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Double bounded Kumaraswamy-power series class of distributions

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Abstract

In this paper, we will introduce the new Kumaraswamy-power series class of distributions. This new class is obtained by compounding the Kumaraswamy distribution of Kumaraswamy (1980) and the family of power series distributions. The new class contains some new double bounded distributions such as the Kumaraswamy-geometric, -Poisson, -logarithmic and -binomial, which are used widely in hydrology and related areas. In addition, the corresponding hazard rate function of the new class can be increasing, decreasing, bathtub and upside-down bathtub. Some basic properties of this class of distributions such as the moment generating function, moments and order statistics are studied. Some special members of the class are also investigated in detail. The maximum likelihood method is used for estimating the unknown parameters of the members of the new class. Finally, an application of the proposed class is illustrated using a real data set.

MSC: 60E05, 62E10.

Keywords: Kumaraswamy distribution, Maximum likelihood estimation, Power series distributions, Uniform-power series distributions.

1. Introduction

Many times, the data are modelled by the finite range distributions. For many years, the beta distribution has been used as one of the most basic and useful distributions supported on finite range $(0,1)$ which has been utilized widely in both practical and theoretical aspects of Statistics. This distribution is very flexible to model data which are restricted to any finite interval in view of the fact that it can take an amazingly great

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variety of forms depending on the values of the index parameters (cf. Lemonte and Barreto-Souza, 2013). In econometrics, hydrological processes and related areas several types of data can be modelled by the beta distribution.

An alternative distribution like the beta distribution, which is easier to work with it, is the K distribution proposed by Kumaraswamy (1980). Unlike the beta distribution, the K distribution has a simple closed form of cumulative distribution function (cdf) given by

$$G(x) = 1 - (1 - x^a)^b; \quad 0 < x < 1, \tag{1}$$

where $a > 0$ and $b > 0$ are the shape parameters. The K distribution, similar to the beta distribution, can be unimodal, uniantimodal, increasing, decreasing or constant depending on the values of its parameters. In addition, one can easily show that the K distribution has the same basic shape properties of the beta distribution. But, because of the cdf of the K distribution, which has a simple closed form, it has received much attention in simulating hydrological data and related areas. For more detailed properties of the K distribution see Kumaraswamy (1980) and Jones (2009).

To model data with the finite range on $(0,1)$, we can only address a few distributions in the literature. Here, we attempt to introduce a new family of distributions in this connection. Indeed, to obtain some new double bounded distributions, we compound the K distribution with the family of power series distributions and construct the *Kumaraswamy-power series (KPS)* class of distributions. Compounding a continuous distribution with a discrete one is a known method to introduce new continuous distributions. In recent years, many authors have been interested using this method for constructing new models. For example, the four compound classes proposed by Chahkandi and Ganjali (2009), Morais and Barreto-Souza (2011), Mahmoudi and Jafari (2012) and Silva et al. (2013) are some researches in this regard.

The rest of the paper is organized as follows. In Section 2, we introduce the KPS class of distributions. The density, survival, hazard rate and moment generating functions as well as the moments, quantiles and order statistics are given in this section. In Section 3, we obtain some special distributions and study some of their distributional properties in detail. In addition, the stress-strength parameter is obtained for a special member of the family of KPS distributions in this section. Estimation of the parameters involved using the maximum likelihood method and some related inferences are discussed in Section 4. An application of the new class, using a real data set, is illustrated in Section 5. Finally, some concluding remarks are given in Section 6.

2. The KPS class of distributions

Given N, let $X_1, X_2, \ldots, X_N$ be independent and identically distributed (iid) random variables following a K distribution with cdf (1). Here, N is independent of X_i's and it is a member of the family of power series distributions, truncated at zero, with the probability mass function

$$\pi_n = P(N=n) = \frac{a_n\theta^n}{C(\theta)}; \quad n = 1,2,\ldots,$$

where $a_n \geq 0$ depends only on n, $C(\theta) = \sum_{n=1}^{\infty} a_n\theta^n$ and $\theta \in (0,s)$ (s can be $+\infty$). $C(\theta)$ is finite and $C'(.)$, $C''(.)$ and $C'''(.)$ denote its first, second and third derivatives, respectively. Useful quantities of some power series distributions, truncated at zero, such as geometric, Poisson, logarithmic and binomial (with m being the number of replicates) distributions are shown in Table 1. For more detailed properties of the power series class of distributions, see Noack (1950).

Table 1: *Useful quantities for some power series distributions.*

Model	a_n	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C'''(\theta)$	s
Geometric	1	$\theta(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$3(1-\theta)^{-4}$	1
Poisson	$n!^{-1}$	$e^{\theta}-1$	e^{θ}	e^{θ}	e^{θ}	$+\infty$
Logarithmic	n^{-1}	$-\log(1-\theta)$	$(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	1
Binomial	$\binom{m}{n}$	$(\theta+1)^m-1$	$\frac{m}{(\theta+1)^{1-m}}$	$\frac{m(m-1)}{(1+\theta)^{2-m}}$	$\frac{m(m-1)(m-2)}{(1+\theta)^{3-m}}$	1

Now, let $X_{(1)} = \min\{X_i\}_{i=1}^{N}$. Then, the conditional cdf of $X_{(1)}|N=n$ is given by

$$G_{X_{(1)}|N=n}(x) = 1-[\overline{G}(x)]^n = 1-(1-x^a)^{nb}; \quad 0<x<1,$$

where $\overline{G}(.)$ is the survival function of K distribution associated to cdf (1). As we see, $X_{(1)}|N=n$ follows a K distribution with parameters a and nb. The marginal cdf of $X_{(1)}$, that is,

$$\begin{aligned} F(x) = \sum_{n=1}^{n} \frac{a_n\theta^n}{C(\theta)}\{1-[\overline{G}(x)]^n\} &= 1-\frac{C(\theta\overline{G}(x))}{C(\theta)} \\ &= 1-\frac{C(\theta(1-x^a)^b)}{C(\theta)}; \quad 0<x<1, \end{aligned} \tag{2}$$

defines the cdf of the family of KPS distributions. We denote a random variable X following the KPS distribution with parameters a, b, and θ by $KPS(a,b,\theta)$.

2.1. Density, survival and hazard rate functions

The probability density function (pdf) of a random variable X following a $KPS(a,b,\theta)$ distribution is given by

$$f(x) = \theta abx^{a-1}(1-x^a)^{b-1}\frac{C'(\theta(1-x^a)^b)}{C(\theta)}; \quad 0 < x < 1. \tag{3}$$

Proposition 2.1 *The pdf of KPS distributions has at least a mode, for $a > 1$ and $b > 1$. It is increasing, for $a > 1$ and $b < 1$, and decreasing or bathtub elsewhere.*

Proof. See Appendix A.

