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ONE MORE FACET OF A MAPPING THEOREM FOR LUSTERNIK SCHNIRELMANN CATEGORY

H. Scheerer

Let $f: X \to Y$ be a map of simply connected CW-spaces. When is $cat(X) \leq cat(Y)$? In [2] an answer has been given within rational homotopy category and in [3] within rational or tame homotopy theory. Here we prove a corresponding result using the theory of [1].

Let R be a subring of $\mathbb{Q}$ containing 1/2, 1/3 throughout.

<u>Proposition</u>: Let $f: X \to Y$ be a map of simply connected R-local CW-spaces. Let ΩY be decomposable over R ; assume that the homomorphism $(\Omega f)_*: \pi_*(\Omega X) \to \pi_*(\Omega Y)$ of M-Lie algebras (in the sense of [1]) has a left inverse in the category of M-Lie algebras.

Then $cat(X) \leq cat(Y)$.

We first explain the notions used in the proposition.

We work in the homotopy category of pointed spaces.

A connected complex X is called "R-local", if the reduced homology $\tilde{H}_*(X;\mathbb{Z})$ is an R-module. For X nilpotent we denote by $X \to X_R$ the localization of X with respect to the set of primes not invertible in R .

For $n > 0$ we set

$$\Omega_R^n := \begin{cases} S_R^n & n \text{ odd,} \\ \Omega\Sigma S_R^n & n \text{ even,} \end{cases}$$

where Σ, Ω denotes the suspension resp. loop space functor.

For any connected finite dimensional complex X let $M^i(X) := R$ for $i = 0$ and $M^i(X) := [X, \Omega_R^i]$ for $i > 0$.

An H-space E is called "decomposable over R", if E is homotopy equivalent to a weak direct product ${}^w\prod_{i\in I} \Omega_R^{n_i}$.

Let E be a connected grouplike R-local H-space. Then the Lie algebra $\pi_*(E)$ (with the Samelson product as Lie bracket) has an additional structure as an M-Lie algebra (see [1], chap. V, (2.12) and [4], section 7), i.e. there is an operation $(i,r > 0)$

$$\pi_i(E) \times M^i(S^r) \to \pi_r(E) \quad , \quad (\alpha,\zeta) \to \alpha \odot \zeta \; ,$$

defined by the formula

$$\alpha \odot \zeta := \begin{cases} \alpha\zeta & \text{for } i \text{ odd,} \\ r_E(\Omega\Sigma\alpha)\zeta & \text{for } i \text{ even,} \end{cases}$$

where $r_E\colon \Omega\Sigma E \to E$ is an H-retraction (see [4], section 7). (Note that we do not notationally distinguish between maps and there homotopy classes). This operation obeys certain laws ([1], loc.cit.) which we do not need here.

<u>Proof of the proposition</u>: Let ΩY be homotopy equivalent to ${}^w\prod_{i\in I} \Omega_R^{n_i}$. Let $\alpha_i\colon S^{n_i} \to \Omega Y$ represent a generator (over R) of the direct summand $\pi_{n_i}(\Omega_R^{n_i})$ of $\pi_{n_i}(\Omega Y)$. For $m > 0$ let $M^{m,*} := \bigoplus_{j>0} M^m(S^j)$; the maps α_i induce maps $M^{n_i,*} \to \pi_*(\Omega Y)$, $\zeta \to \alpha_i \odot \zeta$, such that the map $\bigoplus_{i\in I} M^{n_i,*} \to \pi_*(\Omega Y)$, $(\zeta_i)_{i\in I} \to \Sigma\, \alpha_i \odot \zeta_i$, is an isomorphism of R-modules. (This is proved in [1], chap. V, (3.13) in case $H_*(\Omega Y;R)$ is of finite type over R ; but this assumption is not needed).

Let now ϕ be a left inverse to $(\Omega f)_*\colon \pi_*(\Omega X) \to \pi_*(\Omega Y)$.

We define maps $\beta_i: \Omega_R^{n_i} \to \Omega X$ using $\phi(\alpha_i)$ as follows:

$$\beta_i := \begin{cases} \phi(\alpha_i) & \text{for } n_i \text{ odd,} \\ r_{\Omega X}(\Omega\Sigma(\phi(\alpha_i))) & \text{for } n_i \text{ even.} \end{cases}$$

Since ΩX may be thought of as an associative H-space with unit, the maps β_i can be multiplied together to a map $\beta: \Omega Y = {}^{w}\coprod_{i\in I} \Omega_R^{n_i} \to \Omega X$ with $\beta|\Omega_R^{n_i}$ homotopic to β_i . We have $\beta_{i*}(\alpha_i) = \phi(\alpha_i)$, hence $\beta_{i*}(\alpha_i \circledcirc \zeta_i) = \beta_{i*}(\alpha_i) \circledcirc \zeta_i$. For n_i even this equation follows (see [4], section 7) from the fact that β_i is an H-map; for n_i odd the equality is trivial. (Note that also for n_i odd $\pi_*(\Omega_R^{n_i})$ is an M-Lie algebra, because $\Omega_R^{n_i}$ is group-like (comp. [1], chap. V, (1.9))). We now deduce the formula $\beta_i(\alpha_i) \circledcirc \zeta_i = \phi(\alpha_i) \circledcirc \zeta_i = \phi(\alpha_i \circledcirc \zeta_i)$, because ϕ commutes with the operation " $\circledcirc$ " by assumption. Hence $\beta_{i*}: \pi_*(\Omega_R^{n_i}) \to \pi_*(\Omega X)$ coincides with $\phi|\pi_*(\Omega_R^{n_i})$, hence $\beta_* = \phi$. It follows that there is a left inverse to Ωf up to homotopy. This implies $\mathrm{cat}(X) \leq \mathrm{cat}(Y)$ by [3], lemma 2 .

<u>Remark</u>: By [4], appendix, the H-space ΩX is also decomposable over R .

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Mathematisches Institut

Arnimallee 2 - 6

1000 Berlin 33

West Germany