Lemma 4.5. *If (u, v, r) is the mild solution of (2.1) corresponding to an initial condition $(u_0, v_0, r_0) \in Y^+$, then $v \in W^{1,1}[0, l]$. Moreover,*

$$\|v\|_{W^{1,1}[0,l]} \leq \big(1 + (1 + 2\kappa)Ml \max(1, e^{(\omega+M\kappa)l})\big)\, |(u_0, v_0)|_X,$$

where κ denotes an upper bound of $m_1(r)$ and $m_2(r)$. Finally, for $a \leq t \leq l$ the following relation holds

$$u(a, t) = bv(t - a)e^{-\int_{t-a}^{t} m_1(r(s))ds}$$

and $u(., l) \in W^{1,1}[0, l]$.

Proof. Firstly, by the variation of constants formula (2.6) we have

$$v(t) = S_2(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} - \int_0^t S_2(t - s)\begin{pmatrix} m_1(r(s))u(s) \\ m_2(r(s))v(s) \end{pmatrix} ds \tag{4.16}$$

where (u, v, r) is the mild solution of (2.1) for the initial condition (u_0, v_0, r_0). From the proof of Theorem 2.6 we have that $|(u(t), v(t))|_X \leq Me^{(\omega+M\kappa)t}|(u_0, v_0)|_X$ which implies that

$$\|v\|_{L^1[0,l]} \leq Ml \max(1, e^{(\omega+M\kappa)l})|(u_0, v_0)|_X.$$

On the other hand, from (2.3) we have

$$\left\|\frac{d}{dt}S_2(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\right\|_{L^1[0,l]} = \|u_0\|_{L^1[0,l]}$$

and

$$\int_0^t S_2(t-s)\begin{pmatrix} m_1(r(s))u(s) \\ m_2(r(s))v(s) \end{pmatrix} ds = \int_0^t m_2(r(s))v(s)ds + \int_0^t \int_0^{t-s} m_1(r(s))u(l-\sigma, s)\, d\sigma\, ds.$$

Taking the generalized derivative of the first term of the right hand side of this equality we have

$$\|\frac{d}{dt}\int_0^t m_2(r(s))v(s)ds\|_{L^1[0,l]} = \|m_2(r(t))v(t)\|_{L^1[0,l]} \leq \kappa\|v\|_{L^1[0,l]}.$$

Similarly, using Fubini's theorem one proves that

$$\begin{array}{rcl}\|\frac{d}{dt}\int_0^t\int_0^{t-s} m_1(r(s))u(l-\sigma,s)\,d\sigma\,ds\|_{L^1[0,l]} & = & \|\int_0^t m_1(r(s))u(l-t+s,s)\,ds\|_{L^1[0,l]}\\ & = & \int_0^l\int_0^t m_1(r(s))u(l-t+s,s)\,ds\,dt\\ & \le & \kappa\int_0^l\int_s^l u(l-t+s,s)\,dt\,ds\\ & \le & \kappa\int_0^l \|u(s)\|_{L^1[0,l]}\,ds\\ & \le & \kappa Ml\max(1,e^{(\omega+M\kappa)l})|(u_0,v_0)|_X.\end{array}$$

Therefore

$$\|v\|_{W^{1,1}[0,l]} \le \big(1+(1+2\kappa)Ml\max(1,e^{(\omega+M\kappa)l})\big)\,|(u_0,v_0)|_X$$

and

$$bv(l-a)e^{-\int_{l-a}^{l} m_1(r(s))ds} \in W^{1,1}[0,l].$$

Finally, let us prove the last statement. The variation of constants formula (2.6) gives for $a \le t$,

$$\begin{array}{rcl} u(a,t) & = & S_1(t)\begin{pmatrix}u_0\\ v_0\end{pmatrix}(a) - \displaystyle\int_0^{t-a} S_1(t-s)\begin{pmatrix}m_1(r(s))\,u(a,s)\\ m_2(r(r))\,v(s)\end{pmatrix}ds\\ & & -\displaystyle\int_{t-a}^{t} S_1(t-s)\begin{pmatrix}m_1(r(s))\,u(a,s)\\ m_2(r(r))\,v(s)\end{pmatrix}ds\\ & = & bv(t-a)-\displaystyle\int_{t-a}^{t} m_1(r(s))u(a-t+s,s)ds.\end{array}\tag{4.17}$$

The last equality is a consequence of (2.3), which implies $S_1(t)\binom{u_0}{v_0}(a) = b\,S_2(t-a)\binom{u_0}{v_0}$ for $a \le t$, and (4.16).

