

Bargaining with Claims in Economic Environments*

Carmen Herrero

Universitat d'Alacant. Departament de Fonaments de l'Anàlisi Econòmica.

Recibido: enero de 1997
Aceptado: septiembre de 1997

Abstract

In this paper a reconstruction of the theory of bargaining with claims in economic environments is addressed. The spirit of that reconstruction is similar to that of the standard bargaining theory made by Roemer.

Key words: economic environments; bargaining with claims.

Resumen. *Negociación con derechos en entornos económicos*

En este trabajo se aborda el problema de la reconstrucción de la teoría de negociación con derechos en entornos económicos. El espíritu de la reconstrucción y las técnicas son semejantes a las de Roemer para el problema clásico de negociación.

Palabras clave: entornos económicos, negociación, derechos.

1. Introduction

Chun & Thomson (1992) enriched the traditional model of axiomatic bargaining (Nash (1950)), by adding to the disagreement point and the feasible set a third element, the *claims point*, an unfeasible point which has to be taken into account when proposing a solution to the conflict.

A problem of bargaining with claims is understood as the mathematical abstraction of a physical bargaining problem in which the claims point plays the role of *founded expectations* of the agents. Examples of such a type of situation may be either bankruptcy-like problems, or arbitration problems in which agents'

(*) Thanks are due to Antonio Villar and María del Carmen Marco for helpful comments. My special gratitude to John Roemer who supported wonderfully this work during my visit to the University of California, Davis, in the winter of 1994. Financial support from the IVIE and the DGICYT, project PB92-0342, are gratefully acknowledged.

expectations correspond to previous agreements, environmental variables, etc. Finding solutions to this type of problems entails providing with satisfactory answers to the conflict at hand.

As in any arbitration scheme, two main ingredients are always present when looking for satisfactory answers: efficiency and fairness. The traditional axiomatic approach is particularly useful, since by means of a suitable combination of a small group of *nice* criteria, a *unique* outcome is selected as the solution of each problem.

Despite the formal beauty of the axiomatic approach, it is not immune from criticism. While the motivation for the bargaining with claims model is economic (agents bargain over some kind of pie, with founded expectations on the result), all the relevant economic information is sterilized out of the model, since it concentrates only on utility data. As a consequence, problems which have nothing to do with each other in terms of their economic information (different utility functions, different goods, different physical claims) collapse into the same data in utility terms, and thus become indistinguishable. This fundamental difficulty is particularly apparent if we take into account that individuals' perceptions of what they consider as being *just* vary, depending upon the particular nature of the problem at hand (see Yaari & Bar-Hillel (1984)).

As in the case of bargaining theory, working on a domain of economic environments is entirely different from using the domain of triples $\{S, d, c\}$, where S is a convex, closed, comprehensive set (in \mathcal{R}^n), d is a point in S , and c is a point in \mathcal{R}^n outside S . Notice that some *natural* economic mechanisms are not mechanisms in classical bargaining with claims theory. This is the case, for instance, for the classical *proportional to the claims division* mechanism. It is interesting to observe that, nevertheless, properties (or axioms) in the classical bargaining with claims framework, are justified in terms of some underlying economic environment situation, in order to motivate their rational or their fairness content. In contrast, those properties are much more demanding than intuition suggests.

The said difficulty is shared by both the traditional axiomatic bargaining and the bargaining with claims approaches. In the case of classical bargaining, Roemer (1988) reconstructed the traditional axiomatic bargaining theory focusing on economic environments.

This paper explores the reconstruction of bargaining with claims theory taking economic environments as the domain of the allocation mechanism. Two main questions arise: First, why it is considered to be an interesting exercise, and second, whether or not the technical aspects of such an exercise are different from that in Roemer (1988) for the classical bargaining theory.

In answer to the first question, the interest of the exercise derives from clarifying the limits of concentrating only on utility information in this particular case. If the claims point is understood as indicating property rights, and the disagreement point indicates that unless an agreement is reached the available resources are thrown out, it may be that the classical bargaining with claims theory is closest to its reconstruction from an economic environments perspective than is the case for the classical bargaining theory.

