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MAPS FROM $B\pi$ INTO X

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Let π be a finite group and lets $B\pi$ be its classifying space. With every subgroup $\gamma \subset \pi$ there is associated a covering $i(\gamma,\pi) : B\gamma \to B\pi$. If $g \in \pi$ then multiplication by g on $E\pi$ induces a map $c_g : B\gamma \to B(g^{-1}\gamma g)$. Let E be an infinite loop space. Then there is the following exact sequence

$$(*) \quad 0 \to [B\pi;E] \to \prod_{\pi_p}[B\pi_p;E] \xrightarrow{i_*,j_*} \prod_{\pi_p,g\in\pi} [B(\pi_p \cap g\pi_p g^{-1});E] ,$$

where π_p is a p-Sylow subgroup of π , products $\prod\limits_{\pi_p} \ldots$ and $\prod\limits_{\pi_p,g\in\pi} \ldots$ are over all p-Sylow subgroups for all primes p , $i_* = \prod\limits_{\pi_p,g} i(\pi_p \cap g\pi_p g^{-1};\pi_p)$ and $j_* = \prod\limits_{\pi_p,g} i(g^{-1}\pi_p g \cap \pi_p;\pi_p)\circ c_g$. (see [2]).

From the sequence (*) it follows that a map from $B\pi$ to an infinite loop space is homotopic to zero if and only if its restrictions to classifying spaces of all Sylow subgroups are homotopic to zero. We want to see whether the same statement is true for an arbitrary simply connected space. For example if $\pi = \prod\limits_p \pi_p$ then we have the following proposition

Proposition 1. If X is simply-connected then

$$[B\pi;X] \approx \prod_p [B\pi_p;X] .$$

Proof. The map $\bigvee_p B\pi_p \to B\pi$ is a homological equivalence. Therefore using an obstruction theory we obtain a required isomorphism for any simply connected space X.

In further considerations we restrict our attention to a very small class of groups. Let π_p be a maximal p-Sylow subgroup of π.

Let $N(\pi_p)$ be a normalizor of π_p in π and let $W_p = N(\pi_p)/\pi_p$.

Definition 1. We say that π satisfies W_p-condition if the map $H^*(\pi;Z_{(p)}) \to H^*(\pi_p;Z_{(p)})^{W_p}$ is an isomorphism.

Examples

1. If π_p is a normal divisor in π then W_p-condition is satisfied.
2. If π_p is abelian then W_p-condition is satisfied.
3. W_p-condition is satisfied for the binary icosahedral group I^* and all primes p.
4. If $\pi = GL(n;F_q)$ then W_p-condition is satisfied for some primes p.

Notation. "$\sim$" means "is homotopic to".

We have the following sequence of cofibrations (π_p is a maximal p-Sylow subgroup of π.)

$$(**)\quad B\pi_p \xrightarrow{i} B\pi \xrightarrow{j} \mathrm{Cone}(i) = C \xrightarrow{\delta} S(B\pi_p) \xrightarrow{S(i)} S(B\pi) \to \cdots .$$

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Let $(\ )\hat{}_{(p)}$ denotes the p-completion functor and let $(\ )_{(p)}$ denotes the p-localization functor. After applying $(\ )\hat{}_{(p)}$ to (**) we obtain the following sequence of cofibrations

$$(***)\quad (B\pi_p)\hat{}_{(p)} = B\pi_p \xrightarrow{i_p} (B\pi)\hat{}_{(p)} \xrightarrow{j_p} \mathbb{C}\hat{}_{(p)} = \mathbb{C}_{(p)} \xrightarrow{\delta_p} S(B\pi_p)\hat{}_{(p)} =$$

$$= S(B\pi_p) \xrightarrow{S(i)_p} S(B\pi)\hat{}_{(p)} = S(B\pi)_{(p)} \to \cdots .$$

Further we shall deal only with a case of a fixed prime p and therefore we always drop the index p in $i_p, j_p, \delta_p, \dots$.

