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ON THE LIE ALGEBRA OF A TRANSVERSALLY COMPLETE FOLIATION

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In this short note we consider the Lie algebra of all vector fields which preserve a transversally complete foliation. Our considerations are based on the structural theorem for a transversally complete foliation which was proved by Molino in [4]. The class of transversally complete foliations contains other important classes: fibrations with compact fibers and transversally oriented codimension one foliations without holonomy on compact manifolds (cf.[7], theorem 1.3). In [5], Omori gives an analogue of the classical theorem of Pursell and Shanks (cf.[6]) for a fibration with compact fibers. Recently Fukui and Tomita [2] proved another such an analogue for a transversally oriented codimension one foliation without holonomy. Our theorem 6 can be viewed as a generalization of the both above results.

In this note all objects are of class C^∞. The manifolds are connected, Hausdorff and second countable.

§.1. The theorem of Amemiya for foliations.

For any manifold M we denote by $\mathfrak{X}(M)$ the Lie algebra of all vector fields on M. Let (M,F) be a foliated manifold and let $J(M)$ be the Lie algebra of all leaf preserving vector fields on M, i.e. vector fields which are tangent to the leaves of F. Let $n = \dim M$ and $q = \operatorname{codim} F$. A local coordinate system $(U, x^1,\ldots,x^{n-q},y^1,\ldots,y^q)$ is said to be distinguished by the foliation F, if for fixed $y^1,\ldots,y^q$ the coordinates $x^1,\ldots,x^{n-q}$ are coordinates of a leaf. Let us denote $x = (x^1,\ldots,x^{n-q})$, $y = (y^1,\ldots,y^q)$. The proof of the following proposition is obvious.

Proposition 1. For any $X \in \mathfrak{X}(M)$ the following conditions are equivalent:

(1) $[X,Y] \in J(M)$, if $Y \in J(M)$,

(2) the flow of X preserves F

(3) $X = \Sigma\, \xi^i(x,y)\partial_i + \Sigma\, \eta^j(y)\tilde{\partial}_j$ on the domain of a distinguished chart, where $\partial_i = \partial/\partial x^i$, $\tilde{\partial}_j = \partial/\partial y^j$.

By $\mathfrak{X}_F$ we denote the Lie algebra of all vector fields satisfying the above conditions.

Lemma 2. If $X \in \mathfrak{X}_F$ satisfies $X(p) \neq 0$, then there is a distinguished chart (U,x,y) at p such that one of the following identities is satisfied on U: (i) $X = \tilde{\partial}_1$, or (ii) $X = \partial_1 + \Sigma\eta^i(y)\tilde{\partial}_i$, where $\eta^i(0,\ldots,0) = 0$ for $i = 1,\ldots,q$.

In fact, the vector $X(p)$ is either tangent to the leaf or not.

<u>Theorem 3</u> (Amemiya [1]). Let $(M,\mathcal{F})$ and $(M',\mathcal{F}')$ be foliated manifolds. If Φ is a Lie algebra isomorphism of $J(M)$ onto $J(M')$, then there is a foliation preserving diffeomorphism φ of M onto M' such that $\Phi = \varphi_*$ on $J(M)$.

The following corollary from Theorem 3 will be useful in the sequel.

<u>Corollary 4.</u> Let $(M,\mathcal{F})$ and $(M',\mathcal{F}')$ be foliated manifolds. If Φ is a Lie algebra isomorphism of $\mathfrak{X}_{\mathcal{F}}$ onto $\mathfrak{X}_{\mathcal{F}'}$ such that $\Phi(J(M)) = J(M')$, then there is a foliation preserving diffeomorphism φ of M onto M' such that $\Phi = \varphi_*$ on $\mathfrak{X}_{\mathcal{F}}$.

