On the derivation of point neutron kinetic equations

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RESUMEN

En los últimos años ha cobrado cierta importancia el tema de la reducción de las ecuaciones cinéticas del transporte de neutrones a una forma independiente del espacio y de la energía. También se ha discutido ampliamente la ecuación de la hora-inversa a que conduce aquella forma reducida. Entre otros autores, Ussachoff (1), Henry (2, 3, 4), Cohen (5), Gross y Marable (6) y recientemente Becker (7), han deducido rigurosamente de las ecuaciones generales otras en las que la única variable independiente es el tiempo. El objeto del presente artículo es presentar un tratamiento general del problema en el que las funciones de peso pueden ser funciones no negativas, cualesquiera dentro de ciertos límites, acaso dependientes del tiempo. Los resultados más importantes están contenidos en un artículo (8) que el autor presentó en la reunión del European-American Committee on Reactor Physics celebrada en Hankø, Noruega.

ABSTRACT

In recent years, the subject of reduction of the kinetic neutron transport equations to a space-energy independent form has become of some importance. The related inhour equation has also been largely discussed. Rigurous derivations have been provided, among other authors, by Ussachoff (1), Henry (2, 3, 4), Cohen (5), Gross and Marable (6), and quite recently by Becker (7). The subject of the present paper is a general treatment of the problem, in which the weight density functions might be rather arbitrary non-negative functions and are allowed to depend on the time. The main results were already contained in an earlier paper by the author (8) presented to the Hancø Meeting of the European-American Reactor Physics Committee.

I. Notation

The basic equations describing the time behaviour of the neutron distribution for a reactor system with stationary fuel will be taken to be the following:

$$\frac{1}{v}\dot{\mathrm{F}}(\mathbf{x},\mathbf{u},t)=(\mathrm{H}_a+\mathrm{K}_s+\mathrm{K}_{\mathrm{F}_i})\,\mathrm{F}(\mathbf{x},\mathbf{u},t)+\sum_m \lambda_m f_m(u)\,\mathrm{C}_m(\mathbf{x},t)+$$

$$+\,\mathrm{Q}(\mathbf{x},\mathbf{u},t),\ [1.\mathrm{F}],\quad \dot{\mathrm{C}}_m(\mathbf{x},t)=\frac{1}{f_m(u)}\,\mathrm{K}^m_{\mathrm{F}_r}\,\mathrm{F}(\mathbf{x},\mathbf{u},t)-\lambda_m\,\mathrm{C}_m(\mathbf{x},t),\ [1.\mathrm{C}]$$

where: $\mathrm{F}(\mathbf{x}, \mathbf{u}, t)$ is the directional flux density at the point $(\mathbf{x}, \mathbf{u})$ of the phase space ($\mathbf{u}$ is the vector lethargy, $\mathbf{u} = u\,\Omega$), and time t; $\mathrm{C}_m(\mathbf{x}, t)$ is the concentration of the m-th group of delayed neutron precursors ($m = 1, 2, \ldots, g$), and λ_m is the precursor decay constant; $\mathrm{Q}(\mathbf{x}, \mathbf{u}, t)$ is the external source density. The external source density for precursors is suppossed to be zero.

The linear operators H_a, K_s, $\mathrm{K}_{\mathrm{F}_i}$, and $\mathrm{K}^m{}_{\mathrm{F}_r}$ are defined as follows:

$$\mathrm{H}_a\,\mathrm{F}=-\,\Omega\cdot\operatorname{grad}\mathrm{F}-\Sigma_T\,\mathrm{F},\qquad[2]$$

$$\mathrm{K}_s\,\mathrm{F}=\int_{\mathrm{U}} d\,\mathrm{U}'\,\Sigma_s(\mathbf{x},\mathbf{u}'\to\mathbf{u},t)\,\mathrm{F}\,\mathbf{x},\mathbf{u}',t),\quad(d\,\mathrm{U}'=d\,u'\,d\,\Omega')\qquad[3]$$

$$\mathrm{K}_{\mathrm{F}_i}\,\mathrm{F}=\sum_{\mu} f^{\mu}(u)\,(1-\beta^{\mu})\int_{\mathrm{U}} d\,\mathrm{U}'\,\nu^{\mu}(u')\,\Sigma^{\mu}_{\mathrm{F}}(\mathbf{x},u',t)\,\mathrm{F}(\mathbf{x},\mathbf{u}',t)\qquad[4]$$

$$\mathrm{K}^m_{\mathrm{F}_r}\,\mathrm{F}=\sum_{\mu} f_m(u)\,\beta^{\mu}_m\int_{\mathrm{U}} d\,\mathrm{U}'\,\nu^{\mu}(u')\,\Sigma^{\mu}_{\mathrm{F}}(\mathbf{x},u',t)\,\mathrm{F}(\mathbf{x},\mathbf{u}',t),\qquad[5]$$

