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LINEARY COMPACT INJECTIVE MODULES AND A THEOREM OF VAMOS

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A ring R will denote a commutative associative ring with unit. After Vámos, R is a <u>SISI Ring</u> if every subdirectly irreducible factor ring is self-injective.

Let M be a maximal ideal of R and let $E = E(R/M)_R$ denote the injective hull of the simple R-module R/M, and let $A(M)$ denote the endomorphism ring. Now E is canonically a module over the local ring R_M of R at M, and the unique simple R_M-module embeds in E canonically. Moreover:

$$E = E(R/M)_R = E(R_M/MR_M)_{R_M} \tag{A}$$

We call the module (A) the <u>local injective hull of</u> R <u>at</u> M, and its endomorphism ring

$$A(M) = \text{End } E(R/M)_R = \text{End } E_{R_M} \tag{B}$$

the <u>local endomorphism ring of</u> R at M.

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R is <u>Vamosian</u> (classical in [1]) if every local injective hull of R is linearly compact (in the discrete topology). In [1] Vamos proved that every Vamosian ring is SISI, and that every local endomorphism ring of a SISI ring is commutative. We shall prove the converse here, and number of subsidiary results:

1. Any SISI chain ring is Vamosian, in fact, an almost maximal valuation ring (Theorem 9).

As a consequence, we prove:

2. A ring R that is locally a SISI chain ring is Vamosian (Theorem 10).

Von Neumann regular rings are locally Noetherian rings, and are examples of Vamosian rings ([1]); we show that polynomial rings over them are also Vamosian (Theorem 12 and Corollary).

A number of unsolved problems are listed. One of the main ones asks if R Vâmos (SISI) is inherited by the polynomial ring R[x]. This is unknown even for an almost maximal valuation ring R.

1. THEOREM. The following are equivalent conditions on a ring R:

(1) R is SISI.

(2) Every local endomorphism ring of R is commutative.

(3) Every R-submodule of every local injective hull of R is quasi-injective.

(4) Every R-submodule of every local injective module

E is an End_R E-submodule, i.e. is fully invariant. When any of these hold, then R_M is SISI for every maximal ideal M.

Remark: When this is so, then every local endomorphism ring of R is "almost" SISI; See Proposition 4.

For the proof, we need a result implicit in [1].

2. PROPOSITION. Let E be an indecomposable injective R-module.

The following are equivalent conditions.

(1) Every submodule of E is quasi-injective.

(2) Every submodule of E is fully invariant (FI).

(3) Every cyclic submodule of E is quasi-injective.

(4) Every cyclic submodule of E is FI.

(5) If a cyclic module R/I embeds in E then R/I is a self-injective ring.

(6) For each $a \in A = \mathrm{End}\, E_R$ and $x \in E$, there exists $r \in R$ such that $a(x) = xr$.

When this is so, then $A = \mathrm{End}_R E$ is commutative.

PROOF. (1) $\Leftrightarrow$ (2) by a theorem of Johnson and Wong ([7], p.63. Cor 19.3), which states that an R-module M is quasi-injective iff M is fully invariant in $E(M_R)$. If every cyclic submodule of E is fully-invariant (quasi-injective) then every submodule is, hence (1) - (4) are equivalent.

Obviously (4) $\Leftrightarrow$ (6). Furthermore (5) $\Rightarrow$ (3), because

R/I self-injective implies R/I is quasi-injective (since every R-submodule is an R/I-submodule).

(3) ⇒ (5). If R/I is quasi-injective qua R-module, it is quasi-injective qua R/I-module, equivalently, self- injective by Baer's criterion ([6], p.157, Theorem 3.41.)

Evidently, (6) implies that A is commutative.

3. COROLLARY. (Vámos) A ring R is SISI iff every local injective module $E = E(R/M)_R$ satisfies any of the equivalent conditions of the proposition.

PROOF. This follows since every subdirectly irreducible factor ring R/I embeds in $E(R/M)_R$, where R/M = socle R/I; and conversely, if $R/I \to E(R/M)$, where M is maximal, then R/I is subdirectly irreducible.

PROOF OF THEOREM 1.

(1) ⇒ (2) by Vámos [1], and (1) ⇔ (3) by Corollary 3. Moreover in view of (A) and (B), Corollary 3 also yields (3) ⇒ (4).