Proof. Let φ be the diffeomorphism obtained in Theorem 3. We use Proposition 1. Since

$$\Phi[X,Y] = [\Phi X,\Phi Y] \in J(M'), \quad \text{if} \quad X \in \mathfrak{X}_J,\ Y \in J(M)$$

$$\varphi_*[X,Y] = [\varphi_* X,\varphi_* Y] \in J(M'), \quad \text{if} \quad X \in \mathfrak{X}_{\mathcal{F}},\ Y \in J(M)$$

and $\Phi = \varphi_*$ on $J(M)$, we have

$$[\varphi_* X,Y'] = [\Phi X,Y'] \ , \text{ if } \ Y' \in J(M'). \qquad (\star)$$

In particular, $\Phi_* X \in \mathfrak{X}_{\mathcal{F}}$. Let us denote $X_1 = \varphi_* X - \Phi X$. We shall show that $X_1 = 0$. Suppose $X_1(p) \neq 0$. By Lemma 2 there is (U,x,y) a distinguished chart at p such that $X_1 = \tilde{\partial}_1$ or $X_1 = \partial_1 + \Sigma\eta^i(y)\tilde{\partial}_i$ in a neighborhood of p. Let $Y' = y^1\partial_1$ in the first case or $Y' = x^1\partial_1$ in the second. In both cases we obtain $[X_1, Y'] \neq 0$ which contradicts (☆).

§2. The structural theorem for transversally complete foliations.

Let (M,F) denote a foliated manifold with codimension q. Suppose M is a compact manifold. The foliation F is said to be transversally complete, if for any $p \in M$ the evaluation map

$$ev_p : X \in \mathfrak{X}_F \longrightarrow X(p) \in T_p(M)$$

is a surjection. In particular, if M is connected, then the group of all foliation preserving diffeomorphisms acts transitively on M. The examples of the transversally complete foliations are the total spaces of fibrations, the transversally parallelisable foliations and the Lie foliations (c.f.[4]).

We need the following fundamental result :

Theorem 5 (Molino [4]). If (M,F) is a transversally complete foliated manifold, then the closures of the leaves of F are the fibers of a fibration $\pi: M \longrightarrow W$. Moreover, the local trivialisations $\psi : \pi^{-1}(U) \longrightarrow U\times F$ of the fibration π preserve foliation, if $U\times F$ is foliated by $\{pt\}\times L$, where L is a leaf of F.

Remark. The fibration $\pi: M \longrightarrow W$ is called basic. $r = \dim W$ is called the basic dimension of the transversally complete foliation.

Now we introduce a special kind of coordinates. A local coordinate system $(U, x^1, \dots, x^{n-q}, \bar{y}^1, \dots, \bar{y}^{q-r}, y^1, \dots, y^r)$ is called distinguished by the foliation F, if for any fixed $(y^1, \dots, y^r)$ the coordinates $(x^1, \dots, x^{n-q}, \bar{y}^1, \dots, \bar{y}^{q-r})$ are coordinates of a fiber of π and for any fixed $(\bar{y}^1, \dots, \bar{y}^{q-r}, y^1, \dots, y^r)$ the coordinates $(x^1, \dots, x^{n-q})$ are coordinates of a leaf of F. The existence of the distinguished coordinates follows from Theorem 5. In the sequel, it shall be denoted $x = (x^1, \dots, x^{n-q})$, $\bar{y} = (\bar{y}^1, \dots, \bar{y}^{q-r})$, $y = (y^1, \dots, y^r)$.

§3. The analogue of Pursell-Shanks Theorem for transversally complete foliations.

We want to prove the following

Theorem 6. Let (M,F) and (M',F') be transversally complete foliated manifolds with M, M' compact and codim $F = q > 0$, codim $F' = q' > 0$.
If there is a Lie algebra isomorphism Φ of $\mathfrak{X}_F$ onto $\mathfrak{X}_{F'}$, then there is a foliation preserving diffeormorphism φ of M onto M' such that $\Phi = \varphi_*$.