<u>Theorem 1</u>. (F. Cohen [1]) If π satisfies W_p-condition then

$s(i) : S(B\pi_p) \to S(B\pi)_{(p)}$ has a left inverse k ,

$\delta \vee k : \mathbb{C}_{(p)} \vee S(B\pi)_{(p)} \to S(B\pi_p)$ is a homotopy equivalence and

$j : (B\pi)\hat{}_p \to \mathbb{C}_{(p)}$ is homotopic to zero.

<u>Proof</u>. Let $k = |W_p|$. Every element $g \in W_p$ induces a map $h_g : B\pi_p \to B\pi_p$ (conjugation by g). Let $N = \sum_{g \in W_p} S(h_g) : S(B\pi_p) \to S(B\pi_p)$ and let $k-N = k\cdot id-N : S(B\pi_p) \to S(B\pi_p)$. One easily checks that the natural map $r = r_1 + r_2 : S(B\pi_p) \to Tel(N) \vee Tel(k-N)$ is a homotopy equivalence. Every element $g \in W_p$ induces also a map $\tilde{h}_g : B\pi \to B\pi$ homotopic to the identity. Let $\tilde{N} = \sum_{g \in W_p} S(\tilde{h}_g) = k : (SB\pi)_{(p)} \to (SB\pi)_{(p)}$. The maps $\ell : Tel(N) \to Tel(\tilde{N})$ and $\tilde{r}_1 : (SB\pi)_{(p)} \to Tel(N)$ are homotopy equivalences. Let $i_1 : Tel(N) \to Tel(N) \vee Tel(k-N)$ be the natural inclusion. One can check that $k = (r_2 + r_1)^{-1} \circ i_1 \circ \ell^{-1} \circ \tilde{r}_1$ is a left inverse to $S(i)$. Therefore $\delta \vee k$ is a homotopy equivalence. It rests to show that $j \sim 0$. δ has a right inverse t . This implies that $j \sim j \circ \delta \circ t$. Hence we have that $j \sim 0$. ■

Corollary 1. If π satisfies W_p-condition and X is simply-connected and p-local then the map $f : B\pi \to X$ is homotopically trivial if and only if its restriction to $B\pi_p$ is homotopically trivial.

Proof. If $f \circ i \sim 0$ then there is $f' : C \to X$ such that $f' \circ j \sim f$. This implies that $f \sim 0$. □

Let us suppose that we have a map $f : SB\pi_p \to X$. We want to understand its restrictions to $Tel(N)$ and $Tel(k-N)$.

Lemma 1. Let us suppose that $X = \Omega Y$. Then there is an isomorphism

$$[Tel(N)\ (\text{resp. } Tel(k-N));X] \approx \varprojlim_{N(\text{resp. } k-N)} [SB\pi_p;X] \quad .$$

Proof. We have a direct system of spaces $SB\pi_p \xrightarrow{N(\text{resp. } k-N)} SB\pi_p \to \cdots$. There is the following exact sequence of Milnor

$$0 \to \varprojlim^1_{N(\text{resp. } k-N)} [SB\pi_p, X] \to [Tel(N)(\text{resp.} k-N));X] \to \varprojlim_{N(\text{resp. } k-N)} [SB\pi_p;X] \to 0$$

Let us notice that $N \circ N = k \cdot N$ (resp. $(k-N)\circ(k-N) = k(k-N)$) implies that our inverse systems satisfy the Mitag-Leffler conditions. This implies that $\lim^1$ terms vanish. □

If $f \in [SB\pi_p;\Omega Y]$ and Y is p-local then for any $n \in Z_{(p)}$ we can define $n \cdot f$ in the following two ways.

i) $Maps.(S^1;Y) = \Omega Y$ has the same homotopy type as $Maps.(S^1_{(p)};Y)$. For any $n \in Z_{(p)}$ there is a map $n : S^1_{(p)} \to S^1_{(p)}$ of degree n and we define $n \cdot f$ as a composition $n \circ f$.

ii) $S^1 \wedge B\pi_p \approx S^1_{(p)} \wedge B\pi_p$. The map $n : S^1_{(p)} \to S^1_{(p)}$ induces $\tilde{n} : S^1_{(p)} \wedge B\pi_p \to S^1_{(p)} \wedge B\pi_p$. We define $n \cdot f$ as a composition $f \circ \tilde{n}$.

