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On Some General Properties of Static Solutions of Schiff's Equation.

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Summary. — Some general properties of the static solutions of Schiff's equation are derived in Section **1** from the structure of the non-linear equation and the behavior of the source distribution. Section **2** is devoted to the proof of the existence of the solution, and to a derivation of the many-body forces implied in the classical scalar theory of Schiff potentials.

Introduction.

In an attempt to account for both saturation and the independent-nucleon model of atomic nuclei, SCHIFF (1) has described the interactions between nucleons as arising from mesons which obey a non-linear wave equation. We are led to the same equation as a result of the renormalization process. Now many difficulties arise in dealing with the corresponding quantum field theory which have not yet been solved. On the other hand, it is clear by now that other terms than the Φ^3 term must also play a role in explaining saturation. For both reasons we shall consider Schiff's equation from the phenomenological standpoint and treat the non-linear field as a classical field.

In the present paper we shall deal with the nature and some general properties of the static solutions of Schiff's equation which can be derived in a rather simple way from the structure of both the non-linear equation and source distribution. Most of the results we shall arrive at are quite insensitive

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(1) J. L. SCHIFF: *Phys. Rev.*, **84**, 1 and 10 (1951); **86**, 856 (1952); **92**, 766 (1953).

to the particular analytical form of the source distribution, being mere mathematical consequences of the type of non-linearity exhibited by the partial differential equation. The assumptions we shall accordingly make as to the behavior of the source distribution will be large enough to include all cases of physical interest, with the only exception of point sources. Though in the theory appear two parameters, namely the coupling constant g and the coefficient α^2 of the Φ^3 term, only the product $\lambda = \alpha g$ of the two plays an essential role in determining the behavior of the field for a given nucleon distribution. The field thus appears as a function of λ and a functional of the nucleon distribution $f(x)$, being everywhere proportional to the coupling constant for given λ and $f(x)$. In Section 1 we define the mathematical problem and examine the general properties of the static solutions of Schiff's equation assuming that such solutions do exist. Section 2 is devoted to the proof of the existence of the solution, and to a derivation of the many-body forces among nucleons implied in the classical scalar theory of Schiff potentials.

1. – Behavior of the Static Solution of Schiff's Equation.

1·1. – In this first paper on non-linear meson theory of nuclear forces, Schiff describes the interactions between nucleons as arising from mesons which obey the non-linear wave equation

$$\frac{\partial^2\Phi}{\partial t^2} = \Delta\Phi - \Phi - \alpha^2\Phi^3 + gf\,. \tag{1}$$

where Φ is the meson field amplitude, f the nucleon distribution, g the coupling constant and α^2 a constant. Units are chosen in such a way that c, $\hbar$, and the meson mass are all equal to unity.

We shall assume that the distribution $f(x)$ does not depend on time and that $f(x)$ tends to zero u.d. (that is, uniformly with regard to the direction) as $r = |x|$ tends to infinity. However, most of the properties can be extended to the case in which $f(x) \to f_N$ u.d. as $r \to \infty$, where f_N is a non zero constant. This is the situation that would result from the superposition of a more or less fluctuating nuclear distribution, which is zero at infinity, and a constant « nucleon background » f_N. To put it still another way, we are faced with a distribution of this type when one or more nucleons are embedded in a sea of nuclear matter represented by the constant density f_N, as opposed to the case in which the nucleon or nucleons are considered in empty space.

In this paper we shall mainly be concerned, however, with source distributions that are zero at infinite distance, and only incidentally the possible extensions to the more general case will be pointed to. We shall in particular discuss nucleon distributions of the form

$$f(x) = \sum_{i=1}^{A} \varrho(x - x_i)\,, \tag{2}$$

where $\varrho(x - x_i)$ represents the nucleon density for *one* nucleon. It will be assumed that $\varrho(x - x_i)$ is a positive monotonic decreasing function of the distance $|x - x_i|$ from the fixed point x_i to the variable point x, and that $\varrho(x - x_i)$ goes to zero when $|x - x_i| \to \infty$ in such a way that it is practically zero for $|x - x_i| > a$, where a is a small quantity. We will then speak of a nucleon *at* x_i and radius a. We thus consider the nucleon as a source of finite extent But it is well known that the theory runs into difficulties when attempting to deal with point sources, i.e., when the function ϱ behaves as a δ-function. Further, we shall assume the function $\varrho(x - x_i)$ to be continuous throughout the space. This assumption excludes sources of the type $\varrho(x - x_i) = 0$ for $|x - x_i| > a$, $\varrho(x - x_i) \neq 0$ for $|x - x_i| \leqslant a$, i.e. the case of nucleons of sharply defined extent. This hypothesis does not entail any essential loss of generality from the physical point of view, since we can always smooth out any discontinuous dropping to zero of the density at $|x - x_i| = a$. As a matter of fact, the assumption of a continuous nucleon distribution $f(x)$ simplifies the reasonings without severely restricting the theory. All the results we shall arrive at are also true when the source distribution $f(x)$ is discontinuous, with *finite* discontinuities, on a finite number of closed regular surfaces, thus in particular for a finite assembly of sharply defined nucleons. We shall also admit that $\int \varrho(x - x_i)\,dx$ is finite, and will normalize $\varrho(x - x_i)$ in such a way that $\int \varrho(x - x_i)\,dx = 1$. All the nucleons of an assembly of A nucleons will be considered as equivalent, and so the functions $\varrho(x - x_i)$ in Eq. (2) will differ only in the values of the parameter x_i. Under this conditions and for nucleon distributions of the form Eq. (2), the coupling constant g in Eq. (1) is a well defined quantity, and $\int f(x)\,dx = A$. Finally, we shall find it convenient to change the notation by writing

$$V(x) = \frac{1}{g}\,\Phi(x)\,. \tag{3}$$

The boundary problem of which $V(x)$ must be the solution takes then the form (see Eq. (1))

$$(\mathrm{P}) \quad \begin{cases} \Delta V - V - \lambda^2 V^3 + f(x) = 0\,, \\ V(x) \to 0 \text{ u.d. as } |x| \to \infty\,, \end{cases} \tag{4}$$

where

$$\lambda = \varkappa g\,. \tag{5}$$

The value of $V(x)$ at a point x depends on all the values of $f(x)$ and on the particular value of the parameter λ. In the following, we shall accordingly deal with some simple properties of $V(x)$ considered as a functional of the nucleon distribution and as a function of the parameter λ.

By a solution of problem (P) we shall always mean a function $V(x)$ that satisfies the partial differential equation (4) and the boundary condition at infinity, the function $V(x)$ itself being continuous together with its partial derivatives of the first two orders throughout the space. However, when $f(x)$ exhibits discontinuities of the type referred to above, the laplacian ΔV will also be discontinuous on the same surfaces and with the same discontinuities as the source distribution itself, though of opposite sign. The function $f(x)$ will not necessarily be everywhere one-signed, but if it is so, it will be assumed to be positive — or at least non negative — throughout the space. The relations obtained in the case $f(x) \geqslant 0$ are, in fact, readily translated into those holding in the case $f(x) \leqslant 0$.

1·2. – We proceed now to formulate without proof two theorems which will be extremely useful for the subsequent analysis. Simple cases of them appear obvious from physical intuitions, and formal proofs can be found in any standard book on partial differential equations. Some care must be taken, however, when they are proved under the assumption that the region of space involved is bounded, since this is not the case here.

We formulate the first theorem as follows:

Theorem 1. Let $\varphi(x)$ be a *positive* continuous function of x throughout the space. The boundary problem

$$(6) \qquad (\mathrm{P}_1) \qquad \begin{cases} \Delta V - \varphi(x) V = 0 \,, \\ V(x) \to 0 \text{ u.d. as } |x| \to \infty \,. \end{cases}$$

has no other solution, continuous with continuous partial derivatives of the first two orders, than the $V(x) \equiv 0$.

From this theorem we can immediately deduce two corollaries of great importance for our problem. First observe that, in the absence of sources ($f(x) \equiv 0$), the only solution (cfr. § **1**·1) of problem P, Eq. (4), is $V(x) \equiv 0$. For any solution of problem (P) with $f(x) \equiv 0$ is a solution of problem (P_1) with

$$\varphi(x) = 1 + \lambda^2 V(x)^2 \geqslant 1$$

and hence $V(x) \equiv 0$. As pointed out by Schiff, we may interpret this result by saying that mesons cannot be permanently localized when there is no source. The same lack of permanent locability in the absence of sources would be true for *any* meson field governed by a wave equation of the form

$$\frac{\partial^2 \Phi}{\partial t^2} = \Delta\Phi - \Gamma(x, \Phi)\Phi + gf,$$

where $\Gamma(x, \Phi)$ is any *positive* continuous function of x and Φ for all values of x and Φ. On the other hand, it is by now clear that the nucleon distribution determines uniquely the meson field, since if there were two fields V_1 and V_2 associate to the same distribution $f(x)$, the difference $V_1 - V_2$ would be a solution of problem (P_1) with

$$\varphi(x) \equiv 1 + \lambda^2(V_1^2 + V_1V_2 + V_2^2) \geqslant 1$$

and hence necessarily $V_1 - V_2 \equiv 0$. In other words, if we get a solution of problem (P), this is *the* solution and defines the field caused by the given source distribution.

