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A THEOREM ON SCHAUDER DECOMPOSITIONS IN BANACH SPACES

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Abstract. In this paper we prove that in a Banach space all Schauder decompositions are shrinking iff all Schauder decompositions are boundedly complete.

1. Definitions and preliminary results

A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X is called a Schauder basis if for every $x \in X$ there exists a unique sequence $(\alpha_n)_{n=1}^{\infty}$ in R such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$, and this series converges with respect the norm of X. A sequence $(y_n)_{n=1}^{\infty}$ is called a basic sequence if it is a basis of his closed linear span.

A Schauder decomposition of X is a sequence $(X_i)_{i=1}^{\infty}$ of closed subspaces of X such that for every x in X there exists a unique sequence $(x_i)_{i=1}^{\infty}$ with $x_i \in X_i$ for all i and $x = \sum_{i=1}^{\infty} x_i$. Every Schauder decomposition of X is related with a sequence of continuous projections $P_n : X \longrightarrow X$ defined by

$$P_n(x) = P_n\left(\sum_{i=1}^{\infty} x_i\right) = \sum_{i=1}^{n} x_i$$

In all this paper, the linear span of an element $x \in X$ is denoted by $[x]$ and the closed linear span of the subspaces $(X_i)_{i=n}^{m}$ $(1 \leq n < m \leq \infty)$ is denoted by $[X_i]_{i=n}^{m}$.

The following theorem characterizes the Schauder decompositions and it can be found in [5].

1. Theorem: Let X be a Banach space and $(X_n)_{n=1}^{\infty}$ a sequence of closed subspaces of X. The following are equivalent:

i) $(X_n)_{n=1}$ is a Schauder decomposition of X.

ii) There exists a sequence $(P_n)_{n=1}^{\infty}$ of continuous projections P_n: $X \longrightarrow [X_i]_{i=1}^{n}$ such that $P_n P_m = P_{\min(m,n)}$ and $\lim_{n\to\infty} P_n(x)=x$ for every x in X.

iii) There exists a sequence $(P_n)_{n=1}^{\infty}$ of continuous projections P_n: $X \longrightarrow [X_i]_{i=1}^{n}$ such that $P_n P_m = P_{\min(m,n)}$ and $(P_n)_{n=1}^{\infty}$ is uniformly bounded.

To $\sup_n \|P_n\|$ is called norm of the decomposition.

A Schauder decomposition $(X_n)_{n=1}^{\infty}$ in a Banach space X is called boundedly complete if for every sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in X_n$ such that $\sup_n \|\sum_{i=1}^{n} x_i\| < \infty$, the sequence $(\sum_{i=1}^{n} x_i)_{n=1}^{\infty}$ converges towards an element x in X. And it is called shrinking if for every $x^* \in X$, $\lim_{n\to\infty} \|x^*\|_n = 0$, where

$$\|x^*\|_n = \sup\{|x^*(x)| \text{ with } x \in [X_i]_{i=n+1}^{\infty} \text{ and } \|x\| \leq 1\}.$$

Boundedly complete and shrinking basis and basic sequences are defined in a similar way.

Singer (cf. [6]) has proved that in a Banach space all basic sequences are boundedly complete if and only if all basic sequences are shrinking. Afterwards Zippin (cf. [7] and [3]) proved a similar theorem for Schauder basis of X. Our purpose in this paper is to prove that in a Banach space all Schauder decompositions are boundedly complete iff all Schauder decompositions are shrinking.

If X is a locally bounded F-space, then there exists p ($0<p<1$) such that the topology of X is originated by a p-norm. In this case X is called p-Banach space (cf. [1] and [4]). Let X be a p-Banach space such that X^* separates points of X and let $J: X \longrightarrow X^{**}$ be the canonical imbedding of X into its bidual. We define in X the norm $\|.\|^{**}$:

$$\|x\|^{**} = \|J(x)\| \qquad \text{if} \quad x \in X.$$

The Mackey topology of X is originated by this norm (cf. [2]) and it is called the Mackey norm of X. The Mackey completion of X is denoted by $\overline{J(X)}$.

All the above definitions for Banach spaces can be extended to p-Banach spaces.