Let $f : SB\pi_p \to X = \Omega Y$. Let us set $f_1 = \frac{1}{k} \cdot (f \circ N)$ and $f_2 = \frac{1}{k} \cdot (f \circ (k-N))$. Then $f_1^* = \{\frac{1}{k^n} f_1\}_{n \in \{1,2\ldots\}} \in \varprojlim_{N}[SB\pi_p;X]$ and $f_2^* = \{\frac{1}{k^n} f_2\}_{n \in \{1,2\ldots\}} \in \varprojlim_{k-N}[SB\pi_p;X]$. Therefore by Lemma 1 f_1^* and f_2^* define maps $f_1^* : \mathrm{Tel}(N) \to X$ and $f_2^* : \mathrm{Tel}(k-N) \to X$. $f_1^* \vee f_2^*$ restricted to $SB\pi_p$ (i.e. $(f_1^* \vee f_2^*) \circ r$ where $r = r_1 + r_2 : SB\pi_p \to \mathrm{Tel}(N) \vee \mathrm{Tel}(k \setminus N)$ is a sum of inclusions onto the first segments of the mapping telescopes) is homotopic to $\frac{1}{k} f \circ N + \frac{1}{k} f \circ (k-N) = f$.

<u>Proposition 1</u>. The natural isomorphism

$$r^* : \varprojlim_{N}[SB\pi_p;X] \oplus \varprojlim_{k-N}[SB\pi_p;X] \to [SB\pi_p;X]$$

is given by $((f_n);(g_n)) \to f_1 + g_1$. The inverse map is given by $f \to (f_1^*;f_2^*)$.

<u>Proof</u>. The map $r : SB\pi_p \to \mathrm{Tel}(N) \vee \mathrm{Tel}(k-N)$ induces a map $[\mathrm{Tel}(N);X] \oplus [\mathrm{Tel}(k-N);X] \to [SB\pi_p;X]$ which is given by the sum of restrictions to the first segments of the telescopes. This shows the first part of the proposition. By the previous discussions $f \to (f_1^*, f_2^*)$ defines a map in the opposite direction which is the inverse of r^*.

<u>Corollary 2</u>. If $f \circ N$ is homotopic to $k \cdot f$ then

i) $f_2^* \sim 0$,

ii) $f \circ \delta \sim 0$,

iii) for any $g \in W_p$ we have that $f \circ S(h_g) \sim f$.

Proof. i) follows from the definition of f_2^*. We have that $f\circ\delta \sim (f_1^* \vee f_2^*)\circ r\circ\delta \sim f_1^*\circ r\circ\delta \sim f_1^*\circ\ell^{-1}\circ\ell\circ r_1\circ\delta \sim f_1^*\circ\ell^{-1}\circ\tilde{r}_1\circ S(i)\circ\delta \sim 0$.

ii) implies that there is $f'\colon SB\pi \to X$ such that $f'\circ S(i) \sim f$. This implies that $f\circ S(h_g) \sim f$. ▣

Corollary 3. If $X = \Omega^2 Y$ and X is simply connected then $i : B\pi_p \to B\pi$ induces an isomorphism

$$[(B\pi)^{\wedge}_p;X] \cong [B\pi_p;X]^{W_p} .$$

Proof. We have that $\tilde{r}_1\circ S(i) \sim \ell\circ r_1$. $\tilde{r}_1$ and ℓ are homotopy equivalences. Therefore it is enough to show that

$$r_1^*\colon [\mathrm{Tel}(N);X] = \varprojlim_{N}[SB\pi_p;\Omega Y] \to [SB\pi_p;\Omega Y]^{W_p}$$

is an isomorphism. Let us suppose that $f \in [SB\pi_p;\Omega Y]^{W_p}$. Then $f_1^* = \{\frac{1}{k^n} f\circ N\}_{n\in\{1,2,\dots\}} \in \varprojlim_{N}[SB\pi_p;\Omega Y]$ and $r_1^*(f_1^*) = f$. This implies that r_1^* is an epimorphism. r_1^* is also a monomorphism and therefore it is an isomorphism. ▣