The second theorem refers to some inequalities ... boundary problem

$$(7) \qquad (P_2) \qquad \begin{cases} \Delta V - \varphi(x)V + f(x) = 0\,, \\ V(x) \to 0 \text{ u.d. as } |x| \to \infty\,, \end{cases}$$

where $f(x)$ and $\varphi(x)$ are given. Both are assumed to be continuous for all values of x, and $\varphi(x) > 0$ everywhere. The source distribution $f(x)$ can take positive and negative values, but if it is one-signed, it will be assumed that $f(x) \geqslant 0$ throughout the space.

Now: *Theorem 2* states that:

1) if $V(x)$ is the solution of problem (P_2), there are only two possible alternatives, namely

$$9a) \qquad \sup_x V(x) > 0\,, \qquad \inf_x V(x) < 0;$$

$$9b) \qquad \sup_x V(x) > 0\,, \qquad \inf_x V(x) = 0;$$

2) depending on which of the two alternatives above holds, we shall necessarily have one of the two following inequalities

$$(9a') \qquad \frac{f(x_m)}{\varphi(x_m)} \leqslant V(x) \leqslant \frac{f(x_M)}{\varphi(x_M)}\,, \qquad \text{with } f(x_m) < 0 \text{ and } f(x_M) > 0\,;$$

$$(9b') \qquad 0 \leqslant V(x) \leqslant \frac{f(x_M)}{\varphi(x_M)}\,,$$

these inequalities holding for any value of x.

Here we mean by $\sup_x V (\inf_x V)$ the upper bound (the lower bound) of $V(x)$ throughout the space, and $x_M(x_m)$ is any point at a *finite* distance at which the field $V(x)$ takes its true maximum $\max_x V(x)$ (its true minimum $\min_x V(x)$).

Observe that in case a) both x_M and x_m exist, but that case b) does not ensure the existence of the true minimum. All these results are also true when $f(x)$ is discontinuous, with finite discontinuities, on a finite number of regular closed surfaces.

From the theorem above it immediately follows that, if $f(x) \geqslant 0$ everywhere, then we shall also have $V(x) \geqslant 0$ for all values of x, though the converse is not in general true. Under the same hypothesis $f(x) \geqslant 0$, $V(x)$ can be zero only at those points at which $f(x) = 0$, too, so that $f(x) > 0$, all x, entails $V(x) > 0$ everywhere.

Another fairly obvious consequence of *Theor. 2* is that, *if* $\varphi(x) \geqslant 1$, then the fields V_1 and V_2 determined by the source distributions f_1 and f_2, with $|f_1(x) - f_2(x)| < k$ throughout the space, are such that $|V_1(x) - V_2(x)| < k$ for all values of x. This follows from the fact that the difference $V_1 - V_2$ is the solution of problem (P_2) for $f = f_1 - f_2$ as source distribution and from relations ($9a'$) and ($9b'$). The physical interpretation of this result is rather trivial. It states that, if two source distributions differ *uniformly* in less than, say, a very small quantity ε, the associate fields will also differ in less than *the same* quantity throughout the space.

Finally, it is easily seen that, if $f(x) \geqslant 0$, then *increasing* $\varphi(x)$ everywhere *decreases* the potential $V(x)$ throughout the space, with the only possible exception of those points at which $V(x)$ cannot decrease remaining non-negative, i.e., the points at which $V(x) = 0$. In particular, the static Neumann-Yukawa potential (2)

$$V_0(x) = \int G(x - x_0) f(x_0)\, dx_0 ,$$

with $G(x - x_0) = \exp[-|x - x_0|]/4\pi|x - x_0|$ and $f(x_0) \geqslant 0$, is everywhere positive and will be greater than any static potential that obeys a field equation of the form

$$\Delta V - \Gamma(x, V) V + f = 0 ,$$

where $\Gamma(x, V) > 1$ for all values of x and V. For example, in the case of Schiff potentials we have $\Gamma(x, V) = 1 + \lambda^2 V(x)^2 > 1$ with the only possible exception of the points at which $V(x) = 0$. But at these points $V_0(x) > 0$, Hence, if $f(x) \geqslant 0$, then $V(x) < V_0(x)$ everywhere. In other words, given a static non-negative nucleon distribution and if mesons obey Schiff non-linear equation, the field amplitude must be everywhere less than the amplitude

(2) These potentials were considered extensively by C. NEUMANN: *Allgemeine Untersuchungen über das Newtonische Prinzip der Fernwirkungen mit besonderer Rücksicht auf die elektrischen Wirkungen* (Teubner, 1896).

of the classical Yukawa field for the *same* nucleon distribution and meson mass. More precisely, if g and g_0 are the coupling constants for the Schiff and Yukawa fields, respectively, we shall have for all values of x [see Eq. (3)]

$$\Phi(x) < \frac{g}{g_0}\,\Phi_0(x)\,.$$

We shall come back to this point in § **1·5**.

1·3. – We shall now apply all these results to the non-linear meson field $V(x)$ that obeys Eq. (4). We have already seen that problem (P) has either only one solution or no solution at all. To put it still another way, given the nucleon distribution $f(x)$ and the parameter λ, either we shall have a well determined field $V(x)$, or the nucleon distribution and/or the value of λ are not compatible with the theory. For the time being, we shall therefore assume that *the* solution of problem (P) exists for all $f(x)$ and λ which will come in play.

The essential point is that the solution $V(x)$ of problem (P) is also the solution of problem (P_2) with the same source and $\varphi(x) \equiv 1 + \lambda^2 V(x)^2 \geqslant 1$. But now we can be a little more precise than before, since we know the form of the function $\varphi(x)$. For example, since $V(x)$ and $\varphi(x)V(x) \equiv V(x) + \lambda^2 V(x)^3$ are now maximum or minimum *at the same points*, we can write, instead of ($9a'$) and ($9b'$),

$$(10a) \qquad f(x_m) \leqslant V(x) + \lambda^2 V(x)^3 \leqslant f(x_M), \qquad \text{with } f(x_m) < 0 \text{ and } f(x_M) > 0\,,$$

$$(10b) \qquad 0 \leqslant V(x) + \lambda^2 V(x)^3 \leqslant f(x_M)\,,$$

respectively. The first relation holds when $V(x)$ is not one-signed, the second when $V(x) \geqslant 0$. As in *Theor. 2*, $x_M(x_m)$ is any point at which the field takes its true maximum (its true minimum). We see then that the static field described by Eq. (4) is everywhere less than (greater than) the value of the nucleon density at the point at which *the field* exhibits its true maximum (its true minimum) and *a fortiori* less than (greater than) the upper bound F (lower bound f) of the nucleon density, which *does not depend* on λ. If the nucleon distribution is one-signed, the field is also one-signed and both have the same sign.

Now, from ($10a$) it follows that

$$(11a) \qquad 0 \leqslant |V(x)|^3 < \frac{1}{\lambda^2}\sup_x |f(x)|\,, \qquad \text{all } x\,,$$

and from ($10b$)

$$(11b) \qquad 0 \leqslant V(x)^3 < \frac{F}{\lambda^2}\,, \qquad \text{all } x\,.$$

Hence a positive constant K, independent of λ, always exists such that $|V(x)| < K\lambda^{-\frac{2}{3}}$ for *all* values of x and λ. In other words, the field $V(x)$ tends to zero as $\lambda \to \infty$ in such a way that

$$V(x) = 0(\lambda^{-\frac{2}{3}}), \tag{12}$$

uniformly throughout the space as $\lambda \to \infty$. This result has a simple, though important, physical interpretation. Given a *static* nucleon distribution $f(x)$ and the value of the coupling constant g, we cannot enhance the non-linearity of the theory by increasing the constant α^2 beyond any limit, since then either there is no solution, or the field will become vanishingly small tending everywhere to zero not less rapidly than $\alpha^{-\frac{2}{3}}$. For example, in the case $f(x) \geqslant 0$ we shall have, by Eqs. (12) and (5),

$$0 \leqslant \Phi(x, \lambda) < \frac{(gF)^{\frac{1}{3}}}{\alpha^{\frac{2}{3}}}. \tag{13}$$

Consider now the fields $V_1(x)$ and $V_2(x)$ determined by the nucleon distributions $f_1(x)$ and $f_2(x)$, respectively, and suppose that $f_1(x) > f_2(x)$ throughout a region R, $f_1(x)$ being equal to $f_2(x)$ at all points which do not belong to R. Then by considering as before the difference $V_1 - V_2$ as the solution of problem (P_2) with the source $f_1 - f_2$ and $\varphi(x) = 1 + \lambda^2(V_1^2 + V_1V_2 + V_2^2) \geqslant 1$, it is easily seen from what was said in § 1·2 that *at least throughout* R is $V_1(x) > V_2(x)$ and that we shall have for all values of x

$$0 \leqslant V_1(x) - V_2(x) \leqslant \left(\frac{f_1(x) - f_2(x)}{1 + \lambda^2(V_1^2 + V_1V_2 + V_2^2)}\right)_{x=x_M} < \sup_x (f_1(x) - f_2(x)),$$

where x_M is a point at which $V_1 - V_2$ takes its true maximum. That is, if the nucleon density is increased throughout a region R, the field amplitude increases *at least* everywhere throughout R, but the increment of the field at *any* point is always less than the maximum increase of the nucleon density.