Here $\Sigma_T = \Sigma_e + \Sigma_\mathrm{F} + \Sigma_s$ is the total macroscopic cross-section; $\Sigma_s(\mathbf{x}, \mathbf{u}' \to \mathbf{u}, t)$ is the macroscopic cross-section for the scattering process $\mathbf{u}' \to \mathbf{u}$; $\Sigma^{\mu}{}_{\mathrm{F}}(\mathbf{x}, u, t)$ is the macroscopic fission cross-section of the μ-th fissionable nuclide ($\mu = 1, 2, \ldots, \mathrm{N}$), ν^{μ} is the total

number of neutrons emitted from a fission of the μ-th isotope, $f^{\mu}(u)$ the corresponding spectrum (normalization:

$$\int_{U} d\,U\, f^{\mu}(u) = 1);$$

β^{μ}_{m} is the fraction of the total number of neutrons emitted from the m-th group of precursors due to a fission in nuclide μ; $f_m(u)$ is the spectrum of delayed neutrons from the *m-th* precursors (normalization:

$$\int_{U} d\,U\, f_m(u) = 1).$$

The linear operator Π_a is diagonal in $(\mathbf{x}, \mathbf{u})$ space. The scattering and fission operators K_s, K_{F_i}, and $K^{m}{}_{F_r}$ are diagonal in $\mathbf{x}$-space, but integral operators in $\mathbf{u}$-space. It will be convenient to introduce the (total) fission operator

$$K_F = K_{F_i} + \sum_{m} K^{m}_{F_r}, \qquad [6]$$

as well as the operator

$$\mathbf{L} = \Pi_a + K_s + K_F. \qquad [7]$$

II. The general weighting procedure

Let D be the domain of $(\mathbf{x}, \mathbf{u})$ space defined by the reactor system, R, and the domain, U, of lethargy space, D = R × U, S the (spatial) boundary of R. The directional flux density F is a non-negative function with domain D which must satisfy adecuate boundary conditions on S (e. g., $F(\mathbf{x}, \mathbf{u}\, t) = 0$ if $\mathbf{x} \in S$, $\mathbf{n} \cdot \Omega < 0$, $\mathbf{n}$ being the outward normal vector on S at $\mathbf{x}$) and continuity conditions in R. Let {F} be the class of all functions with domain D which satisfy the same boundary and continuity conditions as F, and {F+} the class of all functions with domain D which satisfy the adjoint

boundary conditions (e. g., $F^+(\mathbf{x}, \mathbf{u}, t) = 0$ if $\mathbf{x} \in S$, $\mathbf{n} \cdot \Omega > 0$) and the same continuity conditions as F. Finally, let $W^+(\mathbf{x}, \mathbf{u}/t)$ be a non-negative function of class $\{F^+\}$, depending parametrically on the time t in some interval $0 \leqslant t \leqslant T$. The function W^+ defines a linear functional on $\{F\}$.

$$W^+[F] = \int_a W^+(\mathbf{x}, \mathbf{u}/t)\, d\,V\, d\,U\, \frac{1}{v} F(\mathbf{x}, \mathbf{u}, t) \qquad [8]$$

which will be called the *total weight* of the neutron population, $N(\mathbf{x}, \mathbf{u}, t) = \frac{1}{v} F(\mathbf{x}, \mathbf{u}, t)$, at the time t (*). $W^+[F]$ is positive for all non-negative F, and $W^+(\mathbf{x}, \mathbf{u}/t)$ is called the directional *weight density* or, simply, the *neutron weight*. The local destruction rate of neutron weight at $(\mathbf{x}, \mathbf{u})$ is obviously given by

$$-W^+(\mathbf{x}, \mathbf{u}/t)\, \Pi_a F(\mathbf{x}, \mathbf{u}, t)$$

and the scattering creation rate by

$$W^+(\mathbf{x}, \mathbf{u}/t)\, K_s F(\mathbf{x}, \mathbf{u}, t).$$

Therefore, for the total destruction rate of neutron weight we find

$$p_W = \int_D W^+\, d\,V\, d\,U\, (\Omega \cdot \mathbf{grad} + \Sigma_a) F +$$

$$+ \int_D F(\mathbf{x}, \mathbf{u}, t)\, d\,V\, d\,U \int_U d\,U'\, \Sigma_s(\mathbf{x}, \mathbf{u} \to \mathbf{u}', t))\} W^+(\mathbf{x}, \mathbf{u}'/t) - W^+(\mathbf{x}, \mathbf{u}/t)\} \qquad [9]$$

Thus, only for those weight function which are $\mathbf{u}$-independent the destruction rate can be characterized by the operator $\Omega \cdot \mathrm{grad} + \Sigma_a$, that is, by leakage and absorption. This result is trivial from the physical point of view, for only when W^+ does not depend on $\mathbf{u}$

(*) To avoid unnecessary details it is supposed that all the mathematical objects concerned do exist, in particular that the integrals do exist and the operations are valid.

the scattering process is irrelevant for the weighting procedure. In particular, by taking $W^+ = 1$ (*) the total weight reduces to the total number of neutrons in R, and since the scattering process does not modify the neutron number, the characteristic destruction operator is defined by $\Omega \cdot \text{grad} + \Sigma_a$.