Proposition 2.2 *The K distribution with parameters a and bc is a limiting distribution of the KPS distribution when $\theta \to 0^+$, where $c = \min\{n \in \mathbb{N} : a_n > 0\}$.*

Proof. See Appendix B.

Proposition 2.3 *The pdf of KPS distributions can be written as a mixture of the K distribution with parameters a and nb.*

Proof. Using a conditional argument on N, the proof is completed.

The survival and hazard rate functions of KPS distributions are given by

$$\overline{F}(x) = \frac{C(\theta(1-x^a)^b)}{C(\theta)} \tag{4}$$

and

$$h(x) = \theta abx^{a-1}(1-x^a)^{b-1}\frac{C'(\theta(1-x^a)^b)}{C(\theta(1-x^a)^b)}, \tag{5}$$

respectively. To see the density and hazard rate functions shapes of KPS distributions, let $C(\theta) = \theta + \theta^{20}$ (see also Mahmoudi and Jafari, 2012; Morais and Barreto-Souza, 2011). Then, for $\theta = 1$, we have $f(x) = \frac{ab}{2}x^{a-1}(1-x^a)^{b-1}[1+20(1-x^a)^{19b}]$ and $h(x) = abx^{a-1}(1-x^a)^{b-1}\frac{1+20(1-x^a)^{19b}}{(1-x^a)^b+(1-x^a)^{20b}}$. The plots of this density and the corresponding hazard rate function are given in Figure 1 for some selected values of parameters.

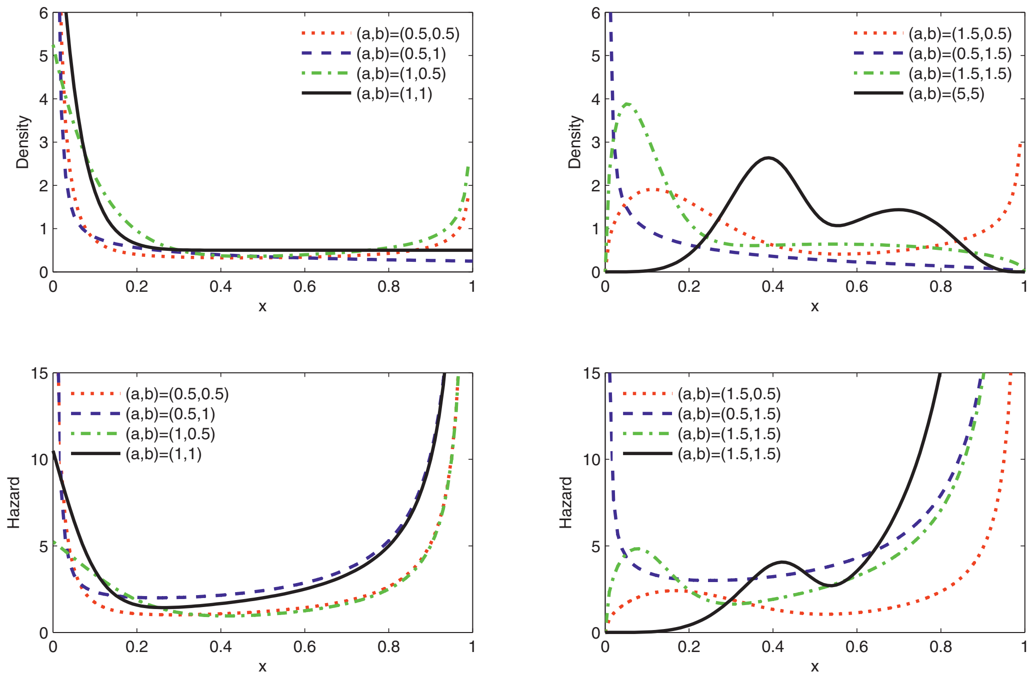

Figure 1: *Plots of density and hazard rate functions of* $KPS(a,b,1)$ *distribution with* $C(\theta) = \theta + \theta^{20}$.

2.2. Quantiles and median

The q-th quantile, say x_q, of the KPS distributions is given by

$$x_q = \{1 - [\frac{1}{\theta}C^{-1}((1-q)C(\theta))]^{1/b}\}^{1/a},$$

where $C^{-1}(.)$ is the inverse function of $C(.)$. In particular, the median is immediately obtained by

$$m = \{1 - [\frac{1}{\theta}C^{-1}(\frac{C(\theta)}{2})]^{1/b}\}^{1/a}.$$

2.3. Moment generating function and moments

Let Y be a random variable following the K distribution with parameters a and b. Lemonte and Barreto-Souza (2013) obtained the moment generating function (mgf) of the random variable Y as follows:

$$M_Y(t) = b\sum_{s=0}^{\infty}\frac{\Gamma(b)(-1)^s}{\Gamma(b-s)(s+1)!}\,{}_1F_1(a(s+1),a(s+1)+1;t), \tag{6}$$

where ${}_1F_1$ denotes the confluent hypergeometric function defined by

$${}_1F_1(a,b;t) = \sum_{m=0}^{\infty}\frac{a_{(m)}}{b_{(m)}m!}t^m$$

in which $a_{(m)} = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)...(a+m-1)$ is the ascending factorial. Combining Eq. (6) and Prop. 2.3 yields the mgf of the random variable $X \sim KPS(a,b,\theta)$ as follows:

$$M_X(t) = b\sum_{n=1}^{\infty}\sum_{s=0}^{\infty}\frac{n\Gamma(nb)(-1)^s}{\Gamma(nb-s)(s+1)!}\,{}_1F_1(a(s+1),a(s+1)+1;t)\pi_n. \tag{7}$$

The r-th moment of the K distribution is given by $bB(1+\frac{r}{a},b)$ (see Jones, 2009), where $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ denotes the beta function. Thus, the r-th moment of $X \sim KPS(a,b,\theta)$ is given by

$$E(X^r) = b\sum_{n=1}^{\infty} nB(1+\frac{r}{a},nb)\pi_n, \quad r = 1,2,\ldots . \tag{8}$$

2.4. Order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from a KPS distribution and $X_{i:n},\ i = 1,2,...,n$, denote its i-th order statistic. The pdf of $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{1}{B(i,n-i+1)}f(x)[F(x)]^{i-1}[1-F(x)]^{n-i}, \tag{9}$$

where F and f are the cdf and pdf of KPS distributions given by (2) and (3), respectively. Eq. (9) can be written as the following forms

$$f_{i:n}(x) = \frac{1}{B(i,n-i+1)}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^k f(x)[F(x)]^{k+i-1} \tag{10}$$

or

$$f_{i:n}(x) = \frac{1}{B(i,n-i+1)}\sum_{k=0}^{i-1}\binom{i-1}{k}(-1)^k f(x)[1-F(x)]^{k+n-i}. \tag{11}$$

