Pub. Mat. UAB
Vol. 30 Nº 1 Maig 1986

ON F' - CLOSURE OF $\tilde{F}$ - HOMOGENEOUS GROUPS

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ABSTRACT: Given a homomorph F, a finite group G is a $\tilde{D}_F$ group if G has an F-projector F such that every solvable F-subgroup is contained in some conjugate of F. G is $\tilde{F}$-homogeneous if $N_G(X)/C_G(X) \in F$ for every solvable F-subgroup of G. The following theorem is proved. Assume that F is an s-closed extensible homomorph and G is a $\tilde{D}_F$ group which is $\tilde{F}$-homogeneous, then $G \in F' F$.

This theorem generalizes results about D_π π-homogeneous groups and π'-closure.

Introduction

All groups considered in this paper are finite. In [1], it is shown that if π is a set of prime numbers, then every π- homogeneous D_π-group is π'-closed. The following equivalence is then trivial: $G/0_{\pi'}(G)$ is a solvable π-group if and only if G is a π-homogeneous D_π -group with solvable Hall π-subgroups.

1980 Mathematics Subject Classification 20D99
Key words and phrases, s-closed extensible homomorph, $D_{\tilde{F}}$ group, $\tilde{F}$-homogeneity.
* The second author was partially supported by a National Science Foundation Grant.

In this paper, as in [4], we consider an extensible ans s-closed homomorph F and generalize these results.

We recall that a non-empty class of groups F is a homomorph if whenever $G \in F$, then all homomorphic images of G are contained in F. A homomorph is s-closed if whenever $G \in F$, then all subgroups of G are contained in F. A homomorph F is extensible if whenever both G/N and N are contained in F, then $G \in F$. Let F denote any homomorph which is closed under normal subgroups; then F', the derived class of F, is defined by $F' = \{G \mid S/N \in F$ implies that $S = N$ for each subgroup S of $G\}$. (See [4]). For such a homomorph F, the radical $G_{F'}$ is defined for a group G by $G_{F'} = \langle N \trianglelefteq G \mid N \in F' \rangle$ is defined for a group G. A group G is defined to be F'-closed if $G/G_{F'} \in F$ or equivalently if $G \in F' F$.

Let F be a homomorph, a group G is defined to be a D_F group if G has an F-projector F such that every solvable F-subgroup of G is contained in some conjugate of F.

G is defined to be a $\underline{D}_F$ group if G has exactly one conjugacy class of F-projectors and every F-subgroup of G is contained in an F-projector.

G is defined to be F-homogeneous if $N_G(X)/C_G(X) \in F$ for every solvable F-subgroup X of G, is F-homogeneous if $N_G(X)/C_G(X) \in F$ for every F-subgroup X of G.

We note that if G is a π-homogeneous D_π-group, then it is direct to see that G is an F-homogeneous D_F-group where F is s-closed extensible homomorph of π-groups. We show

(Lemma 3) that $G_{F'} = O_{\pi'}(G)$ for this F. Thus the following theorem is a generalization of the result mentioned in the first paragraph.

Theorem A

Assume that F is an s-closed extensible homomorph and G is a $\tilde{D}_F$ group which is $\tilde{F}$-homogeneous, then $G \in F' F$.

The following corollary generalizes the second remark in the first paragraph and characterizes the product class $F'(F \cap H)$ when H is a solvable class.

Corollary B

Let F be an s-closed and extensible homomorph and H a class of solvable groups. Then the following are equivalent:

(i) $G \in F'(F \cap H)$.

(ii) G is a D_F-group with F-projectors belonging to H and G is F-homogeneous.

(iii) G is a $\tilde{D}_F$-group with an F-projector F belonging to H an G is F-homogeneous.

Corollary B strengthens Theorem III 2.6 of [4] by replacing the nilpotency hypothesis in the F-projectors of that theorem by solvability.

Section One

Lemma 1

Let F be an s-closed homomorph. Every subgroup H of an $\tilde{F}$-homogeneous group G is an $\tilde{F}$-homogeneous group.

Proof:

Let X be a solvable F-subgroup of H, then X is an F-subgroup of G. Whence $N_G(X)/C_G(X) \in F$. Now $N_H(X)/C_H(X) =$ $= (N_G(X) \cap H)/(C_G(X) \cap H) \cong (N_G(X) \cap H)C_G(X)/C_G(X)$ implies that $N_H(X)/C_H(X)$ is isomorphic to a subgroup of $N_G(X)/C_G(X)$. Since F is s-closed, $N_H(X)/C_H(X) \in F$.

Lemma 2

Let F an extensible homomorph of finite groups and let G be an F-homogeneous group. Then:

(i) G/K is an $\tilde{F}$-homogeneous group for each normal solvable F-subgroup K of G.