Now, let us suppose we are given a weight W^+. In general, W^+ will depend on the time. The directional flux density $F(\mathbf{x}, \mathbf{u}, t)$ can always be factorized into a function $\varphi(\mathbf{x}, \mathbf{u}, t)$ and a function $T(t)$, and this is an infinity of manners. But, for a given W^+ (non-negative) the function $T(t)$-and, therefore, $\varphi(\mathbf{x}, \mathbf{u}, t)$- is uniquely determined by the conditions

$$\left(W^+, \frac{1}{v}\dot{\varphi}\right) = 0, \quad T(0) = 1, \qquad [10]$$

where (g^+, f) represents the inner product (in D) (**):

$$(g^+, f) = \int_D g^+(\mathbf{x}, \mathbf{u})\, d\,V\, d\,U f(\mathbf{x}, \mathbf{u}).$$

To prove this, we observe from the first condition [10] and $F = \varphi T$ that

$$T\left(W^+, \frac{1}{v}\dot{F}\right) = \dot{T}\left(W^+, \frac{1}{v}F\right),$$

and since $\left(W^+ \frac{1}{v}F\right) > 0$ for all values of $t \geqslant 0$, we shall have

$$\frac{\dot{T}}{T} = \frac{\left(W^+, \frac{1}{v}\dot{F}\right)}{\left(W^+, \frac{1}{v}T\right)}.$$

(*) $W+ = 1$, $W+ \in \{F+\}$. means: a) $W+ = 1$ in R, all $\mathbf{u}$; b) $W+ = 1(0)$ on S and for those $\mathbf{u} = u\,\Omega$ such that $\mathbf{n} \cdot \Omega < 0$ $(\mathbf{n} \cdot \Omega > 0)$.

(**) All functions are supposed to be real.

Therefore

$$T(t) = \exp \int_0^t d\tau \frac{\left(W^+, \frac{1}{v}\dot{F}\right)}{\left(W^+, \frac{1}{v}F\right)}, \qquad [11]$$

It is important to observe that this expression for $T(t)$ is invariant under any gauge transformation of the form

$$W^+ \to w^+(t)\, W^+, \quad F \to a\, F, \qquad [12]$$

where $w^+(t)$ is any positive function of t and a is an arbitrary constant.

The factorized form of $F(\mathbf{x}, \mathbf{u}, t) = \varphi(\mathbf{x}, \mathbf{u}, t)\, T(t)$, with $\varphi(\mathbf{x}, \mathbf{u}, t)$ defined by

$$\varphi(\mathbf{x}, \mathbf{u}, t) = \frac{F(\mathbf{x}, \mathbf{u}, t)}{T(t)} \qquad [13]$$

and $T(t)$ given by [11], will be called the *normalized shape* of $F(\mathbf{x}, \mathbf{u}, t)$ with respect to the weight W^+, and $T(t)$ will be called the *normalized amplitude*.

We now proceed to establish another important property. We have just seen that $T(t)$, and therefore the normalized shape, is invariant under the gauge transformation $W^+ \to w^+(t)\, W^+$. As a consequence, we shall prove that it is always possible to choose $w^+(t)$ in such a way that the weight $W_n^+ = w^+(t)\, W^+$ satisfies the identity

$$\left(\dot{W}_n^+, \frac{1}{v}\varphi\right) = 0. \qquad [14]$$

For, if we substitute the product $w^+(t)\, W^+$ in [14], the condition that the equation should be satisfied becomes

$$\dot{w}^+\left(W^+, \frac{1}{v}\varphi\right) + w^+\left(\dot{W}^+, \frac{1}{v}\varphi\right) = 0.$$

It follows that the condition to be satisfied reduces to

$$\frac{\dot{w}^+}{w^+} = -\frac{\left(\dot{W}^+, \frac{1}{v}\varphi\right)}{\left(W^+, \frac{1}{v}\varphi\right)}.$$

Hence

$$w^+(t) = \exp \int_t^T d\tau \frac{\left(\dot{W}^+, \frac{1}{v}\varphi\right)}{\left(W^+, \frac{1}{v}\varphi\right)}. \qquad [15]$$

The weight $W_n^+ = w^+(t)\, W^+$, with $w^+(t)$ given by [15], will be called the *normalized weight* with respect to the flux density F. From the relations [10] and [14] it follows at once that:

The total normalized weight of the neutron population distributed according to the normalized shape is a constant of the motion.

For, differentiating $\left(W_n^+, \frac{1}{v}\varphi\right)$ with respect to the time, we get

$$\frac{d}{dt}\left(W_n^+, \frac{1}{v}\varphi\right) = \left(\dot{W}_n^+, \frac{1}{v}\varphi\right) + \left(W_n^+, \frac{1}{v}\dot{\varphi}\right) = 0. \qquad [16]$$

This is the result which had to be proved.

Consequently, we see that the total normalized weight of the neutron population $\frac{1}{v} F(\mathbf{x}, \mathbf{u}, t)$ depends on the time t only through the normalized amplitude $T(t)$ defined by [11]:

$$\left(W_n^+, \frac{1}{v} F\right) = \left(W_n^+, \frac{1}{v}\varphi\right) T(t), \qquad [17]$$

with $\left(W_n^+, \frac{1}{v}\varphi\right) = \text{Const.}$ The relations [16] and [17] generalize the conservation requirement postulated by Henry (2, Eq. 5) to the case of a general, time depending weight density.