In view of the fact that

$$f(x)[F(x)]^{k+i-1} = \frac{1}{k+i}\frac{d}{dx}[F(x)]^{k+i},$$

the corresponding cdf of $f_{i:n}(x)$, denoted by $F_{i:n}(x)$, becomes

$$\begin{aligned} F_{i:n}(x) &= \frac{1}{B(i,n-i+1)}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}(-1)^k}{k+i}[F(x)]^{k+i} \\ &= \frac{1}{B(i,n-i+1)}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}(-1)^k}{k+i}[1-\frac{C(\theta(1-x^a)^b)}{C(\theta)}]^{k+i} \qquad (12) \\ &= \frac{1}{B(i,n-i+1)}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}(-1)^k}{k+i}F_W(x;a,b,\theta,k+i), \end{aligned}$$

where W follows an *exponentiated* KPS (EKPS) distribution with parameters a, b, θ and $k+i$. For more details of *exponentiated F* distributions or, equivalently, *resilience parameter families*, see Marshall and Olkin (2007).

An alternative expression for $F_{i:n}(x)$, using Eq. (11), is

$$\begin{aligned} F_{i:n}(x) &= 1-\frac{1}{B(i,n-i+1)}\sum_{k=0}^{i-1}\frac{\binom{i-1}{k}(-1)^k}{k+n-i+1}[1-F(x)]^{k+n-i+1} \\ &= 1-\frac{1}{B(i,n-i+1)}\sum_{k=0}^{i-1}\frac{\binom{i-1}{k}(-1)^k}{k+n-i+1}[\frac{C(\theta(1-x^a)^b)}{C(\theta)}]^{k+n-i+1}. \end{aligned}$$

Expressions for moments of the i-th order statistic $X_{i:n},\ i=1,2,...,n$, with cdf (12), can be obtained using a result of Barakat and Abdelkader (2004) as follows:

$$\begin{aligned} E(X_{i:n}^r) &= r\sum_{k=n-i+1}^{n}(-1)^{k-n+i-1}\binom{k-1}{n-i}\binom{n}{k}\int_0^\infty x^{r-1}[\overline{F}(x)]^k dx \\ &= r\sum_{k=n-i+1}^{n}\frac{(-1)^{k-n+i-1}}{C(\theta)^k}\binom{k-1}{n-i}\binom{n}{k}\int_0^\infty x^{r-1}[C(\theta(1-x^a)^b)]^k dx, \end{aligned}$$

for $r=1,2,\ldots$ and $i=1,2,\ldots,n$, where $\overline{F}(x)$ is the survival function given by (4); see also Morais and Barreto-Souza (2011). An application of the first moments of order statistics can be considered in calculating the L-moments which are in fact the linear combinations of the expected order statistics. See Hosking (1990) for details.

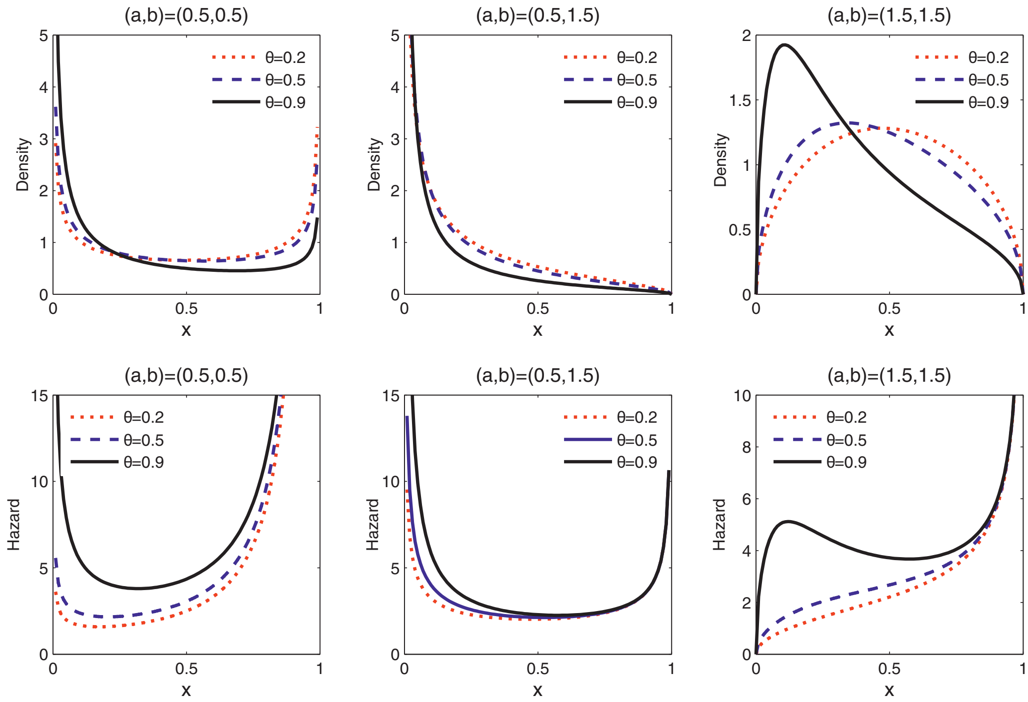

Figure 2: *Plots of $KG(a,b,\theta)$ density and hazard rate functions for some parameter values.*

3. Special cases of the KPS family

In this section, we study basic distributional properties of the Kumaraswamy-geometric (KG), Kumaraswamy-Poisson (KP), Kumaraswamy-logarithmic (KL) and Kumaraswamy-binomial (KB) distributions as special cases of KPS family. In addition, expressions for the pdf and moments of order statistics as well as the stress-strength parameter of the KG distribution are obtained. First, to illustrate the flexibility of the distributions, plots of the density and hazard rate functions are presented in Figures 2, 3, 4 and 5 for some selected values of the parameters.

3.1. Basic distributional properties

Using Table 1 and Eqs. (4-8) given in Section 2, basic distributional properties of the four special distributions of KPS family are immediately obtained. Table 2 contains the survival function, pdf, hazard rate function, mgf and the moments of KG, KP, KL and KB distributions.