As for the second question, some remarks are in order:

1. The literature on bargaining problems with claims was initiated by Chun & Thomson (1992), meaning that it is still a novel topic. Somehow, we may think of a bargaining with claims problem as an extension of both the classical bargaining theory and the bankruptcy theory. This explains why there are two different types of mechanisms analyzed so far: those adapted from the classical bargaining theory and those adapted from the bankruptcy literature.
2. The reconstruction of those mechanisms inspired in classical bargaining mechanisms, such as the *proportional solution* (Chun & Thomson (1992)), and the *extended-claims egalitarian solution* (Bossert (1993), Marco (1994), (1995)) can be done by suitably adapting Roemer's techniques. Notice that two main ingredients of these techniques are Billera & Bixby (1973) and Howe (1987). As a consequence, we also need the commodity space to be of unbounded dimension.
3. This is not the case for those mechanisms inspired in the bankruptcy model, especially when *minimal concession points* are involved, as they are in Herre-ro (1994), (1998). These solutions are out of the scope of this paper.

In Section 2 we present the set-up, as well as the main axioms used in the aforementioned reconstruction of the theory. Section 3 presents characterization results for a fixed number of agents. Section 4 is devoted to the corresponding characterizations for a variable population. The proofs are relegated to an Appendix.

2. The set-up

Let U^h be the set of all real valued monotone concave continuous functions defined on \mathfrak{R}_+^h . An *economic environment* is a tuple

$$\xi = \langle n, u, h, w, c \rangle$$

where $n, h \geq 1$ are integer numbers, $w \in \mathfrak{R}_+^h$, $c \in \mathfrak{R}_+^{hn}$, $c = (c_1, \dots, c_n)$, such that $c_i \in \mathfrak{R}_+^h$ for all $i = 1, \dots, n$, and $u = (u_1, \dots, u_n)$, where $u_i \in U^h$ for all $i = 1, \dots, n$. The interpretation is that we deal with a problem involving n agents, each one characterized by her utility function u_i , having claims on h commodities c_i , but the available amounts of the commodities to be divided among them is w . If for some $j = 1, \dots, h$, $\sum_{i=1}^n c_{ij} > w_j$, then a conflict arises, which shall be called a conflict of bargaining with claims. Agents may agree upon any feasible distribution of the available commodities, otherwise they will end up with no goods at all. The question is how to allocate w among the n individuals in the previous problem, namely, to suggest an agreement.

We start by dealing with a family of problems in which n (the number of agents) is fixed, but h (the number of goods to be distributed) varies. Let us call

Σ^n the class of all admissible environments for n agents. In this section, when we specify an economic environment $\xi \in \Sigma^n$, no explicit mention to the number of agents will be made, that is, $\xi = \langle u, h, w, c \rangle$.

If $\xi \in \Sigma^n$, an allocation is a vector $x = (x_1, \dots, x_n)$, where $x_i \in \mathfrak{R}_+^h$. Let us denote $u(x) = [u(x_1), \dots, u_n(x_n)]$. Let $Z(\xi) = \{x \mid \sum_{i=1}^n x_i \leq w\}$ be the set of feasible allocations, and $A(\xi) = \{(\bar{u}_1, \dots, \bar{u}_n) \in \mathfrak{R}^n \mid \exists x \in Z(\xi), u_i(x_i) \geq \bar{u}_i\}$ the utility possibility set of ξ . $A(\xi)$ is a closed, comprehensive, convex set in \mathfrak{R}^n , containing the point $u(0)$. Convexity comes from concavity of the utility functions; comprehensiveness and closedness come from free disposability of the goods and continuity of the utility functions, respectively.

An *allocation mechanism* F is a correspondence which associates to each economic environment $\xi \in \Sigma^n$, a set of feasible allocations. It will be assumed that F induces a function in utility space, namely, if $x, y \in F(\xi)$, then $u(x) = u(y)$.

Call the induced utility function $\mu_F(\xi) = u(F(\xi))$, where $F(\xi)$ is the set of bundles which F assigns to every agent. Furthermore, it is assumed that F chooses all the feasible allocations associated with a given point in the utility space. Roemer (1988) call this type of correspondence a «full» correspondence.