(ii) If F is also an s-closed homomorph, with $G/K \in F' F$ and $K \in F$ where K is solvable, then $G \in F' F$.

Proof:

(i) Let X/K be a solvable F-subgroup of G, then X is solvable so $N_G(X)/C_G(X) \in F$. Now $\rho : N_G(X) \longrightarrow N_{G/K}(X/K)/$ $/ C_{G/K}(X/K)$ defined by $\rho(g) = \overline{gK}$ is an epimorhism whose kernel contains $C_G(X)$.

Therefore, $N_{G/K}(X/K) / C_{G/K}(X/K)$ is an epimorphic image of the F-group $N_G(X)/C_G(X)$. Since F is a homomorph, $N_{G/K}(X/K) / C_{G/K}(X/K)$ lies in F. Hence G/K is a $\widetilde{F}$-homogeneous group.

(ii) Let M/K be the F'-radical $(G/K)_{F'}$ of G/K. Since $K \in F$ and $M/K \in F'$, $(|K|, |M:K|) = 1$ as F is s-closed. Now the Schur-Zassenhaus theorem yields a subgroup L of M such that $M = KL$ and $M/K \cong L$. Let K_p denote a Sylow p subgroup of K. By the Frattini argument $M = N_M(K_p)K$. Thus $|L| \mid |N_M(K_p)|$.

Further, $N_M(K_p) \cap K$ is a normal Hall subgroup of $N_M(K_p)$. Hence by the Schur-Zassenhaus theorem, $N_M(K_p)$ has a Hall subgroup L_1 of order $|L|$ and L_1 and L are conjugate in M. Thus, by Sylow theory, we may choose notation so that $L \subseteq N_M(K_p)$.

Therefore, $LC_M(K_p)/C_M(K_p)$ is contained in $N_M(K_p)/C_M(K_p)$. Since G is $\widetilde{F}$-homogeneous, Lemma 1 implies that $N_M(K_p)/C_M(K_p) \in F$. Now $LC_M(K_p)/C_M(K_p)$ must be an F-group because F is s-closed. However, $L \in F'$ implies that $LC_M(K_p)/C_M(K_p) \in F'$.

Thus, $|LC_M(K_p)/C_M/K_p)| = 1$ and $[L,K_p] = 1$. Repeating the argument for all primes p dividing $|K|$, we conclude that $M = L \times K$ and $L = M_{F'}$.

Now L is a characteristic subgroup of M so $L \Delta G$. Finally, $G/L / M/L \cong G/M$, $M/L \cong K \in F$, and $G/M \cong G/K / M/K \in F$. Since F is extensible, $G/L \in F' F$.

We state Lemma 3 and Proposition 4 in greater generality than needed for independent interest. We note that every s-closed extensible homomorph F is an s-closed and saturated formation by [5, I (1.2), (2.1), I 2.5)] and the proof of [5, I (1.14)].

Lemma 3

Assume F is an s-closed and saturated formation and G is a $\tilde{D}_F$-group with F-projector F such that every solvable F-group in G lies in some F^g, $g \in G$. Let π denote the set of prime divisors of $|F|$, then F is a Hall π-subgroup of G and $G_{F'} = O_{\pi'}(G)$.

Proof:

Let F_p denote a non-trivial Sylow p-subgroup of F. If F_p is not a Sylow p-subgroup of G, there is a p-group K such that $F_p \Delta K$ and $[K : F_p] = p$.

Since F is s-closed and saturated, K must belong to F following [5, I: (3.1)]. But K is solvable so $K \subseteq F^g$ which is a contradiction. Hence F is a Hall π-subgroup of G.

Let R be any π'-subgroup of G. If $R \notin F'$, there is $N \Delta T$ with $T/N \in F$ and $R \supseteq T \supset N$. Let v be a prime dividing $[T:N]$, then F contains Z_v, a cyclic group of order v. However, R also contains $\langle x \rangle$ a cyclic group of order v and $\langle x \rangle \in F$.

Since $\langle x \rangle$ is solvable, $\langle x \rangle \subseteq F^g$ which contradicts $(|F|, |R|) = 1$. Thus $R \in F'$ and in particular $O_{\pi'}(G) \subseteq G_{F'}$. If $v | (|G_{F'}|, |F|)$, then both F and F' contain a cyclic group of order v since $\mathcal{F}$ and $\mathcal{F}'$ are s-closed. This contradicts $\mathcal{F} \cap \mathcal{F}' = \{1\}$. Hence, $G_{\mathcal{F}'} = 0_{\pi'}(G)$.

Proposition 4

Let $\mathcal{F}$ be an s-closed saturated formation. Assume G is a $\tilde{D}_{\mathcal{F}}$-group with $\mathcal{F}$-projector F such that every solvable $\mathcal{F}$-subgroup of G lies in some F^g. If whenever two elements in F are conjugate in G then they are conjugate in F, then $G \in \mathcal{F}'\ F$.