(2) ⇒ (1). It suffices to prove that R_M is SISI for every maximal ideal M, hence suppose R is local with maximal ideal M. In this case $E = E(R/M)_R$ is an injective cogenerator of mod-R.

Now let M = R/I be a subdirectly irreducible factor ring, and let $\bar{E} = E(M_R)$ be its injective hull taken in E.

This can be done because $\bar{E}$ is also essential over M as an R-module, and $E = E(M_R)$, since $M \longrightarrow E$ and E is indecomposable. Furthermore $\bar{M} = \mathrm{ann}_E I \supseteq \bar{E}$, and $\bar{M}$ is essential over M as an R, whence as an $\bar{R}$-module, so therefore $\bar{M} = \bar{E}$. This shows that $\bar{E}$ is a fully invariant submodule of E since obviously $a\bar{M} \subseteq \bar{M}\ \forall\, a \in A$. Thus

$$\bar{A} = A/K \approx \mathrm{End}\ \bar{E}_{\bar{R}}$$

where $K = \mathrm{ann}_A \bar{E}$.

The fact that $M \longrightarrow \bar{E}$ implies that $\bar{E}$ is a cyclic $\bar{A}$-module (e.g., see [8], p.15, Prop. 5.5), whence $\bar{E} \approx \bar{A}$ in mod-$\bar{A}$. Furthermore,

$$E \approx \bar{A} \approx \mathrm{Biend}\ \bar{E}_{\bar{R}} = Q \tag{C}$$

so that $Q = Q_{max}(\bar{R}) \hookrightarrow \bar{A}$ canonically (e.g. [7], p.81, Prop. 19.21.).

Since $\bar{E}_{\bar{R}}$ is injective, then it is quasi-injective over $Q = \mathrm{Biend}\ \bar{E}_{\bar{R}}$ by Corollary 5.6A, p.15 of [8]. Using (C) we see that $\bar{A}$ is self-injective. We also need the fact that $\bar{A}$ is an injective cogenerator over $\bar{R}$ (since $\bar{E}$ is). Thus every ideal H of $\bar{R}$ is the annihilator of an $\bar{A}$-submodule of $\bar{E} \approx \bar{A}$, say $H = \mathrm{ann}_{\bar{R}} G$ for an ideal G of $\bar{A}$. (See [7], p. 190, Corollary 23.23.)

But if H is chosen to be a dense ideal of $\bar{R}$ (= $\bar{R}$ is a rational extension of H), then the only ideal in $\bar{A} = Q_{max}(\bar{R})$

that annihilates it is zero. See, e.g. [7], p.80, 19.32(b), which implies that

$$\mathrm{ann}_{\bar{E}}\mathrm{ann}_{\bar{A}}H = \bar{E}$$

for any dense ideal H of $\bar{R}$. In our context, this means that $\mathrm{ann}_A H = 0$, so $G = 0$ whence $H = \bar{R}$.

But, then $\bar{R} = \bar{A}$, since for every $q \in Q$ ther exists a dense ideal I of $\bar{R}$ with $qI \longrightarrow \bar{R}$.

This proves that $\bar{R}$ is self-injective, and hence R is SISI.

<u>Partially Different Proof</u>

Remark: (1) A proof of $(2) \Rightarrow (1)$ has kindly been provided by Professor Vámos, who also supplied the following example (2) of a locally Vamosian ring that is not SISI.

Suppose (2) holds but (1) fails. As before, we may assume that R is a local subdirectly irreducible ring embedded in $E = E(R)$. Since $E \neq R$ there exists $x \in E\backslash R$, and since E is an injective cogenerator, there exist homomorphisms

$$\alpha : E \longrightarrow E \quad \text{and} \quad \beta : E \longrightarrow E$$

such that

$$\alpha(1) = x, \quad \beta(1) = 0, \quad \beta(x) \neq 0.$$

Then

$$0 \neq \beta(x) = \alpha(1) \neq \alpha\beta(1) = 0$$

contradicting commutativity of $\mathrm{End}_R E$.