We shall next consider the effect of superposing two « absolutely rigid » nucleon distributions $f_1(x)$ and $f_2(x)$, both non-negative. We are thus ignoring the possible correlational changes of $f_1(x)$ and $f_2(x)$ brought about by the superposition. Let $V_1(x)$ and $V_2(x)$ be the fields determined by $f_1(x)$ and $f_2(x)$, respectively, and let U be defined by the identity

$$U \equiv V_{12} - (V_1 + V_2),$$

where V_{12} is the field determined by the source $f_1(x) + f_2(x) \geqslant 0$. It is easily seen from the partial differential equations and the common boundary condition for V_1, V_2 and V_{12}, that U is a solution of problem (P_2) with

$$\varphi(x) \equiv 1 + \lambda^2\{(U + \tfrac{3}{2}(V_1 + V_2))^2 + \tfrac{3}{4}(V_1 + V_2)^2\} \geqslant 1\,,$$
$$f(x) \equiv -3\lambda^2(V_1^2 V_2 + V_1 V_2^2)\,.$$

But $f_1(x) \geqslant 0$ and $f_2(x) \geqslant 0$. Hence $V_1(x) \geqslant 0$ and $V_2(x) \geqslant 0$, and so $f(x) \leqslant 0$, i.e., $U(x) \leqslant 0$ for all values of x. Further, $U(x)$ can only be zero at the points at which at least one of the two potentials $V_1(x)$ and $V_2(x)$ is zero. Therefore, with the only possible exceptions of those points [see § 1·5], we shall have

$$V_{12} < V_1 + V_2$$

throughout the space if $\lambda > 0$. As was to be expected, U vanishes identically when $\lambda = 0$, i.e., in the linear case and *only* in this case. On the other hand, we know that $V_{12} > V_1$ and $V_{12} > V_2$. Hence, we can assert that

$$\tfrac{1}{2}(V_1 + V_2) < V_{12} < V_1 + V_2 \tag{14}$$

and that $V_{12} = V_1 + V_2$ if and only if $\lambda = 0$.

1·4. – It might be well to discuss here briefly the more general case $f \to f_N \neq 0$ u.d. as $|x| \to \infty$, with f_N a constant. When the nucleon distribution does not go to zero at infinity, but tends to a constant, say, positive value f_N, the analysis becomes somewhat more complicated. However, the general method of approach is the same as that already discussed.

We first introduce the functions U and h defined by

$$U \equiv V - V_N\,, \qquad h \equiv f - f_N\,, \tag{15}$$

where V_N is the only real root of the equation $V_N + \lambda^2 V_N^3 = f_N$ and V the Schiff field determined by the given nucleon distribution f with the boundary condition $V \to V_N$ u.d. as $|x| \to \infty$. Clearly, U must be solution of the boundary problem

$$\Delta U - [1 + \tfrac{3}{4}\lambda^2 V_N^2 + \lambda^2(U + \tfrac{3}{2}V_N)^2]U + h = 0, \qquad U \to 0 \text{ u.d. as } |x| \to \infty. \tag{16}$$

Comparison with problem (P_2) shows that, in the present case,

$$\varphi(x) \equiv 1 + \tfrac{3}{4}\lambda^2 V_N^2 + \lambda^2(U + \tfrac{3}{2}V_N)^2 \geqslant 1 + \tfrac{3}{4}\lambda^2 V_N > 1\,,$$

and we can apply all theorems above. For example, it follows immediately that, if $h(x) \geqslant 0$, then $U(x) \geqslant 0$. Further, since $U(x)$ and $U(x)\varphi(x)$ are maximum (or minimum) at the same points, we shall have

$$(17)\qquad 0 \leqslant |U|[1 + \tfrac{3}{4}\lambda^2 V_N^2 + \lambda^2(U + \tfrac{3}{2}V_N)^2] < \sup_x |h(x)|,$$

as before [see Eqs. (10)], with $U(x) \geqslant 0$ if $h(x) \geqslant 0$. From the inequality (17) it follows that, for *all* values of x,

$$(18)\qquad |U(x)| < \frac{1}{1 + \frac{3}{4}\lambda^2 V_N^2} \sup_x |h(x)|,$$

where, if we *a priori* know that $U \geqslant 0$ (as would be the case if $h \geqslant 0$), the first factor on the right could be susbstituted by $1/(1 + 3\lambda^2 V_N^2)$. We see, therefore, that the contribution of the supplementary nucleon distribution $h(x)$ to the total field becomes vanishingly small when the density of the background f_N, and therefore V_N, increases tending to infinity. Of course, this effect vanishes in the linear case, when U does not depend on V_N. It is also readily proved that, when $h(x) \geqslant 0$, the fields V_v and U caused by the *same* nucleon distribution h in empty space and in a sea of nuclear matter represented by the constant $f_N > 0$, respectively, are such that the field U in nuclear matter is everywhere less than the field in empty space V_v, both being non-negative. In a certain sense, the presence of a background tends to suppress the effect of any superimposed nucleon distribution zero at infinity.

Relation (18) suggests the possibility of approximately computing U when λ, V_N and h are such that

$$(19)\qquad \frac{1}{1 + \frac{3}{4}\lambda^2 V_N^2} \sup_x |h(x)| \ll V_N .$$

For then we shall have, by Eq. (18), $|U(x)| \ll V_N$ throughout the space, and we can expect the solution of the *linear* partial differential equation

$$\Delta U - (1 + 3\lambda^2 V_N^2) U + h = 0 ,$$

which is zero at infinity to be a good approximation for the exact solution of Eq. (16), at least asymptotically, for very high values of λV_N. This means in physical terms that, as long as condition (19) holds, the supplementary field U behaves as the ordinary Yukawa field determined by the same nucleon distribution $h(x)$, *but with a reduced range* $\varkappa^{-1} = (1 + 3\lambda^2 V_N^2)^{-\frac{1}{2}}$ instead of unity.

On the other hand, the energy density of the *total* field $V = V_N + U$ is easily seen to be

$$(20)\qquad \mathcal{H} = -\tfrac{1}{2}g^2(V_N^2 + \tfrac{3}{2}\lambda^2 V_N^4) - \tfrac{1}{2}g^2(2hV_N + hU + \lambda^2 V_N U^3 + \tfrac{1}{2}\lambda^2 U^4) .$$

The first term on the right represents the contribution of the background of uniformly distributed nuclear matter. The second term represents the (exact) contribution of the nucleon fluctuation $h(x)$. The first component $-g^2V_Nh(x)$ can be interpreted as arising from the interaction of the supplementary nucleons *with the background* and tends *to infinity* as $V_N \to +\infty$. The other three components represent either the interaction between supplementary nucleons, $-\frac{1}{2}g^2(hU+\frac{1}{2}\lambda^2U^4)$, or many-body interactions between these nucleons and the background, $-\frac{1}{2}g^2\lambda^2V_NU^3$. All of them tend *to zero* as $V_N \to +\infty$, if $\lambda \neq 0$. Only if $\lambda \neq 0$ is $\lim_{V_N\to\infty} hU = 0$. As was expected, the non-linearity ($\lambda \neq 0$) tends to suppress all these interactions for sufficiently high values of V_N, i.e., of f_N, and we are left with only the two-body interactions between supplementary nucleons and background $-g^2V_Nh$. Now, if we take for $h(x)$ an expression of the form Eq. (2), the *total* contribution of the nucleon distribution $h(x)$ to the energy of the field is *asymptotically* independent of the « positions » x_i of the nucleons and equal to $-g^2V_NA$, where A is the number of supplementary nucleons. In other words, nucleons embedded in a sea of nuclear matter of sufficiently high uniform density do not practically interact between them, behave as if they were absolutely free, and each contributes to the energy of the field with a constant amount $-g^2V_N$. This result is quite general. In particular it does not depend on the « size and shape » of the nucleon $\varrho(x-x_i)$ under the only condition of this function being everywhere finite. As a matter of fact V_N can be relatively small, and even then the main interaction can be that of the nucleons with the background. All then depends on the value of λ. Take, for example, the values given by SCHIFF [v. 1, p. 6], $g=1.49$, $\lambda=\alpha g=11.86$, $V_N=0.1$, and suppose the nucleons be represented by homogeneous, nonoverlaping spheres of radius $\varepsilon=1$ in our units. The ratios of the second, third and fourth terms to the first in the expression for the supplementary energy density are then of order of magnitude or less than 23%, 2.8%, and 0.6%, respectively.