Proof:

Let π denote the set of prime divisors of $|F|$. By Lemma 3, F is a Hall π-subgroup of G. Let E be an elementary subgroup of G such that $|E| \mid |F|$, then $E = Z \times P$ where P is a p-group and Z is cyclic. Following [5,I.(3.1)], P and every Sylow subgroup of Z lie in F. Now $\mathcal{F}$ a formation yields $E \in \mathcal{F}$. Hence, $E \subseteq F^g$ for some $g \in G$. The Brauer-Suzuki Theorem [3, Th. (8.22)] implies that $G = 0_{\pi'}(G)F$. By Lemma 3, $0_{\mathcal{F}'}(G) = G_{\mathcal{F}'}$ whence $G \in \mathcal{F}'\ F$.

Proof of Theorem A:

The proof is divided into three parts. Let G be a minimal counterexample to the theorem, F be a $\mathcal{F}$-projector such that every solvable $\mathcal{F}$-subgroup lies in a conjugate of F, and let π denote the set of prime divisors of $|F|$.

(A) There are no normal non-trivial solvable $\mathcal{F}$-subgroups of G.

Assume K is a nontrivial normal solvable $\mathcal{F}$-subgroup of G, then $K \subseteq F$ and F/K is a $\mathcal{F}$-projector of G/K. Suppose X/K is any solvable $\mathcal{F}$-subgroup of G/K, then X is solvable and $X \in \mathcal{F}$. Thus, $X \subseteq F^g$ and $X/K \in (F/K)^{gK}$. Hence, G/K is a $\widetilde{D}_{\mathcal{F}}$-group. By Lemma 2, G/K is $\widetilde{\mathcal{F}}$-homogeneous. The minimality of G yields $G/K \in \mathcal{F}'\,\mathcal{F}$. Now $G \in \mathcal{F}'\,\mathcal{F}$ follows from Lemma 2.

(B) Let S be a non-trivial p-subgroup of F. Then

(i) $N_G(S)$ is $\mathcal{F}'$-closed.

(ii) $N_G(S) = N_F(S)0_{\pi'}(C_G(S))$, and

(iii) $S \subseteq F^w$ implies that $F^w = F^y$ where $y \in 0_{\pi'}(C_G(S))$.

We first show that (B)(i) and (ii) hold for any $1 \neq S$ such that

$$(*) S \subseteq F^w \text{ implies } N_{F^w}(S) = (N_F(S))^r$$

for $r \in N_G(S)$.

Assume $(*)$ holds, we will show $N_G(S)$ is a $\widetilde{D}_{\mathcal{F}}$-group and that $N_F(S)$ is an $\mathcal{F}$-projector of $N_G(S)$. IF K is any Sylow v subgroup of $N_G(S)$ for v a prime in π, then KS is a p group if $v = p$ or a (p,v)-group. In particular, KS is solvable so $KS \subseteq F^w$. By $(*)$ $KS \subseteq (N_F(S))^r$ for some $r \in N_G(S)$. Thus, $N_F(S)$ is a Hall π-subgroup of $N_G(S)$.

Let U be a subgroup of $N_G(S)$ which contains $N_F(S)$ with $W \Delta U$ and $U/W \in F$. If t is any prime dividing $[U:W]$, then $\mathcal{F}$ s-closed implies that F contains a cyclic group of order t. However, U also contains a cyclic subgroup $<x>$ of order t. Since $<x>$ is a solvable F-group, $<x> \subseteq F^g$. Therefore, U/W is a π-group. Since $N_F(S)$ is a Hall π-subgroup of $N_G(S)$, $U = WN_F(S)$. Hence, $N_F(S)$ is an F-projector. It T is a solvable F-subgroup of $N_G(S)$, then TS is solvable and $TS \subseteq F^w$ for some $w \in G$.

Now (*) implies that $TS \subseteq (N_F(S))^r$ for some $r \in N_G(S)$. Hence $N_G(S)$ is a $\tilde{D}_F$-group. By Lemma 1, $N_G(S)$ is $\tilde{F}$-homogeneous. Using (A), $|N_G(S)| < |G|$ so $N_G(S)$ is F'-closed. Lemma 3 implies that $N_G(S)_{F'} = 0_{\pi'}(N_G(S))$. However, $0_{\pi'}(N_G(S)) \subseteq C_G(S))$ since $N_G(S)/C_G(S) \in F$ and is thus a π-group. Hence $N_G(S)_{F'} =$ $= 0_{\pi'}(C_G(S))$ and $N_G(S) = 0_{\pi'}(C_G(S))N_F(S)$ follows directly.