Theorem 2. If X is a nilpotent, p-local space and if π satisfies W_p-condition then the natural map

$$[B\pi;X] \twoheadrightarrow [B\pi_p;X]^{W_p} \text{ is a surjection.}$$

If X is a loop space then

$$[B\pi;X] \xrightarrow{\approx} [B\pi_p;X]^{W_p} \text{ is a bijection.}$$

Proof. We have already proved theorem when X is a double loop space. Let us suppose that X is a loop and that X has only a finite number of non-trivial homotopy groups. Let us consider a part of the Postnikoff tower of X ,

$$\to \Omega X_{n-1} \xrightarrow{a} K(\pi_n, n) \xrightarrow{b} X_n \xrightarrow{c} X_{n-1} \xrightarrow{d} K(\pi_n, n+1) \ .$$

Let us suppose that the theorem is true for X_{n-1}. We have the following commutative diagram

$$\begin{array}{ccccccccc}
[B\pi, \Omega X_{n-1}] & \xrightarrow{a} & [B\pi; K(\pi_n, n)] & \xrightarrow{b} & [B\pi; X_n] & \xrightarrow{c} & [B\pi, X_{n-1}] & \xrightarrow{d} & [B\pi; K(\pi_n, n+1)] \\
\cap\downarrow i & & \cap\downarrow j & & \cap\downarrow k & & \cap\downarrow \ell & & \cap\downarrow m \\
[B\pi_p, \Omega X_{n-1}]^{W_p} & \xrightarrow{a_1} & [B\pi_p; K(\pi_n, n)]^{W_p} & \xrightarrow{b_1} & [B\pi_p; X_n]^{W_p} & \xrightarrow{c_1} & [B\pi_p; X_{n-1}]^{W_p} & \xrightarrow{d_1} & [B\pi_p; K(\pi_n; n+1)]^{W_p}
\end{array}$$

We must show that k is a bijection. If $k(x) = k(y)$ then $c(x) = c(y)$. Hence there exists $z \in [B\pi; K(\pi_n; n)]$ such that $z = x^{-1} \cdot y$. This implies that $j(z) = k(x)^{-1} \cdot k(y)$. Therefore there is $\omega \in [B\pi_p; \Omega X_{n-1}]$ such that $a_1(\omega) = j(z)$. Let $\omega_1 = \frac{1}{k} \sum_{g \in W_p} \omega \circ h_g$. Then $a_1(w_1) = j(z)$ and $w_1 \in [B\pi_p; \Omega X_{n-1}]^{W_p}$. There is $v \in [B\pi; \Omega X_{n-1}]$ such that $i(v) = w_1$. We have $j(a(v)) = a_1(i(v)) = a_1(w_1) = j(z)$. This implies that $a(v) = z$ and therefore $x = y$.

Let us suppose that $x \in [B\pi_p; X_n]^{W_p}$ and let $y \in \ell^{-1}(c_1(x))$.

There exists z such that $c(z) = y$ because $d(y) = 0$. We have that $c_1(k(z)) = c_1(x)$. Therefore there is $\omega \in [B\pi_p; K(\pi_n; n)]$ such that $b_1(\omega) = x \cdot k^{-1}(z)$. Let $w_1 = \sum_{g \in W_p} w \circ h_g$. Then $b_1(w_1) = (x \cdot k^{-1}(z))^k$. It follows from the standard properties of fibrations that $(x \cdot k(z)^{-1})^k$ lies in the center of $[B\pi_p; X_n]^{W_p}$. Therefore $b_1(\frac{1}{k} w_1) = x \cdot k(z)^{-1}$. We have also that $b_1(\frac{1}{k} w_1) = k(b(\frac{1}{k} w_1))$. This implies that $x \in \operatorname{im} k$.