Associated to any *economic environment* $\xi \in \Sigma^n$, we can define a *classical bargaining with claims problem* in a natural way: If $\xi = \langle u, h, w, c \rangle$, take $A(\xi)$ as the feasible set, $u(0)$ as the disagreement point and $u(c)$ as the claims point. Thus, we may look at ξ as a classical bargaining problem with claims.

Two main ingredients are necessary in order to reconstruct the classical bargaining with claims theory as a theory on economic environments: (1) To ensure that the information contained in the utility possibilities set, the utilities of the origin and the utilities of the claims point are all that are needed, (2) To ensure that any triple (S, d, c) , where S is a convex comprehensive subset of \mathfrak{R}^n , $d \in S$ and $c \in \mathfrak{R}^n, c \notin S$ comes from a suitable economic environment $\xi \in \Sigma^n$.

Point (2) is answered affirmatively by Billera & Bixby (1973), whenever the commodities space is of unbounded dimension. In order to fulfill point (1), we need the following axiom:

Axiom of Welfarism (W).— Let $\xi = \langle u, h, w, c \rangle$, and $\xi' = \langle u', h', w', c' \rangle$ be two economic environments in Σ^n such that $A(\xi) = A(\xi')$, $u(0) = u'(0)$ and $u(c) = u'(c')$. Then, $\mu_F(\xi) = \mu_F(\xi')$.

Axiom W says that an allocation mechanism F must treat identically (in terms of utility) any two economic environments which have the same utility possibilities set and in which the utility levels of the origin and of the claims point are identical for all individuals. W is extremely strong, since it has to hold even in circumstances in which the dimension of the goods space is different. The welfarist axiom is thus named because it requires the allocation mechanism to ignore all

information about an environment which is not summarized both in the utility possibilities set and in the utilities of the disagreement and of the claims point.

In order to present the next axiom we need the concept of *personal good*. Let $\xi \in \Sigma^n$ be an economic environment, $\xi = \langle u, h, w, c \rangle, i \in N, j \in \{1, 2, \dots, h\}$. We shall say that good j th is a *personal good* for agent i th if for any $k \in N, k \neq i$, for any $x \in \mathfrak{R}_+^h$, and for any $\alpha \geq 0$, if $\bar{\alpha}^j$ stands for a vector in \mathfrak{R}^h having all components equal to zero but the j th component, equal to α , we get that $u_k(x) = u_k(x + \bar{\alpha}^j)$.

Axiom of Consistency (CON). Let $\xi \in \Sigma^n, \xi = \langle u, h+h', (w, w'), (c, c') \rangle$ be an environment such that $(\bar{x}, \bar{y}) \in F(\xi)$, where each of the y -goods ($k = h+1, \dots, h+h'$) is a personal good for at most one of the agents. Define v by $v_i(x_i, \bar{y}_i) = u_i(x_i, \bar{y}_i)$. Consider now the environment $\xi' = \langle v, h, w, c \rangle$. If $\xi' \in \Sigma^n$, and $A(\xi) = A(\xi'), v(0) = u(0, 0), v(c) = u(c, c')$, then $\bar{x} \in F(\xi')$.

CON asks for a particular type of consistency in the allocation procedure. First, the mechanism allocates those commodities which are only liked by a particular agent (personal goods). Next, the remaining commodities are allocated. The mechanism satisfies this property whenever the allocation is immune to this two-step procedure. The interest of the axiom CON is that it will imply the axiom of welfarism, even though it is an axiom relying on the economic data of the problem.

The main result of this section is the next lemma, which is a straightforward adaptation of Lemma 5 in Roemer (1988):

Lemma 1. CON and fullness of F imply W.

3. Bargaining with Claims and economic information with a fixed number of agents

For any $\xi \in \Sigma^n, \xi = \langle u, h, w, c \rangle$, consider the sets of efficient and weakly efficient allocations: $PO(\xi) = \{x \in Z(\xi) \mid u(y) > u(x) \Rightarrow y \notin Z(\xi)\}, WPO(\xi) = \{x \in Z(\xi) \mid u(y) \gg u(x) \Rightarrow y \notin Z(\xi)\}$.

Now, consider the following axioms:

Axiom of Efficiency (PO). $\forall \xi \in \Sigma^n, F(\xi) \subset PO(\xi)$.

Axiom of Weak Efficiency (WPO). $\forall \xi \in \Sigma^n, F(\xi) \subset WPO(\xi)$.