Example. Let $M = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$. For any $\alpha \in \mathbb{R}$ M admits a well-known linear foliation F_α which is generated by the differential form $\omega = \alpha dx^1 + dx^2$, where (x^1, x^2) are coordinates on $\mathbb{R}^2$. If $\alpha \neq \beta$, then by the theorem of Denjoy there does not exist any diffeomorphism $\varphi: M \longrightarrow M$ such that $\varphi_*(F_\alpha) = F_\beta$. Hence, the Lie algebras $\mathfrak{X}_{F_\alpha}$ and $\mathfrak{X}_{F_\beta}$ are not isomorphic.

The proof of the theorem consists of several lemmas. Let $\pi : M \longrightarrow W$ be the basic fibration of F. By F_π we denote the foliation on M by the fibers of π. Let $J_\pi(M)$ denote $\ker \pi_*$ and let $\overline{J}(M)$ denote $J_\pi(M) \cap \mathfrak{X}_F$.

Lemma 7. Let $(U,x,\overline{y},y)$ be a distinguished chart of (M,F). Then

(1) $X = \Sigma\xi^i(x,\overline{y},y)\partial_i + \Sigma\eta^j(x,\overline{y},y)\overline{\partial}_j + \Sigma\zeta^k(y)\widetilde{\partial}_k$ on U for $X \in \mathfrak{X}_{F_\pi}$,

(2) $X = \Sigma\xi^i(x,\overline{y},y)\partial_i + \Sigma\eta^j(y)\overline{\partial}_j + \Sigma\zeta^k(y)\widetilde{\partial}_k$ on U for $X \in \mathfrak{X}_F$,

where, $\partial_i = \partial/\partial x^i$, $\overline{\partial}_j = \partial/\partial\overline{y}^j$, $\widetilde{\partial}_k = \partial/\partial y^k$.

Proof. (1) follows immediately from Proposition 1. In order to prove (2), let $X \in \mathfrak{X}_F$ and, in view of Proposition 1, $X = \Sigma\xi^i(x,\overline{y},y)\partial_i + \Sigma\eta^j(\overline{y},y)\overline{\partial}_j + \Sigma\zeta^k(\overline{y},y)\widetilde{\partial}_k$ on U. Note that $\mathfrak{X}_F \subset \mathfrak{X}_{F_\pi}$ since the leaves of F_π are the closures of the leaves of F. Hence, again in view of Proposition 1, we have $\zeta^k(\overline{y},y) = \zeta^k(y)$. Next, let us denote by F an arbitrary fiber of the basic fibration of F and by F_F the foliation induced by F on F. Let $\mathfrak{X}_F$ be the Lie algebra of all vector fields on F preserving F_F. Suppose L is a dense leaf in F. It is easily seen that if $X \in \mathfrak{X}_F$ is such that $X(p)$ is tangent to L, $p \in L$, then $X \in J(F)$, i.e. X is tangent to the foliation F_F. Therefore $\dim \mathfrak{X}_F/J(F) \leqslant \operatorname{codim} F_F = q-r$. Hence $\eta^j(\overline{y},y)$ cannot depend on $\overline{y}$ and (2) is satisfied.

Lemma 8. Let m be a proper ideal of $\mathfrak{X}_F$ such that for any point $p \in M$ there is $X \in m$ satisfying $(\pi_* X)(x) \neq 0$, where $x = \pi(p)$. Then $m + \overline{J}(M) = \mathfrak{X}_F$.