It rest to show the theorem for an arbitrary nilpotent, p-local space X. We use once more the Postnikoff tower of X and the same diagram as before. The map i is an isomorphism because ΩX_{n-1} is a loop space. We assume that ℓ is surjective. To show that k is surjective we must use the following lemma.

<u>Lemma 2</u>. Let M be a finitely generated $Z_{(p)}$-module. Let us suppose that the abelian group M acts on a set X in such a way that isotropy subgroups are $Z_{(p)}$-submodules of M. We denote this action by $*$. Let us suppose further that a finite group G acts on M and on X, the action of G on M is $Z_{(p)}$-linear, the order of G is $k \in Z^*_{(p)}$ and $h^g * x^g = (h*x)^g$.
If $x, x_1 \in X^G$ and $\omega * x = x_1$ then $\left(\sum_{g\in G} \frac{1}{k}\,\omega^g\right) * x = x_1$.

<u>Proof</u>. $\omega * x = x_1$ and $x, x_1 \in X^G$ imply that $\omega^g * x = x_1$ for each $g \in G$. $\omega^g * ((\omega-\omega^g) * x)) = \omega^g * x$ implies that $(\omega-\omega^g) * x = x$ for each $g \in G$. Therefore $\left(\frac{1}{k} \sum_{g\in G} (\omega-\omega^g)\right) * x = x$. We have that $\frac{1}{k} \sum_{g\in G} \omega^g + \frac{1}{k} \sum_{g\in G} (\omega-\omega^g) = \omega$. This implies $\left(\frac{1}{k} \sum_{g\in G} \omega^g\right) * x = x_1$. ∎

The action of $[B\pi_p; K(\pi_n; n)]$ on $[B\pi_p; X_n]$ satisfies the assumptions of Lemma 2. We prove that k is surjective in the same way as for a double loop space. We have that $c_1(k(z)) = c_1(z)$. Therefore there is ω such that $\omega * k(z) = x$. It follows from Lemma 2 that

$\left(\sum_{g\in W_p} \frac{1}{k}(\omega\circ h_g)\right) * k(z) = x$. $\left(\sum_{g\in W_p} \frac{1}{k}(\omega\circ h_g')\right) = j(\omega_1)$ implies that $k(\omega_1 * z) = x$. The spaces $B\pi$ and $B\pi_p$ have only finite homology groups therefore we have isomorphisms

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$[B\pi;X] \overset{\approx}{\to} \varprojlim_n [B\pi, X_n]$ and $[B\pi_p;X] \overset{\approx}{\to} \varprojlim_n [B\pi_p, X_n]$. (If $\{X_n\}_{n\in N}$ is an inverse system of p-complete spaces then the functor $\varprojlim_n [\ ;X_n]$ is representable by Sullivan i.e. $\varprojlim_n [\ ;X_n] = [\ ;Z]$ and $Z = \operatorname{holim}_{\leftarrow} X_n$. In our case $\operatorname{holim}_{\leftarrow}(X_n)^\wedge_p = X^\wedge$ and

$[B\pi \text{ (or } B\pi_p); X \text{ (or } X_n)] = [B\pi \text{ (or } B\pi_p); X^\wedge_p \text{ (or} (X_n)^\wedge_p)]$

because $B\pi$ and $B\pi_p$ have finite homotopy groups.)

We have the following commutative diagram

$$\begin{array}{ccc} [B\pi;X] & \xrightarrow[\approx]{pr} & \varprojlim_n [B\pi;X_n] \\ a\downarrow & & b\downarrow\wr \\ [B\pi_p,X]^{W_p} & \xrightarrow{pr_1} & \varprojlim_n [B\pi_p;X_n]^{W_p} \end{array} .$$

pr is an isomorphism, b is an epimorphism (resp. isomorphism if X is a loop space) and pr_1 is a monomorphism. This implies that a is an epimorphism (resp. isomorphism if X is a loop space). This finishes the proof of Theorem 2. ■

If we analize the proofs carefully then it appears that in fact we have proved much more general result.