1·5. – We can now turn to the question as to how does the field $V(x)$, Eq. (4), behave considered as a function of the parameter λ, with $f(x)$ fixed. We shall indicate the explicit dependence of the field on λ by writing $V(x,\lambda)$ for $V(x)$. Suppose $V'\equiv V(x,\lambda')$ and $V''\equiv V(x,\lambda'')$ are the solutions of problem (P) for two values λ' and $\lambda''(\lambda'<\lambda'')$ of the parameter λ and the same nucleon distribution. By considering $V'-V''$ as the solution of problem (P_2) with

$$\varphi(x)\equiv 1+\lambda''^2(V'^2+V'V''+V''^2)\geqslant 1\,,$$
$$f(x)\equiv(\lambda''^2-\lambda'^2)V'^3,$$

it is readily proved that, if V' is not one-signed,

$$(21)\qquad (\lambda''^2-\lambda'^2)\inf_x V(x,\lambda')^3 < V(x,\lambda')-V(x,\lambda'') < (\lambda''^2-\lambda'^2)\sup_x V(x,\lambda')^3$$

and that, if $V' \geqslant 0$ everywhere,

$$(22) \qquad 0 \leqslant V(x, \lambda') - V(x, \lambda'') < (\lambda''^2 - \lambda'^2) \sup_x V(x, \lambda')^3.$$

From Eqs. (21), (22) and (11) it follows that: *a)* $V(x, \lambda)$ is a continuous function of λ throughout the space and the continuity is *uniform* with regard to x, in particular

$$(23) \qquad \lim_{\lambda \to 0} V(x, \lambda) = V_0(x) = \frac{1}{4\pi}\int \frac{\exp[-|x - x_0|]}{|x - x_0|} f(x_0)\, dx_0;$$

b) if the derivative $\partial V(x, \lambda)/\partial\lambda$ exists, it will be everywhere finite, equal to zero for $\lambda = 0$, and will tend to zero as $\lambda \to +\infty$.

Consider now the particular case $f(x) \geqslant 0$. The Neumann-Yukawa potential $V_0(x)$, Eq. (23), is everywhere positive and, by Eq. (22), $V(x, \lambda)$ is a positive *decreasing monotonic* continuous function of λ [this ensures that the derivative $\partial V(x, \lambda)/\partial V$ will exist for almost all values of λ]. At each point x, $V(x, \lambda)$ reaches its maximum at $\lambda = 0$ and is equal there to the associate Neumann-Yukawa potential $V_0(x)$. Observe that from (22) it only follows that $\lambda' < \lambda''$ entails $V(x, \lambda') \geqslant V(x, \lambda'')$, i.e., that $V(x, \lambda)$ *cannot increase*. Now, the equality sign can hold only if $V(x, \lambda') = 0$, and then $V(x, \lambda) = 0$ for all $\lambda \geqslant \lambda'$. However, as we shall presently see, $V(x, \lambda)$ cannot be zero if $f(x) \geqslant 0$. Hence $V(x, \lambda)$ decreases monotonically and is everywhere positive.

Again, let $f(x)$ be everywhere non-negative. We know that $\lambda V(x, \lambda)$ is a continuous non-negative function of λ. Now, for two values $\lambda' < \lambda''$ of λ we shall have

$$\Delta(\lambda'' V'' - \lambda' V') - [1 + \lambda'^2 V'^2 + \lambda' \lambda'' V' V'' + \lambda''^2 V''^2](\lambda'' V'' - \lambda' V') + (\lambda'' - \lambda') f(x) = 0,$$

The bracket [ ] is $\geqslant 1$ and $(\lambda'' - \lambda') f(x) \geqslant 0$. Hence, $\lambda'' V'' - \lambda' V' \geqslant 0$, i.e., $\lambda V(x, \lambda)$ does not decrease as λ increases. But, for sufficiently small values of $\lambda > 0$, $\lambda V(x, \lambda)$ is positive. Hence $\lambda V(x, \lambda) > 0$ for *all* positive values, i.e., $V(x, \lambda)$ cannot be zero. On the other hand, it is easily proved that $\partial(\lambda V)/\partial\lambda = V + \lambda(\partial V/\partial\lambda)$ is positive for all values of λ, decreases monotonically with increasing λ, and tends to zero as $\lambda \to +\infty$. We can therefore assert that $\lambda V(x, \lambda)$ is a positive increasing monotonic function of λ. However, by Eq. (11*b*), $\lambda V(x, \lambda)$ cannot increase more rapidly than $(F\lambda)^{\frac{1}{3}}$, where F is the maximum value of the nucleon density $f(x)$. We thus have the situation qualitatively shown in fig. 1.

As a simple example, let us apply the results above to the field $\Phi(x, \lambda) = gV(x, \lambda)$. We still assume $f(x) \geqslant 0$. The value of $\lambda = \alpha g$ increases by increasing λ and/or g. Given the value of the coupling constant g, we now see that the field Φ is everywhere positive and decreases monotonically as α increases, i.e., as the non-linearity is enhanced, tending to zero uniformly throughout the space as $\alpha \to +\infty$ [see Eq. (13)]. Consider now α fixed. We have

$$\Phi(x, \lambda) \equiv \frac{1}{\alpha} \cdot \lambda V(x, \lambda) .$$

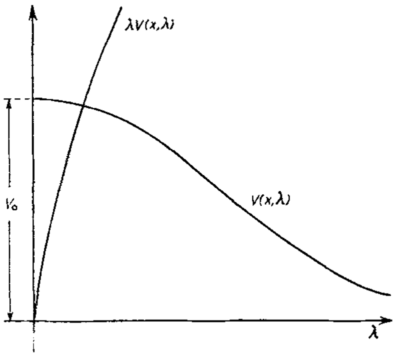

Fig. 1.

As g increases, λ increases and $\lambda V(x, \lambda)$ increases monotonically. Hence, $\Phi(x)$ is an *increasing monotonic* function of the coupling constant g, though Φ can never increase more rapidly than $\alpha^{-\frac{2}{3}}(Fg)^{\frac{1}{3}}$. In the linear case, $\Phi(x)$ is everywhere proportional to g.

It may be interesting to note in this connection that there is a class of positive nucleon distributions $f(x)$ for which, *if* the solution of problem (P) exists for arbitrarily large values of λ, one has $\lim_{\lambda\to\infty} (V(x, \lambda) - f(x)/\lambda^{\frac{2}{3}}) = 0$ uniformly with regard to x, namely the class of positive functions $f(x)$ which tend to zero u.d. as $|x| \to \infty$ and for which

$$M \equiv \sup_x \left| f(x)^{-\frac{2}{3}} \Delta f - \tfrac{2}{3} f(x)^{-\frac{5}{3}} |\operatorname{grad} f|^2 - 3 f(x)^{\frac{1}{3}} \right| \tag{24}$$

exists and is finite. For example, in the simple case $f(x) = \exp[-r^2]$ the expression between the bars is equal to $(\frac{4}{3}r^2 - 9)\exp[-\frac{1}{3}r^2]$ and $M = 9$.

Assuming M exists and is finite for the given positive nucleon distribution $f(x)$, let $U(x, \lambda)$ be defined by

$$U(x, \lambda) \equiv \frac{f(x)}{\lambda^{\frac{2}{3}}} .$$

The function $U(x, \lambda)$ is the solution of problem (P) with the source distribution

$$f(x) - \frac{1}{3\lambda^{\frac{2}{3}}} \left[f(x)^{-\frac{2}{3}} \Delta f - \frac{2}{3} f(x)^{-\frac{5}{3}} |\operatorname{grad} f|^2 - 3 f(x)^{\frac{1}{3}} \right] .$$

Hence, according to what was said at the end of § 1·2, if $V(x, \lambda)$ is the field determined by the source $f(x)$, then

$$|V(x, \lambda) - U(x, \lambda)| < \frac{M}{3\lambda^{\frac{2}{3}}} ,$$

where the upper bound on the right does not depend on the point x. Therefore if $V(x, \lambda)$ exists for arbitrarily large values of λ and the given $f(x)$, for any given $\varepsilon > 0$ there always exists λ_0, *independent* of x, such that $\lambda > \lambda_0$ entails

$$\left| V(x, \lambda) - \frac{f(x)^{\frac{1}{3}}}{\lambda^{\frac{2}{3}}} \right| < \varepsilon ,$$

throughout the space. In other words, if we increase the non-linearity by increasing α (i.e., λ), then the meson field, though everywhere small, will follow very closely the spacial behavior of the nucleon distribution, being smaller where f is small and larger where f is large: the field concentrates all around the sources [see also § **1·4**].

2. – Non-linear Meson Theory and Many-Body Interactions.

