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ON A LOCALLY CONVEX SPACE ADMITTING A FUNDAMENTAL SEQUENCE OF STRONGLY BOUNDED SUBSETS

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ABSTRACT. Let $E(t)$ be a locally convex space admitting a fundamental sequence $\{B_n\}$ of strongly bounded subsets. In this paper, we consider a characterization of the subspace $\bigcup_{n=1}^{\infty} \tilde{B}_n$ of the completion $\tilde{E}$ of $E(t)$, where each $\tilde{B}_n$ denotes the completion of B_n. As a result of this, the space $\bigcup_{n=1}^{\infty} \tilde{B}_n$ is characterized as the smallest subspace of all subspaces of $\tilde{E}$ that contain E and have the property that every closed strongly bounded subset is complete. Furthermore, by using this result, we concretely study the L^p-spaces under some weak topologies.

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INTRODUCTION.

Let $E(t)$ be a locally convex space admitting a fundamental sequence of bounded subsets $\{B_n\}$. Then many authors have considered this space in general or under some additional conditions and have obtained various important results.(for instance, see M. De Wilde [3] and D. J. H. Garling [4].) As one of these results, we can give the result concerned with the subspace $\bigcup_{n=1}^{\infty} \tilde{B}_n$ of the completion $\tilde{E}$ of $E(t)$, where each $\tilde{B}_n$ is the completion of B_n in $E(t)$. For example, if $E(t)$ is distinguished, denoting by $\overline{E}$ the quasi-completion of $E(t)$, then $\overline{E} = \bigcup_{n=1}^{\infty} \tilde{B}_n$. Furthermore in case $E(t)$ is a (DF)-space, then $\tilde{E} = \bigcup_{n=1}^{\infty} \tilde{B}_n$. But comparatively little attension has been paid to a locally convex space $F(t')$ admitting a fundamental sequence of strongly bounded subsets $\{C_n\}$.

The main purpose of this paper is to consider a characterization of the subspace $\bigcup_{n=1}^{\infty} \tilde{C}_n$ of the completion $\tilde{F}$. To answer this problem, we introduce a certain completion of a locally convex space, which has the property that every closed strongly bounded subset is complete. We call this completion the β-quasi-completion. The property of the β-quasi-completion is weaker than that of the quasi-completion. However, as is shown later, under an appropriate condition we can easily obtain the β-quasi-completion(see Proposition 5.) and this completion of a non-complete locally convex space is frequently such a delicate subspace that is not obtained by considering the quasi-completion. (see the results in section 4.) In these respects, the notion of β-quasi-completion seem to be useful in application.

Here follows explanation of each section. In section 1, we explain some definitions and notations used through this paper. In section 2, we introduce the β-quasi-completion of a locally convex space and, under an appropriate condition, we give a representation of this completion. In section 3, by using the result in section 2, we give an answer of the above problem. In the final section, we use the results in section 3 to study the β-quasi-completions of the L^p-spaces with some weak topologies, and give a few propositions with the Cauchy sequences in this completion.

1. Preliminaries.

Mostly we shall follow the definitions and notations in H. Jarchow [6] and H. H. Schaefer [10]. Through this paper we deal with Hausdorff locally convex spaces over the real field R. Let E(t) be a locally convex space. We denote by E' the topological dual of E(t) and simply call this space the dual of E(t). Further we write $\overline{E}^{seq}(\overline{t}^{seq})$, $\overline{E}(\overline{t})$ and $\tilde{E}(\tilde{t})$ for the sequential completion, the quasi-completion and the completion of E(t) respectively. Let B be any subset of E, then to specify a locally convex topology we represent $\hat{B}[t]$ and $\bar{B}[t]$ for the completion and the closure of B respectively. If $\{A_\alpha\}$ is a family of subsets of E, then we express $AC(A_\alpha)$ for the absolutely convex cover of $\bigcup_\alpha A_\alpha$. As for bounded subsets in $E(t)$, we denote by B_{st} the family of all closed absolutely convex strongly bounded subsets. If there exists a sequence $B_1 \subset B_2 \subset \cdots$ of absolutely convex bounded (strongly bounded) subsets such that every bounded (strongly bounded) subset is contained in some B_k, then we call this sequence a fundamental sequence of bounded (strongly bounded) subsets. Let (E, F) be a dual pair, then we express $\langle u, v\rangle$, $u \in E$, $v \in F$ for a bilinear form on E and F. For the dual pair (E, F), we often use the following locally convex topologies;

$\sigma(E, F)$ = the topology of uniform convergence on the set of all finite subsets of F on E,

$\beta(E, F)$ = the topology of uniform convergence on the set of all $\sigma(F, E)$-bounded subsets on E and

$\beta^*(E, F)$ = the topology of uniform convergence on the set of all $\beta(F, E)$-bounded subsets on E.

Finally we give some definitions of locally convex spaces. $E(t)$ is said to be quasi-$\aleph_0$-barrelled if every bornivorous barrel in E which can be represented as the intersection of a sequence of closed, absolutely convex o-neighbourhoods is itself a o-neighbourhood in $E(t)$.(see H. Jarchow [6].) Supposing that every $\sigma(E', E)$-bounded subset of E' is $\beta(E', E)$-bounded, then we call this space W-space.(see P. K. Kamthan and M. Gupta [7].) $E(t)$ is said to be β-semi-Montel if every closed strongly bounded subset is compact.(see K. Kitahara [8].)

2. On a locally convex space.

Definition 1. A locally convex space $E(t)$ is said to be a β-quasi-complete space if for every $\beta(E, E')$-bounded subset B, $\overline{B}[t]$ is complete.

Definition 2. Let $\tilde{E}(\tilde{t})$ be the completion of a locally convex space $E(t)$ and $\{E_\lambda\}_{\lambda\in\Lambda}$ the family which consists of all the β-quasi-complete subspaces of $\tilde{E}(\tilde{t})$ with $E \subset E_\lambda \subset \tilde{E}$. Then we set the linear subspace $\overline{E}^\beta = \bigcap_{\lambda\in\Lambda} E_\lambda$ of $\tilde{E}$ and call this space the β-quasi-completion of $E(t)$. We denote by $\overline{t}^\beta$ the induced topology from $\tilde{E}(\tilde{t})$ on $\overline{E}^\beta$. To specify a locally convex topology, we often use the notation $\overline{E}^\beta(\overline{t}^\beta)$ for the β-quasi-completion of $E(t)$.

