A Note on Numerical Representations for Weak-Continuous Acyclic Preferences*

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Abstract

In this note we obtain a continuous utility function for acyclic preferences by using a weaker than usual continuity condition. Then we extend some results on representability of continuous quasiorders and acyclic preferences.

Key words: utility functions, weak-continuity, acyclicity.

Resumen. *Una representación numérica para preferencias acíclicas débilmente continuas*

En este trabajo se extienden algunos resultados sobre representaciones numéricas de órdenes parciales y relaciones acíclicas, utilizando una condición de continuidad más débil que la usual.

Palabras clave: funciones de utilidad, continuidad débil, aciclicidad.

1. Introduction

It is well known that classical utility functions ($u: X \to \Re$ such that $x \, P \, y \Leftrightarrow u(x) > u(y)$) only exist for preorders (complete, reflexive and transitive binary relations); however, several authors consider that transitivity (especially transitivity of the indifference relation) is very unrealistic [see, for instance, the seminal paper of Luce (1956)]. More general binary relations as quasiorders or acyclic relations, can only be represented in a weaker way, being the most usual a real-valued function, $u: X \to \Re$ such that

$$x \, P \, y \Rightarrow u(x) > u(y)$$

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this function is also called a utility function [see, for instance, Peleg (1970) for quasiorders, Subiza (1994) and Peris and Subiza (1995) for acyclic preference relations]. Although this general type of utility functions does not give complete information about the binary relation, they are useful because they exist for a wide class of binary relations and the maxima of such functions are their maximal elements.

It should be noticed that we are considering numerical representations based on a real-valued function. There are other kind of numerical representations such as set-valued functions, hemisymmetric functions, two real-valued functions, which exist for more general relations than preorders [see Subiza (1993), for a survey on numerical representations of preferences].

In this paper we obtain a continuous utility function by measuring the lower contour sets of the relation. The concept of measure has already been used in the literature to obtain utility representations (Neuefeind (1972), Chichilnisky (1980), for continuous separable preorders; Candeal-Haro and Induráin-Eraso (1993, 1994) for continuous and separable quasiorders). By using this idea we weaken the transitivity and continuity conditions, for the case of weak-continuous acyclic binary relations. The result obtained generalizes Theorem 3 in Candeal and Induráin (1993) and moreover it provides numerical representations for acyclic preferences which are not spacious and then the result in Peris and Subiza (1995) cannot be applied.

2. Preliminaries

Throughout the paper let X be a topological space, P an asymmetric [$x\, P\, y$ implies not ($y\, P\, x$)] binary relation defined on X, R the weak relation associated with P [$x\, R\, y$ if and only if (not $y\, P\, x$)] and I the indifference relation [$x\, I\, y$ if and only if $x\, R\, y$ and $y\, R\, x$]. Note that, from the way in which it has been defined, the relation R is a complete [$\forall\, x, y,\; x\, R\, y$ or $y\, R\, x$] and reflexive [$\forall\, x,\, x\, R\, x$] binary relation. The following type of binary relations will be used: we will say that P is

acyclic if for any $x_1, x_2, \ldots, x_n$, $x_1Px_2, x_2Px_3\ldots, x_{n-1}Px_n$ implies x_1Px_n,

a *quasiorder* if it is transitive [$x\, P\, y$ and $y\, P\, z$ implies $x\, P\, z$]

We will denote by $L_P\,(x)$ and $U_P\,(x)$ the lower and upper contour sets of x with respect to the relation P:

$$L_P\,(x) = \{\, z \in X \;|\; x\, P\, z \,\}$$
$$U_P\,(x) = \{\, y \in X \;|\; y\, P\, z \,\}$$

The relation P is said to be *continuous* if $L_P\,(x)$ and $U_P\,(x)$ are open sets for all $x \in X$; P is said to be *separable* if there is a countable subset $D = \{d_i,\, i \in \mathrm{N}\}$ of X such that whenever $x\, P\, y$ then there is some $d_i \in D$ such that x Pd_i, $d_i\, P\, y$.

We will denote by PP the *transitive closure* of P, defined by

$$x\, PP\, y \Rightarrow \exists\; z_1, z_2, \ldots, z_n \in X \;|\; x = z_1\, Pz_2\, P \ldots Pz_n = y$$

When P is an acyclic relation, PP is a quasiorder.

When the set of alternatives is a topological space, the continuity of the function which represents the binary relation is a very desirable property. Some authors define utility functions from this point of view, as numerical representations satisfying continuity [see, for instance, Peleg (1970)]. We also consider this point of view and we define a *utility* function representing the binary relation P as a continuous real-valued function $u: X \to \mathrm{R}$ such that $x\,P\,y$ implies $u(x) > u(y)$.