Now, let $P_m^+(\mathbf{x}/t)$ be g non-negative functions ($m = 1, 2, \ldots, g$) with domain R, depending parametrically on the time t ($0 \leqslant t \leqslant T$)

and satisfying in R the usual continuity conditions. The linear functional

$$P_m^+ [C_m] = \int_R P_m^+ (\mathbf{x}/t)\, d\,V\, C_m (\mathbf{x}, t) \qquad [18]$$

will be called the *total weight* of the m-th group of delayed neutron precursors population at the time t. The same as $W^+ [F]$, the total weigth $P_m{}^+ [C_m]$ is positive for all non-negative C_m, and $P_m{}^+ (\mathbf{x}/t)$ is called the weight density of the m-th group of delayed neutron precursors, or simply the *m-precursor weight*. The weights $P_m{}^+ (\mathbf{x}/t)$ are for the moment quite arbitrary, *given* functions. If we represent the inner product (in R) by

$$\{ g^+, f \{ = \int_R g^+ (\mathbf{x})\, d\,V f (\mathbf{x}),$$

the linear functional [18] may be written

$$P_m^+ [C_m] = \left\{ P_m^+,\ C_m \right\}. \qquad [18']$$

The notions of normalized shapes and normalized amplitudes may now be extended to cover the precursor concentrations. Thus, we will have the normalized amplitudes $c_m (t)$ defined by (cf. equation [11])

$$c_m (t) = \exp \int_0^t d\,\tau \frac{\left\{ P_m^+,\ \dot{C}_m \right\}}{\left\{ P_m^+,\ C_m \right\}} \qquad [19]$$

and the normalized shapes

$$\gamma_m (\mathbf{x}, t) = \frac{1}{c_m (t)} C_m (\mathbf{x}, t) \qquad [20]$$

all with the same properties as $T (t)$ and $\varphi (\mathbf{x}, \mathbf{u}, t)$, respectively. In particular, the amplitudes $c_m (t)$ are invariant under any gauge transformation (cf. Eq [12])

$$P_m^+ \to p_m^+ (t)\, P_m^+, \qquad C_m \to a_m C_m, \qquad [21]$$

where $p_m^+(t)$ is any positive function of t and a_m a constant ($m = 1, 2, \ldots, g$). As a consequence, to each given set of precursor weights corresponds a set of *normalized precursor weights* $P_{m,n}$ such that

$$\left\{ \dot{P}^+_{m,n}, \gamma_m \right\} = 0. \qquad [22]$$

III. The general kinetic equations

We are now able to proceed to establish the basic formulation of the space-lethargy independent kinetic equations. The characteristic reactor operators are supposed to depend on the time, as well as the weight density, in general. We write the first Boltzmann equation [1] in the form

$$\frac{1}{v}\dot{F} = L\,F - \sum_m K^m_{F,r}\,F + \sum_m f_m\,C_m + Q, \qquad [23]$$

where L is the linear operator defined in [7]. Taking the inner product of W_n^+ with the Eq. [13], we get

$$\frac{dT}{dt} = \frac{(W_n^+, L\,\varphi)}{\left(W_n^+, \frac{1}{v}\varphi\right)}\,T - \sum_m \frac{(W_n^+, K_F^m\,\varphi)}{\left(W_n^+, \frac{1}{v}\varphi\right)}\,T + \\ + \sum_m \frac{(W_n^+, f_m\,C_m)}{\left(W_n^+, \frac{1}{v}\varphi\right)} + \frac{(W_n^+, Q)}{\left(W_n^+, \frac{1}{v}\varphi\right)}, \qquad [24]$$

Let the magnitudes $\bar{\rho}$, $\bar{\beta}_m$ and $\bar{l}$ be defined by the relations

$$\frac{\bar{\rho}}{\bar{l}} = \frac{(W_n^+, L\,\varphi)}{\left(W_n^+, \frac{1}{v}\varphi\right)}, \qquad \frac{\bar{\beta}_m}{\bar{l}} = \frac{(W_n^+, K^m_{F,r}\,\varphi)}{\left(W_n^+, \frac{1}{v}\varphi\right)}. \qquad [25]$$

It is convenient to observe that: a) the ratios $\bar{\rho}/\bar{l}$ and $\bar{\beta}_m/\bar{l}$ depend only on the shape φ and on the weight W^+, being independent of the normalization factor $w^+(t)$; b) in fact the validity of the re-

duction of the first transport equation [1] to the space-energy independent form [24] depends only on [10]. The same is true for the magnitudes

$$c'_m = \frac{(W_n^+, f_m C_m)}{\left(W_n^+, \frac{1}{v}\varphi\right)}, \qquad q = \frac{(W_n^+, Q)}{\left(W_n^+, \frac{1}{v}\varphi\right)}. \qquad [26]$$

In other words, the subscript n can be dropped in Eqs. [24], [25], and [26].

Let us introduce

$$N_\mu(\mathbf{x}, t) = \int_u \nu^\mu(u)\, \Sigma_F^\mu(\mathbf{x}, u, t)\, d\,U\, \varphi(\mathbf{x}, \mathbf{u}, t) \qquad [27]$$

and

$$W_m^+(\mathbf{x}/t) = \int_u W^+(\mathbf{x}, \mathbf{u}/t)\, d\,U\, f_m(u). \qquad [28]$$

Given the neutron weight W^+, Eq. [28] defines a family of precursor weights —the precursor weights associated to the given neutron weight W^+. A clue to the significance of $W_m^+(\mathbf{x}/t)$ can be obtained by consideration of the equations to which obey the adjoint precursor concentrations (*). As is well known, in the absence of an importance source for the m-group of delayed neutron emitters (m-precursor), the adjoint precursor concentration C_m^+ satisfies the adjoint equation

$$\dot{C}_m^+(\mathbf{x}, t) = -\lambda_m \int_u F^+(\mathbf{x}, \mathbf{u}/t)\, d\,U f_m(u) + \lambda_m\, C_m^+(\mathbf{x}, t),$$

where $F^+(\mathbf{x}, \mathbf{u}, t)$ is the adjoint flux (the importance). For the time-independent problem, therefore, we can write

$$C_m^+(\mathbf{x}) = \int_u F^+(\mathbf{x}, \mathbf{u})\, d\,U f_m(u).$$

(*) Concerning the adjoint equations in nuclear reactor theory, see, for example, refs. (9) (12).