Proposition 1. *The β-quasi-completion $\overline{E}^\beta$ of a locally convex space $E(t)$ is β-quasi-complete under the topology $\overline{t}^\beta$.*

Proof. Let B be an arbitrary $\beta(\overline{E}^\beta, E')$-bounded subset(The dual of $\overline{E}^\beta(\overline{t}^\beta)$ is E'.) and $\{E_\lambda\}_{\lambda\in\Lambda}$ the family in Definition 2. Since $\overline{E}^\beta \subset E_\lambda$ and $\beta(\overline{E}^\beta, E')$ is finer than $\beta(E_\lambda, E')$ on $\overline{E}^\beta$ for each $\lambda \in \Lambda$, B is $\beta(E_\lambda, E')$-bounded. Thus $\overline{B}[\overline{t}^\beta] = \tilde{B}[t] \cap (\bigcap_{\lambda\in\Lambda} E_\lambda) = \bigcap_{\lambda\in\Lambda}(\tilde{B}[t] \cap E_\lambda) = \tilde{B}[t]$. Hence we obtain $\tilde{B}[t] \subset \overline{E}^\beta$.

Remark 1. For a locally convex space $E(t)$, if we set $E^\beta = \bigcup_{B\in B_{st}} \tilde{B}[t]$, then it holds that

$$E \subset E^\beta \subset \overline{E}^\beta \subset \overline{E} \subset \tilde{E}.$$

Now we show that under what conditions $\overline{E}^\beta$ coincides with $\overline{E}$.

Proposition 2. *Let $E(t)$ be a locally convex space, then $\overline{E}^\beta$ coincides with $\overline{E}$ if and only if $\overline{E}^\beta(\overline{t}^\beta)$ is a W-space.*

Proof. If $\overline{E}^\beta = \overline{E}$, then $\overline{E}^\beta(\overline{t}^\beta)$ is a W-space from its quasi-completeness. Conversely let $\overline{E}^\beta(\overline{t}^\beta)$ be a W-space. Since, in $\overline{E}^\beta(\overline{t}^\beta)$, $\sigma(\overline{E}^\beta, E')$-boundedness is identical with $\beta(\overline{E}^\beta, E')$-boundedness, $\overline{E}^\beta(\overline{t}^\beta)$ is a quasi-complete space. Hence we obtain $\overline{E}^\beta = \overline{E}$ from Remark 1.

Definition 3. Let $E(t)$ be a locally convex space and $E''_{\beta*}$ the dual of $E'(\beta^*(E', E))$. If for an arbitrary $\beta(E''_{\beta*}, E')$-bounded subset B of $E''_{\beta*}$, there exists a $\beta(E, E')$-bounded subset $B_0 \subset E$ such that $\overline{B}_0[\sigma(E''_{\beta*}, E')] \supset B$, then $E(t)$ is said to have (*)-property.

Proposition 3. *A locally convex space $E(t)$ has (*)-property if and only if $E'(\beta^*(E', E))$ is quasi-barrelled.*

Proof. If $E(t)$ has (*)-property and B is an arbitrary $\beta(E''_{\beta*}, E')$-bounded subset in $E''_{\beta*}$, then there exists a $\beta(E, E')$-bounded subset B_0 in E such that $\overline{B}_0[\sigma(E''_{\beta*}, E')] \supset B$. Since $\overline{B}_0[\sigma(E''_{\beta*}, E')]$ is a $\beta^*(E', E)$-equicontinuous subset, so is B. Conversely suppose that $E'(\beta^*(E', E))$ is quasi-barrelled and B is an arbitrary $\beta(E''_{\beta*}, E')$-bounded subset. Then the polar B^o of B in E' is a $\beta^*(E', E)$-neighbourhood of o. Hence we take a $\beta(E, E')$-bounded subset B_1 satisfying that the polar B_1^o of B_1 in E' is contained in B^o. Consequently the polar B^{oo} of B^o in $E''_{\beta*}$ is contained in the polar B_1^{oo} of B_1^o in $E''_{\beta*}$. Thus we get the conclusion from the property of polarity.

Using Proposition 3, first we show

LEMMA 4. *Let $E(t)$ be a locally convex space with (*)-property, then the β-quasi-completion $\overline{E}^{\beta}(\overline{\sigma(E, E')}^{\beta})$ of $E(\sigma(E, E'))$ coincides with $E''_{\beta*}$.*

Proof. Since $E''_{\beta*}$ is the dual of $E'(\beta^*(E', E))$, $E''_{\beta*} = \bigcup_{B \in B_{st}} B^{oo}$, where B^{o} is the polar of B in E' and B^{oo} is the polar of B^{o} in $E''_{\beta*}$. Every B^{oo} is equal to $\overline{B}[\sigma(E''_{\beta*}, E')]$ and $\sigma(E''_{\beta*}, E')$-compact, hence $B^{oo} = \tilde{B}[\sigma(E, E')]$. So $\overline{E}^{\beta}(\overline{\sigma(E, E')}^{\beta}) \supset E''_{\beta*}$ follows from Remark 1. Let B be an arbitrary $\beta(E''_{\beta*}, E')$-bounded subset, then by the assumption there exists a $\beta(E, E')$-bounded subset B_0 with $B \subset B_0^{oo} \subset E''_{\beta*}$. From this fact, we can state that $E''_{\beta*}(\sigma(E''_{\beta*}, E'))$ is a β-quasi-complete space, which implies $\overline{E}^{\beta}(\overline{\sigma(E, E')}^{\beta}) = E''_{\beta*}$.

Then we prove

PROPOSITION 5. *Let $E(t)$ be a locally convex space with (*)-property, then the β-quasi-completion $\overline{E}^{\beta}(\overline{t}^{\beta})$ of $E(t)$ coincides with $\bigcup_{B \in B_{st}} \tilde{B}[t]$.*

Proof. Since clearly $\overline{E}^{\beta}(\overline{t}^{\beta}) \supset \bigcup_{B \in B_{st}} \tilde{B}[t]$, it is sufficient to show the converse inclusion. First we consider the identity map $i : E(t) \to E(\sigma(E, E'))$. As this map satisfies the filter condition(see Sec. 6, N. Adasch, B Ernst and D. Keim [1].), the continuous extension $\tilde{i} : \tilde{E}_t(\tilde{t}) \to \tilde{E}_{\sigma}(\widetilde{\sigma(E, E')})$ of i is one to one, where $\tilde{E}_t$ and $\tilde{E}_{\sigma}$ are the completions of $E(t)$ and of $E(\sigma(E, E'))$ respectively. Hence we regard $\tilde{E}_t$ as a subspace of $\tilde{E}_\sigma$. Further if we set $F = \tilde{i}^{-1}(\overline{E}_\sigma^\beta)$, where $\overline{E}_\sigma^\beta$ is the β-quasi-completion of $E(\sigma(E, E'))$, then the subspace F of $\tilde{E}_t$ is β-quasi-complete under the induced topology from $\tilde{E}_t(\tilde{t})$. So we also regard the β-quasi-completion $\overline{E}_t^\beta$ of $E(t)$ as a subspace of $\overline{E}_\sigma^\beta$. Now let B be an arbitrary absolutely convex $\beta(\overline{E}_t^\beta, E')$-bounded subset of $\overline{E}_t^\beta$. Since B is $\beta(\overline{E}_\sigma^\beta, E')$-bounded, there exists an absolutely convex $\beta(E, E')$-bounded subset B_1 with $\tilde{B}_1[\sigma(E, E')] \supset B$ by Lemma 4. From this fact, we obtain $\overline{E}_t^\beta \cap \tilde{B}_1[\sigma(E, E')] = \overline{B}_1[\sigma(\overline{E}_t^\beta, E')] \supset B$. On the other hand, the locally convex topologies $\sigma(\overline{E}_t^\beta, E')$ and $\overline{t}^\beta$ are compatible with the dual pair $(\overline{E}_t^\beta, E')$. Hence we have $\overline{E}_t^\beta \cap \tilde{B}_1[\sigma(E, E')] = \overline{B}_1[\overline{t}^\beta] \supset \overline{B}[\overline{t}^\beta] = \tilde{B}[\overline{t}^\beta]$. This means that $\overline{E}_t^\beta$ is contained in $\bigcup_{B \in \mathcal{B}_{st}} \tilde{B}[t]$.