Proof. Let $W = \bigcup_{\alpha=1}^{s} U'_\alpha$ be a finite open covering of W such that for each α the fibration $\pi: M \longrightarrow W$ is trivial over U'_α, there is $(U'_\alpha, y^1, \ldots, y^r)$ a chart of W and there is $X \in m$ such that $\pi_* X = \widetilde{\partial}_1$ on U'_α. Suppose $W = \bigcup_{\alpha=1}^{s} U_\alpha$ is a new open covering of W such that $U_\alpha \subset \overline{U}_\alpha \subset U'_\alpha$. We consider a distinguished coordinate system $(V \times U'_\alpha, x, \overline{y}, y)$, where $(V, x, \overline{y})$ is a distinguished coordinate system for the foliation F_F. Let us define $X_1 \in \mathfrak{X}_F$ such that $X_1 = \widetilde{\partial}_1$ on $F \times U_\alpha$ and $X - X_1 \in \overline{J}(M)$. There is $\{\lambda, \mu\} \subset C^\infty(W)$ a partition of unity subordinated to $M = U'_\alpha \cup \backslash \overline{U}_\alpha$. Then we define $X_1 = \lambda \widetilde{\partial}_1 + \mu X$. It is easily seen that $X - X_1 \in \overline{J}(M)$ and $X_1 \in m + \overline{J}(M)$.

Let Y be an arbitrary element of $\mathfrak{X}_F$. If $\{\psi_\alpha\}$ is a partition of unity on W subordinated to the covering $M = \cup U_\alpha$, then $Y = Y_1 + \ldots + Y_s$, where $Y_\alpha = \psi_\alpha Y \in \mathfrak{X}_F$. Thus it suffices to show that $Y \in m + \overline{J}(M)$ for $Y \in \mathfrak{X}_F$ such that supp $Y \subset F \times U = \pi^{-1}(U)$, where $U = U_\alpha$ for some α. Let $Z \in \mathfrak{X}_F$ be such that $Z = \pi_* Y$ on $F \times U$ and supp $Z \subset F \times U$. Hence $Y - Z \in \overline{J}(M)$ and it suffices to show that $Z \in m + \overline{J}(M)$. Let χ denote a C^∞-function on W such that supp $\chi \subset U$ and $\chi = 1$ on a neigborhood of $\pi(\text{supp } Z)$ and let μ be another C^∞-function such that supp $\mu \subset U$ and $\mu = 1$ on a neighborhood of supp χ. Let $Z = \Sigma \zeta^j(y) \widetilde{\partial}_j$, where supp $\zeta^j \subset U$. Since $X_1 = \widetilde{\partial}_1$ on $F \times U$,

$$[X_1, (\mu \int_{-\infty}^{Y^1} \chi dy^1) \widetilde{\partial}_j] = Z_1 + \chi \widetilde{\partial}_j$$

for any j, where supp $Z_1 \subset \setminus$ supp χ. Next

$$[Z_1 + \chi\tilde{\partial}_j, (\chi \int_{-\infty}^{y^j} \zeta^j dy^j)\tilde{\partial}_j] = [\chi\tilde{\partial}_j, (\chi \int_{-\infty}^{y^j} \zeta^j dy^j)\tilde{\partial}_j] =$$

$$= \chi^2 \zeta^j \tilde{\partial}_j = \zeta^j \tilde{\partial}_j .$$

Since $X_1 \in m + \bar{J}(M)$, we have $\zeta^j\tilde{\partial}_j \in m + \bar{J}(M)$ and $Z \in m+\bar{J}(M)$. Lemma 8 is then proved.

Note that the assumption of Lemma 8 is satisfied only if the basic dimension r is nonzero.

Lemma 9. Under the assumption of Lemma 8, if m is maximal then m is one codimensional.

Proof. First we show that $J(M) \subset m$. Let $M = \cup V_\alpha$ be a finite open covering of M such that $(V_\alpha, x, \vec{y}, y)$ is a distinguished chart of M and a suitable extension of $\tilde{\partial}_1 = \partial/\partial y^1$ is contained in m. Let $\{\psi_\alpha\}$ be a partition of unity subordinated to the above covering. We define $X_\alpha = \psi_\alpha X$ for $X \in J(M)$. By an argument similar to that in the proof of Lemma 8, one can see that $X_\alpha \in m$. Hence $X \in m$ and $J(M) \subset m$. Next observe that $\bar{J}(M)/J(M)$ is abelian. Indeed, every $X \in \bar{J}(M)$ satisfies

$$X = \Sigma\, \xi^i(x, \vec{y}, y)\partial_i + \Sigma\, \eta^j(y)\bar{\partial}_j$$

on the domain of a distinguished chart. Hence, it is easily seen that $[X,Y] \in J(M)$ for $X, Y \in \bar{J}(M)$. In particular, $[X,Y] \in m$ for $X, Y \in \bar{J}(M)$. Thus, in view of Lemma 8, m is a maximal vector subspace of $\mathfrak{X}_F$.