2·1. – Before going to the proof of the existence of the solution of problem (P), it might be convenient to recall a well known theorem in the theory of the Neumann-Yukawa potential, namely the theorem that for every integrable function $f(x)$, and if $f(x) \to 0$ u.d. as $|x| \to \infty$, the potential

$$V_0(x) \equiv \int G(x - x_0) f(x_0)\, \mathrm{d}x_0 \to 0 \qquad \text{u.d.} \qquad \text{as } |x| \to \infty , \tag{25}$$

with $G(x - x_0) = \exp[-|x - x_0|]/4\pi|x - x_0|$. It is also well known that, under well defined conditions of continuity of $f(x)$ and its first partial derivatives, the function $V_0(x)$ defined in Eq. (25) is the only solution of problem (P) with $\lambda = 0$. We formulate next the following

Lemma. For every function $f(x)$, with $|f(x)|$ integrable, such that

$$f(x) = O\left(\frac{e^{-r}}{r^{\beta}}\right) \quad \text{u.d.} \qquad (\beta > 2) \tag{26}$$

as $r \equiv |x| \to \infty$, we shall have

$$V_0(P) \equiv \int G(P - P_0) f(P_0) \mathrm{d}P_0 \sim \frac{e^{-R}}{4\pi R} \int \exp[\overrightarrow{OP_0} \cdot \boldsymbol{n}] f(P_0)\, \mathrm{d}P_0 \quad \text{as } R \to \infty , \tag{27}$$

where O is a fixed arbitrarily chosen point, $R \equiv |\overrightarrow{OP}|$ and $\boldsymbol{n} \equiv (1/R)\overrightarrow{OP}$(*). We get in this way a more precise information as to the asymptotic behavior of $V_0(x)$ than that furnished by the theorem Eq. (25). The asymptotic behavior (27) is nearly trivial when $f(x) = 0$ outside a *finite* region of space. But that

(*) This result holds *a fortiori* if $f(x) = 0(e^{-r}/r^{\beta})$, $\beta > 2$.

$V_0(x)$ will not in *general* behave as required by (27) is made clear by simply considering the source distribution $f(x) = 1/r$. The associate potential is then

$$V_0(R) = \frac{1 - e^{-2R}}{R} \sim \frac{1}{R}\,.$$

Further, since the integral on the right of the asymptotic Eq. (27) is not in general zero — certainly not when $f(x) \geqslant 0$ everywhere —, the principal part of the asymptotic expansion of $V_0(x)$ will *always* be of Yukawa type, irrespective of the analytical form of $f(x)$, as long as Eq. (26) is valid. The value of the integral on the right in Eq. (27) depends, of course, on the point O chosen and the direction $\boldsymbol{n}$ along which P tends to infinity. As a function of $\boldsymbol{n}$, the integral is a continuous function on the sphere of radius unity $|\boldsymbol{n}| = 1$, and so it is bounded. When the source distribution is spherically symmetric and we take O at the center of symmetry, Eq. (27) reduces to

$$\int G(x - x_0)\, f(x_0)\, \mathrm{d}x_0 \sim \frac{e^{-R}}{4\pi R}\cdot 4\pi \int_0^\infty f(r)\, r\, \mathrm{sh}\, r\, \mathrm{d}r\,.$$

A proof of this lemma is sketched in *Appendix I.*

2·2. – Let us turn back to the non-linear field $V(x)$ of problem (P). We know [see § **1·5**] that the potential $V(x)$ determined by a non-negative nucleon distribution $f(x) \geqslant 0$ is everywhere positive and bounded by the Neumann-Yukawa potential due to *the same* distribution. We have namely

$$(28)\qquad 0 < V(x) < V_0(x) = \int G(x - x_0)\, f(x_0)\, \mathrm{d}x_0\,.$$

This upper bound does not depend on λ. One can easily find a lower bound for $V(x)$, as can be seen in the following way. Consider the function

$$(29)\qquad V_0(x, \lambda) = V_0(x) - \lambda^2 \int G(x - x_0)\, V_0(x_0)^3\, \mathrm{d}x_0\,.$$

This function tends to zero u.d. ad $|x| \to \infty$ and is the solution of the boundary problem

$$\Delta V_1 - V_1 - \lambda^2 V_0(x)^3 + f(x) = 0\,,$$

$$V_1(x) \to 0 \quad \text{u.d. as } |x| \to \infty\,.$$

The difference $V - V_1$, where V is the solution of problem (P), is the solution of problem (P_2) with $\varphi(x) = 1$ and the source $\lambda^2(V_0^3 - V^3) > 0$ [see Eq. (28]). Hence, for all values of x,

$$V_1(x) < V(x) . \tag{30}$$

Of course, $V_1(x)$ is of no use as a lower bound if it happens to be negative, since we know that $V(x)$ must be positive. Now, $V_1(x)$ can be negative at some points. In fact, given arbitrarily a point x we can always make $V_1(x) < 0$ at this point by taking λ sufficiently large. Hence the case $V_1(x) < 0$ cannot be excluded. However, let us proceed to investigate under which conditions can be $V_1(x) \geqslant 0$ for all values of x. We shall then have

$$0 \leqslant V_1(x) < V(x) < V_0(x) , \tag{31}$$

throughout the space. Now, it follows immediately from the definition Eq. (29) of $V_1(x, \lambda)$ that the necessary and sufficient condition that V_1 should be non-negative everywhere is that

$$\lambda^2 \leqslant \inf_x \frac{V_0(x)}{\int G(x - x_0) V_0(x_0)^3 \, dx_0} = m . \tag{32}$$

It might be thought at first that the expression on the right will be zero, even when $V_0(x) > 0$ for all *finite* values of x, since $V_0(x) \to 0$ as $|x| \to \infty$. But there is a pretty large class of source distributions for which m is certainly not zero, namely that of all non-negative source distributions which fulfil the condition Eq. (26). We prove this statement in *Appendix II*. We thus have the result that, at least for a physically reasonable class of source distributions, $V_1(x)$ will be non-negative for sufficiently small values of λ, so that Eq. (31) will hold. But then the energy of the field V

$$H = -\frac{1}{2} g^2 \int \left(fV + \frac{1}{2} \lambda^2 V^4 \right) dx \tag{33}$$

will be greater than the value of *the same integral* computed with the potential $V_0(x)$,

$$H_0 = -\frac{1}{2} g^2 \int \left(fV_0 + \frac{1}{2} \lambda^2 V_0^4 \right) dx \tag{34}$$

and less than the integral

$$H_1 = -\frac{1}{2} g^2 \int \left(fV_1 + \frac{1}{2} \lambda^2 V_1^4 \right) dx . \tag{35}$$

Note that the first statement is *always* true if $f(x) \geqslant 0$, but that, if $V_1(x)$ could be negative, we could not infer from $V_1(x) < V(x)$ that $V_1(x)^4 < V(x)^4$ and therefore we could not assert that $H < H_1$.

2·3. – Let us consider now the sequence $\{V_n(x)\}$ of functions $V_n(x)$ determined in order by

$$V_{n+1}(x) = V_0(x) - \lambda^2 \int G(x - x_0) V_n(x_0)^3 \, dx_0 \,, \tag{36}$$

with $n = 0, 1, 2, \ldots$ and $G(x - x_0)$ defined as in Eq. (25). The functions $V_n(x)$ exist under quite general conditions if $f(x) \to 0$ u.d. as $|x| \to \infty$ and $V_{n+1}(x)$ is the solution of the boundary problem

$$\Delta V_{n+1} - V_{n+1} + f - \lambda^2 V_n^3 = 0 \,, \qquad V_{n+1} \to 0 \quad \text{u.d.} \quad \text{as } |x| \to \infty \,. \tag{37}$$

From *Theor.* 2, Coroll. 1. [see § **1**·2], we can in general only infer that, for $n = 0, 1, 2, \ldots,$

$$V_{2n} > V_{2n+1} < V_{2n+2} \,. \tag{38}$$

But suppose a positive number k can be found such that either $V_{2k-1}(x) < V_{2k+1}(x)$ or $V_{2k-2}(x) > V_{2k}(x)$ for *all* values of x. Then, from the value k on the sequence $\{V_{2n+1}(x)\}$ will increase monotonically and the sequence $\{V_{2n}(x)\}$ will be decreasing monotonic. Further, from that value on, any element of the former will be less than any element of the latter, and so both sequences are bounded and, being monotonic, both have a finite limit for all values of x. The first statement follows from Eq. (37) and the above mentioned corollary. The second statement follows from the first and the general relation (38).

Many questions arise now:

a) will the number k *always* exist?;

b) if k can be found, will be $\lim\limits_{n\to\infty} V_{2n+1}(x) = \lim\limits_{n\to\infty} V_{2n}(x)$ for all values of x?;

c) if both sequences have the same limit, say, $V(x)$, will $V(x)$ satisfy the integral equation

$$V(x) = V_0(x) - \lambda^2 \int G(x - x_0) V(x_0)^3 \, dx_0 ? \tag{39}$$

The answer to the first question is no. The number k does not *always* exist. For suppose $f(x) \geqslant 0$, so that $Q(x)$ [see Appendix II] is everywhere positive and suppose moreover that Eq. (26) holds. Let λ be such that

$$\lambda^2 \geqslant \sup_x Q(x) = M.$$

We shall then have $V_1(x) \leqslant 0$ everywhere, and so, for all values of x,

$$V_2(x) - V_0(x) = -\lambda^2 \int G(x - x_0) V_1(x_0)^3 > 0 ,$$

which entails $V_1(x) > V_3(x)$, which entails $V_2(x) < V_4(x)$, and so on: Hence if $f(x) \geqslant 0$ and λ is large enough, so that (40) holds, the sequence $\{V_{2n+1}(x)\}$ will be decreasing monotonic and $\{V_{2n}(x)\}$ will increase monotonically from the value $n = 0$ on and for all values of x (fig. 2a).