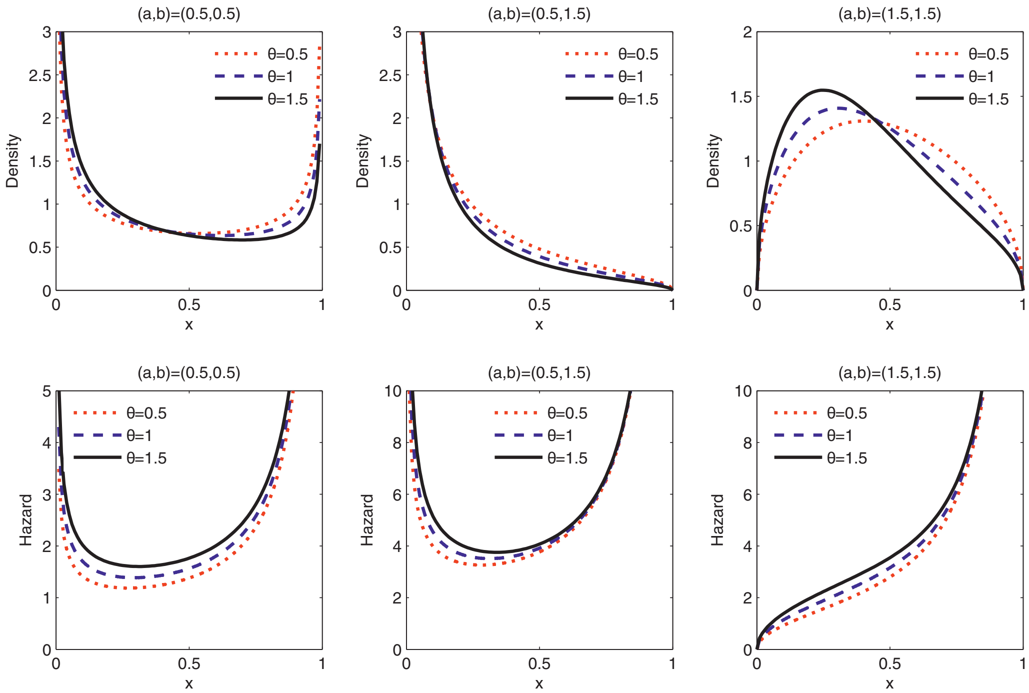

Figure 3: *Plots of* $KP(a,b,\theta)$ *density and hazard rate functions for some parameters values.*

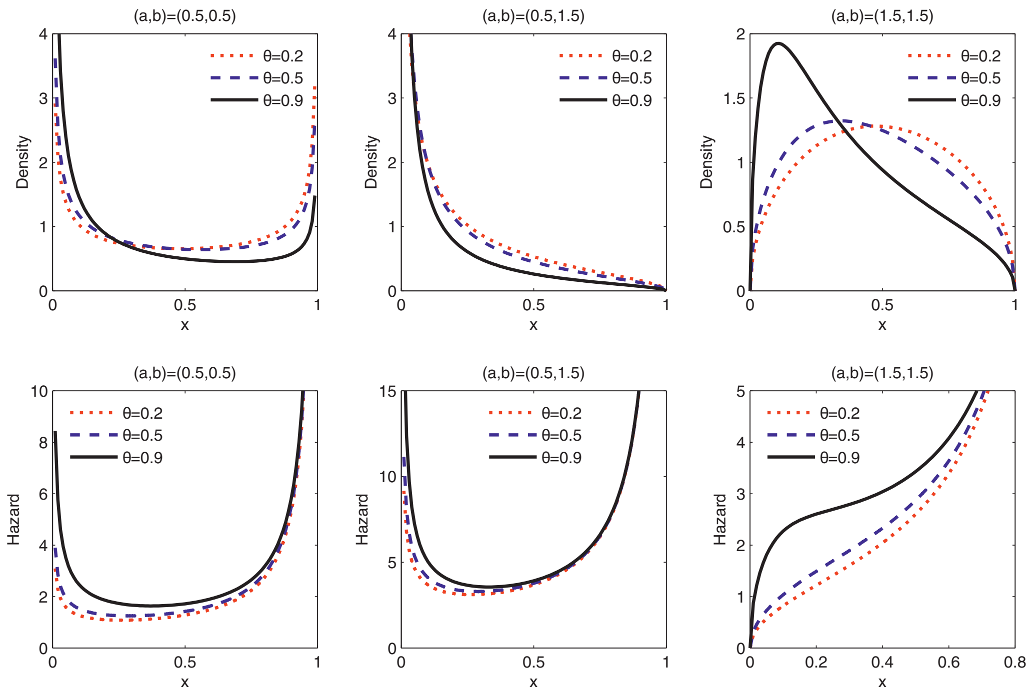

Figure 4: *Plots of* $KL(a,b,\theta)$ *density and hazard rate functions for some parameter values.*

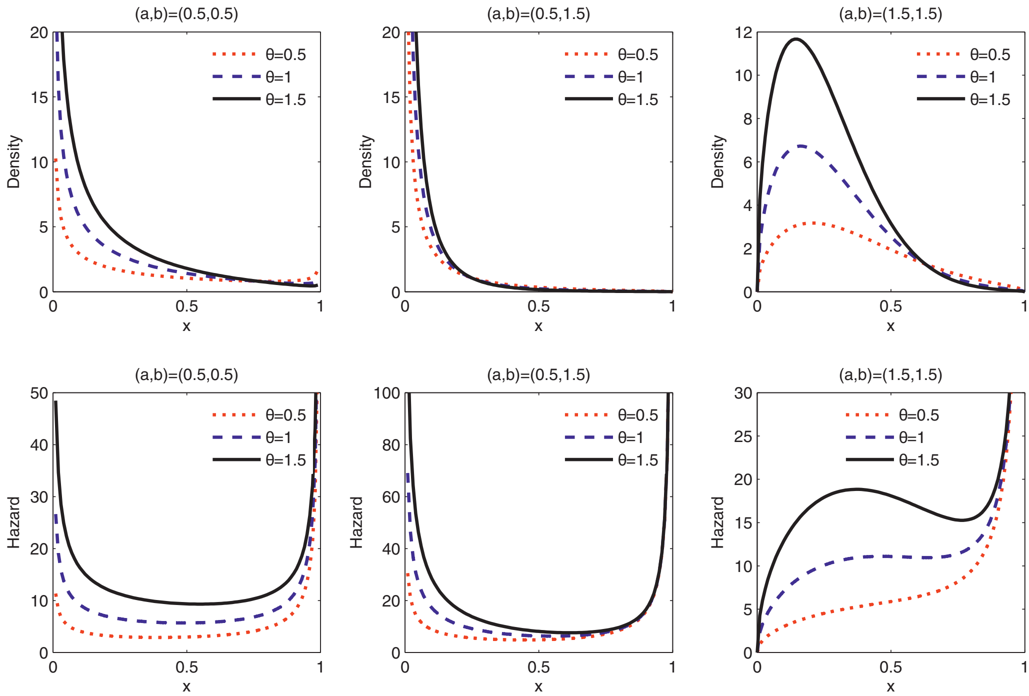

Figure 5: *Plots of $KB(a,b,\theta)$ density and hazard rate functions for some values of (a,b,θ) and $m=5$.*