(2) Let R be a subdirectly irreducible almost maximal torch ring, i.e. a ring R with: (i) at least two maximal ideals such that R_M is an almost maximal valuation ring for each $M \in \max R$, (ii) a waist P, where P is a minimal prime and a uniserial module, and such that R/P is an h-local domain [i.e. every nonzero prime is contained in a unique maximal ideal, and every nonzero ideal of R/P is contained in just finitely many maximal ideals.] Such rings exist (see [15]) but can not be SISI since R is not self-injective. (An indecomposable self-injective ring is local.)

The next result shows that a local endomorphism ring of a SISI ring is "almost" SISI.

4. PROPOSITION. If E is an injective R-module with commutative endomorphism ring A, any A-submodule of E is quasi-injective, and A modulo any ideal I such that $A/I \hookrightarrow E$ is self-injective; equivalently A/I is self injective for any ideal $I = \mathrm{ann}_A x$ for some $x \in E$.

PROOf. As stated in the proof of Theorem 1, E_A is quasi-injective, and $A = \mathrm{End}_A E$. If $S \subseteq M$ are A-submodules of E, and $f : S \to M$ an A-map, then by quasi-injectivity of E over A, f is induced by $a_f \in A$. Since M is an A-submodule, $a_f M \subseteq M$, hence f extends to an endomorphism of M_A.

This proves quasi-injectivity of any A-submodule M of E. The self-injectivity of A/I follows from its quasi-injectivity as in the proof of Proposition 2. If $f : A/I \longrightarrow E$ is an embedding of A-modules, then $I = \mathrm{ann}_A x$, where $x = f(1+I)$. Conversely, if $I = \mathrm{ann}_A x$ then there is an embedding $A/I \longrightarrow E$ sending $A + I \longrightarrow ax$ $\forall a \in A$.

Note, if R is SISI, then every local endomorphism ring, $A = \mathrm{End}\, E(R/M)_R$, is commutative, and the unique simple A-module W embeds in E and coincides with $V = R/M$. Thus, the proposition would imply that A is SISI provided only that E is injective over A. This is not in general true for a SISI ring R. In fact, Vámos singles out a class of rings (called classical in [1]) to rectify this deficiency.

We say that R is a Vámos ring, or Vamosian (formerly classical) provided that every local injective hull is linearly compact (l.c.) over R. We employ the terminology injectivendo to indicate when a module F over R is injective over its endomorphism ring A. An ideal I is co-subdirectly irreducible (co-SDI) if R/I is a subdirectly irreducible ring.

VÁMOS THEOREM [1]. If R is Vamosian then:

(V1) R is SISI.

(V2) The local endomorphism ring A at any maximal ideal M is the completion of R_M in the topology generated by the co-SDI ideals of R_M, and is a l.c. ring.

(V3) Every local injective hull E is injectivendo, and l.c. over its endomorphism ring A, equivalently $\mathrm{Hom}_A(,E)$ induces a Morita duality in mod-A (on the full subcategory of l.c. A-modules).

(V3) Follows from theorems of Morita [4] and Mueller [3], which imply that a commutative ring A has a Morita duality iff the least injective cogenerator E over A satisfies $A = \mathrm{End}_A E$. By Mueller [3] this is equivalent to requiring that both A and E be l.c. A-modules.

5. THEOREM. The following are equivalent donditions on a ring R

(1) R is Vamosian.

(2) R is SISI and every local endomorphism ring is Vamosian.

(3) R is SISI and every local injective module is injectivendo.

PROOF. (1) ⇒ (2). By Vamos' theorem, R is SISI, and every local endomorphism ring $A = \mathrm{End}\ E_R$ has l.c. injective hull E by (V3).

(2) ⇒ (1). Let E be a local injective module of R, and $A = \mathrm{End}_R E$. Since A is Vamosian, then the injective hull F of its unique simple module W is l.c. over A. But, $W \hookrightarrow E$ and, in fact, coincides with the unique simple R-mod-

ule V embedded in E, hence $E \hookrightarrow F$ in mod-A. This implies that E is l.c. over A. But, by (5) of Theorem 1, every R-submodule of E is an A-submodule of E, hence E is l.c. over R. This proves that R is Vamosian.

(1) ⇒ (3) follows from Vámos' theorem.

(3) ⇒ (1). Let E be a local injective R-module, and $A = \text{End } E_R$. By the proof of (2) ⇒ (1), E is the lesast injec tive cogenerator over A (assuming injectivendo) hence l.c. over A by Mueller's theorem, But, since R is SISI, every R-submodule of E is an A-submodule, so E is l.c. over R. This proves R is Vamosian.