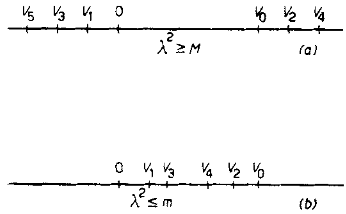

Fig. 2.

But suppose now that, under the same conditions holding for $f(x)$, we have $\lambda^2 \leqslant m$. Then $V_1(x) \geqslant 0$ and hence $V_2(x) < V_0(x)$ throughout the space. The second alternative above holds with $k = 1$, and so $\{V_{2n+1}(x)\}$ increases monotonically, $\{V_{2n}(x)\}$ is decreasing monotonic, and

$$(41) \qquad 0 < \lim_{n\to\infty} V_{2n+1}(x) \equiv \underline{V}(x) \leqslant \lim_{n\to\infty} V_{2n}(x) \equiv \overline{V}(x) < V_0(x)$$

for all values of x (fig. 2b). In this case, therefore, k exists.

We see therefore that, though k does not always exist, there is at least a case in which it does exist, namely when $f(x)$ ($\geqslant 0$) and λ satisfy Eqs. (26) and (32), respectively. But if these conditions actually hold, we can say much more. The reason will become apparent in the course of the subsequent section.

2·4. – Let $f(x)$ and λ satisfy the above mentioned conditions, so that for any positive integers m and n

$$(42) \qquad 0 < V_{2m+1}(x) < \underline{V}(x) \leqslant \overline{V}(x) < V_{2n}(x) < V_0(x) \quad \text{everywhere.}$$

Suppose further that

$$(43) \qquad K \equiv 3\lambda^2 \max_x \int G(x - x_0) V_0(x_0)^2 \, \mathrm{d}x_0 < 1 .$$

This can always be achieved by suitably chosing λ, since $\max_x \int G(x - x_0) \cdot V_0(x_0)^2 \, \mathrm{d}x_0$ is finite and depends only on the structure of the source distribution, *but not* on λ. We have

$$V_{2n}(x) - V_{2n+1}(x) = \lambda^2 \int G(x - x_0)[V_{2n}(x_0)^2 + V_{2n}(x_0)V_{2n-1}(x_0) + V_{2n-1}(x_0)^2] + \\ + [V_{2n}(x_0) - V_{2n-1}(x_0)] \, \mathrm{d}x_0 < \\ < 3\lambda^2 \int G(x - x_0) V_0(x_0)^2 [V_{2n-2}(x_0) - V_{2n-1}(x_0)] \, \mathrm{d}x_0 ,$$

and so

$$(44) \qquad \max_x \left(V_{2n}(x) - V_{2n+1}(x)\right) < K \max_x \left(V_{2n-2}(x) - V_{2n+1}(x)\right),$$

where the quantities on both sides exist and are finite for all values of n. From (44) it follows that

$$\max_x (V_{2n} - V_{2n+1}) < K^n \max_x (V_0 - V_1),$$

where the expression on the right tends to zero as $n \to \infty$ $(K < 1)$.

Now

$$0 \leqslant \overline{V}(x) - V(x) < V_{2n}(x) - V_{2n+1}(x) \leqslant \max_x (V_{2n} - V_{2n+1}) < K^n \max_x (V_0 - V_1).$$

Hence, for all values of x,

$$(45) \qquad \overline{V}(x) = V(x) = V(x),$$

and so both limits $\overline{V}$ and V coincide. Further, since

$$0 \leqslant V_{2n}(x) - V(x) \leqslant V_{2n}(x) - V_{2n+1}(x) < K^n \max_x (V_0 - V_1),$$

where the last expression on the right tends to zero as $n \to \infty$ and does not depend on x, the sequence $V_{2n}(x)$ tends to the limit $V(x)$ *uniformly* with regard to x throughout the space, and so

$$(46) \quad \lim_{n\to\infty} \int G(x - x_0) V_{2n}(x_0)^3 \, dx_0 = \int G(x - x_0) \left\{\lim_{n\to\infty} V_{2n}(x_0)^3\right\} dx_0 = \int G(x - x_0) V(x_0)^3 \, dx_0 \quad (*).$$

But, by Eqs. (36) and (45)

$$(47) \qquad V(x) = \lim_{n\to\infty} V_{2n+1}(x) = V_0(x) - \lambda^2 \lim_{n\to\infty} \int G(x - x_0) V_{2n}(x_0)^3 \, dx_0 .$$

Hence, by Eqs. (46) and (47),

$$V(x) = V_0(x) - \lambda^2 \int G(x - x_0) V(x_0)^3 \, dx_0$$

and $V(x)$ is *the* solution of problem (P). We thus have the following

Theorem 3. If the source distribution $f(x)$ is non-negative, if $f(x) =$

(*) Note that $\int G(x - x_0)(V_{2n}^2 + V_{2n}V + V^2)dx < 3 \max_x V_0(x)^2$, and so the uniformity of the convergence allows us to assert that $\lim_{n\to\infty} \int = \int \lim_{n\to\infty}$.

$= 0(\exp[-r]/r^\beta)$ with $\beta > 2$ as $r = |x| \to \infty$, uniformly with regard to the direction, and if

$$\lambda^2 < \inf\left(\inf_x \frac{V_0(x)}{\int G(x-x_0)V_0(x_0)^3\,\mathrm{d}x_0}, \frac{1}{3\max\limits_x \int G(x-x_0)V_0(x_0)^2\,\mathrm{d}x_0}\right),$$

where $V_0(x) = \int G(x-x_0)f(x_0)\,\mathrm{d}x_0$, the solution of problem (P) exists and is given by the common limit of the sequences $\{V_{2n+1}(x)\}$ and $\{V_{2n}(x)\}$, with the functions $V_n(x)$ determined in order by Eq. (36).

Consider now the integral

$$H_n = \tfrac{1}{2} g^2 \int (fV_n + \tfrac{1}{2}\lambda^2 V_n^4)\,\mathrm{d}x\,, \tag{48}$$

with n even or odd. This integral exists for all values of n, since $0 < V_n(x) < V_0(x)$ and the integral H_0 Eq. (34) is convergent. Now from Eqs. (42) and (45) it follows that

$$H_{2n} < H < H_{2n+1}\,,$$

where H_{2n} increases monotonically and H_{2n+1} decreases monotonically as $n \to \infty$. On the other hand

$$0 < H_{2n+1} - H_{2n} = \tfrac{1}{2} g^2 \int [f + \tfrac{1}{2}\lambda^2(V_{2n}^2 + V_{2n+1}^2)(V_{2n} + V_{2n+1})](V_{2n} - V_{2n+1})\,\mathrm{d}x <$$

$$< \left\{\tfrac{1}{2} g^2 \int (f + 2\lambda^2 V_0^3)\,\mathrm{d}x\right\}\cdot \max_x (V_{2n} - V_{2n+1})\,.$$

Hence, the integral $\frac{1}{2}g^2\int(f + 2\lambda^2 V_0^3)\,\mathrm{d}x$ being convergent,

$$\lim_{n\to\infty} H_{2n} = H = \lim_{n\to\infty} H_{2n+1}$$

and the error $H - H_{2n}$ (or $H_{2n+1} - H$) cannot be greater than a well defined quantity, namely

$$0 < H - H_{2n} < \left\{\tfrac{1}{2} K^n g^2 \int (f + 2\lambda^2 V_0^3)\,\mathrm{d}x\right\}\cdot \max_x (V_0 - V_1)\,.$$

As in all cases when one has to do with ***sufficient*** conditions, the requirements to be met are perhaps too stringent from the physical point of view, particularly those to be met by λ. In fact, the series of iterations Eq. (36) may converge, for a particular source distribution $f(x)$, even when some of the sufficient conditions are not satisfied. In other words, our conclusions are too « pessimistic ». But we know by now that, at ***least*** for sufficiently small values of λ, the iteration procedure will lead to the solution of our problem.