From the comparison with Eq. [28] it follows that the m-precursor weight associated to the given neutron weight $W^+(\mathbf{x}, \mathbf{u}/t)$ might be *formally* interpreted as the «adjoint» m-precursor concentration determined by the neutron «importance» $W^+(\mathbf{x}, \mathbf{u}/t)$ at the time t. Of course, since $W^+(\mathbf{x}, \mathbf{u}/t)$ is not in general a solution of the time independent adjoint problem, the interpretation is only a formal one.

In terms of the associated precursor weights, the magnitudes c'_m, Eq. [26], may be written

$$c'_m(t) = b_m(t)\, c_m(t), \qquad [29]$$

where the $b_m(t)$ are defined by

$$b_m(t) = \frac{\left\{ W^+_m \cdot \gamma_m \right\}}{\left(W^+, \frac{1}{v}\varphi \right)}, \qquad [30]$$

with the normalized shapes φ and γ_m, and the normalized amplitudes c_m given by Eq. [19].

Now, denote by r the common value of the ratios

$$r = \frac{\bar{l}}{\left(W^+, \frac{1}{v}\varphi \right)} = \frac{\bar{\rho}}{(W^+, L\varphi)} = \frac{\bar{\beta}_m}{(W^+, K^m_{F_r}\varphi)}. \qquad [31]$$

The magnitudes $\bar{l}$, $\bar{\rho}$ and $\bar{\beta}_m$ are not determined by the weight and the shape only, and are altered by changes in r. On the other hand, the kinetic equation [24] is not r-depending and can be written in the quasi-conventional form

$$\dot{T}(t) = \frac{\bar{\rho} - \bar{\beta}}{\bar{l}} T + \sum_m \lambda_m b_m c_m + q. \qquad [32]$$

The kinetic equation [28] differs from the conventional form due to the presence of the factors b_m, in general a function of time.

For a given reactor system, and given the neutron weight and the shape, the value of r can be defined only by independently defining one of the magnitudes involved, for instance, the reactivity $\bar{\rho}$. In terms of $\bar{\rho}$, we get

$$\bar{l} = \bar{\rho}\,\frac{\left(W^+, \frac{1}{v}\,\varphi\right)}{(W^+, L\,\varphi)}\,, \qquad \bar{\beta}_m = \bar{\rho}\,\frac{(W^+, K^m_{F_r}\,\varphi)}{(W^+, L\,\varphi)} = \sum_{\mu} \bar{\beta}^{\mu}_m\,, \tag{33}$$

with

$$\frac{\bar{\beta}^{\mu}_m}{\bar{l}} = \frac{\left\{W^+_m, N_{\mu}\right\}}{\left(W^+, \frac{1}{v}\,\varphi\right)}\,\bar{\beta}^{\mu}_m\,. \tag{34}$$

These are the general expressions for the «formal» lifetime $\bar{l}$ and m-th delayed neutron fraction, $\bar{\beta}_m$, independent of the weight normalization.

We have hitherto considered only the first Boltzmann equation [1] and its reduction to the (quasi) conventional form [28]. We proceed now to find the reduced form of the m-th equation for the second set of transport equations [1.C]. Taking the inner product (in R) of $P_m^+(\mathbf{x}/t)$ (the *given* m-precursor weight) with the Eq. [1.C], we get

$$\left\{P^+_m, \gamma_m\right\} c_m(t) = \left[\sum_{\mu} \left\{P^+_m, N_{\mu}\right\} \beta^{\mu}_m\right] T(t) - \lambda_m \left\{P^+_m, \gamma_m\right\} c_m(t),$$

or defining

$$g^{\mu}_m(t) = \frac{\left\{P^+_m, N_{\mu}\right\}}{\left\{P^+_m, \gamma_m\right\}}\,\frac{\left(W^+, \frac{1}{v}\,\varphi\right)}{\left\{W^+_m, N_{\mu}\right\}}\,, \tag{35}$$

$$c_m(t) = \left[\sum_{\mu} \frac{\bar{\beta}^{\mu}_m}{\bar{l}}\,g^{\mu}_m\right] T(t) - \lambda_m c_m(t). \tag{36}$$

As in the case of the first reduced kinetic equation [32], the kinetic Eq. [36] differs from the conventional one due to the presence

of some extra factors —in the present case, the coefficients g^{μ}_{m}. It is interesting to note that the definition Eq. [35] considerably simplifies in the important case in which we take $P_m^+ \equiv W_m^+$. It is then easily seen that g^{μ}_{m} does not depend on μ, and that $g_m\, b_m = 1$ for all values of m:

$$g^{\mu}_{m} = \frac{\left(W^+, \frac{1}{v}\varphi\right)}{\left\{W^+_m, \gamma_m\right\}} = b_m^{-1} \equiv g_m. \qquad [37]$$