By Proposition 3, every distinguished locally convex space does not have (*)-property and conversely every locally convex space with (*)-property is not distinguished. Here we give a normed space which does not have (*)-property by using a counterexample in G. Köthe [9].(see P. 435.)

EXAMPLE. First we use the following notations. As linear subspaces of $R^N \times R^N$, we set

$$\ell^1 = \{ (x_{i,j}) \mid \sum_{i,j} |x_{i,j}| < +\infty \},$$

$$\ell^\infty = \{ (x_{i,j}) \mid \sup_{i,j} |x_{i,j}| < +\infty \},$$

$\lambda = \{ (x_{i,j}) \mid \sum_{i,j} |a_{i,j}^{(n)} \cdot x_{i,j}| < +\infty$ for each $n \in N$, where $a_{i,j}^{(n)} = j$ for $i \leq n$ and $a_{i,j}^{(n)} = 1$ for $i > n \}$ and

$\lambda' = \{ (x_{i,j}) \mid$ there exist an $n \in N$ and a $c > 0$ such that $|x_{i,j}| \leq c \cdot a_{i,j}^{(n)}$ for $(i, j) \in N \times N \}$.

Further we set

t = the locally convex topology on λ generated by the semi-norms $\{p_n\}$, where $p_n(x) = \sum_{i,j} |a_{i,j}^{(n)} \cdot x_{i,j}|$ for all $(x_{i,j}) \in \lambda$ and all $n \in N$.

Then $\lambda(t)$ is an (F)-space with its dual λ'. By G. Köthe [9], in the strong dual $\lambda'(\beta(\lambda', \lambda))$ of $\lambda(t)$ the following facts hold.

(1) For the sequence $B_n = \{ (x_{i,j}) \mid (x_{i,j}) \in \lambda', |x_{i,j}| \leq a_{i,j}^{(n)}$ for all $(i, j) \in N \times N \}$ $n \in N$ of the subsets of λ', $\{n \cdot B_n\}$ forms a fundamental sequence of bounded subsets.

(2) The family consisting of subsets of the form $U_{(c_n)} = (\overline{AC(c_n \cdot B_n)})[\sigma(\lambda', \lambda)]$, where each c_n is a positive number, is a base of neighbourhoods of o.

(3) Every o-neighbourhood U has an element $u = (u_{i,j})$ such that $u \in \ell^{\infty}$ and for each $i \in N$ there exists a $k_i \in N$ with $u_{i,k_i} = 2$.

(4) The subset $V = AC(2^{-n} \cdot B_n)$ is borniborous and does not have an element satisfying the condition in (3). Hence V is not a neighbourhood of o.

To give an example, we need the following two lemmas.

Lemma 6. *$\overline{V}[\beta(\lambda', \lambda)]$ does not have an element satisfying the condition in* (3).

Proof. By 4-(1), P. 399 in G. Köthe [9], it is sufficient to show that the algebraic hull V^a of V satisfies the

the condition in (3). Then there is a $v \in V$ with $[v, u) \subset V$, where $[v, u)$ denotes the real line segment joining v and u, including v and excluding u. On the other hand, for each element z of V it holds that $|z_{i,j}| \leq 1$ for all $i \geq i_0$ and all $j \in N$, where i_0 is a sufficient large positive integer. If we put $w = t_0 \cdot v + (1-t_0) \cdot u$, and if t_0 is a sufficiently small positive number, since $|w_{i,j}| \geq (1-t_0) \cdot |u_{i,j}| - t_0 \cdot |v_{i,j}|$, then we obtain $|w_{i,k_i}| > 1$ for each $i \geq i_1$ and some $k_i \in N$, where i_1 is sufficiently large positive integer. This contradicts the fact that w belongs to V.

LEMMA 7. *In the dual pair* (λ, ℓ^∞), $\beta^*(\ell^\infty, \lambda)$ *is identical with* $\beta(\lambda', \lambda)$ *on* ℓ^∞.

Proof. It is sufficient to show that $\beta(\lambda, \ell^\infty)$ is identical with the topology t. Since $\ell^\infty(\sigma(\ell^\infty, \lambda))$ is a subspace of $\lambda'(\sigma(\lambda', \lambda))$, $\ell^\infty(\sigma(\ell^\infty, \lambda))$ admits a sequence $C_n = \{ (x_{i,j}) \mid (x_{i,j}) \in \ell^\infty, |x_{i,j}| \leq a_{i,j}^{(n)} \}$ $n \in N$ such that $\{n \cdot C_n\}$ is fundamental in bounded subsets. On the other hand, for an arbitrary $z = (z_{i,j}) \in B_n = \{ (x_{i,j}) \mid |x_{i,j}| \leq a_{i,j}^{(n)}$ for all $(i, j) \in N \times N \}$ and for an arbitrary $y = (y_{i,j}) \in \lambda$, if we set $z^{(k,\ell)} = (z_{i,j}^{(k,\ell)})$, where $z_{i,j}^{(k,\ell)} = z_{i,j}$ for $i \leq k$, $j \leq \ell$, $z_{i,j}^{(k,\ell)} = 0$ otherwise, then we have

$$|\langle y, z - z^{(k,\ell)} \rangle| = \Big| \sum_{(i,j) \notin \{ (i,j) \mid i \leq k,\ j \leq \ell \}} y_{i,j} \cdot z_{i,j} \Big| \leq \sum_{(i,j) \notin \{ (i,j) \mid i \leq k,\ j \leq \ell \}} |y_{i,j} \cdot z_{i,j}|.$$

By the above inequality, each C_n $n \in N$ is $\sigma(\lambda', \lambda)$-dense in

B_n. Hence the bipolar of C_n in λ' coincides with B_n. Since the polar of C_n in λ coincides with the polar of B_n in λ, we obtain the conclusion.