2·5. – We shall in the following assume that $f(x)$ and λ are such that the considerations of the preceeding section apply. It follows from the definition Eq. (36) that $V_n(x)$ is a polinomial in λ^2 of degree $\nu(n) = (3^n - 1)/2$,

$$V_n(x) = \sum_{m=0}^{\nu(n)} \lambda^{2m} V_n^{(m)}(x) . \tag{49}$$

The coefficients $V_n^{(m)}(x)$, which could be called *partial potentials* of the n-tieth approximation, have several properties that are readily proved by induction from Eqs. (36) and (38) and the properties assumed for the source distribution $f(x)$ [see *Theor.* 3]. Without going into the details of the proofs, we shall merely state these properties.

a) $V_n^{(0)}(x) \equiv V_0(x)$ for all values $n \geqslant 0$ and

$$V_n^{(1)}(x) \equiv V^{(1)}(x) \equiv -\int G(x-x_0) V_0(x_0)^3 \, \mathrm{d}x_0 \text{ for all } n \geqslant 1.$$

b) $V_n^{(m)}(x) \equiv 0$ for $m > \nu(n)$, by definition Eq. (49).

c) Given the partial potentials of the n-tieth approximation, those of the $(n+1)$-tieth approximation are given by

$$V_{n+1}^{(m)}(x) = -\int \mathrm{d}x_0 \, G(x-x_0) \sum_{m_1+m_2+m_3=m-1} V_n^{(m_1)}(x_0) V_n^{(m_2)}(x_0) V_n^{(m_3)}(x_0) , \quad (m \geqslant 1) \tag{50}$$

where (m_1, m_2, m_3) is *any* partition of $m-1$ into three sumands, all of them non-negative integers, and the $\sum$ is extended to all these partitions.

d) $V_n^{(m)}(x)$ is either identically zero or has the same sign as $(-1)^m$ for all values of n.

e) $V_n^{(m)}(x)$ does not depend on n for $n \geqslant m$ and is equal to $V_m^{(m)}(x)$. For simplicity we shall write $V^{(m)}(x)$ for this common value.

f) $|V_n^{(m)}(x)| < |V_{n+1}^{(m)}(x)|$ for all values $n < m$. In other words, in going from the n-tieth approximation to the approximation of order $n+1$, the absolute values of the partial potentials increase if $m > n$, and remain constant if $m \leqslant n$. In particular we find in the $(n+1)$-tieth approximation non-zero partial potentials which were identically zero in the preceding approximation, namely those for which $\nu(n) < m \leqslant \nu(n+1)$.

g) In any approximation and if $V_n^{(m)}(x)$ is not identically zero, then

$$V_n^{(m)}(x) \sim \frac{e^{-R}}{4\pi R} F_n^{(m)}(\boldsymbol{n}) \quad \text{u.d. as} \quad |x| \to \infty \quad \left(R \equiv |x|, \boldsymbol{n} \equiv \frac{\boldsymbol{R}}{R}\right), \tag{51}$$

with $F_n^{(m)}(\boldsymbol{n})$ a continuous function on the surface of the sphere $|\boldsymbol{n}|=1$. This result follows immediately from the *Lemma* in § 2·1, the asymptotic behavior of $f(x)$, and Eqs. (49) and (50).

Summarizing, the n-tieth approximation $V_n(x)$ is of the form

$$V_n(x) = V_0(x) + \lambda^2 V^{(1)}(x) \ldots + \lambda^{2n} V^{(n)}(x) + \lambda^{2(n+1)} V_n^{(n+1)}(x) + \\ + \ldots + \lambda^{2\nu(n)} V_n^{(\nu(n))} . \tag{52}$$

The partial potentials are alternately positive and negative, the first $V_0(x)$ being positive, and all behave as described by Eq. (51) as $|x|\to\infty$. The $n+1$ first partial potentials are common to the n-tieth approximation and all the subsequent approximations. It can easily be seen that this partial potentials are the $n+1$ first coefficients of a *formal* expansion of $V(x, \lambda)$ in powers of λ^2. The first partial potentials are as follows:

$$\left\{\begin{aligned}
V_0(x) &= \int G(x-x_0) f(x_0)\,\mathrm{d}x_0 > 0 . \\
V^{(1)}(x) &= -\int G(x-x_0) V_0(x_0)^3\,\mathrm{d}x_0 < 0 , \\
V^{(2)}(x) &= -3\int G(x-x_0) V_0(x_0)^2 V^{(1)}(x_0)\,\mathrm{d}x_0 > 0 , \\
V_2^{(3)}(x) &= -3\int G(x-x_0) V_0(x_0) V^{(1)}(x_0)^2\,\mathrm{d}x_0 < 0 , \\
V_2^{(4)}(x) &= -\int G(x-x_0) V^{(1)}(x_0)^3\,\mathrm{d}x_0 > 0 , \\
V^{(3)}(x) &= -3\int G(x-x_0)\left\{V_0(x_0)^2 V^{(2)}(x_0) + V_0(x_0) V^{(1)}(x_0)^2\right\} < 0 , \\
V_3^{(4)}(x) &= -\int G(x-x_0)\left\{3V_0(x_0)^2 V_2^{(3)}(x_0) + 6V_0(x_0)V^{(1)}(x_0)\cdot\right. \\
&\qquad \left.\cdot V^{(2)}(x_0) + V^{(1)}(x_0)^3\right\}\mathrm{d}x_0 > 0 .
\end{aligned}\right. \tag{53}$$

The number of terms in each approximation increases very rapidly with n and the complexity of each term increases considerably with m. The present approach is thus not very promising from the practical point of view.

2·6. – Let us now introduce the expression Eq. (49) for $V_n(x)$ into the integral Eq. (48). We get for the energy in the n-tieth approximation

$$(54)\qquad H_n = -\frac{1}{2} g^2 \int f(x)\, G(x-x_0)\, f(x_0)\, \mathrm{d}x\, \mathrm{d}x_0 +$$

$$+\frac{1}{2} g^2 \sum_{m=1}^{4\nu(n)+1} \lambda^{2m} \int \mathrm{d}x \Big\{ V_0(x) \sum_{m_1+m_2+m_3=m-1} V_{n-1}^{(m_1)}(x)\, V_{n-1}^{(m_2)}(x)\, V_{n-1}^{(m_3)}(x) -$$

$$-\frac{1}{2} \sum_{m_1+m_2+m_3+m_4=m-1} V_n^{(m_1)}(x)\, V_n^{(m_2)}(x)\, V_n^{(m_3)}(x)\, V_n^{(m_4)}(x) \Big\}.$$

Note that the first sum will be zero for all values of m such that $\nu(n) < m \leqslant 4\nu(n)+1$ and that, as we said before, $V_n^{(m_i)} \equiv 0$ ($V_{n-1}^{(m_i)} \equiv 0$) if $m_i > \nu(n)$ ($m_i > \nu(n-1)$). The explicit forms of the two first approximations are

$$(55)\qquad H_0(x) = -\frac{1}{2} g^2 \int f(x)\, G(x-x_0)\, f(x_0)\, \mathrm{d}x\, \mathrm{d}x_0 - \frac{1}{4} g^2 \lambda^2 \int V_0(x)^4\, \mathrm{d}x ,$$

$$(56)\qquad H_1(x) = -\frac{1}{2} g^2 \int f(x)\, G(x-x_0)\, f(x_0)\, \mathrm{d}x\, \mathrm{d}x_0 + \frac{1}{4} g^2 \lambda^2 \int V_0(x)^4\, \mathrm{d}x -$$

$$- g^2 \lambda^4 \int V_0(x)^3 V^{(1)}(x) \mathrm{d}x - \frac{3}{2} g^2 \lambda^6 \int V_0(x)^2 V^{(1)}(x)^2\, \mathrm{d}x -$$

$$- g^2 \lambda^8 \int V_0(x)\, V^{(1)}(x)^3\, \mathrm{d}x - \frac{1}{4} g^2 \lambda^{10} \int V^{(1)}(x)^4\, \mathrm{d}x .$$

Observe that in going from H_0 to H_1 the coefficient of λ^2 changes sign. As we shall presently see, the term in λ^2 in H_0 gives rise to an attraction, and so the term in λ^2 in the approximation H_1 shall determine a repulsion. As a matter of fact, the coefficient of λ^{2m} for given m will vary with n until $m \leqslant \nu(n-1)+1$. From this critical value of n on, the coefficient of λ^{2m} does not depend on n. For example, the coefficient of λ^2 will be the same as in Eq. (56) for all subsequent values of n, and the coefficient of λ^4 is equal to

$$\frac{1}{2} g^2 \int V_0(x)^3 V^{(1)}\, \mathrm{d}x$$

from $n = 2$ on.