EXAMPLES OF VÁMOS RINGS

The following examples of Vámos rings are culled from [1].

(E1) R locally Noetherian (at maximal ideals) that is, R_M Noetherian for all maximal ideals M.

(E2) (Matlis - Vámos) Any almost maximal valuation ring (=AMVR).

(E3) Any commutative ring A with a Morita duality.

In [1] Vámos proved that a ring R (E2) satisfies (E3), provided that R is either not a domain, or R is a complete local in. (For domains, this belongs to Matlis). In (E3), if A has a duality then the least injective cogenerator is l.c. by Mueller [3]. Then every local injective hull is l.c.

In connection with (E1), consult Beck [9]: R_M is Noetherian iff $E(R/M)_R$ is Σ-injective.

APPLICATION TO FPF RINGS

Commutative FPF rings are studied in [10] (and in articles cited ther , where we raised a question: does the existence of injective module E over a commutative ring $A = \text{End}_A E$, imply A is FPF? Following Corollary 8, we show the answer is no in general.

First consider any Vamosian ring R, and local endomorphism ring

$$A = \text{End } E(R/M)_R.$$

e.g., let R be any local Noetherian ring, and A be its completion. We next remark that a domain A is FPF iff A is Prufer, and hence a local domain A is FPF iff A is a chain domain (= ideals form a chain). Hewever, a complete local domain need not be a chain ring, in fact:

6. PROPOSITION. If $A = \text{End}_R E$ is commutative and E injective over R, then A is a chain ring iff E is a chain module over R (= uniserial = the lattice of submodules is a chain).

This is a special case, namely (3) and (6) of the next theorem. To prove it, we employ the following so-called double annihilator conditions which hold for a module E quasi-injective over a ring R, with $A = \text{End } E_R$.

(dac 1) (N. Jacobson - R.E. Johnson) for a finitely generated A-submodule S of E,

$$\mathrm{ann}_E \mathrm{ann}_R S = S$$

(dac 2) (Harada-Ishii) for a finitely generated (left) ideal L of A:

$$\mathrm{ann}_A \mathrm{ann}_E L = L$$

For (dac 1) consult [2] p.66, Prop. 19.10, and for (dac 2), consult [11].

7. THEOREM. Let E be a quasi-injective right R-module, $A = \mathrm{End}\, E_R$ and $S = \mathrm{End}_A E$.

(1) If R is a right chain ring, then E is a chain left A-module.

(2) If E is an injective cogenerator in mod-R, then conversely R is a right chain ring if E is a chain left A-module.

(3) If R is commutative, then any chain R-module E is a chain A-module (without assuming quasi-injectivity).

(4) If A is commutative, then A is a chain ring iff E is a chain A-module.

(5) If R is a right chain ring, and A is commutative, then A is a chain ring, and, i.a., E is

indecomposable

(6) If R and A are commutative, then the f.a.e.:

(a) E is a chain A-module.

(b) E is a chain R-module.

(c) A is a chain ring.

When this is so, then every A or R submodule of E is quasi-injective.

(7) If R and A are commutative, and R is a chain ring, then 6(a)-(c) hold.

PROOF. (1) follows from the (dac 1), since the finitely generated A-submodules of E form a chain, hence the lattice of all A-submodules do too; (2) follows from the fact that $\mathrm{ann}_R\mathrm{ann}_E I = I$ for any right ideal I when E cogenerates mod-R; (3) is trivial since every A-submodule is an R-submodule when R is commutative; (4) is an immediate consequence of (dac 2), and the fact E is quasi-injective over A by the proof of Propositon 4. This implies (5) via (1). In (6), (3) shows that (b) $\Rightarrow$ (a), and (a) $\Leftrightarrow$ (c) by (4). Then E is indecomposable, so every R-submodule S of E is quasi-injective by the proof of Proposition 4, whence is an A-module as stated in the proof of Proposition 2. Thus, (a) $\Rightarrow$ (b). Moreover, any A-submodule is quasi-injective by Proposition 4. Finally (1) implies (7) via (6).