If we consider the normed space $(\lambda, \|\cdot\|_1)$, where $\|\cdot\|_1$ denotes the ℓ^1-norm, then we can establish the following

PROPOSITION 8. *$(\lambda, \|\cdot\|_1)$ does not have (*)-property.*

Proof. Since the dual of $(\lambda, \|\cdot\|_1)$ is ℓ^∞, we need only show that $\ell^\infty(\beta^*(\ell^\infty, \lambda))$ is not quasi-barrelled. By Lemma 7, the family consisting of the subsets of the form $U_{(c_n)} \cap \ell^\infty$, where $U_{(c_n)}$ denotes the subset in (2), forms a base of neighbourhoods of o in $\ell^\infty(\beta^*(\ell^\infty, \lambda))$. Further every $U_{(c_n)} \cap \ell^\infty$ contains an element satisfying the condition in (3). On the other hand, $\overline{V}[\beta(\lambda', \lambda)] \cap \ell^\infty$ is a bornivorous barrel in $\ell^\infty(\beta^*(\ell^\infty, \lambda))$ and does not have an element satisfying the condition in (3) by Lemma 6. Thus $\overline{V}[\beta(\lambda', \lambda)] \cap \ell^\infty$ is not a neighbourhood of o in $\ell^\infty(\beta^*(\ell^\infty, \lambda))$.

3. ON A LOCALLY CONVEX SPACE ADMITTING A FUNDAMENTAL SEQUENCE OF STRONGLY BOUNDED SUBSETS.

Now we draw the main theorem from the results in section 2.

THEOREM 9. *Let $E(t)$ be a locally convex space admitting a fundamental sequence $\{B_n\}$ of strongly bounded subsets, then, for the subspace $\bigcup_{n=1}^{\infty} \tilde{B}_n[t]$ of the completion $\tilde{E}(\tilde{t})$, the following facts hold:*

(1) *$\bigcup_{n=1}^{\infty} \widetilde{B}_n[t]$ coincides with $\overline{E}^{\beta}(\overline{t}^{\beta})$.*

(2) $\bigcup_{n=1}^{\infty} \tilde{B}_n[t]$ *coincides with* $\overline{E}(\overline{t})$ *if and only if* $\bigcup_{n=1}^{\infty} \tilde{B}_n[t]$ *is a W-space under the induced topology from* $\tilde{E}(\tilde{t})$.

Proof. As to (1). Since E(t) admits a fundamental sequence of strongly bounded subsets, $E'(\beta^*(E', E))$ is quasi-barrelled. Hence we obtain the conclusion (1) from Proposition 5.

As to (2). By Proposition 2 and the conclusion (1), we can verify this.

Then we have several corollaries under the same condition as in Theorem 9.

COROLLARY 10. $\bigcup_{n=1}^{\infty} \tilde{B}_n[\sigma(E, E')] = \overline{E}^{\beta}(\overline{\sigma(E, E')}^{\beta}) = E''_{\beta^*}$.

Proof. It is clear from Lemma 4 and Theorem 9.

COROLLARY 11. *Suppose that each* B_n *is precompact in* $E(t)$. *Then, under the induced topology from* $\tilde{E}(\tilde{t})$, $\bigcup_{n=1}^{\infty} \tilde{B}_n[t]$ *is β-semi-Montel. Furthermore* $\bigcup_{n=1}^{\infty} \tilde{B}_n[t]$ *is semi-Montel if and only if* $\bigcup_{n=1}^{\infty} \tilde{B}_n[t]$ *is a W-space.*

Proof. For an arbitrary strongly bounded subset B in $\bigcup_{n=1}^{\infty} \tilde{B}_n[t]$, as in the proof of Proposition 5, there exists a B_{n_0} such that $B \subset \tilde{B}_{n_0}[t]$. Thus the conlusion follows immediately.

COROLLARY 12. *If* t *is the finest locally convex topology that on each* B_n *induces the same neighbourhoods of* o *as* t, *then* $\overline{E}^{\beta}(\overline{t}^{\beta})$ *coincides with* $\tilde{E}(\tilde{t})$.

Proof. Noting that $\{2^n \cdot B_n\}$ is also an absorbing sequence, from (3) and (12) of section 16 in N. Adasch, B. Ernst and D. Keim [1], we get the conclusion.

REMARK 2. Let $E(t)$ be a quasi-$\aleph_0$-barrelled space admitting a fundamental sequence $\{B_n\}$ of strongly bounded subsets. If $\bigcup_{n=1}^{\infty} \tilde{B}_n[t]$ is a W-space under the induced topology from $\tilde{E}(\tilde{t})$, then, by the proof of Proposition 5 and Theorem 9, $\bigcup_{n=1}^{\infty} \tilde{B}_n[t]$ is a quasi-complete and quasi-$\aleph_0$-barrelled space admitting a fundamental sequence $\{\tilde{B}_n[t]\}$ of bounded subsets. Thus $\bigcup_{n=1}^{\infty} \tilde{B}_n[t]$ coincides with $\tilde{E}(\tilde{t})$.

4. ON THE L^p-SPACES.

First we prepare the notations.

NOTATION 1. (i) Let X be a set, $A(X)$ a σ-algebra of subsets of X and $\mu : A(X) \to [0, \infty]$ a σ-finite measure. Then we set the following function spaces on the measure space $(X, A(X), \mu)$; for each p with $0 < p \leq +\infty$,

$L^{(p)}(X) = \{ f(x) \mid f: X \to R$ is an $A(X)$-measurable function such that $|f(x)|^p$ is μ-integrable. $\}$,

$L^{(\infty)}(X) = \{ f(x) \mid f: X \to R$ is an $A(X)$-measurable function and μ-essentially bounded. $\}$,

$Sim.(X) = \{ f(x) \mid f = \sum_i \alpha_i \cdot \chi_{E_i}$ (finite sum), where $\alpha_i \in R$ and each χ_{E_i} is a characteristic function of a subset $E_i \in A(X)$. $\}$ and

$Sim._f(X) = \{ f(x) \mid f = \sum_i \alpha_i \cdot \chi_{E_i}$ (finite sum) and for each $E_i \in A(X)$, $\mu(E_i)$ is finite. $\}$.

Furthermore we denote by $L^p(X)$, $L^\infty(X)$, $S(X)$ and $S_f(X)$ the spaces consisting of the equivalence classes of functions in the above four function spaces respectively, where the equivalence relation $\sim$ means that $f \sim g \leftrightarrow f = g$ μ-a.e..

(ii) For any p with $1 \leq p \leq +\infty$, we denote by $\| \cdot \|_p$ the usual norm on $L^p(X)$ such that, for an arbitrary $f \in \overline{f} \in L^p(X)$,

$$\| \overline{f} \|_p = (\int_X |f|^p \, d\mu(x))^{1/p} \quad \text{for} \quad 1 \leq p < +\infty \quad \text{and}$$

$$\| \overline{f} \|_\infty = \operatorname*{ess.sup}_{x \in X} |f(x)|.$$

The topology generated by the norm $\| \cdot \|_p$ is denoted by t_p.