Equation [36] may then be put in the quasi-conventional form

$$c_m(t) = \frac{\overline{\beta}_m}{\overline{l}}\, g_m\, T(t) - \lambda_m\, c_m(t). \qquad [38]$$

Under the same assumption ($P_m^+ \equiv W_m^+$), suppose that the time dependence of $W^+(\mathbf{x}, \mathbf{u}/t)$ is separable,

$$W^+(\mathbf{x}, \mathbf{u}/t) \equiv f(t)\, W_0^+(\mathbf{x}, \mathbf{u}). \qquad [39]$$

The normalized weights will then reduce to the following time-independent expressions:

$$W_n^+ = W_o^+(\mathbf{x}, \mathbf{u}), \qquad P^+_{m,n} = W^+_{o,m}(\mathbf{x}, \mathbf{u}).$$

From the definition of normalized weights it then follows that the coefficients b_m and g_m ($= b_m^{-1}$) will be constants. These constants may always be taken equal to unity, that is, Eqs. [32], [38] reduce to the conventional form. A particular case of Eq. [39] is that in which W^+ does not depend on time.

IV. Some particular cases

As we have seen, the values of $\overline{l}$ and $\overline{\beta}_m$ (Eq. [33]) depend on the definition of the «reactivity» $\overline{\rho}$. We shall take as standard value for $\overline{\rho}$ the value

$$\overline{\rho} = \frac{k_0 - 1}{k_0} = \rho, \qquad [40]$$

where k_0 is the so called static multiplication factor, i. e., the value of k for which the boundary value problem

$$\left(\Pi_a + K_s - \frac{1}{k} K_F\right) F = 0, \qquad [41]$$

$$F(\mathbf{x}, \mathbf{u}) = 0 \quad \text{if} \quad \mathbf{x} \in S \quad \text{and} \quad \mathbf{n} \cdot \boldsymbol{\Omega} < 0,$$

has a non-negative solution. We shall represent this solution by F_0. According to this definition, between F_0 and the standard reactivity ρ there exists the relation

$$L F_o = \rho K_F F_o, \qquad [42]$$

with L. defined as in [7]. The corresponding solution, F_0^+, for the adjoint problem is such that

$$L^+ F_o^+ = \rho K_F^+ F_o^+. \qquad [42^+]$$

The standard definition of the multiplication factor depends only on the *instantaneous* characteristics of the reactor system, not on the actual neutron population.

On the other hand, let ω_0 be the dominant period for the *instantaneous* characteristics of the reactor. This period is the greatest value of ω for which the boundary value problem

$$L \varphi = \left(\frac{\omega}{v} + \sum_m \frac{\omega}{\lambda_m + \omega} K_{F_r}^m\right) \varphi, \qquad [43]$$

$$\varphi(\mathbf{x}, \mathbf{u}) = 0 \quad \text{if} \quad \mathbf{x} \in S \quad \text{and} \quad \mathbf{n} \cdot \boldsymbol{\Omega} < 0,$$

has a non-negative solution, φ_0. The corresponding solution, φ_0^+, for the adjoint problem is such that

$$L^+ \varphi_o^+ = \left(\frac{\omega_o}{v} + \sum_m \frac{\omega_o}{\lambda_m + \omega_o} K_{F_r}^{m+}\right) \varphi_o^+. \qquad [43^+]$$

From the comparison between [42] and [43] it follows that in Eq. [43] the operator

$$\frac{\omega}{v} + \sum_m \frac{\omega}{\lambda_m + \omega} K^m_{F_r}$$

plays a rôle similar to that plaid by $\rho\, K_F$ in [42]. In particular, we have the relation

$$\rho\,(F^+_o, K_F \varphi_o) = \omega_0 \left(F^+_o . \frac{1}{v} \varphi_o \right) + \sum_m \frac{\omega_o}{\lambda_m + \omega_o} (F^+_o K^m_{F_r} \varphi_o). \qquad [44]$$

We next proceed to consider some particular forms of density weight.

1. W^+ = constant.

The expressions [31] and [33] reduce to

$$\overline{l} = \rho \frac{\int d\,V\, d\,U \frac{1}{v} \varphi}{\int d\,V\, d\,U\, L\, \varphi}, \qquad \overline{\beta}_m = \rho \frac{\int d\,V\, d\,U\, K^m_{F_r} \varphi}{\int d\,V\, d\,U\, L\, \varphi} \qquad [45]$$

where ρ is the (standard) reactivity. Let k_d be the dynamic multiplication factor (Ref. (5), Eq. [44])

$$k_d = - \frac{\int d\,V\, d\,U\, K_F \varphi}{\int d\,V\, d\,U\, (\Pi_a + K_s) \varphi}, \qquad [46]$$

and ρ_d the dynamical reactivity

$$\rho_d = \frac{k_d - 1}{k_d} = \frac{\int d\,V\, d\,U\, L\, \varphi}{\int d\,V\, d\,U\, K_F \varphi}. \qquad [47]$$

Defining

$$l^* = - \frac{\int d\,V\, d\,U \frac{1}{v} \varphi}{\int d\,V\, d\,U\, (\Pi_a + K_s) \varphi}, \qquad \beta^*_m = \frac{\int d\,V\, d\,U\, K^m_{Fr} \varphi}{\int d\,V\, d\,U\, K_F \varphi}, \qquad [48]$$

we get the following relations:

$$\bar{l} = \frac{\rho}{\rho_d}\frac{l^*}{k_d}, \quad \bar{\beta}_m = \frac{\rho}{\rho_d}\beta^*_m, \tag{49}$$