Now, for any fixed $c \in [0,t]$ and $t \le l$, let $w_c(t) := u(t-c,t)$. Using (4.17) we have

$$w_c(t) = bv(c) - \int_c^t m_1(r(s))w_c(s)\,ds,$$

and so

$$\begin{array}{rcl} w_c'(t) & = & -m_1(r(t))w_c(t)\\ w_c(c) & = & bv(c).\end{array}$$

This is an initial value problem for a linear differential equation whose solution, even in the generalized derivatives frame, is

$$w_c(t) = bv(c)e^{-\int_c^t m_1(r(s))}\,ds.$$

Hence, for $a \le t \le l$ we have, using the definition of $w_c(t)$,

$$u(a,t) = bv(t-a)e^{-\int_{t-a}^{t} m_1(r(s))}\, ds. \quad \square$$

Let us call $\tilde{T}(t), t \ge 0$ the nonlinear C^0-semigroup which yields the solution at time t when it is applied to the initial value. Notice that joint continuity with respect to t and the initial condition follows from Proposition 2.1.

Theorem 4.8. *The nonlinear semigroup $\tilde{T}(t)$ is compact for any $t \ge l$.*

Proof. Since $\tilde{T}(t-l)$ is bounded and $\tilde{T}(t) = \tilde{T}(l)\tilde{T}(t-l)$, it suffices to prove that $\tilde{T}(l)$ is compact. On the other hand, using Rellich's theorem on compact embeddings (see Theorem X-16 of [14]) we need only to show that

$$\tilde{T}(l) : L^1_+[0,l] \times \mathbb{R}^2_+ \longrightarrow W^{1,1}_+[0,l] \times \mathbb{R}^2_+$$

is bounded.

In the proof of Theorem 2.6 we have already shown that the function $r(t)$ is bounded for all $t \ge 0$.

The previous lemma gives the following expression for the generalized derivative of u in the set $\{(a,t) : 0 \le a \le t \le l\}$.

$$u_a(a,t) = -bv'(t-a)e^{-\int_{t-a}^{t} m_1(r(s))ds} - bv(t-a)m_1(r(t-a))e^{-\int_{t-a}^{t} m_1(r(s))ds}$$

Therefore

$$\|u_a(.,l)\|_{L^1[0,l]} \le b\|v'\|_{L^1[0,l]} + b\kappa\|v\|_{L^1[0,l]}.$$

and, using again Lemma 5.4 the proof is complete. $\square$

We now start the study of the properties of dissipativeness of the nonlinear semigroup $\tilde{T}(t), t \ge 0$, under some extra hypotheses on the functions h and L.

In the sequel we will use the following notation. Given a subset R of $\mathbb{R}^2_+$ we call $R^\#$ the subset of Y^+ of the points (u,v,r) such that $(r, p+v) \in R$ where $p = \int_0^l u(a)\, da$.

Theorem 4.9. *Let us assume that there exist numbers $r_b > 0$, $\alpha > 0$ and $\beta > 0$ satisfying respectively $m_2(r_b) = b$; $h(x) \ge \alpha x$ for any $x \ge 0$; and for any $u \ge 0, v \ge 0$,*

$$L(u,v) \ge \beta|(u,v)|_X = \beta(p+v).$$

Then for any $r_1 \geq \max(r_b, r_c)$ *there exists*

$$q(r_1) = \begin{cases} \max\limits_{r\in[r_b,r_1]} \dfrac{r_1 - r_b}{\alpha\beta} \left(\dfrac{g(r)}{r_1 - r} + \dfrac{b - m_2(r)}{r} \right) & \text{if } r_1 > r_b \\ \dfrac{g(r_b)}{\alpha\beta} & \text{if } r_1 = r_b. \end{cases}$$

Moreover let $B_{r_1,q}$ *be the convex hull of vertices* $(0,0), (r_1, 0), (r_b, q)$ *and* $(0, q)$ *where* $q > \max(0, q(r_1))$. *Then* $B^{\#}_{r_1,q}$ *is positively invariant by the semigroup* $\tilde{T}(t), t \geq 0$, *and bounded in* Y.