Now we consider the measure space on the set of all positive integers N such that $A(N)$ is the family of all subsets of N and μ is the counting measure. In this case, $L^p(N)$ is identical with ℓ^p for $0 < p \leq +\infty$. Then in the dual pair (ℓ^p, ψ), where ψ is the space of all finite sequences, $\ell^p(\sigma(\ell^p, \psi))$ is β-Montel for $1 \leq p \leq +\infty$ by Proposition 2 in K. Kitahara [8], hence the β-quasi-completion of $\ell^p(\sigma(\ell^p, \psi))$ is ℓ^p for $1 \leq p \leq +\infty$. For the β-quasi-completion of $\ell^p(\sigma(\ell^p, \psi))$ $0 < p < 1$, the following holds.

PROPOSITION 13. *For any p with $0 < p < 1$, the β-quasi-completion of $\ell^p(\sigma(\ell^p, \psi))$ coincides with ℓ^1.*

Proof. $\psi(\sigma(\psi, \ell^p))$ admits a fundamental sequence $B_n = \{ x \mid x \in \psi \subset \ell^\infty, \| x \|_\infty \leq n \}$ $n \in N$ of bounded subsets. Since each B_n is t_∞-dense in $\{ x \mid x \in c_0 \subset \ell^\infty, \| x \|_\infty \leq n \}$, $\ell^p(\beta(\ell^p, \psi))$ admits a fundamental sequence C_n $n \in N$ of bounded subsets such that $C_n = \{ x \mid x \in \ell^p \subset \ell^1, \| x \|_1 \leq n \}$ $n \in N$. Hence by using Theorem 9, $\overline{\ell^p}^\beta(\overline{\sigma(\ell^p, \psi)}^\beta) = \bigcup_{n=1}^\infty \tilde{C}_n[\sigma(\ell^p,$

$\psi)] = \ell^1$.

REMARK 3. Let X be a set. We put $\ell^p(X) = \{ (Z_x)_{x\in X} \mid (Z_x)_{x\in X} \in R^X, \sum_{x\in X} |Z_x|^p < +\infty \}$ for $0 < p \leq 1$ and $\psi(X) = \oplus_{x\in X} R_x$, where each $R_x = R$. Then, by the similar proof of Proposition 13, we obtain $\overline{\ell^p(X)}^\beta(\overline{\sigma(\ell^p(X), \psi(X))}^\beta) = \ell^1(X)$ for $0 < p < 1$.

Let $(X, A(X), \mu)$ be any σ-finite measure space. Then in the dual pair $(L^p(X), S_f(X))$, $1 < p \leq +\infty$, we consider the β-quasi-completion of $L^p(X)(\sigma(L^p(X), S_f(X)))$.

THEOREM 14. *For any p with $1 < p \leq +\infty$, $L^p(X)(\sigma(L^p(X), S_f(X)))$ is β-semi-Montel, hence the β-quasi-completion of this space is itself.*

Proof. It is sufficient to show that every strongly bounded subset is relatively compact in $L^p(X)(\sigma(L^p(X), S_f(X)))$. $S_f(X)(\sigma(S_f(X), L^p(X)))$ admits a fundamental sequence B_n $n \in N$ of bounded subsets such that $B_n = \{ f \mid f \in S_f(X), \| f \|_q \leq n \}$ $n \in N$, where $1/p + 1/q = 1$ and if $p = \infty$, then $q = 1$. Since each B_n is t_q-dense in $\{ f \mid f \in L^q(X), \| f \|_q \leq n \}$, $L^p(X)(\beta(L^p(X), S_f(X)))$ admits a fundamental sequence $C_n = \{ f \mid f \in L^p(X), \| f \|_p \leq n \}$ $n \in N$ of bounded subsets. Each C_n is $\sigma(L^p(X), S_f(X))$-compact by the reflexivility of $L^p(X)(t_p)$ and the fact that $L^\infty(X)$ is the dual of $L^1(X)(t_1)$. Hence $L^p(X)(\sigma(L^p(X), S_f(X)))$ is β-semi-Montel.

Using Theorem 14, we obtain a corollary related with Vitali-Hahn-Saks theorem.

COROLLARY 15. *Let* $(X, A(X), \mu)$ *be any finite measure space and* H *the space which consists of all sequences converging to* o *in* $L^1(X)(\sigma(L^1(X), S(X)))$. *If* $\{f_n\}$ *is a Cauchy sequence in* $L^1(X)(\sigma(L^1(X), S(X)))$, *then* $\{f_n\}$ *has a limit in* $L^p(X)$, $1 < p \leq +\infty$, *if and only if there exists a subsequence* $\{f_{n_k}\}$ *such that* $\inf_{\{h_k\}\in H} \sup_k \| f_{n_k} - h_k \|_p$ *is finite. However in case* g *does not belong to* $L^p(X)$, *we put* $\| g \|_p = +\infty$.

Proof. By Vitali-Hahn-Saks theorem, $\{f_n\}$ converges to an $f_0 \in L^1(X)$. If $f_0 \in L^p(X)$, then $\{f_n - f_0\} \in H$ by putting $f_n = (f_n - f_0) + f_0$ $n \in N$. Hence the necessary condition holds.

Conversely suppose that there exist a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a $\{h_k\} \in H$ satisfying that $\sup_k \| f_{n_k} - h_k \|_p < +\infty$. Then the sequence $\{g_k\}$, $g_k = f_{n_k} - h_k$ $k \in N$ is a Cauchy sequence in $L^p(X)(\sigma(L^p(X), S(X))$. Thus by Theorem 14 there exists a $g_0 \in L^p(X)$ to which $\{g_k\}$ converges. Since $\{f_n\}$ is a Cauchy sequence, $\{f_n\}$ and $\{g_k\}$ have the same limit.

REMARK 4. (1) The fact that $L^\infty(X)(\sigma(L^\infty(X), S_f(X)))$ is β-semi-Montel is a generalization of bounded convergence theorem.

(2) In general since $L^p(X)$ $0 < p < 1$ and $S_f(X)$ do not form a dual pair, we can not consider the weak topology on $L^p(X)$ by $S_f(X)$ unlike the case of ℓ^p $0 < p < 1$.

Next we consider the β-quasi-completion of $L^1(X)(\sigma(L^1(X), S(X)))$ and $L^1(X)(\sigma(L^1(X), S_f(X)))$. Before giving the results, we need the following notations.

NOTATION 2. (i) Let $(X, A(X), \mu)$ be any σ-finite measure space and $F(X)$ the set of real-valued functions ξ on $A(X)$ such that

(a) $\sup \{ |\xi(A)| \mid A \in A(X) \} < +\infty$,

(b) $\xi(A \cup B) = \xi(A) + \xi(B)$ for $A, B \in A(X)$ and $A \cap B = \emptyset$ and

(c) $\xi(A) = 0$ if $A \in A(X)$ and $\mu(A) = 0$.

For such a ξ, we define $|\xi|$ on $A(X)$ by the rule

$|\xi|(A) = \sup \{ \sum_{j=1}^{n} |\xi(A_j)| \mid A_j \in A(X),\ j = 1, 2, \cdots, n,\ A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{j=1}^{n} A_j = A \}$.