Let us suppose that a finite group G acts homotopically on a space X ,i.e. there is a homomorphism $G \to \pi_0(\varepsilon(X))$ where $\varepsilon(X)$ is the space of all homotopy equivalences of X . Let us suppose that $|G| = k$, X is p-local and $k \in Z^*_{(p)}$. By the result of Cooke there is a space X_1 with a free action of G and a homotopy equivalence

$i : X \to X_1$ which is homotopy equivariant with respect to the homotopy action of G .

Theorem 3. Let Z be a p-complete, nilpotent space. The natural map

$$[X_1/G;Z] \twoheadrightarrow [X,Z]^G \quad \text{is a surjection.}$$

If Z is a loop space then

$$[X_1/G;Z] \xrightarrow{\approx} [X;Z]^G \quad \text{is a bijection.}$$

Application to maps between classifying spaces. From Theorems 2 and 3 we deduce some corollaries concerning maps between classifying spaces.

Corollary 4. Let X be a nilpotent space and let π satisfies W_p-condition for every prime p. Suppose that there are maps $f_p : B\pi_p \to X_p^{\wedge}$ homotopy equivariant with respect to an action of W_p. Then there is $f : B\pi \to X$ such that $B\pi_p \xrightarrow{f|B\pi_p} X \to X_p^{\wedge}$ is homotopic to f_p.

Corollary 5. Let π satisfies W_p-condition for every prime p. Let $x \in \tilde{K}_0(B\pi)$ be such that $x_{|B\pi_p} \in \operatorname{Im}(R^+(\pi_p)^{W_p} \to \tilde{K}_0(B\pi_p))$ where $R^+(\pi_p)$ is the set of honest representations. Then there is $f : B\pi \to BU(n)$ such that $B\pi \xrightarrow{f} BU(n) \to BU$ is homotopic to x.

Both these corollaries follows easily from Theorem 2 and the arithmetic square of Sullivan.

Let G be a connected, compact Lie group, T maximal torus in G and let $W = N(T)/T$. W acts homotopically on BT. Using the Cooke result we can construct a honest action of W on $BT\frac{1}{|W|}$ such that the map $BT \to BT\frac{1}{|W|}$ is homotopy equivariant. The standard result about cohomology of BG implies that $BT\frac{1}{|W|}/W$ is

$Z[\frac{1}{|W|}]$-homologically equivalent to $BG \frac{1}{|W|}$.

Corollary 6. If X is nilpotent, p-complete and $(p;|W|) = 1$ then

i) the natural map $[BG \frac{1}{|W|};X] \twoheadrightarrow [BT \frac{1}{|W|};X]^W$ is surjective,

ii) if X is a loop space then we have an isomorphism

$$[BG \frac{1}{|W|};X] \overset{\approx}{\to} [BT \frac{1}{|W|};X]^W \,.$$

iii) $H^*(BT;Z/p) = H^*(BG;Z/p) \oplus M(p)$ as a A_p-module, A_p is the Steenrod algebra.

The points i), ii) of Corollary 6 are consequence of Theorem 3. The point iii) follows from the suitable generalization of Theorem 1.

Let us suppose that $p = 2$ and $G = U(2)$. Then one can check that $H^*(BU(2);Z/2)$ is not a direct summand of $H^*(BT;Z/2)$ in the category of A_2-modules. This implies the following.

Corollary 7. Let $i : BT \to BU(2)$ be the natural map. The map $BU(2) \to \mathrm{Cone}(i)$ is stably non-trivial. This map is zero on cohomology.

What does this map induce on stable homotopy?

In a subsequent paper using quite different method we are able to show much stronger results than Theorem 3. We decided to publish this paper to show what one can get in this direction using the most natural way i.e. an induction on the Postnikoff system.

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