Table 2: *Survival function, pdf, hazard rate function, mgf and moments of KG, KP, KL and KB dists.*

Model	survival function	pdf	hazard rate function
KG	$1-\frac{1-(1-x^a)^b}{1-\theta(1-x^a)^b}$	$\frac{(1-\theta)abx^{a-1}(1-x^a)^{b-1}}{\{1-\theta(1-x^a)^b\}^2}$	$\frac{abx^{a-1}(1-x^a)^{b-1}}{\{1-\theta(1-x^a)^b\}(1-x^a)}$
KP	$\frac{e^{\theta(1-x^a)^b}-1}{e^{\theta}-1}$	$\frac{\theta abx^{a-1}(1-x^a)^{b-1}e^{\theta(1-x^a)^b}}{e^{\theta}-1}$	$\frac{\theta abx^{a-1}(1-x^a)^{b-1}e^{\theta(1-x^a)^b}}{e^{\theta(1-x^a)^b}-1}$
KL	$\frac{\log(1-\theta(1-x^a)^b)}{\log(1-\theta)}$	$-\frac{\theta abx^{a-1}(1-x^a)^{b-1}}{\log(1-\theta)(1-\theta(1-x^a)^b)}$	$-\frac{\theta abx^{a-1}(1-x^a)^{b-1}}{\log(1-\theta(1-x^a)^b)(1-\theta(1-x^a)^b)}$
KB	$\frac{(\theta(1-x^a)^b+1)^m-1}{(\theta+1)^m-1}$	$\frac{m\theta abx^{a-1}(1-x^a)^{b-1}(\theta(1-x^a)^b+1)^{m-1}}{(\theta+1)^m-1}$	$\frac{m\theta abx^{a-1}(1-x^a)^{b-1}(\theta(1-x^a)^b+1)^{m-1}}{(\theta(1-x^a)^b+1)^m-1}$

mgf	moments
$b(1-\theta)\sum_{n=1}^{\infty}\sum_{s=0}^{\infty}\frac{n\Gamma(nb)(-1)^s\theta^{n-1}}{\Gamma(nb-s)(s+1)!}{}_1F_1(a(s+1),a(s+1)+1;t)$	$b(1-\theta)\sum_{n=1}^{\infty}nB(1+\frac{r}{a},nb)\theta^{n-1}$
$\frac{b}{e^{\theta}-1}\sum_{n=1}^{\infty}\sum_{s=0}^{\infty}\frac{\Gamma(nb)\theta^n(-1)^s}{\Gamma(nb-s)(s+1)!(n-1)!}{}_1F_1(a(s+1),a(s+1)+1;t)$	$\frac{b}{e^{\theta}-1}\sum_{n=1}^{\infty}B(1+\frac{r}{a},nb)\frac{\theta^n}{(n-1)!}$
$\frac{-b}{\log(1-\theta)}\sum_{n=1}^{\infty}\sum_{s=0}^{\infty}\frac{\Gamma(nb)\theta^n(-1)^s}{\Gamma(nb-s)(s+1)!}{}_1F_1(a(s+1),a(s+1)+1;t)$	$-\frac{b}{\log(1-\theta)}\sum_{n=1}^{\infty}B(1+\frac{r}{a},nb)\theta^n$
$\frac{b}{(\theta+1)^m-1}\sum_{n=1}^{\infty}\sum_{s=0}^{\infty}\frac{n\Gamma(nb)\binom{m}{n}\theta^n(-1)^s}{\Gamma(nb-s)(s+1)!}{}_1F_1(a(s+1),a(s+1)+1;t)$	$\frac{b}{(\theta+1)^m-1}\sum_{n=1}^{\infty}n\binom{m}{n}\theta^nB(1+\frac{r}{a},nb)$

3.2. Order statistics of the KG distribution

By inserting the pdf and cdf of KG distribution into Eq. (10), we obtain the pdf of the i-th order statistic of KG distribution as follows:

$$f_{i:n}(x) = \frac{1}{B(i,n-i+1)} \sum_{k=0}^{n-i} \frac{\binom{n-i}{k}(-1)^k(1-\theta)abx^{a-1}(1-x^a)^{b-1}\{1-(1-x^a)^b\}^{k+i-1}}{\{1-\theta(1-x^a)^b\}^{k+i+1}}.$$

Expanding the binomial term $\{1-\theta(1-x^a)^b\}^{k+i+1}$ by the series representation

$$(1-z)^{-k} = \sum_{i=0}^{\infty} \frac{\Gamma(k+i)}{\Gamma(k)i!} z^i; \quad k>0,\ |z|<1, \tag{13}$$

the pdf of the i-th order statistic can be rewritten as

$$f_{i:n}(x) = \frac{1-\theta}{B(i,n-i+1)} \sum_{j=0}^{\infty} \sum_{k=0}^{n-i} \frac{\binom{n-i}{k}(-1)^k \theta^j}{k+i} f_{BK}(x;k+i,j+1,a,b),$$

where

$$f_{BK}(x;\alpha,\beta,a,b) = \frac{1}{B(\alpha,\beta)} abx^{a-1}(1-x^a)^{b\beta-1}\{1-(1-x^a)^b\}^{\alpha-1} \tag{14}$$

is the density function of beta-Kumaraswamy (BK) distribution of Carrasco et al. (2012).

An alternative expression for the pdf of the i-th order statistic of KG distribution can be obtained by Eq. (11). Hence,

$$f_{i:n}(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{i-1} \frac{\binom{i-1}{k}(-1)^k \binom{k+n-i+j}{j} \theta^j (1-\theta)^{k+n-i+1}}{(k+n-i+1)B(i,n-i+1)} f_K(x;a,b(k+n-i+j+1)), \tag{15}$$

where f_K is the density function of K distribution. As we see, the pdf of order statistics of KG distribution can be expressed as a linear combination of the pdf of BK or K distributions. Therefore, some properties of the i-th order statistic, such as the mgf and moments, can be obtained directly from those of BK or K distributions. For example, from Eq. (15), the moments of the i-th order statistic of KG distribution are given by

$$E(X_{i:n}^r) = \sum_{j=0}^{\infty} \sum_{k=0}^{i-1} \frac{\binom{i-1}{k}(-1)^k \binom{k+n-i+j}{j} \theta^j (1-\theta)^{k+n-i+1}}{(k+n-i+1)B(i,n-i+1)}$$
$$\times b(k+n-i+j+1)B(1+r/a, b(k+n-i+j+1)), \quad r=1,2,\ldots.$$

3.3. Stress-strength parameter of the KG distribution

The stress-strength parameter $R = P(X > Y)$ is a measure of component reliability and its estimation problem when X and Y are independent and follow a specified distribution has been discussed widely in the literature. Let X be the random variable of the strength of a component which is subjected to a random stress Y. The component fails whenever $X < Y$ and there is no failure when $X > Y$. Here, we obtain an expression for the stress-strength parameter of the KG distribution.