2·7. – Suppose now that the source distribution $f(x)$ is of the form Eq. (3), namely

$$f(x) = \sum_{i=1}^{A} \varrho(x - x_i) ,$$

with the function $\varrho(x - x_i)$ as defined in § **1·1** and $\varrho(x) = 0(\exp[-r]/r^\beta)$ at least ($\beta > 2$), so that also $f(r) = 0(\exp[-r]/r^\beta)$. We shall have

$$\int f(x)\,G(x-x_0)f(x_0)\,\mathrm{d}x\,\mathrm{d}x_0 = \sum_{i=1}^{A}\int \varrho(x-x_i)\,G(x-x_0)\,\varrho(x_0-x_i)\,\mathrm{d}x\,\mathrm{d}x_0 + \\ + 2\sum_{i<j}^{A}\int \varrho(x-x_i)\,G(x-x_0)\,\varrho(x_0-x_j)\,\mathrm{d}x\,\mathrm{d}x_0\,, \tag{57}$$

$$\int V_0(x)^4\,\mathrm{d}x = \sum_{i=1}^{A}\int V_0(x-x_i)^4\,\mathrm{d}x + \tag{58}$$

$$+ 2\sum_{i<j}^{A}\int \big\{2V_0(x-x_i)^3V_0(x-x_j) + \\ + 2V_0(x-x_i)V_0(x-x_j)^3 + 3V_0(x-x_i)^2V_0(x-x_j)^2\big\}\,\mathrm{d}x + \\ + 12\sum_{i<j<k}^{A}\int \big\{V_0(x-x_i)^2V_0(x-x_j)V_0(x-x_k) + V_0(x-x_i)V_0(x-x_j)^2V_0(x-x_k) + \\ + V_0(x-x_i)V_0(x-x_j)V_0(x-x_k)^2\big\}\,\mathrm{d}x + \\ + 24\sum_{i<j<k<l}^{A}\int \big\{V_0(x-x_i)V_0(x-x_j)V_0(x-x_k)V_0(x-x_l)\big\}\mathrm{d}x\,,$$

where

$$V_0(x-x_i) \equiv \int G(x-x_0)\,\varrho(x_0-x_i)\,\mathrm{d}x_0\,.$$

The first term in Eqs. (57) and (58) are self-energy terms and are the only ones which do not tend to zero when the nucleons are an infinite distance apart each other. But the potential energy of the system of nucleons in each approximation is given by the difference $H_n - H_n(\infty)$, where $H_n(\infty)$ refers to the limit of H_n when *all* the distances between nucleons tend to infinity. The self-energy terms will therefore drop out, since they do not depend on the mutual distances. The second term in Eq. (57) represents the ordinary Yukawa two-body forces. A second kind of two-body forces proportional to $g^2\lambda^2$ is given by the second term Eq. (58). To be more precise, one would say that these forces are of two types as given by the sum of the first two terms and the third term of the integral, respectively. There is only one type of 3-body forces in this approximation and the same holds for the 4-body forces. It was assumed, of course, that $A \geqslant 4$.

We see therefore that in the approximation H_0 we have to deal with 2, 3 and 4-body forces, all of them attractive. The same forces contribute to the H_1 approximation, but are then repulsive as pointed out before. New types of many body forces of increasing degree of complexity appear in the H_1 approximation. Even for the simplest case, that of the term in λ^4, one finds three types of 2-body forces, three types of 3-body forces, two types of 4-body forces, one type of 5-body forces and one type of 6-body forces. The results are too involved to be quoted here. All these forces are attractive in the H_1 approximation and repulsive in the H_2 approximation. However, in the latter case the absolute value is $\frac{1}{2}$ of the absolute value in the H_1 approximation. Absolute value and the repulsive character are conserved in all subsequent approximations. That is, the effect of both terms, that proportional to λ^2 and that proportional to λ^4, is to diminish the attraction of the ordinary Yukawa two-body forces from the third approximation on.

The author is greatly indebted to Professor W. HEISENBERG, who suggested this problelem and offered helpful advice during its study.

APPENDIX I

Proof of the Lemma on page 145.

Take for simplicity the origin of the coordinate system at the point O. By a suitable choice of the constant R_0, we can secure, by Eq. (26), that for all values of x such that $|x| \geqslant R_0$ we shall have $|f(x)| < K(e^{-r}/r^\beta)$, where K is a positive constant independent of the direction of the vector x. Take $R \geqslant 2R_0$ and write

$$\text{(A)} \qquad Re^R \int G(x - x_0) f(x_0)\, \mathrm{d}x_0 = Re^R \int_0^{R_0} r^2 \mathrm{d}r \int_\Omega \frac{e^{-\varrho}}{4\pi\varrho} f(x_0)\, \mathrm{d}\Omega +$$

$$+ Re^R \int_{R_0}^{R} r^2 \mathrm{d}r \int_\Omega \frac{e^{-\varrho}}{4\pi\varrho} f(x_0)\, \mathrm{d}\Omega + Re^R \int_R^\infty r^2\, \mathrm{d}r \int_\Omega \frac{e^{-\varrho}}{4\pi\varrho} f(x_0)\, \mathrm{d}\Omega,$$

where $R \equiv |x|$ and $\varrho \equiv |x - x_0|$. Call for brevity I_1, I_2 and I_3 the first, second and third term on the right, respectively. It can be shown that in I_1

$$\frac{\exp[R - \varrho]}{4\pi(\varrho/R)} \to \frac{\exp[r \cos\psi]}{4\pi}, \qquad (r \equiv |x_0|)$$

as $R \to \infty$, uniformly with regard to x_0 throughout the sphere E_0 of center at the origin and radius R_0. ψ is the angle between the direction $\boldsymbol{n}$ along which x tends to infinity and that of the vector x_0. Now, by hypothesis,

$|f(x_0)|$ is integrable in E_0. Hence $\lim_{R\to\infty} I_1$ exists and is equal to

$$\frac{1}{4\pi}\int_0^{R_0} r^2\,\mathrm{d}r\int_\Omega \exp[r\cos\psi]f(x_0)\,\mathrm{d}\Omega\,.$$

Consider now I_2. It is easily proved from Eq. (26) that $F(R_0)\equiv\lim_{R\to\infty} I_2$ exists and is such that $|F(R_0)|<(K/(\beta-2))R_0^{2-\beta}$. Finally, $|I_3|<(K/2(\beta-2))R^{2-\beta}$ and, since $\beta>2$, $I_3\to 0$ as $R\to\infty$. We can thus assert that I_1, I_2 and I_3 tend to a finite limit as $R\to\infty$. Hence the expression on the left, Eq. (A), has a limit, too:

$$\text{(B)}\qquad \lim_{R\to\infty}\left\{R\,e^R\int G(x-x_0)f(x_0)\,\mathrm{d}x_0\right\}=\frac{1}{4\pi}\int_{E_0}\exp[r\cos\psi]f(x_0)\,\mathrm{d}x_0+F(R_0)\,.$$

Now, the quantity on the left, Eq. (B), does not depend on R_0. We can therefore take the limit on the right and we get

$$\lim_{R\to\infty}\left\{R\,e^R\int G(x-x_0)f(x_0)\,\mathrm{d}x_0\right\}=\frac{1}{4\pi}\int \exp[r\cos\psi]f(x_0)\,\mathrm{d}x_0\,.$$

since $\lim_{R\to\infty} F(R_0)=0$, and where the integral on the right is extended to all the space. This completes the proof of the *Lemma*.

APPENDIX II

If $f(x)\geqslant 0$ *is at least* $0(e^{-r}/r^\beta)$, *with* $\beta>2$, *uniformly with regard to the direction as* $r\equiv|x|\to\infty$, *then*

$$m\equiv\inf_x\frac{V_0(x)}{\int G(x-x_0)V_0(x_0)^3\,\mathrm{d}x_0}>0\,,$$

where $V_0(x)$ and $G(x-x_0)$ are defined as in Eq. (25).

To see this observe first that

$$Q(x)\equiv\frac{V_0(x)}{\int G(x-x_0)V_0(x_0)^3\,\mathrm{d}x_0}\,,$$

is a bounded continuous function of x throughout any finite domain and that, if $f(x)\geqslant 0$, $V_0(x)$ cannot be zero, so that $Q(x)>0$ for all finite values of x. Now, by the *Lemma* above, $f(x)=0(e^{-R}/R^\beta)$, with $R\equiv|x|$ and $\beta>2$, entails $V_0(x)^3=0(e^{-3R}/R^3)$, so that $V_0(x)^3=O(e^{-R}/R^3)$. But then, by the same lemma (see Note on p. 145),

$$\lim_{R\to\infty} Q(x) = \lim_{R\to\infty} \frac{V_0(x)}{\int G(x-x_0)V_0(x_0)^3\,\mathrm{d}x_0} = \frac{\int \exp[r\cos\psi]f(x_0)\,\mathrm{d}x_0}{\int \exp[r\cos\psi]V_0(x_0)^3\,\mathrm{d}x_0},$$

where the expression on the right is a function of the direction $\boldsymbol{n}$ along which x tends to infinity, everywhere *positive and continuous* on the surface $|\boldsymbol{n}| = 1$. Further, $Q(x)$ tends to its limit u.d.. Hence two positive numbers K and R_0 exist such that $Q(x) > K > 0$ for all $|x| > R_0$. Now, since $Q(x)$ is continuous and *positive* throughout the sphere $|x| \leqslant R_0$, we shall have $\min_{|x|\leq R_0} Q(x) > 0$ and therefore

$$m = \inf_x Q(x) \geqslant \inf\big(K, \min_{|x|\leq R_0} Q(x)\big) > 0.$$

RIASSUNTO (*)

Nel primo paragrafo si derivano alcune proprietà generali delle soluzioni statiche dell'equazione di Schiff dalla struttura dell'equazione non lineare e dal comportamento della distribuzione delle sorgenti. Nel secondo paragrafo si dà la prova dell'esistenza della soluzione e si derivano le forze di più corpi implicite nella teoria scalare classica dei potenziali di Schiff.

(*) *Traduzione a cura della Redazione.*