Axiom of Symmetry (SY). Let $\xi \in \Sigma^n, \xi = \langle u, h, w, c \rangle$ be such that $\forall i, j \in N, u_i = u_j, c_i = c_j$. Then, $(w/n, \dots, w/n) \in F(\xi)$.

PO and WPO ask the solution allocations for different degrees of efficiency. SY establishes that whenever we face identical individuals, then the equal division of the available resources is a solution allocation.

Let A^n be the class of transformations $\lambda: \mathfrak{R}^n \rightarrow \mathfrak{R}^n$ such that for any $i \in N$, there exist a $a_i \in \mathfrak{R}_{++}, b_i \in \mathfrak{R}$, such that for all $x \in \mathfrak{R}^n, \lambda_i(x) = a_i x_i + b_i$. Let T^n be the subclass of A^n with $b_i = 0$ for all $i \in N$.

Consider now the following axioms:

Axiom of Cardinality and Non Comparability (CNC). Let $\xi, \xi' \in \Sigma^n$.
 $\xi = \langle u, h, w, c \rangle$, $\xi' = \langle u', h, w, c \rangle$, with $u' = \lambda u$, $\lambda \in A^n$. Then, $F(\xi) = F(\xi')$.

Axiom of Ordinal Level Comparability (OLC). Let $\xi, \xi' \in \Sigma^n$, $\xi = \langle u, h, w, c \rangle$,
 $\xi' = \langle u', h, w, c \rangle$, with $u' = \tau u$, $\tau \in T^n$. Then, $F(\xi) = F(\xi')$.

CNC and OLC deal with two problems in which all the physical data are identical and the only difference relies in the utilities. If these utilities are related by allowed transformations, then the solutions must be identical. In CNC the allowed transformations belong to A^n , and therefore we are confined to a universe in which utilities are cardinal and no interpersonal comparability can be performed. Under OLC, the transformations belong to T^n , and in consequence, interpersonal comparability is possible, and utilities are ordinal. These axioms play the role of Scale and Translation Invariance, respectively.

Axiom of Boundedness (BDD). For all $\xi \in \Sigma^n$, $\xi = \langle u, h, w, c \rangle$, $u(0) \leq \mu_F(\xi) \leq u(c)$.

The previous axiom has the same meaning as in traditional bargaining problems with claims. BDD requires that at any solution allocation no agent is either better off than he is at the claims point or worse off than he is at the disagreement point.

The axioms of consistency we present below are two different ways of strengthening CON.

Axiom of Strong Consistency (CON).* Let $\xi' \in \Sigma^n$, $\xi = \langle u, h+h', (w, w'), (c, c') \rangle$ be an environment such that $(\bar{x}, \bar{y}) \in F(\xi)$, where each of the y -goods ($k = h+1, \dots, h+h'$) is a personal good for at most one of the agents. Define v by $v_i(x_i) = u_i(x_i, \bar{y}_i)$. Consider now the environment $\xi' = \langle u, h, w, c \rangle$. If $\xi' \in \Sigma^n$, and $v(0) = u(0,0)$, $v(c) = u(c, c')$, then $\bar{x} \in F(\xi')$.

Axiom of Rational Strong Consistency (RCON).* Let $\xi \in \Sigma^n$, $\xi = \langle u, h+h', (w, w'), (c, c') \rangle$ be an environment such that $(\bar{x}, \bar{y}) \in F(\xi)$, where each of the y -goods ($k = h+1, \dots, h+h'$) is a personal good for at most one of the agents. Define v by $v_i(x_i) = u_i(x_i, \bar{y}_i)$. Consider now the environment $\xi' = \langle v, h, w, c \rangle$. If $\xi' \in \Sigma^n$, and $v(0) \geq u(0,0)$, $v(c) = u(c, c')$, $u(\bar{x}, \bar{y}) \geq v(0)$, then $\bar{x} \in F(\xi')$.

The axiom below is the version of the monotonicity axiom in economic environments:

Axiom of Resource Monotonicity (R.MON). Let $\xi, \xi' \in \Sigma^n$, $\xi = \langle u, h, w, c \rangle$,
 $\xi' = \langle u, h, w', c \rangle$, with $w' \geq w$. Then $\mu_F(\xi') \geq \mu_F(\xi)$.