2. Shrinking and boundedly complete Schauder decomposition.

2. Lemma. Let $(X_n)_{n=1}^{\infty}$ be a Schauder decomposition of a Banach space X and let $(P_n)_{n=1}^{\infty}$ be its sequence of projections. We suppose that each X_n admits a topological decomposition $X_n = Y_n \oplus Z_n$. The following are equivalent:

i) $(Y_1, Z_1, \ldots, Y_n, Z_n, \ldots)$ is a Schauder decomposition of X.

ii) If A_n is the continuous projection from X_n into Y_n, then $\sup_n \|A_n\| < \infty$.

Proof: i $\Rightarrow$ ii. If $(Q_n)_{n=1}^{\infty}$ is the sequence of projections of $(Y_1, Z_1, \ldots$ $\ldots, Y_n, Z_n, \ldots)$, as $A_n = Q_{2n-1}|_{X_n}$, the statement ii is proved.

ii ⇒ i. If $\sup_n \|A_n\| < \infty$, we define

$$Q_{2n} = P_n$$
$$Q_{2n+1} = P_n + A_n(P_n - P_{n-1}) \qquad n > 1$$
$$Q_1 = A_1 P_1$$

and thus $(Q_n)_{n=1}^{\infty}$ is a uniformly bounded sequence of projections which defines the decomposition $(Y_n, Z_n)_{n=1}^{\infty}$ because of theorem 1. //

Remark that if any of the previous subspaces is 0, it must be taken away in the decomposition.

<u>3. Corollary</u>. <u>Let</u> $(X_n)_{n=1}^{\infty}$ <u>be a Schauder decomposition of a Banach space</u> X <u>and</u> $(x_n)_{n=1}^{\infty}$ <u>a normalized sequence in</u> X <u>with</u> $x_n \in X_n$. <u>For every</u> n <u>there exists an hyperplane</u> W_n <u>of</u> X_n <u>such that</u> $([x_1], W_1, \ldots, [x_n], W_n, \ldots)$ <u>is a Schauder decomposition of</u> X.

Proof: As $\|x_n\| = 1$, we can define $A_n(x) = u_n^*(x)x_n$, where $u_n^* \in X_n^*$ and $u_n^*(x_n) = \|u_n^*\| = 1$. //

<u>4. Lemma</u>. <u>Let</u> X <u>be a Banach space and a Schauder decomposition of the form</u> $([y_1], W_1, \ldots, [y_n], W_n, \ldots)$ <u>where</u> $(y_n)_{n=1}^{\infty}$ <u>satisfies</u> $\inf_n \|y_n\| = C > 0$ <u>and</u> $\sup_n \|\sum_{i=1}^{n} y_i\| = M < \infty$. <u>We define the sequence</u> $(v_n)_{n=1}^{\infty}$ <u>by</u> $v_n = \sum_{i=1}^{n} y_i$. <u>Then</u> $([v_1], W_1, \ldots, [v_n], W_n, \ldots)$ <u>is a Schauder decomposition of</u> X.

Proof: Let $(P_n)_{n=1}^{\infty}$ be the sequence of projections of $(X_n)_{n=1}^{\infty}$ and let K be its norm. Each $P_{2n-1} - P_{2n-2}$ (the projection over $[y_n]$) is originated by a $y_n^* \in X^*$ according to

$$(P_{2n-1} - P_{2n-2})(x) = y_n^*(x)y_n \quad \text{if} \quad x \in X.$$

and thus

i) $y_n^*(y_m) = \delta_{n,m}$

ii) $\| y_n^* \| \| y_n \| \leq 2K$ for every n, and

iii) $y_n^*|_{W_m} = 0$ for every n and m.

As $\inf_n \| y_n \| = C > 0$, from ii) we obtain that $\sup_n \| y_n^* \| \leq 2K/C$. Let $(v_n^*)_{n=1}^{\infty}$ be defined be $v_n^* = y_n^* - y_{n+1}^*$. It is easy to check that $v_n^*(v_m) = \delta_{n,m}$.