8. COROLLARY. If R is a SISI ring, then a local endomorphism ring $A = \mathrm{End}\, E(R/M)_R$ is a chain ring iff E(R/M) is

a chain module. In this case, R_M is an almost maximal valuation ring (=AMVR), and A is its completion (and an AMVR).

PROOF. As stated, E = E(M/M) is a canonical R_M-module, and by a result in 1 , Prop. 4.4, the f.a.e.:

(i) E is a chain module over R.

(ii) R_M is an AMVR

(iii) R_M is FGC ring (= finitely generated modules are direct sums of cyclic modules).

(iv) R_M is a Vamosian chain ring.

In this case, the completion of R_M is End E_R and is an AMVR.

PROOF. Assume R is SISI. Then A is commutative and by the propositon, E is a chain module iff A is a chain ring. Then, assuming this, the rest follows from Vamos' equivalences (i) ↔ (iv).

Remark the equivalence of (i)- (iii) constitutes a theorem of D.T. Gill. (See [1]).

Thus, if R is any local Noetherian domain not a valuation domain, with local injective mudule E then A = $End_R E$ is the completion of R, a complete local domain but not FPF.

9. THEOREM. If R is a SISI ring, and if R is locally a chain ring, then R is locally an AMVR, hence R is locally Vamosian.

PROOF. By Theorem 8, every local injective hull is a chain module and hence by Corollary 9, every local ring R_M is an AMVR, equivalently Vamosian (since R_M is a chain ring).

A theorem of Kaplansky and Warfield (see. e.g.[7], p.131, Theorem 20.45) characterizes a locally chain ring R by the property that finitely presented modules are direct summands of direct sums of cyclics. Any flat-ideal ring is an example, in particular, any semi-hereditary ring.

10 COROLLARY. A SISI flat-ideal ring (e.g. semi-hereditary SISI ring) is locally Vamosian.

PROBLEMS.

1. If R is Vámos, is the polynomial ring R[x]?
2. If R is SISI or Vámos, is R[x] SISI?
3. If R is an AMVR, is R[x] Vámos? (SISI?).
4. If R is linearly compact (l.c.), does R have a duality, equivalently, by ([3]), is the minimal injective cogenerator over R also l.c.?

   Note: #4 is a question of Mueller [3], and partially solved in the affirmative by Vámos in [2], so its affirmation is denoted MVC (the Mueller-Vámos "conjecture").
5. Does MVC imply that R[x] is (a) Vamosian, (b) SISI assuming R is l.c.?
6. If R a SISI ring such that every factor ring is of finite uniform dimension, is R Vamosian?

11 LEMMA. If P is a prime ideal of $R[x]$, and P_0 is the contracted ideal in R, then

$$R[x]_P \approx R_{P_0}[x]_{P^{ex}}$$

where P^{ex} is the extension of P to $R_{P_0}[x]$ (i.e. $P^{ex}=PR_{P_0}[x]$).

PROOF. P^{ex} consists of all $g(x)$ in $R_{P_0}[x]$ with coefficients in PR_{P_0}, and P^{ex} is prime since, in general, for any ring A and prime ideal L of A, we have $A[x]/L[x] \approx A/L[x]$ is a domain.

Let $f(x) = h(x)/g(x)$ denote an element of the right side, i.e. Let $h(x), g(x) \in R_{P_0}[x]$, with $g(x) \notin P^{ex}$. We can write

$$h(x) = h_0(x)/c \quad \text{and} \quad g(x) = g_0(x)/d$$

with $c,d \in R\backslash P_0$, and $g_0(x), h_0(x) \in R[x]$. Since $c,d \notin P_0$, then $cdg_0 \notin P$, hence $h(x) = h_0(x)/cdg_0(x) \in R[x]_P$. The reverse inclusion is proved similarly, i.e., if $h, g \in R[x]$, and $g \notin P$, then we may view h and g as elements of $R_{P_0}[x]$, and moreover, $g \notin P^{ex}m$ so $h/g \in R_{P_0}[x]_{P^{ex}}$

12. THEOREM. If R is locally Noetherian, then so is any polynomial ring over R in finitely many variables $x_1,\dots,x_n$. In particular, then $R[x_1,\dots,x_n]$ is Vamosian.

PROOF. Since R is locally Noetherian, then $R_{P_0}[x]$ is Noetherian for any prime ideal P of R[x], and hence, by Lemma 13, so is the local ring at P.

