\chi I_{n,i})$ and by linearity we obtain (3.3) and finish the proof.

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