<u>Lemma 10.</u> If m is a maximal ideal of $\mathfrak{X}_F$ satisfying the assumption of Lemma 8, then $\mathcal{J}(M) \subset m$.

Proof. Let $Y \in \mathcal{J}(M)$. As in the proof of Lemma 8, we can assume that supp $Y \subset F \times U = \pi^{-1}(U)$, where (U,y) is a chart of W. Let $\partial \in \mathfrak{X}_F$ denote some extension of $\tilde{\partial} = \partial/\partial y^1$ on $F \times U$. We can assume $\partial \in m$. In fact, if $\partial \notin m$ and $y^1\partial \notin m$ (otherwise $[\partial, y^1\partial] = \partial \in m$), then there is $\alpha \neq 0$ such that $\alpha\partial + y^1\partial \in m$ (Lemma 9) and we get

$$[\partial, \alpha\partial + y^1\partial] = \partial \in m.$$

Next we have

$$[\partial, Y] = \partial Y \in m,$$

$$[\partial, y^1 Y] = Y + y^1\partial Y \in m,$$

$$[y^1\partial Y + y^1\partial Y] = 2y^1\partial Y + (y^1)^2\partial^2 Y \in m,$$

$$[(y^1)^2\partial, \partial Y] = (y^1)^2\partial^2 Y \in m.$$

Hence we have $y^1\partial Y \in m$ and $Y \in m$. Lemma 10 is then proved.

Suppose dim $W = r > 0$ and $x \in W$. By m_x we denote the ideal of $\mathfrak{X}_F$ which consits of all $X \in \mathfrak{X}_F$ such that $\pi_* X$ and all its derivatives vanish at x.

<u>Lemma 11.</u> Let m be an ideal of $\mathfrak{X}_F$ such that $\pi_* m$ vanishes

at $x \in W$. Then the ideal $\pi_* m$ vanishes at x with all its derivatives, i.e. $m \subset m_x$.

Proof. Let $(V, y^1, \ldots, y^r)$ be a chart at x such that π is trivial over V. Let $X \in m$ be such that $\pi_* X = \Sigma \eta^j(y)\tilde{\partial}_j$ and suppose there is $k > 0$ such that

$$\partial^k \eta^i(x)/\partial y^I \neq 0 \quad \text{for some } i \text{ and } I = (i_1, \ldots, i_k).$$

Then $[\pi_* X, \tilde{\partial} i_k] = -\Sigma(\eta^j(x)/\tilde{\partial} y^{i_k})\tilde{\partial}_j \in \pi_* m$. This vector field has a coefficient with derivative of order $k-1$ wich does not vanish at x. Repeating this procedure $k-1$ times we get $X' \in m$ such that $\pi_* X$ does not vanish at x.

<u>Corollary 12.</u> Every ideal m_x is maximal. Every proper ideal of $\mathfrak{X}_F$ must be contained in some m_x.

<u>Corollary 13.</u> The ideal $\bar{J}(M)$ is the intersection of all maximal ideals of $\mathfrak{X}_F$.

<u>Lemma 14.</u> The ideal $J(M)$ can be characterized as a minimal ideal of $\bar{J}(M)$ such that $\bar{J}(M)/J(M)$ is abelian.