13. COROLLARY. If R is von Neumann regular, then $R[x_1,\ldots,x_n]$ is Vamosian.

14. REMARK. R[x] is then semihereditary, and conversely, if the polynomial ring R[x] over a (not necessarily commutative) ring is semihereditary, then R must be von Neumann regular. (See [12, 13, and 14]. (However, a general von Neumann regular ring R does not imply R[x] semihereditary).

15 PROPOSITION. If R is Vamosian (resp. SISI), then so is every factor ring.

PROOF. If R is SISI, then every factor ring obviously is, so suppose that R is Vamosian, and I is an ideal, and V a simple R/I module, and E the injective hull of V in mod-R. It is easy to see that the annihilator $\bar{E}$ of I in E is the injective hull of V in mod-R/I (cf. the proof of (2) ⇒ (1) of Theorem 1). It follows that $\bar{E}$ is l.c. over R/I, since E is l.c. over R, hence R/I is also Vamosian.

16. COROLLARY. If R[x] is Vamosian (SISI), then so is R.

R[x] is <u>monic</u> if it contains a monic polynomial

An ideal I of $R[x]$ is monic if I contains a monic polynomial, equivalently, $R[x]/I$ is a finitely generated R-module. A ring R is called a Monica ring if every co-subdirectly irreducible ideal of $R[x]$ is monic.

An ideal I of R is colocal if R/I is a local ring.

Example. Any co-SDI ideal I of a SISI ring is colocal, since R/I is then indecomposable injective, hence has local endomorphisin ring which is isomorphic to R/I.

In this example I is also co-PF in the sense that R/I is PF. Thus, R/I has a Morita duality, and hence R/I is Vámos.

If P is a subcategory of the category RINGS, then for any ideal H of a ring A, we say that H is a co-P-ideal if $A/H \in P$. In this paper interalia we have been interested in subcategories of RINGS consisting of: irreducible (i.e. uniform) rings, local rings, semilocal rings, semiperfect rings, self-injective rings, PF-rings, and (locally) Noetherian rings.

17. Theorem. If R is l.c., then every monic ideal I of $R[x]$ is co-semiperfect, i.e.

$$R[x]/I = R[x]/I_1 x \ldots x R[x]/I_t$$

for co-local ideals $I_i \supseteq I$, $i = 1,\ldots,t$, and $t \geqslant 1$. Consequently, any monic co-irreducible ideal of $R[x]$ is co-local.

Proof. Since I is monic, then $R[x]/I$ is finitely generated over R, and hence by [1] or [3], is l.c. as an R-module. By [16], any l.c. ring is semiperfect, so $R[x]/I$ has the stated decomposition.

18. THEOREM (VÁMOS [2]). If R is a Morita ring (i.e., has a Morita duality), then so does any algebra A over R that is l.c. over R, in particular, that is a finitely generated R-module.

19. COROLLARY. If R is a Morita ring, then $R[x]/I$ is a Morita ring for any monic ideal I, and hence $R[x]/I$ is self-injective for any monic co-SDI ideal I.

Proof. Obvious from the above theorem of Vamos and the proof of Theorem 17.

20. COROLLARY. If R is a Monica Morita ring, then $R[x]$ is SISI.

Proof. By Corollary 19, $R[x]/I$ is Morita, hence Vamos, and therefore SISI, for every co-SDI ideal.

21. COROLLARY. If R is a l.c. Vamosian Monica ring, then $R[x]$ is SISI.

Proof. A ring R is Morita iff R is l.c. and Vamosian, according to Mueller's Theorem stated earlier so

R x   is SISI by Corollary 20.

22. PROPOSITION. If R is a l.c. ring, then the Mueller-Vamos conjecture implies that R[x]/I is Morita for any monic ideal I.

Proof. Obvious, since R[x]/I is a finitely generated module over the Morita ring R.

23. COROLLARY. If R is a l.c. Monica ring, then MVC implies that R[x] is SISI.

Proof. Clear from the proof of Corolary 21.

24. QUESTION. Is every Morita, Vámosian, or SISI ring Monica?

25. REMARK. If R is SISI, if I is a co-SDI ideal of R[x] and if $I \cap R$ is a co-SDI ideal of R, then I can show that I is monic iff I is co-local. This result will appear elsewhere.

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