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DUALITY OF INFINITE-DIMENSIONAL SUBSPACES IN AN INDEFINITE INNER PRODUCT SPACE

Robert Fuster Capilla

Abstract: A characterization of the subspaces of an inner product space which admit a dual companion and the relation between duality and orthogonal projections are given.

Following Bognár [1] we mean, by "inner product space", a complex vector space E, endowed with a sesquilinear form (|) -the "inner product on E"- not necessarily positive defined. If A is a subset of E, we symbolize by $A^{\perp}$ the set of all vectors $x \in E$ such that $(a|x)=0$ for every $a \in A$. Two subspaces L and M are "dual companions" (L#M) if $L \cap M^{\perp} = L^{\perp} \cap M = 0$. A locally convex topology τ on E is "admissible" if the inner product is separately τ-continous and, for every linear form τ-continous φ there exists a vector x_o with $\varphi(y) = (y|x_o)$ for every y.

In [1], [2] and [3] duality between finite-dimensional subspaces is studied.

The purpose of this note is to give a characterisation of the subspaces L of E which admit a dual companion (Corollary 1) that generalizes the results of the quoted works. Theorem 2 gives a constructive method of dual companions in a particular case, which extends a result of [1]. Finally, if the subspaces considered are orthocomplemented ($L+L^{\perp}=E$), we express the duality by means of orthogonal projections.

THEOREM 1: Let L and M be subspaces of the inner product space E with $L \cap M^{\perp} = 0$. Then, M contains a dual companion of L.

Proof: The result is obvious when L = 0. In the other case, let Ω be the family of the pairs (L',M') of subspaces of E such that L'#M', $L' \subset L$ and $M' \subset M$. Because 0#0, Ω is not void.

Besides, the relation

$$(L',M') \leq (L'',M'') \iff L' \subset L'' \text{ and } M' \subset M''$$

determines a partial ordering of Ω. If

$$\Omega_1 = \{(L_i,M_i): i \in I\}$$

is a totally ordered subset of Ω, let be

$$L' = \underset{i\in I}{+} L_i \quad M' = \underset{i\in I}{+} M_i.$$

Since Ω_1 is totally ordered, we get the identities

$$L' = \bigcup_{i\in I} L_i \quad M' = \bigcup_{i\in I} M_i.$$

Consequently, if z is a vector of $L' \cap M'^{\perp}$, there must exist an index i_o in I such that

$$z \in L_{i_o} \cap M'^{\perp} = L_{i_o} \cap (\bigcap_{i\in I} M_i^{\perp}) \subset L_{i_o} \cap M_{i_o}^{\perp},$$

i.e., z=0, because $L_{i_o} \# M_{i_o}$.

Similarly, $L'^{\perp} \cap M' = 0$, so (L',M') is an upper bound of Ω_1.

Thus, from the Zorn Lemma, we infer the existence of a maximal element (L_1,M_1) in Ω. The proof will conclude if we get the identity $L = L_1$. Let is assume now $L \neq L_1$; if so, we can consider a vector x in $L-L_1$ and the subspace $L_2 = L_1 + \langle x \rangle \subset L$ ($\langle x \rangle$ symbolizes the linear span of x). By virtue of maximality of (L_1,M_1), there must be $u \neq 0$ in $L_2 \cap M_1^{\perp}$ since $L_2^{\perp} \cap M_1 \subseteq L_1^{\perp} \cap M_1 = 0$. Thus, $u = z+ax$, with $z \in L_1$ and $a \neq 0$, which implies $L_2 = L_1 + \langle u \rangle$.

On the other hand, since $L \cap M^{\perp} = 0$, it is possible to get $y \in M$ with

$$(u|y) = 1 \tag{1}$$

and consider the subspace of M, $M_2 = M_1 + \langle y \rangle$.

Then immediately follows $M_2^{\perp} \cap L_2 = 0$, so, $M_2 \cap L_2^{\perp} \neq 0$ and there exists $v \neq 0$ in $M_2 \cap L_2^{\perp}$. So, $M_2 = M_1 + \langle v \rangle$, since

$$v = m+dy, \; m \in M_1, \; d \neq 0 \tag{2}$$

By considering a not vanishing vector m_1+bv in $M_2 \cap L_2^{\perp}$ (consequently, $b \neq 0$), for every $t \in L_1$, $(m_1+bv|t) = 0$,
since $L_2^{\perp} \subset L_1^{\perp}$. From it, $(m_1|t) = 0$,
and so, $m_1 = 0$.

Finally, since $u \in L_2$, $v \in L_2^{\perp}$,

$$(m_1+bv|u) = b(v|u) = 0.$$

But, taking into account (1) and (2),

$$(v|u) = (m_1+by|u) = b \neq 0. \blacksquare$$

COROLLARY 1: A subspace L of the inner product space E admits a dual companion if and only if $L \cap E^{\perp} = 0$. ∎

COROLLARY 2: Let E be an inner product space and let τ be an admissible topology on E. The following propositions are equivalent:

i) E is non degenerate (i.e., $E^{\perp} = 0$)

ii) Every subspace of E admits a dual companion

iii) There exists a subspace L, τ-closed in E, which admits dual companion.