In order to obtain the existence of a utility function a usual requirement is the continuity of this relation. We will introduce a weaker property with the same purpose.

Definition 1

A binary relation P defined on a topological space X is said to be *weak-continuous* if whenever $x\,P\,z\,P\,y$ there are $z_1, z_2 \in X$ and $U \in E(x)$, $V \in E(y)$ such that

$$\begin{array}{ll} \forall\, a \in U & a\,P\,z_1\,P\,y \\ \forall\, b \in V & x\,P\,z_2\,P\,b \end{array}$$

where $E(x)$ stands for the family of open neighborhoods of x.

It is clear that continuity implies weak continuity. The converse, even by adding separability, is not true. The next example shows this fact, and the following Lemma establishes the relationship between both notions of continuity.

Example 1

Let $X = \{x \in \mathrm{R}^2_+ \mid \Sigma\, x_i > 10\}$ and let P be the binary relation defined by

$$x, y \in X \quad x\,P\,y \;\Leftrightarrow\; \sum x_i > \sum y_i \;\text{ and }\; \|x - y\| \geq 1$$

where $\|a\|$ denotes the euclidean norm of a vector. This relation is acyclic, separable ($D = X \cap (\mathrm{Q} \times \mathrm{Q})$) and weak-continuous. Nevertheless it is not continuous.

Lemma 1

Let X be a topological space and let P be a separable weak-continuous binary relation defined on X. Then PP is continuous.

Proof:

Let $y \in L_{PP}(x)$. Thus there exist $z_1, \ldots, z_{n-1}, z_n$ such that

$$x = z_1 P \ldots P\, z_{n-1} \mathrm{P} z_n = y$$

Note that, by separability, we can obtain such a chain with a length of at least 3 and then applying weak-continuity to the sequence

$$z_{n-2} P z_{n-1} P z_n = y$$

there is some $z \in X$ and some $U \in E(y)$ such that $\forall\, b \in U$

$$z_{n-2}\; P\; z\; P\; b$$

so $x\; PP\; b$, which implies $U \subseteq L_{PP}(x)$, and the lower contour set of PP is open. With an analogous argument, $U_{PP}(x)$ is open and PP is continuous.■

From the above Lemma, as every quasiorder coincides with its transitive closure, it is deduced immediately that for a separable quasiorder continuity and weak-continuity are equivalent conditions.

3. The Result

In order to get a utility function we assume that there is a finite measure μ defined on a topological space such that any open set U is a measurable set and $\mu(U) > 0$, for any non-empty open set. Formally, given a binary relation P, we define the real-valued function $\alpha: X \to \mathrm{R}$ in the following way:

$$\text{for any } x \in X,\ \alpha(x) = \mu[\mathrm{int}(L_P(x))]$$

where *int*(A) stands for the interior of this set. Since open sets are measurable, this function is well-defined.

To obtain the continuity of this function we need to introduce an additional property about the behavior of the lower contour sets. Similar conditions are introduced in the literature in order to avoid the fact that the indifferent classes have positive measures (Neuefeind (1972) directly requests this condition, $\mu[\{ a \mid a\, I\, x \}] = 0$, $\forall x$). The condition we are going to use is taken from Candeal-Haro and Induráin-Eraso (1993).

Definition 2

Let X be a topological space with a finite measure μ, and P a binary relation defined on X. μ is said to satisfy the *regularity condition* if for every net $\{x_j\}_{j \in J}$, $\lim_{j \in J} x_j = x$, the net of real numbers

$$\alpha_j = \mu[\mathrm{int}(L_P(x))\ \Delta\ \mathrm{int}(L_P(x_j))]$$

converges to 0, where Δ denotes the symmetric difference of two sets,

$$A\ \Delta\ B = (A \cup B) - (A \cap B).$$

Candeal-Haro and Induráin-Eraso (1993) use the regularity condition in order to obtain utility functions for continuous quasiorders; Bosi and Isler (1995) have used this property to prove the existence of a pair of continuous real-valued functions representing a strongly separable interval order.

The following result proves the continuity of the function $\alpha(x)$ when P is a quasiorder satisfying the regularity condition.

Lemma 2

Let P be a quasiorder defined on a topological space X with a finite measure μ satisfying the regularity condition. Then the real function $\alpha: X \to \mathrm{R}$ defined as

$$\alpha(x) = \mu[\mathrm{int}(L_P(x))]$$

is continuous.