Then $F(X)$ is a linear space and we can define a norm $\|\cdot\|_F$ such that $\|\xi\|_F = |\xi|(X)$. This norm space $(F(X), \|\cdot\|_F)$ is identical with the strong dual of $(L^{\infty}(X), \|\cdot\|_{\infty})$, where for an arbitrary $\xi \in F(X)$ and an arbitrary equivalence class $I_A \in L^{\infty}(X)$ to which χ_A, $A \in A(X)$, belongs, the following bilinear form holds; $\langle \xi, I_A \rangle = \xi(A)$. (see P. 357 in E. Hewitt and K. Stromberg [5].) Further by considering the injection map $i_1 : L^1(X) \to F(X)$; $\overline{f} \to \xi_{\overline{f}}$ such that $f \in \overline{f}$ and $\xi_{\overline{f}}(A) = \int_A f\, d\mu(x)$ for all $A \in A(X)$, $(L^1(X), \|\cdot\|_1)$ is a subspace of $(F(X), \|\cdot\|_F)$.

(ii) Under the same condition of (i), we denote by $F_0(X)$ the set of real-valued functions η on $A_0(X) = \{ A \mid A \in A(X)$ and $\mu(A) < +\infty \}$ satisfying that the conditions replaced $A(X)$ in (i)-(a), (b) with $A_0(X)$ and the condition (i)-(c) hold. Similarly we can define $|\eta|(A) = \sup \{ \sum_{j=1}^{n} |\eta(A_j)| \mid A_j \in A_0(X),\ j = 1, 2, \cdots, n,\ A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{j=1}^{n} A_j = A \}$ for all $A \in A_0(X)$. Hence if we set $\|\eta\|_{F_0} = \sup_{A \in A_0(X)} |\eta|(A)$, then $(F_0(X), \|\cdot\|_{F_0})$ is a normed space and identical with the strong dual of $(S_f(X), \|\cdot\|_\infty)$. For the bilinear form in $F_0(X)$ and $S_f(X)$, the same formula of (i) holds. Moreover setting the injection map $i_2 : L^1(X) \to F_0(X)$; $\bar{f} \to \eta_{\bar{f}}$ such that $f \in \bar{f}$ and $\eta_{\bar{f}}(A) = \int_A f \, d\mu(x)$ for $A \in A_0(X)$, $(L^1(X), \|\cdot\|_1)$ is a subspace of $(F_0(X), \|\cdot\|_{F_0})$.

Using these notations, we obtain

THEOREM 16. *Let* $(X, A(X), \mu)$ *be any σ-finite measure space, then* $\overline{L^1(X)}^\beta(\overline{\sigma(L^1(X), S(X))}^\beta) = \overline{L^1(X)}(\overline{\sigma(L^1(X), S(X))}) =$ $F(X)$ *and* $\overline{L^1(X)}^\beta(\overline{\sigma(L^1(X), S_f(X))}^\beta) = F_0(X)$.

Proof. First we consider the β-quasi-completion of $L^1(X)$ $(\sigma(L^1(X), S(X)))$. $S(X)(\sigma(S(X), L^1(X)))$ admits a fundamental sequence B_n $n \in N$ of bounded subsets such that $B_n = \{ f \mid f \in S(X), \|f\|_\infty \leq n \}$ $n \in N$. Since each B_n is t_∞-dense in $\{ f \mid f \in L^\infty(X), \|f\|_\infty \leq n \}$, $L^1(X)(\beta(L^1(X), S(X)))$ admits a fundamental sequence $C_n = \{ f \mid f \in L^1(X), \|f\|_1 \leq n \}$ $n \in N$ of bounded subsets. Hence the topology $\beta^*(S(X), L^1(X))$ is identical with t_∞ on $S(X)$. Noting that $(S(X), \|\cdot\|_\infty)$ is a dense subspace of $(L^\infty(X), \|\cdot\|_\infty)$, by Corollary 10 we obtain $\overline{L^1(X)}^\beta(\overline{\sigma(L^1(X), S(X))}^\beta) = F(X)$. Moreover since $F(X)(\sigma(F(X), S(X)))$ is sequentially complete by Main Theorem in T. Andô [2], from Proposition 2 we have $\overline{L^1(X)}^\beta(\overline{\sigma(L^1(X), S(X))}^\beta) = \overline{L^1(X)}$ $(\overline{\sigma(L^1(X), S(X))})$. As for $L^1(X)(\sigma(L^1(X), S_f(X)))$, by the similar argument, it holds that $\beta^*(S_f(X), L^1(X))$ is identical with t_∞ on $S_f(X)$. Hence we obtain $\overline{L^1(X)}^\beta(\overline{\sigma(L^1(X), S_f(X))}^\beta)$ $= F_0(X)$ by Corollary 10.

In Vitali-Hahn-Saks theorem the Cauchy sequences in $L^1(X)$ $(\sigma(L^1(X), S(X)))$ are dealt with and the Cauchy sequences in $F(X)(\sigma(F(X), S(X)))$ are treated in Main Theorem in T. Andô [2]. Here we shall investigate the Cauchy sequences in $F_0(X)(\sigma(F_0(X), S_f(X)))$. If, for an arbitrary $\xi \in F_0(X)$, $\tilde{\xi}$ denotes any continuous extension of ξ on $(S(X), \|\cdot\|_\infty)$ with $\|\xi\|_{F_0} = \|\tilde{\xi}\|_F$, then we can prove

THEOREM 17 *Let* $(X, A(X), \mu)$ *be any σ-finite measure space and* $F_0(X)$ *the dual of* $(S_f(X), \|\cdot\|_\infty)$. *For a Cauchy sequence* $\{\xi_n\}$ *in* $F_0(X)(\sigma(F_0(X), S_f(X)))$ *the following facts hold.*

(1) $\{\xi_n\}$ *converges to a* $\xi_0 \in F_0(X)$ *if and only if* $\sup_{E \in A_0(X)} \lim_n |\xi_n(E)| < +\infty$.

(2) $\{\tilde{\xi}_n\}$ *is a Cauchy sequence in* $F(X)(\sigma(F(X), S(X)))$ *if and only if there exists an increasing sequence* $\{E_n\}$ *of subsets of* X *such that* $E_n \in A_0(X)$ $n \in N$ *and the sequence* $\varepsilon_n = \sup_k(\|\xi_k\|_{F_0} - |\xi_k|(E_n))$ $n \in N$ *converges to* 0. *In this case, there exists a* $\xi_0 \in F(X)$ *with* $\xi_0(E) = \lim_n \tilde{\xi}_n(E)$ *for all* $E \in A(X)$.