We define the sequence of projections by

$$A_{2n}(x) = \sum_{k=1}^{n} (P_{2k} - P_{2k-1})(x) + \sum_{k=1}^{n} v_k^*(x) v_k$$

$$A_{2n+1}(x) = A_{2n}(x) + v_{n+1}^*(x) v_{n+1}.$$

Because of the theorem 1 we only need to prove that $(A_n)_{n=1}^{\infty}$ is uniformly bounded, and, because of the last considerations, it shall be proved if we prove that $\sup_n \| A_{2n} \| < \infty$:

$$\| A_{2n}(x) \| = \| P_{2n}(x) - \sum_{k=1}^{n} y_k^*(x) y_k + \sum_{k=1}^{n} (y_k^*(x) - y_{k+1}^*(x)) (\sum_{i=1}^{k} y_i) \| \leq$$

$$\leq K \| x \| + \| - \sum_{k=1}^{n} y_k^*(x) y_k + \sum_{k=1}^{n} (y_k^*(x) - y_{n+1}^*(x)) y_k \| \leq$$

$$\leq K \| x \| + \| y_{n+1}^* \| \| x \| \| \sum_{k=1}^{n} y_k \| \leq (K + \frac{2K}{C} M) \| x \| .$$

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5. Lemma. Let X be a Banach space and $([y_1],W_1,\dots,[y_n],W_n,\dots)$ a Schauder decomposition of X, where $(y_n)_{n=1}^{\infty}$ satisfies $\sup_n \|y_n\| = M < \infty$. We define the sequence $(v_n)_{n=1}^{\infty}$ by $v_1=y_1$ and $v_n=y_n-y_{n-1}$. Then the following are equivalent:

i) $([v_1],W_1,\dots,[v_n],W_n,\dots)$ is a Schauder decomposition of X.

ii) There exists $x^*\in X^*$ such that

a) $x^*(y_n) = 1$ for every n

b) $x^*|_{W_m} = 0$ for every m

Proof: If, for every n, there is a continuous projection from X into $[v_n]$ parallel to the other subspaces, the existence of $x^*\in X^*$ satisfying a) and b) is necessary. We suppose that there exists a such x^*. We define $(y_n^*)_{n=1}^{\infty}$ as in the preceding lemma, and if we consider the sequence

$$v_1^* = x^* \qquad \text{and} \qquad v_n^* = x^* - \sum_{k=1}^{n-1} y_k^*$$

the orthogonal relations $v_n^*(v_m) = \delta_{n,m}$ hold.

Let $(A_n)_{n=1}^{\infty}$ be a sequence of projections as in the preceding lemma. We must prove that $\sup_n \|A_n\| < \infty$. For every m, $x^*|_{W_m} = 0$, and hence $x^*(x) = \sum_{n=1}^{\infty} y_n^*(x)$ for every x in X and so $(v_n^*)_{n=1}^{\infty}$ converges weakly to 0 and $\sup_n \|\sum_{k=1}^{n} y_k\| = M_1 < \infty$. lso $\sup_n \|v_n\| \leq 2M$, again we must only prove that $\sup_n \|A_{2n}\| < \infty$:

we have

$$\sum_{k=1}^{n} v_k^*(x)v_k = x^*(x) + \sum_{k=2}^{n} x^*(x)(y_k-y_{k-1}) - \sum_{k=2}^{n}\left[\left(\sum_{i=1}^{k-1} y_i^*\right)(x)\right](y_k-y_{k-1}) =$$

$$= x^*(x)y_1 + x^*(x)(y_n - y_1) - \sum_{k=1}^{n-1} y_k^*(x)\ y_n + \sum_{k=1}^{n-1} y_k^*(x)y_k =$$

$$= \sum_{k=1}^{n} y_k^*(x)y_k + x^*(x)y_n - \sum_{k=1}^{n} y_k^*(x)y_n,$$

and thus

$$A_{2n}(x) = P_{2n}(x) + x^*(x)y_n - \sum_{k=1}^{n} y_k^*(x)y_n.$$

And finally:

$$\| A_{2n}(x)\| \le \| P_{2n}\|\|x\| + \|x^*\|\ \|x\|\ \|y_n\| + \| \sum_{k=1}^{n} y_k^* \|\ \|x\|\ \|y_n\|$$

and

$$\| A_{2n}\| \le K + M\|x^*\| + M_1 M$$

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Now we can prove the main result:

6. Theorem. Let X be a Banach space. The following statements are equivalent:

i) All Schauder decompositions of X are shrinking

ii) All Schauder decompositions of X are boundedly complete

Proof. $i \Rightarrow ii$. Let $(X_n)_{n=1}^{\infty}$ be a non boundedly complete Schauder decomposition of X. There exists then a sequence $(x_i)_{i=1}^{\infty}$ with $x_i \in X_i$ such that $\sup_n \| \sum_{i=1}^{n} x_i \| = 1$ and $(\sum_{i=1}^{n} x_i)_{n=1}^{\infty}$ is not a Cauchy sequence, and thus, there exist ε and a strictly increasing sequence $(m_k)_{k=1}^{\infty}$ such that $\varepsilon < \| \sum_{i=m_{k-1}+1}^{m_k} x_i \| \le 2$ for every k. We define $Y_k = [X_i]_{i=m_{k-1}+1}^{m_k}$ and $y_k = \sum_{i=m_{k-1}+1}^{m_k} x_i$ if $k \ge 1$. $(Y_k)_{k=1}^{\infty}$ is a Schauder decomposition of X with $y_k \in Y_k$. Because of the corollary 3, for each k there exists a hyperplane W_k of Y_k such that $([y_1], W_1, \ldots$

$\dots,[y_n],\ W_n,\dots)$ is a Schauder decomposition. Because of the lemma 4, the sequence $(v_k)_{k=1}^{\infty}$ defined by $v_n = \sum_{i=1}^{n} y_i$ originates the Schauder decomposition $([v_1],W_1,\dots,[v_n],W_n,\dots)$ which is not shrinking because of $y_1^*(v_k) = 1$ for every $k \geq 1$.

ii ⇒ i. Let $(X_n)_{n=1}^{\infty}$ be a non shrinking Schauder decomposition of X. There exist then $x^* \in X$ with $\|x^*\| = 1$, $\varepsilon > 0$, a strictly increasing sequence of index $(m_k)_{k=1}^{\infty}$ and a sequence $(y_k)_{k=1}^{\infty}$ with $y_k \in Y_k = [X_i]_{i=m_{k-1}+1}^{m_k}$ such that:

a) $1 \leq \|y_n\| \leq 1/\varepsilon$

b) $x^*(y_n) = 1$.

We can choose the hyperplane $W_k = Y_k \cap \operatorname{Ker} x^*$ and using the lemma 5, if $v_1 = y_1$ and $v_n = y_n - y_{n-1}$, then $([v_1],W_1,\dots,[v_n],W_n,\dots)$ is a Schauder decomposition of X which is not boundedly complete because of

$$\left\| \sum_{k=2}^{n} v_k \right\| = \left\| \sum_{k=2}^{n} (y_k - y_{k-1}) \right\| = \| y_n - y_1 \| \leq 2/\varepsilon$$

while

$$\| v_k \| = \| y_k - y_{k-1} \| \geq \frac{1}{k} \| y_{k-1} \|$$

where K is the norm of $(X_n)_{n=1}^{\infty}$.

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With certain modifications, this theorem has an extension to p-Banach spaces (if its dual separates points). The Mackey topology of this spaces plays an important role in this extension. We need before a definition: we shall say that a Schauder decomposition $(X_n)_{n=1}^{\infty}$ in a p-Banach space is an almost boundedly complete decomposition if for every sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in X_n$ such that $\sup_n \| \sum_{k=1}^{n} x_k \| < \infty$, the sequence $(\sum_{k=1}^{n} x_k)_{n=1}^{\infty}$ converges in $(\overline{J(X)}, \|.\|^{**})$.

We must point out that if $(X_n)_{n=1}^{\infty}$ is boundedly complete then it is also almost boundedly complete. Almost boundedly complete basis of X are defined in a similar way.

7. Theorem. Let X be a p-Banach space. The following are equivalent:

i) All Schauder decompositions of X are shrinking.

ii) All Schauder decompositions of X are almost boundedly complete.

Proof: Similar to the proof of Theorem 6, and it can be found in [2]

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