It is of some interest to point out that, although the definitions Eqs. [46]-[48] are formally the same as those of Gross and Marable, they differ form the latter in that the shape φ ($\mathbf{x}$, $\mathbf{u}$, t) is, in general, a function of time t. Now, suppose that $\varphi = \varphi_0$ corresponds to the dominant period ω_0. Then we shall have

$$\mathrm{L}\,\varphi = \mathrm{L}\,\varphi_o = \left(\frac{\omega_o}{v} + \sum_m \frac{\omega_o}{\lambda_m + \omega_o}\,\mathrm{K}^m_{\mathrm{F}_r}\right)\varphi_o, \tag{50}$$

and therefore

$$\rho_d = \frac{l^*}{k_d}\,\omega_o + \sum_m \frac{\omega_o\,\beta^*_m}{\lambda_m + \omega_o} \tag{51}$$

or, from Eq. [49],

$$\rho = \bar{l}\,\omega_o + \sum_m \frac{\omega_o\,\bar{\beta}_m}{\lambda_m + \omega_o}\,. \tag{52}$$

Equation [51] is the inhour equation derived from the gross neutron conservation condition by Gross and Marable (5). To take W^+ = const. is essentially equivalent, in fact, to define the weight of the neutron population by the total number of neutrons in the system. Although Eqs. [51] and [52] are exact relations, they have been derived from the two assumptions W^+ = const., and stable period behaviour of the system. As we shall see, the same form of inhour equation holds under more general assumptions.

2. $$W^+ = F^+(\mathbf{x}, \mathbf{u}),$$

Let $\overset{\circ}{\mathrm{L}}$ characterize some standard, time-independent reactor configuration and let $\overset{\circ}{\mathrm{F}}{}^+$ be the neutron importance in this configuration, i. e., the non-negative solution of

$$\overset{\circ}{\mathrm{L}}{}^+\,\overset{\circ}{\mathrm{F}}{}^+ = \rho_o\,\overset{\circ}{\mathrm{K}}{}^+_{\mathrm{F}}\,\overset{\circ}{\mathrm{F}}{}^+. \tag{53}$$

With $\overset{\circ}{F}^+$ as a weight, to compute $\overline{l}$ and $\overline{\beta}_m$ we only need the expression for $(\overset{\circ}{F}^+, L\,\varphi)$. Now,

$$(\overset{\circ}{F}^+, L\,\varphi) = (\overset{\circ}{F}^+, (\overset{\circ}{L} + \delta L)\,\varphi),$$

with $\delta L = L - \overset{\circ}{L}$. Therefore (cf. Eq. [53])

$$(\overset{\circ}{F}^+, L\,\varphi) = (\overset{\circ}{L}^+ \overset{\circ}{F}^+, \varphi) + (\overset{\circ}{F}^+, \delta L\,\varphi) = \rho_o\,(\overset{\circ}{F}^+, \overset{\circ}{K}_F\,\varphi) + (\overset{\circ}{F}^+, \delta L\,\varphi) \qquad [54]$$

Following Cohen ((4), Eq. [42]), let us introduce the reactivity relative to the standard configuration

$$\rho_C = \frac{(\overset{\circ}{F}^+, L\,\varphi)}{(\overset{\circ}{F}^+, \overset{\circ}{K}_F\,\varphi)} = \rho_o + \frac{(\overset{\circ}{F}^+, \delta L\,\varphi)}{(\overset{\circ}{F}^+, \overset{\circ}{K}_F\,\varphi)}, \qquad [55]$$

and the magnitudes l and β^m_{eff} defined by

$$l = \frac{\left(\overset{\circ}{F}^+, \frac{1}{v}\,\varphi\right)}{(\overset{\circ}{F}^+, \overset{\circ}{K}_F\,\varphi)}, \quad \beta^m_{\text{eff}} = \frac{(\overset{\circ}{F}^+, K^m_{F_r}\,\varphi)}{(\overset{\circ}{F}^+, \overset{\circ}{K}^F\,\varphi)}. \qquad [56]$$

From Eqs. [55] and [56], and from Eq. [33] it follows that

$$\overline{l} = \frac{\rho}{\rho_o}\,l, \quad \overline{\beta}_m = \frac{\rho}{\rho_c}\,\beta^m_{\text{eff}}. \qquad [57]$$

In particular, for stable period behaviour we have $\varphi = \varphi_0$, with φ_0 non-negative solution of Eq. [50], and therefore

$$(\overset{\circ}{F}^+, L\,\varphi) = \omega_o\left(\overset{\circ}{F}^+, \frac{1}{v}\,\varphi_o\right) = \sum_m \frac{\omega_o}{\lambda_m + \omega_o}\,(\overset{\circ}{F}^+, K^m_{F_r}\,\varphi_o),$$

that is

$$\rho_C = \omega_o\,l + \sum_m \frac{\omega_0\,\beta^m_{\text{eff}}}{\lambda_m + \omega_o} \qquad [58]$$

or

$$= \omega_o \bar{l} + \sum_m \frac{\omega_0 \bar{\beta}_m}{\lambda_m + \omega_0} . \qquad [59]$$