Proof: By using Corollary 1 it is enough to prove that i) follows from iii). If M is a dual companion of the τ-closed subspace L, we obtain,

$$E^{\perp} \subset (L^{\perp}+M)^{\perp} = L^{\perp\perp} \cap M^{\perp} = L \cap M^{\perp} = 0$$

since $L^{\perp\perp}$ coincides with the τ-closure of L given that τ is admissible. ∎

In Corollary 2, iii) the hypothesis of L be closed is necessary as the following example proves: Let $E = \langle e,f \rangle$, $(e|e) = (e|f) = (f|e) = 0$, $(f|f) = 1$. Then $\langle f \rangle \# \langle f \rangle$, but E is degenerate.

DEFINITION: Two families of vectors $\{e_i: i \in I\}$ and $\{f_i: i \in I\}$ in the inner product space E form a "dual pair" if, for every $i,j \in I$, with $i \neq j$, the relations

$$(e_i|f_i) = 1 \quad (e_i|f_j) = 0,$$

are verified.

As is easily checked, if two families of vectors form dual pair each of them is linearly independent and their linear envelopes are dual companions.

THEOREM 2: Let L and M be subspaces of the inner product space E such that $L \cap M^{\perp} = 0$. If L admits a countable Hamel basis then it exists a dual pair of families of vectors, their linear envelopes being L and a subspace of M.

Proof: We will construct recurrently the dual pair.

Let $\{g_n: n=1,2, \ldots\}$ a Hamel basis of L. Since $g_1 \notin M^{\perp}$, it is possible to find f_1 in M with $(g_1|f_1) = 1$. Let $e_1 = g_1$.

Assuming that the vectors $e_1, e_2, \ldots, e_n$ in L and $f_1, f_2, \ldots, f_n$ in M are given verifying

$$(e_i|f_j) = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i\neq j \end{cases} \quad 1\leq i,j\leq n \tag{4}$$

and

$$\langle e_1,e_2,\ldots,e_n\rangle = \langle g_1,g_2,\ldots,g_n\rangle, \tag{4'}$$

we define

$$e_{n+1} = g_{n+1} - \sum_{K=1}^{n} (g_{n+1}|f_k)e_k,$$

and, because $e_{n+1} \in M$, it exists $h \in M$ such that $(e_{n+1}|h) = 1$. Let

$$f_{n+1} = h - \sum_{k=1}^{n} (h|e_k)f_k.$$

Now, the relations (4),(4') also follow when the indexes i,j range form 1 to n+1.

The countable families obtained by means of this process form dual pair and, obviously, the former one is a Hamel basis for L.∎

If the subspace L is orthocomplemented, every vector of E can be expressed (in a not necessarily only way) as the sum of one of L and another of $L^{\perp}$. Thus, in a natural way, the "orthogonal projection" of a subspace M on L, $P_L M$, can be defined as the set of the vectors $x \in L$ such that $x-z \in L^{\perp}$ for some z in M.

For a Hilbert space it is well known the fact that, given two closed subspaces L and M, $L \cap M^{\perp} = 0$ if and only if $P_L M$ is dense in L. The following lemma expresses the best possible generalization of this result for an inner product space.

LEMMA 1: Let L be an orthocomplemented subspace of the inner product space E such that $L \cap L^{\perp} = 0$ (i.e., L is nondegenerate). Then, for every subspace M in E and for every admissible topology τ on E,

$$M^{\perp} \cap L = 0 \iff P_L M \text{ is } \tau\text{-dense in } L.$$

Proof: Since the closures of the subspaces are the same for every admissible topologies it is enough to work with one of them, in particular with the weak topology $\sigma(E)$. Following a result of Scheibe (see [3]) if L is orthocomplemented then the weak topology of L, $\sigma(L)$ coincides with the relative one of $\sigma(E)$. Besides, in [1] it is proved that $(P_L M)^{\perp} \cap L = 0$ if and only if $P_L M$ is $\sigma(L)$-dense in L. Finally, it is easily checked that (for L orthocomplemented) $M^{\perp} \cap L = (P_L M)^{\perp} \cap L$, which concludes the proof.∎

Lemma 1 is a particular case of the following fact: if L is an orthocomplemented subspace of E, then for every admissible topology and for every subspace M, $P_L M$ is dense in L if and only if $L \cap M^{\perp} = L \cap L^{\perp}$, result established in [4].

From Lemma 1 the proof of the next theorem follows straigthforwardly

THEOREM 3: Let L and M be subspaces of the inner product space E and assume that L is orthocomplemented and nondegenerate. Then, if τ is an admissible topology on E,

$$L\#M \iff P_L M \text{ is } \tau\text{-dense in } L \text{ and } M \cap L^{\perp} = 0. \blacksquare$$

COROLLARY 3: If L and M are orthocomplemented nondegenerate subspaces of the inner product space E, then L#M if and only if $P_L M$ is weakly dense in L and $P_M L$ is weakly dense in M.$\blacksquare$

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Departament de Matemàtiques. E.T.S.E. Agrònoms
Universitat Politècnica
Camí de Vera, nº 14
46020 València
ESPANYA