We now prove (B) by induction on $[F_p : S]$ where F_p is a Sylow p subgroup of F. Assume first that $[F_p : S] = 1$, then S is a Sylow p-subgroup of G. Hence, $S \subseteq F \cap F^w$ yields $S = S^{fw}$ where $f \in F$. Therefore, $fw \in N_G(S)$ and (*) is satisfied so (i) and (ii) are proved.

Now $N_G(S) = N_F(S) 0_{\pi'}(C_G(S))$ yields $fw = f_1 y$ where $f_1 \in N_F(S)$ and $y \in 0_{\pi'}(C_G(S))$. Hence $F^w = F^y$ and (iii) follows.

We assume (B) is proved for all p-subgroups T of F_p such that $[F_p : T] < [F_p : S]$ and $|S| > 1$. Let T be a Sylow p subgroup of $N_F(S)$, then $|T| > |S|$ and $S \subseteq T \subseteq S_1 \subseteq F^g$ where S_1 is a Sylow p-subgroup of $N_G(S)$.

By induction $F^g = F^y$ where $y \in 0_{\pi'}(C_G(T))$. Hence, $N_{F^g}(S) = (N_F(S))^y$ so T is a Sylow subgroup of $N_G(S)$. If $S \subseteq F \cap F^w$, let U be a Sylow p-subgroup of $N_{F^w}(S)$.

Then $S \subseteq U = T_1^r \subseteq T^r$ where $r \in N_G(S)$ and $|T_1| > |S|$. Now $T_1 \subseteq F^{wr^{-1}} \cap F$ yields $F^{wr^{-1}} = F^{y_1}$ where $y_1 \in 0_{\pi'}(C_G(T_1))$.

Therefore, $N_{F^w}(S) = (N_F(S))^{y_1 r}$ where $y_1 r \in N_G(S)$ and (*) is satisfied.

Hence, (B) (i) and (ii) are proved. Further, $F^w = F^{y_1 r}$ where $y_1 r \in N_G(S) = N_F(S)0_{\pi'}(C_G(S))$ yields $y_1 r = fy$ where $f \in N_F(S)$ and $y \in 0_{\pi'}(C_G(S))$. Now (B) (iii) follows.

(C) Final Contradiction.

By Lemma 3, F is a Hall π-subgroup of G. Thus, $N_G(F) = FM$ where M is a Hall π'-subgroup of $N_G(F)$. Let $p \in \pi$, then the Frattini argument and the Schur-Zassenhaus theorem imply that there is a Sylow p subgroup F_p of F Such that $N_G(F) = N_G(F_p)F$ and $M \subseteq N_G(F_p)$. By (B) (ii), $M \subseteq C_G(F_p)$. Repeating this argument for all primes p in π, we see that $M \subseteq C_G(F)$ and $N_G(F) = F \times M$.

Suppose $z_1 = z_2^w$ where z_1 and $z_2 \in F^\#$, then $z_1 \in F^w \cap F$. Because of (B), an argument analogous to that used in the proof of [1, Lemma 5] implies that $F^w = F^y$ for some $y \in 0_{\pi'}(C_G(z_1))$. Thus, $wy^{-1} \in N_G(F)$ and by the previous paragrah $w = fmy$ where $f \in F$ and $m \in M$. Hence $z_2 = z_1^{y^{-1}m^{-1}f^{-1}} = z_1^{f^{-1}}$ so z_1 and z_2 are conjugate in F. The theorem now follows from Proposition 4.

Proof of Corollary B:

(i) ⇒ (ii). [4,II. (2.7)] yields that G is a $D_{\mathcal{F}}$-group, and [4, III (2.2)] implies that G is $\mathcal{F}$-homogeneous. Let F be an $\mathcal{F}$-projector of G, then $G/G_{\mathcal{F}'} \cong F$ and $G/G_{\mathcal{F}'} \in \mathcal{F} \cap \mathcal{H}$.

Therefore, $F \in H$.

(ii) ⇒ (iii) It is obvious.

(iii) ⇒ (i) By Theorem A, $G \in \mathcal{F}'\mathcal{F}$.

Now $G/G_{\mathcal{F}'} \cong F \in \mathcal{F}$ implies that $G \in \mathcal{F}'(\mathcal{F} \cap H)$.

As noted in the introduction if a group G is π-homogeneous and $D_{\pi'}$, then G is $\widetilde{\mathcal{F}}$-homogeneous and $\widetilde{D}_{\mathcal{F}}$ where $\mathcal{F}$ is the s-closed extensible homomorph of π-groups.

The following generalization of the theorem in [1] may be obtained easily from Theorem A and Lemma 3.

Corollary C:

Assume G is a finite group which is π-homogeneous and has a Hall π-subgroup which contains a conjugate of every solvable π-subgroup of G, then G is π'-closed.

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Rebut el 6 de juny del 1985

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