Let $X \sim KG(a,b,\theta_1)$ and $Y \sim KG(a,b,\theta_2)$ be independent random variables. The stress-strength parameter is defined as

$$R = P(X > Y) = \int_0^1 f_X(x)F_Y(x)dx$$
$$= \int_0^1 \frac{(1-\theta_1)abx^{a-1}(1-x^a)^{b-1}\{1-(1-x^a)^b\}}{\{1-\theta_1(1-x^a)^b\}^2\{1-\theta_2(1-x^a)^b\}}dx.$$

Expanding the binomial terms $\{1-\theta_1(1-x^a)^b\}^2$ and $\{1-\theta_2(1-x^a)^b\}$ as in Eq. (13), we obtain

$$R = (1-\theta_1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\theta_1^i\theta_2^j(i+1)}{(i+j+1)^2(i+j+2)}\int_0^1 f_{BK}(x;2,i+j+1,a,b)dx$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\theta_1^i\theta_2^j(i+1)}{(i+j+1)^2(i+j+2)},$$

where f_{BK} has been already defined by Eq. (14). It is clear that R can be estimated when the parameters θ_1 and θ_2 are estimated by the maximum likelihood method.

Remark 3.1 *If $a = 1$ $[b = 1]$ in a $KPS(a,b,\theta)$ distribution, then we obtain the beta-PS$(1,b,\theta)$ [beta-PS$(a,1,\theta)$] distribution. In addition, $KPS(a,b,\theta)$ distribution reduces to a standard uniform-PS distribution, when $a = b = 1$. All properties of KPS distribution are valid for these special distributions.*

4. Estimation and inference

Let $x_1, x_2, \ldots, x_n$ be n observations of a random sample from a $KPS(a,b,\theta)$ distribution and $\boldsymbol{\theta} = (a,b,\theta)^{\mathsf{T}}$ be the unknown parameter vector in the rest of the paper. The log-likelihood function is given by

$$\ell_n = \ell_n(\boldsymbol{\theta}; x_1, x_2, \ldots, x_n) = n\log\theta + n\log a + n\log b + (a-1)\sum_{i=1}^{n}\log x_i$$
$$+(b-1)\sum_{i=1}^{n}\log(1-x_i^a) + \sum_{i=1}^{n}\log C'(\theta(1-x_i^a)^b) - n\log(C(\theta)).$$

The associated score function is given by $U_n(\boldsymbol{\theta}) = (\partial\ell_n/\partial a, \partial\ell_n/\partial b, \partial\ell_n/\partial\theta)^T$, where

$$\frac{\partial\ell_n}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n}\log x_i - (b-1)\sum_{i=1}^{n}\frac{x_i^a\log x_i}{1-x_i^a} - \sum_{i=1}^{n}\frac{\theta b x_i^a\log x_i(1-x_i^a)^{b-1}C''(\theta(1-x_i^a)^b)}{C'(\theta(1-x_i^a)^b)},$$
$$\frac{\partial\ell_n}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log(1-x_i^a) + \sum_{i=1}^{n}\frac{\theta(1-x_i^a)^b\log(1-x_i^a)C''(\theta(1-x_i^a)^b)}{C'(\theta(1-x_i^a)^b)}$$

and

$$\frac{\partial\ell_n}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\frac{(1-x_i^a)^b C''(\theta(1-x_i^a)^b)}{C'(\theta(1-x_i^a)^b)} - n\frac{C'(\theta)}{C(\theta)}.$$

The maximum likelihood estimation (MLE) of $\boldsymbol{\theta}$, say $\hat{\boldsymbol{\theta}}$, is obtained by solving the nonlinear system $U_n(\hat{\boldsymbol{\theta}}) = \mathbf{0}$. The solution of this nonlinear system of equations can be found by using a numerical method. We need the Fisher information matrix for interval estimation and hypotheses testing on the model parameters. The 3×3 Fisher information matrix is given by

$$I_n(\boldsymbol{\theta}) = -\begin{bmatrix} I_{aa} & I_{ab} & I_{a\theta} \\ I_{ba} & I_{bb} & I_{b\theta} \\ I_{\theta a} & I_{\theta b} & I_{\theta\theta} \end{bmatrix},$$

whose elements are obtained by the relationship $I_{\theta_i\theta_j} = E[\frac{\partial^2\ell_n}{\partial\theta_i\partial\theta_j}];\ i,j = 1,2,3$ (see Appendix C). However, for usual large sample, the Fisher information matrix can be approximated by its observed matrix. That is,

$$I_n(\hat{\boldsymbol{\theta}}) \approx -[\frac{\partial^2\ell_n}{\partial\theta_i\theta_j}|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}];\ i,j = 1,2,3,$$

where $\hat{\boldsymbol{\theta}}$ is the MLE of θ. See, for example, Cox and Hinkley (1974) for more discussions of MLEs properties.

Table 3: *Phosphorus concentration in leaves data set.*

0.22	0.17	0.11	0.10	0.15	0.06	0.05	0.07	0.12	0.09	0.23	0.25	0.23	0.24	0.20	0.08
0.11	0.12	0.10	0.06	0.20	0.17	0.20	0.11	0.16	0.09	0.10	0.12	0.12	0.10	0.09	0.17
0.19	0.21	0.18	0.26	0.19	0.17	0.18	0.20	0.24	0.19	0.21	0.22	0.17	0.08	0.08	0.06
0.09	0.22	0.23	0.22	0.19	0.27	0.16	0.28	0.11	0.10	0.20	0.12	0.15	0.08	0.12	0.09
0.14	0.07	0.09	0.05	0.06	0.11	0.16	0.20	0.25	0.16	0.13	0.11	0.11	0.11	0.08	0.22
0.11	0.13	0.12	0.15	0.12	0.11	0.11	0.15	0.10	0.15	0.17	0.14	0.12	0.18	0.14	0.18
0.13	0.12	0.14	0.09	0.10	0.13	0.09	0.11	0.11	0.14	0.07	0.07	0.19	0.17	0.18	0.16
0.19	0.15	0.07	0.09	0.17	0.10	0.08	0.15	0.21	0.16	0.08	0.10	0.06	0.08	0.12	0.13

5. Application of the KPS distributions

Fonseca and Franca (2007) studied the soil fertility in influence and the characterization of the biologic fixation of N_2 for the *Dimorphandra wilsonii rizz growth*. For 128 plants, they made measures of the phosphorus concentration in the leaves. The data, which have also been analyzed by Silva et al. (2013), are listed in Table 3.