Proof:

For any convergent net $\{x_j\}_{j\in J}$, $\lim_{j\in J} x_j = x$, then by using measure properties,

$$\begin{gathered}|\alpha(x) - \alpha(x_j)| = |\mu[\mathrm{int}(L_P(x))] - \mu[\mathrm{int}(L_P(x_j))]| = \\ = |\mu[\mathrm{int}(L_P(x)) - \mathrm{int}(L_P(x_j))] - \mu[\mathrm{int}(L_P(x_j)) - \mathrm{int}(L_P(x))]| \leq \\ \leq \mu[\mathrm{int}(L_P(x)) - \mathrm{int}(L_P(x_j))] + \mu[\mathrm{int}(L_P(x_j)) - \mathrm{int}(L_P(x))] = \\ = \mu[\big(\mathrm{int}(L_P(x)) - \mathrm{int}(L_P(x_j))\big) \cup \big(\mathrm{int}(L_P(x_j)) - \mathrm{int}(L_P(x))\big)] = \\ = \mu[\mathrm{int}(L_P(x)) \,\Delta\, \mathrm{int}(L_P(x_j))] = \alpha_j\end{gathered}$$

and the regularity condition implies that $\lim_{j\in J} \alpha(x_j) = \alpha(x)$. Therefore $\alpha(x)$ is continuous.■

If we consider, in the previous Lemma, that the quasiorder P is continuous, then the function $\alpha(x)$ coincides with the measure of the lower contour set, since this set is open. The main result is now obtained as a consequence of Lemma 1 and 2.

Theorem 1

Let P be an acyclic, weak continuous and separable binary relation defined on a topological space with a finite measure μ satisfying the regularity condition with respect to the transitive closure of P. Then the function

$$u(x) = \mu[L_{PP}(x)]$$

is a utility function representing P.

Proof:

By using Lemma 1, relation PP is a continuous quasiorder; the separability of P implies that PP is also separable.

As PP is continuous Lemma 2 provides the continuity of $u(x)$. Let $x, y \in X$ such that $x\ PP\ y$. The continuity and separability of PP imply that there is an open set U such that

$$U \subseteq L_{PP}(x) - L_{PP}(y)$$

and then, by using measure properties

$$u(y) < u(y) + \mu[U] = \mu[L_{PP}(y) \cup U] \leq \mu[L_{PP}(x)] = u(x).$$

Then $u(x)$ is a utility for the relation PP. It is immediate that $u(x)$ is also a utility function for P.■

Theorem 3 in Candeal-Haro and Induráin-Eraso (1993) provides a utility function for continuous quasiorders when the regularity condition is satisfied; therefore our result is a generalization of that theorem. This fact can be shown with binary relation in Example 1 which satisfies the conditions in our Theorem 1 and it is not a quasiorder. Easy computations show that the utility function of the relation in Example 1 is

$$u(x,y) = \frac{(\sum x_i)^2 - 100}{2}$$

On the other hand, Peris and Subiza (1995) prove the existence of a utility function for preferences which are separable, continuous and *spacious* (for every $x, y \in X$ such that $x\,P\,y$, $Cl(L_P(y) \subset L_P(x))$; as Example 1 shows, in this context, the weak-continuity condition is a weaker property. Moreover, Candeal-Haro and Induráin-Eraso (1993) provide an example of an acyclic binary relation which satisfies the hypotheses in Theorem 1 but it is not spacious. So Theorem 1 covers situations where the results in Candeal-Haro and Induráin-Eraso (1993) or in Peris and Subiza (1995) cannot be applied.

4. Final Comments

It would be interesting to obtain a «regularity» condition independent of the measure used in the construction of the utility function. In order to do so, the following condition about convergence of sets could be used:

If $A_1, A_2, \ldots$ are a sequence of subsets we define the following set operations:

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

$$\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

If $\limsup A_n = \liminf A_n = A$, A is said to be the *limit* of the sequence and we write $A = \textit{limit}\, A_n$.

Note that this definition of set convergence does not coincide, in general, with the usual notion of set convergence in the Hausdorff topology. By using this set convergence, we will say that a binary relation defined in a metric space X has *convergent lower sets* if for any convergent sequence $\{x_n\}$, $\lim x_n = x$, then

$$\textit{limit}\,[L_P(x_n)] = L_P(x).$$

It is now obvious, from the properties of the measure (see, for instance Ash (1972)), that if a binary relation P has convergent lower sets then for any finite measure μ defined in the space of alternatives X the function

$$\alpha(x) = \mu[\mathrm{int}(L_P(x))]$$

is continuous.

Requiring this property is, in general, a much stronger assumption than the regularity condition. Although the result we obtain is presented using the regularity condition, it can obviously be rewritten in terms of the convergence of lower sets.

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