(3) $\{\tilde{\xi}_n\}$ *is a Cauchy sequence and* $\lim_n \tilde{\xi}_n \in L^1(X)$ *if and only if there exists a sequence* $\{E_n\}$ *of subsets of* X *satisfying the condition in (2) and for this sequence there is a sequence* $\{\delta_n\}$, $\delta_n > 0$ $n \in N$ *such that the sequence* $c_n = \sup\{\lim_i |\xi_i(F)| \mid F \subset E_n \text{ and } \mu(F) < \delta_n\}$ $n \in N$ *converges to* 0. *In this case, there exists a* $\xi_0 \in L^1(X)$ *with* $\int_E \xi_0 \, d\mu(x) = \lim_n \tilde{\xi}_n(E)$ *for all* $E \in A(X)$.

... As to (1). If we set $\xi_0(E) = \lim_n \xi_n(E)$ for $E \in$ $A_0(X)$, then ξ_0 belongs to $F_0(X)$ if and only if $\sup_{E \in A_0(X)} |\lim_n \xi_n(E)| = \sup_{E \in A_0(X)} \lim_n |\xi_n(E)| < +\infty$ by Notation 2-(ii).

As to (2). Since the measure space $(X, A(X), \mu)$ is σ-finite, we set an increasing sequence $\{X_i\}$ of subsets of X with $\mu(X_i) < +\infty$ and $\bigcup_{i=1}^{\infty} X_i = X$. Suppose that $\{\tilde{\xi}_n\}$ is a Cauchy sequence. Since $\| \tilde{\xi}_n \|_F = \| \xi_n \|_{F_0}$ for all $n \in N$, for each $n \in N$ there exists an increasing sequence $\{Y_i^{(n)}\}$ such that $Y_i^{(n)} \in A_0(X)$ and $\| \xi_n \|_{F_0} - |\xi_n|(Y_i^{(n)}) < 1/i$. Then we consider the following increasing sequence $\{Z_n\}$; $Z_1 = X_1 \cup Y_1^{(1)}$, $Z_2 = X_2 \cup Y_2^{(1)} \cup Y_2^{(2)}, \cdots$, $Z_n = X_n \cup Y_n^{(1)} \cup \cdots \cup Y_n^{(n)}, \cdots$. Clearly each $Z_n \in A_0(X)$ $n \in N$ and $\bigcup_{n=1}^{\infty} Z_n = X$. For this sequence, observing that $|\tilde{\xi}_n|(X) - (|\xi_n|(Z_1) + \sum_{k=1}^{j} |\xi_n|(Z_{k+1} - Z_k)) < 1/j$ for each $n \in N$ and all $j \geq n$, it holds that $|\tilde{\xi}_n|(X) = |\xi_n|(Z_1) + \sum_{k=1}^{\infty} |\xi_n|(Z_{k+1} - Z_k)$ for each $n \in N$. On the other hand, for the sequence $\{Z_n\}$ and the Cauchy sequence $\{\tilde{\xi}_n\}$, by using Lemma 4 in T. Andô [2], there is a $\{n_\ell\} \subset N$ such that $\sup_k |\tilde{\xi}_k|(Z_i - Z_j) < 1/2^{\ell}$ for all $i, j \geq n_\ell$. Now we shall show that for an arbitrary $\varepsilon > 0$ there is a $k(\varepsilon) \in N$ with $\sup_k |\tilde{\xi}_k|(X - Z_{k(\varepsilon)}) < \varepsilon$. For given positive number ε, if we put ℓ_0 with $1/2^{\ell_0} < \varepsilon/2$, then we obtain, for each $k \in N$,

$$|\tilde{\xi}_k|(X - Z_{n_{\ell_0}}) = |\tilde{\xi}_k|(\bigcup_{i=\ell_0}^{\infty}(Z_{n_{i+1}} - Z_{n_i})) = \sum_{i=\ell_0}^{\infty} |\tilde{\xi}_k|(Z_{n_{i+1}} - Z_{n_i})$$

$$\leq \sum_{i=\ell_0}^{\infty} 1/2^i = 1/2^{\ell_0 - 1} < \varepsilon.$$

Hence if we put $Z_{k(\varepsilon)} = Z_{n_{\ell_0}}$, then the conclusion follows.

Conversely suppose that there exists an increasing sequence $\{E_n\}$ satisfying the condition. For an arbitrary $E \in A(X)$ and an arbitrary $\varepsilon > 0$, we take an $n_0 \in N$ with $\varepsilon_n < \varepsilon/3$ for all

$n \geq n_0$. If we set $E = (E \cap E_{n_0}) \cup (E \cap E_{n_0}^c)$, where $E_{n_0}^c$ is the complement of E_{n_0}, then we obtain , for all $n, m \geq n_0$,

$$|\tilde{\xi}_n(E) - \tilde{\xi}_m(E)| \leq |\tilde{\xi}_n(E \cap E_{n_0}) - \tilde{\xi}_m(E \cap E_{n_0})| + |\tilde{\xi}_n(E \cap E_{n_0}^c) - \tilde{\xi}_m(E \cap E_{n_0}^c)|$$
$$\leq |\xi_n(E \cap E_{n_0}) - \xi_m(E \cap E_{n_0})| + 2\cdot\varepsilon/3.$$

Hence taking an $n_1 \in N$ such that $n_1 \geq n_0$ and for all $n, m \geq n_1$ $|\xi_n(E \cap E_{n_0}) - \xi_m(E \cap E_{n_0})| < \varepsilon/3$, we have $|\tilde{\xi}_n(E) - \tilde{\xi}_m(E)| < \varepsilon$ for all $n, m \geq n_1$. Clearly by Theorem 16 the Cauchy sequence $\{\tilde{\xi}_n\}$ has a limit ξ_0 in $F(X)$ and it holds that $\xi_0(E) = \lim_n \tilde{\xi}_n(E)$ for all $E \in A(X)$.

As to (3). Suppose that $\{\xi_n\}$ has a limit in $L^1(X)$. Then from absolute continuity with the measure space $(X, A(X), \mu)$ and the proof of (2), the necessity follows.

Conversely assume that the necessary condition holds. Then we need only show that $\xi_0 = \lim_n \tilde{\xi}_n$ is countably additive. For an arbitrary sequence $\{F_n\} \subset A(X)$ with $F_n \cap F_m = \phi$ for $n \neq m$, it holds that $\xi_0(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{k} \xi_0(F_i) + \xi_0(\bigcup_{i=k+1}^{\infty} F_i)$ for all $k \in N$ from its finitely additivity. For an arbitrary $\varepsilon > 0$, we can take an $n_0 \in N$ with $c_n, \varepsilon_n < \varepsilon/3$ for all $n \geq n_0$, by the assumption. Then we set $D_k = (\bigcup_{i=k+1}^{\infty} F_i) \cap E_{n_0}^c$, $G_k = (\bigcup_{i=k+1}^{\infty} F_i) \cap E_{n_0}$ for all $k \in N$. Since $\{G_k\}$ is a decreasing sequence such that $\mu(G_1) < +\infty$ and $\bigcap_{k=1}^{\infty} G_k = \phi$, there exists a $k_0 \in N$ satisfying that $\mu(G_k) < \delta_{n_0}$ for all $k \geq k_0$. Hence for each $k \geq k_0$ and all sufficiently large n which depends on k, we obtain

$$|\tilde{\xi}_n(\bigcup_{i=k+1}^{\infty} F_i)| = |\tilde{\xi}_n(D_k) + \tilde{\xi}_n(G_k)| \leq |\tilde{\xi}_n(D_k)| + |\tilde{\xi}_n(G_k)|$$
$$\leq \varepsilon/3 + 2\cdot\varepsilon/3 \leq \varepsilon.$$

Thus we have $|\xi_0(\bigcup_{i=k+1}^{\infty} F_i)| = \lim_n |\xi_n(\bigcup_{i=k+1}^{\infty} F_i)| \leq \varepsilon$ for all $k \geq k_0$, which implies the sequence $\{\xi_0(\bigcup_{i=k+1}^{\infty} F_i)\}_k$ converges to 0. This completes the proof.