We can now define the *Proportional Mechanism P* in economic environments: For any $\xi \in \Sigma^n$, $\xi = \langle u, h, w, c \rangle$, $P(\xi) = \{ x \in Z(\xi) \mid u(x) = \lambda u(c) + (1-\lambda)u(0), 0 \leq \lambda \leq 1, x \in \text{WPO}(\xi) \}$.

Similarly, we may consider the *Claim-Egalitarian Mechanism E*, as follows: For any $\xi \in \Sigma^n$, $\xi = \langle u, h, w, c \rangle$, $E(\xi) = \{ x \in Z(\xi) \mid u_i(c_i) - u_i(x_i) = u_j(c_j) - u_j(x_j), i, j = 1, \dots, n, x \in \text{WPO} \}$. Now, the *Extended Claim-Egalitarian Mechanism EE*, is defined in the following way: For any $\xi \in \Sigma^n$, $\xi = \langle u, h, w, c \rangle$, $EE(\xi) = \{ x \in Z(\xi) \mid \text{if for some } i, j \in N, u_i(c_i) - u_i(x_i) > u_j(c_j) - u_j(x_j), \text{ thus } u_j(x_j) = 0, x \in \text{WPO} \}$.

By means of previous axioms the following characterization results are obtained [see Appendix]:

Theorem 1. The proportional mechanism is the unique full mechanism in Σ^n satisfying Weak Efficiency, Consistency, Resource Monotonicity, Symmetry and Cardinality and Non Comparability.

Theorem 2. The extended claim-egalitarian mechanism is the unique full mechanism in Σ^n satisfying Weak Efficiency, Ordinal Level Comparability, Strong Consistency, Resource Monotonicity, Symmetry and Boundedness.

4. Bargaining with Claims in economic environments with variable population

Let us consider now an infinite set of potential agents $I = \{ 1, 2, \dots \}$, such that only a finite group of them are present in every concrete problem. Let M be the class of finite subsets of I . For a given $M \in M$, \mathfrak{R}^M denotes the cartesian product of $|M|$ copies of \mathfrak{R} indexed by the elements of M . Let Σ^M be the class of economic environments for $|M|$ agents indexed by the elements of M . Let $\Sigma = \bigcup_{M \in M} \Sigma^M$. From now on, an economic environment is a tuple $\xi = \langle M, u, h, w, c \rangle$, where $M \in M$, and the environment $\langle u, h, w, c \rangle \in \Sigma^m$, where $m = |M|$.

An allocation mechanism F is a full correspondence defined on Σ associating to any economic environment $\xi \in \Sigma$ a set of feasible allocations.

Consider now the following axiom:

Axiom of Anonymity (AN). For all M, M' , with $|M| = |M'|$, for any $\xi \in \Sigma^M$, $\xi = \langle u, h, w, c \rangle$ and for any one to one mapping $\gamma: M \rightarrow M'$, if $\xi \in \Sigma^M$, is such that $\xi' = \langle \gamma(u), h, w, \gamma(c) \rangle$, then $x \in F(\xi)$ iff $\gamma(x) \in F(\xi')$.

Axiom of Population Monotonicity (POP.MON). For any $M, N \in M$, for any $\xi = \langle M, u, h, w, c \rangle \in \Sigma^M$, $\xi' = \langle N, u', h', w, c' \rangle \in \Sigma^N$, if $M \subset N$, $u'_M = u$, $c'_M = c$, then $\mu_{FM}(\xi') \geq \mu_F(\xi)$.

Axiom of Continuity (CONT). For all N , for all sequence $\{\xi^t\}$, $\xi^t \in \Sigma^n$, $\xi^t = \langle u^t, h, w^t, c^t \rangle$, $\xi = \langle u, h, w, c \rangle \in \Sigma^n$, if $u^t \rightarrow u$ in the pointwise convergence topology, $w^t \rightarrow w$, $c^t \rightarrow c$ in the Hausdorff topology, then $\mu_F(\xi^t) \rightarrow \mu_F(\xi)$ in the pointwise convergence topology.