We fit the KG, KP, KL and K models to the data to show the capability and potentiality of the new class of distributions in data modelling. In addition, we fit the Weibull-geometric (WG) distribution of Barreto-Souza et al. (2011), which is also a member of the proposed class of Silva et el. (2013), and compare it with our models. We first estimate unknown parameters of the models by the maximum likelihood method and, then, we obtain the values of Akaike information criterion (AIC) and Bayesian information criterion (BIC) as well as Kolmogorov-Smirnov (K-S) statistic and their corresponding p-values. A summary of computations is given in Table 4.

Table 4: *MLE, maximized log-likelihood, AIC, BIC and K-S statistic (p-value) for fitted models.*

Model	MLEs of parameters	$\log L$	AIC	BIC	K-S (p-value)
KG	$(\hat{a},\hat{b},\hat{\theta}) = (3.5909, 318.2081, 0.7338)$	196.7994	−387.5989	−380.0127	0.0944 (0.1911)
KP	$(\hat{a},\hat{b},\hat{\theta}) = (3.1424, 73.3827, 5.1828)$	194.4806	−382.9613	−374.4052	0.1110 (0.0792)
KL	$(\hat{a},\hat{b},\hat{\theta}) = (2.6380, 130.8358, 0.0327)$	194.3899	−382.7797	−374.2236	0.0943 (0.1927)
K	$(\hat{a},\hat{b}) = (2.8104, 176.3491)$	194.8007	−385.6015	−379.8974	0.1181 (0.0517)
WG	$(\hat{\alpha},\hat{\gamma},\hat{\theta}) = (2.4471, 4.2041, 0.9995)$	192.2505	−378.5125	−370.0125	0.1208 (0.0461)

As we see from the results presented in Table 4, the KG model with the minimum values of AIC and BIC gives a better fit than the other rival models. However, the KG, KP and KL models (even K model with the two parameters) have better fits than the WG model of Silva et al. (2013). Further, Figures 6 and 7 also confirm these conclusions.

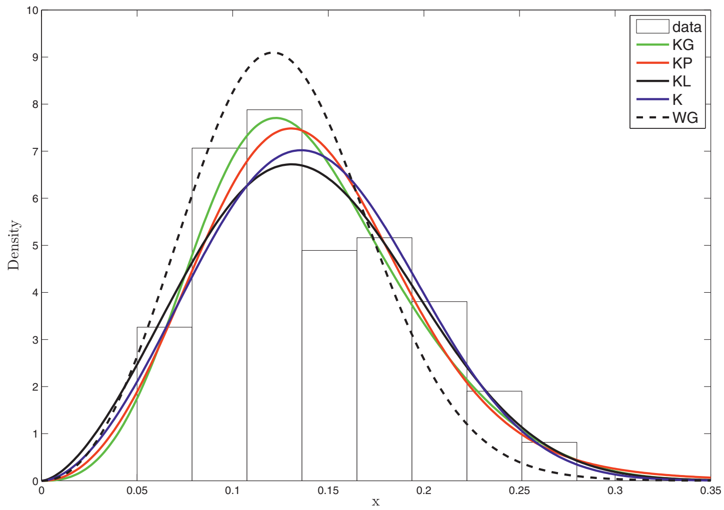

Figure 6: *Plots of the fitted KG, KP, KL, K and WG densities.*

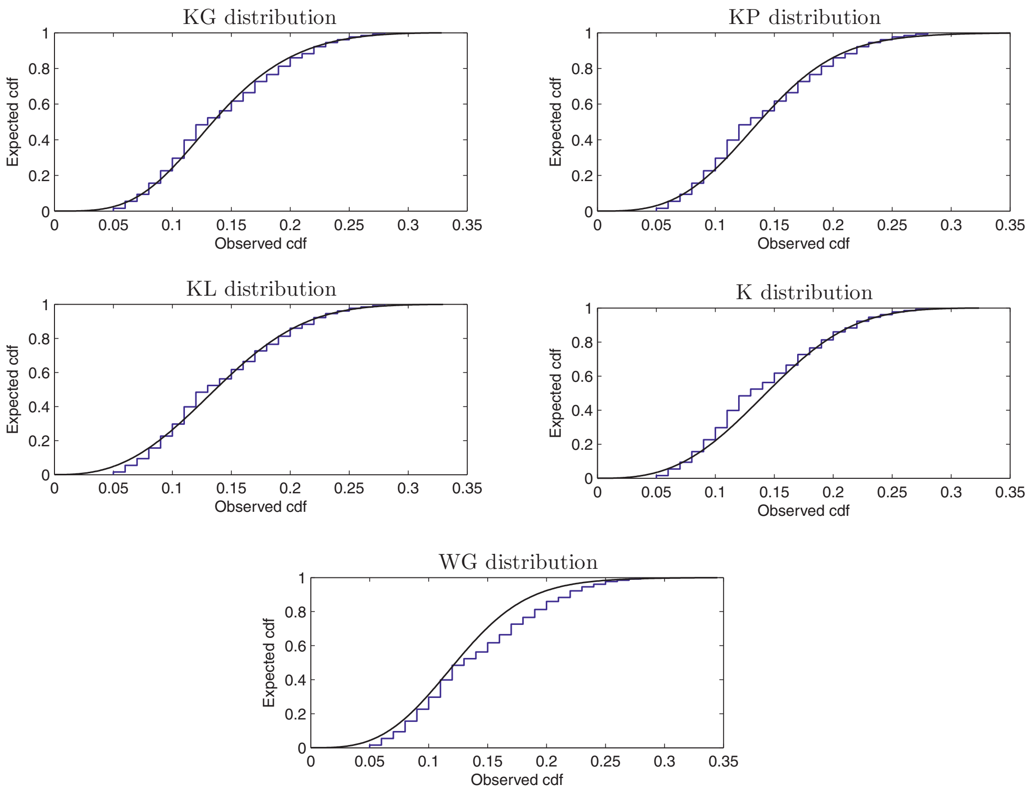

Figure 7: *Empirical cdf plots of the fitted KG, KP, KL, K and WG models.*

6. Concluding remarks

A new compound class of distributions with a finite range on $(0,1)$ is defined by the stochastic representation $X_{(1)} = \min\{X_i\}_{i=1}^{N}$, where X_i's have a Kumaraswamy distribution and N is a member of the family of the power series distributions, independent of X_i's. The new class, namely KPS, contains four new distributions with applications to hydrological areas. We had a comprehensive study on this class of distributions and investigated some their important distributional properties. In the application section, we fitted some special members of the KPS class to a real data set to indicate the potential of the new class in data modelling. As a new family of distributions in this connection, one can establish a new class by considering the stochastic representation $X_{(n)} = \max\{X_i\}_{i=1}^{N}$. In the context of reliability, the stochastic representations $X_{(1)}$ and $X_{(n)}$ have important roles in the series and parallel systems, respectively, which appear in many industrial applications and biological organisms.