REMARK 5. (1) Let $A(R)$ be the family of all Lebesgue measurable subsets on R and μ the Lebesgue measure on $(R, A(R))$. If we consider the sequence $\xi_n(E) = \mu(E \cap [-n, n])$ $n \in N$, $E \in A(R)$, then $\{\xi_n\} \subset F_0(R)$ is a Cauchy sequence in $F_0(R)(\sigma(F_0(R), S_f(R)))$ but $\sup_{E \in A_0(R)} \lim_n |\xi_n(E)| = +\infty$.

(2) Let $(R, A(R), \mu)$ be the same measure space in (1). If we consider the sequence $\nu_n(E) = \mu(E \cap [2n, 2n+1])$ $n \in N$, $E \in A(R)$, then $\{\nu_n\} \subset F_0(R)$ is a Cauchy sequence converging to o in $F_0(R)(\sigma(F_0(R), S_f(R)))$ and $\sup_n \|\nu_n\|_{F_0(R)} = 1$. But this sequence does not have an increasing sequence of subsets of R in Theorem 17-(2). Thus $\{\tilde{\nu}_n\}$ is not a Cauchy sequence in $F(R)(\sigma(F(R), S(R)))$.

Finally we give a slight consideration on the space of all real-valued continuous functions.

NOTATION 3. Let X be a Hausdorff compact topological space. Then we use the following notations:

$B(X)$ = the Borel algebra of X,

$\mu(\cdot)$ = a regular measure on $(X, B(X))$,

$C(X)$ = the space of all real-valued continuous functions on X,

$\tilde{C}(X)$ = the equivalence classes of $C(X)$ with the measure space $(X, B(X), \mu)$,

$\|\cdot\|_u$ = the uniform norm on $C(X)$ and

$\|\cdot\|_\infty$ = the essential supremum norm on $C(X)$.

Further in the dual pair $(\tilde{C}(X), S(X))$, we consider the bilinear form $\langle \overline{f}, \overline{h} \rangle = \int_X f\cdot h \, d\mu(x)$ for $f \in \overline{f} \in \tilde{C}(X)$ and $h \in \overline{h} \in S(X)$.

THEOREM 18. *In the space* $\tilde{C}(X)(\sigma(\tilde{C}(X), S(X)))$, *the following facts hold:*

(1) $\overline{\tilde{C}(X)}^{\beta}(\overline{\sigma(\tilde{C}(X), S(X))}^{\beta}) = L^\infty(X)$,

(2) $\overline{\tilde{C}(X)}^{seq}(\overline{\sigma(\tilde{C}(X), S(X))}^{seq}) = L^1(X)$ *and*

(3) $\overline{\tilde{C}(X)}(\overline{\sigma(\tilde{C}(X), S(X))}) = F(X)$.

Proof. As to (1). For an arbitrary $\overline{f} \in \tilde{C}(X)$ and an arbitrary $\overline{h} \in S(X)$, if $f \in \overline{f}$ and $h = \sum_{i=1}^{n} \alpha_i \cdot \chi_{E_i} \in \overline{h}$, then we obtain

$$\begin{aligned}|\langle \overline{f}, \overline{h} \rangle| &= \left|\int_X f\cdot\left(\sum_{i=1}^{n} \alpha_i\cdot\chi_{E_i}\right) d\mu(x)\right| = \left|\sum_{i=1}^{n}\alpha_i\cdot\left(\int_X f\cdot\chi_{E_i}\, d\mu(x)\right)\right| \\ &\leq \sum_{i=1}^{n} |\alpha_i|\cdot\left|\int_X f\cdot\chi_{E_i}\, d\mu(x)\right| \leq \|\overline{f}\|_\infty\cdot\left(\sum_{i=1}^{n} |\alpha_i|\cdot\mu(E_i)\right) \\ &\leq \|\overline{h}\|_1\cdot\|f\|_u.\end{aligned}$$

By the above inequality, we can regard $S(X)$ as a subspace of the dual of $(C(X), \|\cdot\|_u)$. Hence $S(X)(\sigma(S(X), \tilde{C}(X)))$ admits a fundamental sequence $B_n = \{ \overline{h} \mid \overline{h} \in S(X), \|\overline{h}\|_1 \leq n \}$ $n \in N$ of bounded subsets. Since each B_n is t_1-dense in $\{ \overline{f} \mid \overline{f} \in L^1(X), \|\overline{f}\|_1 \leq n \}$, $\tilde{C}(X)(\beta(\tilde{C}(X), S(X)))$ admits a fundamental sequence $C_n = \{ \overline{g} \mid \overline{g} \in \tilde{C}(X), \|\overline{g}\|_\infty \leq n \}$ $n \in N$ of bounded subsets. On the other hand, since X is compact and μ is a regular measure, each C_n is $\sigma(L^\infty(X), S(X))$-dense in $\{ \overline{h} \mid \overline{h} \in L^\infty(X), \|\overline{h}\|_\infty \leq n \}$. Hence we get the conclusion from Theorem 14 and Corollary 10.

As to (2). For an arbitrary $\overline{f} \in L^1(X)$ and an arbitrary positive number ε, there exists a $\overline{h} \in S(X)$ with $\|\overline{f} - \overline{h}\|_1 < \varepsilon$. For $\overline{h} \in S(X)$, from the assumption of the measure space, there is a $\overline{g} \in \tilde{C}(X)$ with $\|\overline{g} - \overline{h}\|_1 < \varepsilon$. By this, for each $\overline{f} \in L^1(X)$ we can take a sequence $\{\overline{g}_n\} \subset \tilde{C}(X)$ which converges to $\overline{f}$ with the topology $\sigma(L^1(X), S(X))$. Consequently the conclusion (2) follows from Vitali-Hahn-Saks theorem.

As to (3). Since $\overline{\tilde{C}(X)}^{seq} \subset \overline{\tilde{C}(X)}$, we have $\overline{\overline{\tilde{C}(X)}^{seq}} = \overline{\tilde{C}(X)}$. Thus we obtain the conclusion (3) by Theorem 16.

All the propositions with minor changes in the proofs are true for a locally convex space over the field of the complex number.

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