It is not surprising that every time the shape is related to a stable period condition we arrive at an inhour equation, either for $W^+ \leqslant$ const., or $W^+ = F^+(\mathbf{x}, \mathbf{u})$. One can easily see that the inhour equation holds, whatever the weight, when the flux is on a stable period. This follows inmediately from Eq. [43], taking the inner product of this equation with the generai weight W^+, and from Eq. [33]:

$$p = \bar{l} \frac{(W^+, L\,\varphi_o)}{\left(W^+, \frac{1}{v}\,\varphi_o\right)} = \bar{l} \left[\omega_0 + \sum_m \frac{\omega_o}{\lambda_m + \omega_o} \frac{(W^+, K_F^m\,\varphi_o)}{\left(W^+, \frac{1}{v}\,\varphi_o\right)} \right],$$

and therefore

$$p = \bar{l}\,\omega_o + \sum_m \frac{\omega_o \bar{\beta}_m}{\lambda_m + \omega_o} . \qquad [60]$$

All the weights considered thus far were time independent weights. We will now consider two cases of, in general, time-dependent weights. Let L be the characteristic operator, Eq. [7], for the actual reactor. All reactor linear operators will depend on time. At every time t these operators define a particular virtual reactor which is stationary in time by virtue of the (static) multiplication factor k_0. Let F_0 be the directional flux density. There is then a sequence of virtually critical reactors in correspondence with the values t of the time. Similarly, we can associate at every time t a static, real reactor the characteristic operators of which are the same as those of the actual reactor. Let φ_0 be the stable flux shape. Both F_0 and φ_0 depend *parametrically* on time, $F_0 = F_0(\mathbf{x}, \mathbf{u}/t)$, $\varphi_0 = \varphi_0(\mathbf{x}, \mathbf{u}/t)$, and the same holds for the adjoint solutions, F_0^+ and φ_0^+ (cf. equations [42+] and [43+]) and for the multiplication factor, k_0.

Now suppose we choose $W^+ = F_0^+ (\mathbf{x}, \mathbf{u}/t)$. Then $\bar{l}$ and $\bar{\beta}_m$ take the simple form

$$\bar{l} = \frac{\left(F_o^+, \frac{1}{v}\varphi\right)}{(F_o^+, K_F \varphi)}, \qquad \bar{\beta}_m = \frac{(F_o^+, K_{F_r}^m \varphi)}{(F_o^+, K_F \varphi)}. \qquad [61]$$

The two expressions are similar to the magnitudes defined by Henry (2), Eqs. [9] and [10]. There are some differences, though. First, the latter refer to a fixed reactor and, therefore, depend on time only through the shape $\varphi (\mathbf{x}, \mathbf{u}, t)$. Second, the weight chosen by Henry is not F_0^+, but the integral $\int F_0^+ (\mathbf{x}, \mathbf{u})\, d\Omega$. On the other hand, $\bar{l}$ and $\bar{\beta}_m$, Eq. [61], depend on time not only through the shape, but through the parametric dependence of F_0^+ on t. Because of their twofold time dependence, the magnitudes $\bar{l}$ and $\bar{\beta}_m$ might vary with time more slowly than do those defined in the conventional manner.

Finally, suppose we chosse $W^+ = \varphi_0^+ (\mathbf{x}, \mathbf{u}/t)$. Then, the inner product

$$(W^+, L\varphi) = (\varphi^+, L\varphi)$$

takes exactly the same form as if the actual reactor were on a stable period ω_0:

$$(\varphi_o^+, L\psi) = (L^+ \varphi_o^+ \cdot \varphi) = \omega_o \left(\varphi_o^+, \frac{1}{v}\varphi\right) + \sum_m \frac{\omega_o}{\lambda_m + \omega_o} (\varphi_o^+ \cdot K_{F_r}^m \varphi). \qquad [62]$$

But from Eq. [33],

$$\rho = \bar{l} \frac{(\varphi_o^+, L\varphi)}{\left(\varphi_o^+, \frac{1}{v}\varphi\right)},$$

and therefore

$$\rho = \bar{l}\omega_o + \sum_m \frac{\omega_o \bar{\beta}_m}{\lambda_m + \omega_o}, \qquad [63]$$

with

$$\overline{l} = \frac{\left(\varphi_o^+, \frac{1}{v} F_o\right)}{(\varphi_o^+, K_F F_o)}, \quad \overline{\beta}_m = \frac{(\varphi_o^+, K_{F_r}^m F_o)}{(\varphi_o^+, K_F F_o)}. \qquad [64]$$

Here $F_0(\mathbf{x}, \mathbf{u}/t)$ is the directional flux density for the virtual reactor associate to the actual reactor at time t. For a given reactor system, the magnitudes $\overline{l}$, and $\overline{\beta}_m$ defined by [64] are independent of the state of the neutron population. Only the physical structure of the system determines the values of $\overline{l}$, $\overline{\beta}_m$ and ρ. We are therefore let to the following statement:

The inhour equation

$$\rho = \overline{l}\,\omega_o + \sum_m \frac{\omega_o \overline{\beta}_m}{\lambda_m + \omega_0}$$

holds: a) for any weight, if the actual reactor is on a stable period (Eq. [60]); b) for any shape, if the weight is taken equal to the persistent importance φ_0^+ in the associate reactor at each time t (Eq. [63]).

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