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Appendix A

Here, we examine the density shapes of KPS distributions. For this purpose, for all $b > 0$, we have

$$\lim_{x\to 0^+} f(x) = \begin{cases} \infty, & a < 1 \\ \theta b \dfrac{C'(\theta)}{C(\theta)}, & a = 1 \\ 0, & a > 1 \end{cases}$$

and, for all $a > 0$,

$$\lim_{x\to 1^-} f(x) = \begin{cases} \infty, & b < 1 \\ \theta a \dfrac{C'(\theta)}{C(\theta)}, & b = 1 \\ 0, & b > 1. \end{cases}$$

Therefore, as wee see, for $a > 1$ and $b > 1$, the pdf of KPS distributions has at least a mode and for $a > 1$ and $b < 1$, the pdf is increasing.

Appendix B

Below, we give a proof for Proposition 2.2:

$$\begin{aligned}\lim_{\theta\to 0^+} F(x) = \lim_{\theta\to 0^+}\{1-\frac{C(\theta\overline{G}(x))}{C(\theta)}\} &= 1-\lim_{\theta\to 0^+}\frac{\sum_{n=1}^{\infty} a_n\theta^n[\overline{G}(x)]^n}{\sum_{n=1}^{\infty} a_n\theta^n}\\ &= 1-\lim_{\theta\to 0^+}\frac{a_c[\overline{G}(x)]^c+\sum_{n=c+1}^{\infty} a_n\theta^{n-c}[\overline{G}(x)]^n}{a_c+\sum_{n=c+1}^{\infty} a_n\theta^{n-c}}\\ &= 1-[\overline{G}(x)]^c = 1-(1-x^a)^{bc}.\end{aligned}$$

Appendix C

Let $p_i = (1-x_i^a)$. Then, the elements of $I_n(\boldsymbol{\theta})$ are given by

$$I_{a,a} = \frac{\partial^2 \ell_n}{\partial a^2} = -\frac{n}{a^2}-(b-1)\sum_{i=1}^{n} x_i^a[\frac{\log x_i}{p_i}]^2+\sum_{i=1}^{n} z''_{(aa)i},$$

$$I_{a,b} = I_{b,a} = \frac{\partial^2 \ell_n}{\partial a\partial b} = -\sum_{i=1}^{n}\frac{x_i^a\log x_i}{p_i}+\sum_{i=1}^{n} z''_{(ab)i},$$

$$I_{a,\theta} = I_{\theta,a} = \frac{\partial^2 \ell_n}{\partial a\partial\theta} = \sum_{i=1}^{n} z''_{(a\theta)i}, \qquad I_{b,b} = \frac{\partial^2 \ell_n}{\partial b^2} = -\frac{n}{b^2}+\sum_{i=1}^{n} z''_{(bb)i},$$

$$I_{b,\theta} = I_{\theta,b} = \frac{\partial^2 \ell_n}{\partial b\partial\theta} = \sum_{i=1}^{n} z''_{(b\theta)i}$$

and

$$I_{\theta,\theta} = \frac{\partial^2 \ell_n}{\partial \theta^2} = -\frac{n}{\theta^2}-n\frac{C''(\theta)C(\theta)-[C'(\theta)]^2}{[C(\theta)]^2}+\sum_{i=1}^{n} z''_{(\theta\theta)i},$$

where

$$\begin{aligned} z''_{(aa)i} &= \frac{\partial^2}{\partial a^2}\log C'(\theta p_i^b)\\ &= \frac{-b\theta(\log x_i)^2 x_i^a}{[C'(\theta p_i^b)]^2}\{[p_i^{b-1}C''(\theta p_i^b)-(b-1)p_i^{b-2}C''(\theta p_i^b)-\theta b x_i^a p_i^{2b-2}C'''(\theta p_i^b)]C'(\theta p_i^b)\\ &+\theta b x_i^a p_i^{2b-2}[C''(\theta p_i^b)]^2\},\end{aligned}$$

$$z''_{(ab)i} = \frac{\partial^2}{\partial a \partial b} \log C'(\theta p_i^b)$$

$$= \frac{-\theta p_i^{b-1} x_i^a \log x_i \{[b \log p_i C''(\theta p_i^b) - C''(\theta p_i^b) - \theta C'''(p_i^b)] C'(\theta p_i^b) - \theta b p_i^b [C''(\theta p_i^b)]^2\}}{[C'(\theta p_i^b)]^2},$$

$$z''_{(a\theta)i} = -\frac{b x_i^a p_i^{b-1} \log x_i \{[C''(\theta p_i^b) + \theta p_i^b C'''(\theta p_i^b)] C'(\theta p_i^b) - \theta p_i^b [C'(\theta p_i^b)]^2\}}{[C'(\theta p_i^b)]^2},$$

$$z''_{(bb)i} = \frac{\theta \log p_i \{[p_i^b \log p_i C''(\theta p_i^b) + \theta p_i^{2b} \log p_i C'''(\theta p_i^b)] C'(\theta p_i^b)\}}{[C'(\theta p_i^b)]^2}$$

$$- \frac{\theta^2 p_i^{2b} (\log p_i)^2 [C''(\theta p_i)]^2}{[C'(\theta p_i)]^2},$$

$$z''_{(b\theta)i} = \frac{p_i^b \log p_i \{[C''(\theta p_i^b) + \theta p_i^b C'''(\theta p_i^b)] C'(\theta p_i^b) - \theta p_i^b [C''(\theta p_i^b)]^2\}}{[C'(\theta p_i^b)]^2},$$

$$z''_{(\theta\theta)i} = \frac{p_i^{2b} C'''(\theta p_i^b) C'(\theta p_i^b) - p_i^{2b} [C''(\theta p_i^b)]^2}{[C'(\theta p_i^b)]^2}.$$

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