Now, the following characterization results are obtained [see Appendix]:

Theorem 3. The proportional mechanism is the unique full mechanism in Σ satisfying Weak Efficiency, Boundedness, Continuity, Anonymity, Cardinality and non Comparability, Strong Consistency and Population Monotonicity.

Theorem 4. The extended claim-egalitarian mechanism is the unique full mechanism in Σ satisfying Weak Efficiency, Boundedness, Continuity, Symmetry, Ordinal Level Comparability, Rational Strong Consistency and Population Monotonicity.

Appendix: Proofs of the Theorems

We shall prove that the right combination of axioms on economic environments imply the corresponding counterparts in utility terms. Then, by using the characterizations of the proportional and the extended claim-egalitarian solutions in the classical setting, we obtain our characterization results.

For the sake of completeness, we include the definitions of the axioms in utility terms as well as the characterization theorems in the classical setting. For an environment $\xi = \langle u, h, w, c \rangle$, and a permutation π of the set of agents, call $\pi(\xi) = \langle \pi(u), h, w, \pi(c) \rangle$.

U-Symmetry (U-SY). Let $\xi \in \Sigma^n$ such that for any permutation $\pi: N \rightarrow N$, $A(\xi) = A[\pi(\xi)]$. Then, $[\mu_F(\xi)]_i = [\mu_F(z)]_j, \forall i, j \in N$.

U-Anonymity (U-AN). For all M, M' , with $|M| = |M'|$, for any $\xi \in \Sigma^M, \xi' \in \Sigma^{M'}$, $\xi = \langle u, h, w, c \rangle, \xi' = \langle u', h', w', u' \rangle$ and for any one to one mapping $\gamma: M \rightarrow M'$, if $A(\xi') = \gamma[A(\xi)], u'(0) = \gamma[u(0)], u'(c') = \gamma[u(c)]$, then $\mu_F(\xi') = \gamma[\mu_F(\xi)]$.

Scale Invariance (SC.INV). Let $\xi, \xi' \in \Sigma^n$. If there exist $\lambda \in A^n$ such that $z \in Z(\xi)$ iff $\lambda z \in Z(\xi')$, then, $\mu_F(\xi') = \lambda[\mu_F(\xi)]$.

Translation Invariance (T.INV). Let $\xi, \xi' \in \Sigma^n$. If there exist $\lambda \in T^n$ such that $z \in Z(\xi)$ iff $\lambda z \in Z(\xi')$, then, $\mu_F(\xi') = \lambda[\mu_F(\xi)]$.

Strong Monotonicity (ST.MON). If $\xi = \langle u, h, w, c \rangle, \xi' = \langle u', h', w', c' \rangle \in \Sigma^n$ are such that $u(0) = u'(0), u(c) = u'(c'), A(\xi) \subset A(\xi')$, then $\mu_F(\xi') \geq \mu_F(\xi)$.

For $\xi = \langle u, h, w, c \rangle$, call $K(\xi) = \{z \in \text{WPO}(\xi) \mid u(0) \leq u(z) \leq u(c)\}$.

Independence of Pareto irrelevant, non-individually rational and unclaimed alternatives (IND). If $\xi, \xi' \in \Sigma^n$ are such that $A[K(\xi)] = A[K(\xi')]$, then $\mu_F(\xi) = \mu_F(\xi')$.

U-Continuity (U-CONT). For all N , for all sequence $\{\xi^t\}, \xi^t \in \Sigma^n, \xi^t = \langle u^t, h, w^t, c^t \rangle, \xi = \langle u, h, w, c \rangle \in \Sigma^n$, if $A(\xi^t) \rightarrow A(\xi), u^t(0) \rightarrow u(0)$ and $u^t(c^t) \rightarrow u(c)$, then $\mu_F(\xi^t) \rightarrow \mu_F(\xi)$ where the convergences are taken in the Hausdorff topology.

For $\xi = \langle u, h, w, c \rangle \in \Sigma^n$, call $IR(\xi) = \{z \in Z(\xi) \mid u(z) \geq u(0)\}$.

Rational Contraction Independence other than Claims point (RCIC). For any $\xi, \xi' \in \Sigma^n$, if $A(IR(\xi)) \subset A(IR(\xi'))$, $u(c) = u'(c')$, $\mu_F(\xi') \in A(IR(\xi))$, then $\mu_F(\xi) = \mu_F(\xi')$.