Proof. We stated that $\bar{J}(M)/J(M)$ is abelian in the proof of Lemma 9. It suffices to show that $J(M)$ is a unique ideal which is minimal with this property. Suppose a is an ideal of $\bar{J}(M)$ such that $a \not\supset J(M)$ and $\bar{J}(M)/a$ is abelian. Then $J(M)/a \cap J(M)$ $\neq 0$ is also abelian. So it suffices to prove that there does not exist any ideal a such that $a \subsetneq J(M)$ and $J(M)/a$ is abelian. Assume a is an ideal with the above property. Let $X \in J(M)$ and, by a partition-of-unity argument, let supp X be contained in a chart domain U. Let $Z \in \mathcal{J}(M)$ be such that $Z = \partial/\partial x^1$ on a neighborhood of supp X and supp $Z \subset U$. Repeating the reasoning from the proof of Lemma 8, one can see that there are $I(X)$, $J(X) \in \mathcal{J}(M)$ such that

$$[[Z, I(X)], J(X)] = X \quad \text{on } M.$$

Since $\mathcal{J}(M)/a$ is abelian, we have $X \in a$ and $\mathcal{J}(M) \subset a$. This contradiction proves the uniqueness of $\mathcal{J}(M)$.

Corollary 15. $\mathfrak{X}_F$ has an ideal with finite codimension if and only if $r = 0$.

In fact, for $r = 0$ $\mathcal{J}(M)$ is q codimensional in $\mathfrak{X}_F$.

Now we are in a position to conclude the proof of Theorem 6. Let Φ be a Lie algebra isomorphism of $\mathfrak{X}_F$ onto $\mathfrak{X}_{F'}$. If $\mathfrak{X}_F$ and $\mathfrak{X}_{F'}$ have finite codimensional ideals, then $\mathfrak{X}_F{}' = \overline{\mathcal{J}}(M)$, $\mathfrak{X}_{F'}{}' = \overline{\mathcal{J}}(M')$ and trivially $\Phi\ (\overline{\mathcal{J}}(M)) = \overline{\mathcal{J}}(M')$. Otherwise, we get $\Phi(\overline{\mathcal{J}}(M)) = \overline{\mathcal{J}}(M')$ by Corollary 13. Next Lemma 14 implies $\Phi(\mathcal{J}(M)) = \mathcal{J}(M')$. Finally we apply Corollary 4.

Remark. Let us discuss briefly the assumption of Theorem 6. The theorem is no longer true for the trivial case of codimension 0 foliation i.e. the foliation with one leaf M. The foliations $F = \{M\}$, $F' = \{\text{points}\}$ give a simple counterexample to Theorem 6. Lecomte in his thesis [3] considered the Lie algebra of all infinitesimal automorphisms of a vector bundle E with a base B and standard fibre F. It was proved a Pursell-Shanks type theorem. However, if dim $F = 1$ then the Lie algebra of all infinitesimal automorphisms of E is isomorphic to $\mathfrak{X}(B) \times C^\infty(B)$

and obviously it does not determine E. This result suggest that Theorem 6 cannot be proved for the non-compact case.

References

[1] I.Amemiya, Lie algebra of vector fields and complex structure, J. Math. Soc. Japan 27(1975), p.545.

[2] K. Fukui, N. Tomita, Lie algebra of foliation preserving vector fields, J. Math. Kyoto Univ. 22(1983), p.685.

[3] P. Lecomte, Algèbres de Lie d'ordre zero sur une variété, Thesis, Université de Liège, 1980.

[4] P. Molino, Etude des feuilletages transversalement complets et applications, Ann. Sci.Ec.Norm.Sup. 10(1977), p.289.

[5] H. Omori, Infinite dimensional Lie transformation groups, Springer Lecture Notes in Math., New York 1976.

[6] L.E. Pursell, M.E. Shanks, The Lie algebra of smooth manifold, Proc.Amer.Math.Soc. 5(1954), p.468.

[7] H. Rosenberg, R. Roussarie, Reeb foliations, Ann. of Math. 91 (1970), p.1.

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