Fixed Claims Contraction Independence (FCCI). For any $\xi, \xi' \in \Sigma^n$, if $u(0) = u'(0)$, $u(c) = u'(c')$, $A(\xi) \subset A(\xi')$, $\mu_F(\xi') \in A(\xi)$, then $\mu_F(\xi) = \mu_F(\xi')$.

Let us consider a set of agents, N , and a subset of it, M . If $x \in \mathfrak{R}^N$, let x_M be the projection of x onto \mathfrak{R}^M , and given (S, d) such that $S \subset \mathfrak{R}^N$, $d \in S$, let $t_M^d(S) = \{x \in \mathfrak{R}^M \mid (x, d_{N/M}) \in S\}$.

U-Population Monotonicity (U-POP.MON). For all M, N , such that $M \subset N$, for all $\xi = \langle n, u, h, w, c \rangle \in \Sigma^n$, $\xi' = \langle m, u', h', w', c' \rangle \in \Sigma^m$, if $u_M(0) = u'(0)$, $u_M(c) = u'(c')$, and $t_M^{u'(0)}[A(\xi)] = A(\xi')$, then, for all $i \in M$, $u_i[F(\xi)] \leq u'_i[F(\xi')]$.

Chun & Thomson (1992), theorem 1, characterize the proportional solution in utility terms by means of WPO, U-SY, SC.INV. and ST.MON.

Lemma 1. CON and fullness of F imply W.

Proof: Let $\xi, \xi' \in \Sigma^n$, $\xi = \langle u, h, w, c \rangle$, $\xi' = \langle u', h', w', c' \rangle$ such that $A(\xi) = A(\xi')$, $u(0) = u'(0)$, $u(c) = u'(c')$. It must be proven that $\mu_F(\xi) = \mu_F(\xi')$.

As in Roemer (1988, lemma 5), construct $\xi^* = \xi \otimes \xi'$, $\xi^* = \langle v, h+h', (w, w'), (c, c') \rangle$, where $v_i(x_i, y_i) = \min \{u_i(x_i), u'_i(y_i)\}$. By Billera & Bixby (1973), $A(\xi^*) = A(\xi) = A(\xi')$; $v(0) = u(0) = u'(0)$. Furthermore, $v(c, c') = u(c) = u'(c')$.

It has to be proven that $\mu_F(\xi^*) = \mu_F(\xi)$ and $\mu_F(\xi^*) = \mu_F(\xi')$. The previous result is obtained as in Roemer (1988, lemma 5), by means of a series of intermediate problems, suitably modified by adding the appropriate claims point to each one, in order to maintain not only the equality of the utilities possibilities sets, but also the utilities of the claims points. ■

Lemma 2. Fullness, R.MON and CON imply ST.MON.

Proof: Let $\xi = \langle u, h, w, c \rangle$, $\xi' = \langle u', h', w', c' \rangle$ such that $u(0) = u'(0)$, $u(c) = u'(c')$, and $A(\xi) \supseteq A(\xi')$. We have to show that $\mu_F(\xi) \geq \mu_F(\xi')$. Act as in Roemer (1988, lemma 6), constructing the auxiliary problems, suitably modified by adding the claims point in such a way that the utility of the claims points is always preserved. ■

In an immediate way we obtain:

Lemma 3. Fullness, W and SY imply U-SY.

Lemma 4. Fullness, CNC and W imply SC.INV.

Theorem 1 follows from Lemmas 1, 2, 3, 4, and theorem 1 in Chun & Thomson (1992).

Bossert (1993), theorem 1, characterizes the extended claim-egalitarian solution by means of WPO, U-SY, T.INV., ST.MON, BDD and IND.

Lemma 5. CON* and fullness imply IND and FCCI.

Proof: Let us prove that IND follows. The other implication goes similarly.

1. Let $\xi = \langle u, h, w, c \rangle$, $\xi' = \langle u', h', w', c' \rangle$ such that $u(c) = u'(c')$, and $K(\xi) \subseteq K(\xi')$. Let $x' \in F(\xi')$ such that $u'(x') \in K(\xi)$. To prove: $\mu_F(\xi) = \mu_F(\xi')$.
2. Construct $\xi^* = \xi \otimes \xi'$. $\xi^* = \langle v, h+h', (w, w'), (c, c') \rangle$, where $v_i(x_i, y_i) = \min \{u_i(x_i), u'_i(y_i)\}$. $K(\xi^*) = K(\xi) \cap K(\xi') = K(\xi)$, and $v(c, c') = u(c) = u'(c')$.
3. Consider now $\bar{\xi} = \langle \bar{u}, h+h', (w, w'), (c, 0) \rangle$, where $\bar{u}_i(x_i, y_i) = u_i(x_i)$, $i = 1, \dots, n$. Obviously, $K(\xi) = K(\bar{\xi})$, $u(c) = \bar{u}(c, 0)$.
4. Notice that $\bar{u}_i(x_i, y_i) \geq v_i(x_i, y_i)$ for all $i = 1, \dots, n$. Now by Howe (1987, Prop. 3), it is possible to find functions $U_i: \mathfrak{R}^{h+h'+1} \rightarrow \mathfrak{R}$, such that $U_i(x_i, y_i, 1) = \bar{u}_i(x_i, y_i)$, $U_i(x_i, y_i, 0) = v_i(x_i, y_i)$. Now, construct the functions $V_i: \mathfrak{R}^{h+h'+n} \rightarrow \mathfrak{R}$, $V_i(x, y, z) = U_i(x, y, z_i)$, for $z \in \mathfrak{R}^n$, $i = 1, \dots, n$.
5. Consider now the environment: $\zeta = \langle V, h+h'+n, (w, w', 1, \dots, 1), (c, 0, f) \rangle$, where $f_i = e_i$, the i th vector of the canonical basis in \mathfrak{R}^n . Thus, $K(\zeta) = K(\bar{\xi}) = K(\xi)$. Moreover, $V_i(c_i, 0, e_i) = U_i(c_i, 0, 1) = \bar{u}_i(c_i, 0) = c_i(c_i, c'_i)$.
6. Construct the environment $\xi^* = \langle V, h+h'+n, (w, w', 0); (c, c', 0) \rangle$. Thus, $K(\xi^*) = K(\bar{\xi}^*)$. Moreover, $V(c_i, c'_i, 0) = U_i(c_i, c'_i, 0) = v_i(c_i, c'_i)$.
7. From previous constructions, $A(\bar{\xi}^*) = A(\zeta)$. Consider an allocation in $F(\zeta)$, $z_i = (x_i, y_i, e_i)$. Then, there must be an allocation $t_i = (x'_i, y'_i, 0)$ in $Z(\bar{\xi}^*)$, inducing the same utility point, i.e., $V(z_i) = V(t_i)$. Thus, since t_i is feasible in ζ , and because of fullness of F , $t \in F(\zeta)$.

Now, by CON*, $(x'_i, y'_i)_{i=1, \dots, n} \in F(\xi^*)$, obtaining that $\mu_F(\xi^*) = \mu_F(\zeta)$. And therefore $\mu_F(\xi^*) = \mu_F(\xi)$. ■

Lemma 6.- OLC, W and fullness imply T.INV.

Theorem 2 follows from lemmas 1, 2, 3, 5, 6 and theorem 1 in Bossert (1993).

Chun & Thomson (1992), theorem 6, characterize the proportional solution for a variable population by means of WPO, BDD, U-CONT, U-AN, SC.INV, FCCI and U-POP.MON. Marco (1995), theorem 1 characterizes the extended claim-egalitarian solution for a variable population by means of WPO, U-SY, T.INV., U-CONT., BDD, RCIC and U-POP.MON.

In a similar way to previous lemmas we obtain now:

Lemma 7. Fullness of F, W and POP.MON imply U-POP.MON.**Lemma 8. Fullness of F, W and RCON* imply RCIC.****Lemma 9. Fullness of F, W and CONT. imply U-CONT.**

Lemma 10. Fullness of F, AN and W imply U-AN.

Theorem 3 follows from lemmas 1, 4, 5, 7, 9, 10 and theorem 6 in Chun & Thomson (1992). Theorem 4 follows from lemmas 1, 3, 6, 7, 8, 9 and theorem 1 in Marco (1995).

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