Proof. Using Theorem 4.1 and the variation of constants formula (2.6) the following two equations are obtained for the mild solutions of Problem (2.1)

$$\begin{cases} r' &= (g(r) - h(L(u, v)))r, \\ (p + v)' &= -m_1(r)p - (m_2(r) - b)v. \end{cases}$$

In order to prove Theorem 4.9 we will show that the scalar product of this tangent

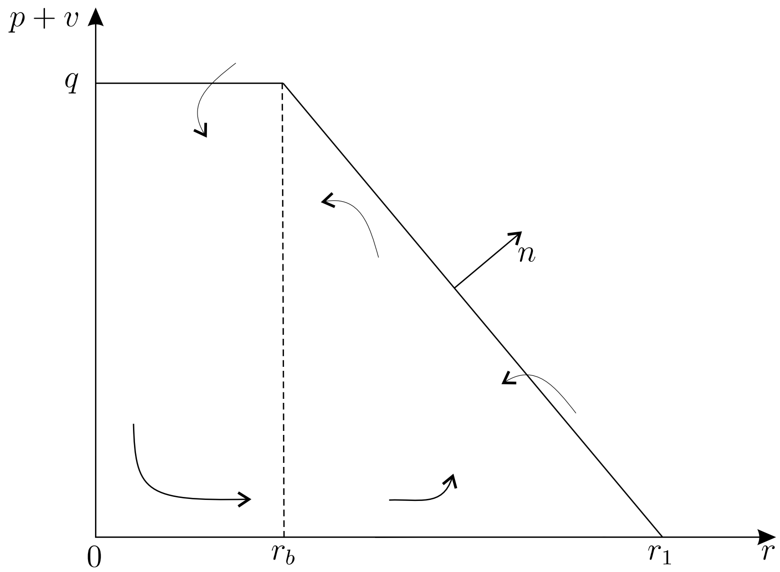

Figure 4.6: projected positively invariant region $B_{r_1,q}$

vector and an orthogonal vector to the boundary of $B_{r_1,q}$ is negative. So, let $n =$

(n_1, n_2) be a vector orthogonal to the segment L of extremities $(r_1, 0)$ and (r_b, q) pointing to the exterior of $B_{r_1,q}$; for instance $n = (q, r_1 - r_b)$. Let $t \geq 0$ be such that $(r(t), (p+v)(t)) \in L$; so $q(r_1 - r) = (p+v)(r_1 - r_b)$. The assumptions of Theorem 4.9 imply that

$$h(L(u,v)) \geq \alpha\, L(u,v) \geq \alpha\beta(p+v);$$

and therefore

$$\left\{ \begin{array}{rcl} r' & \leq & (g(r) - \alpha\beta(p+v))r, \\ (p+v)' & \leq & (b - m_2(r))(p+v). \end{array} \right. \tag{4.18}$$

If $r_1 = r_b$ then $(r', (p+v)') \cdot n = qr(g(r) - h(L(u,v))) \leq 0$. Otherwise, for any $r \in [r_b, r_1)$ it follows that

$$\begin{array}{rcl} (r', (p+v)') \cdot n & \leq & q(g(r) - \alpha\beta(p+v))r + (b - m_2(r))(p+v)(r_1 - r_b) \\ & = & q\left((g(r) - \alpha\beta q\frac{r_1 - r}{r_1 - r_b})r + (b - m_2(r))(r_1 - r)\right) \\ & = & qr\alpha\beta\frac{r_1 - r}{r_1 - r_b}\left(\frac{r_1 - r_b}{\alpha\beta}\left(\frac{g(r)}{r_1 - r} + \frac{b - m_2(r)}{r}\right) - q\right) \\ & < & 0. \end{array}$$

At the point $(r_1, 0)$ the previous scalar product equals $g(r_1)r_1$ which is strictly negative whenever $r_1 > r_c$ and vanishes for $r_1 = r_c$. On the other hand, if $t \geq 0$ is such that $r(t) \in [0, r_b]$ and $(p+v)(t) = q$, then $(p+v)'(t) \leq 0$ with equality only if $r(t) = r_b, p(t) = 0$ and $v(t) = q$. The statement follows using in addition the positivity of the solutions. $\square$

When $b \leq m_2(\infty)$, without assuming the hypotheses of the previous theorem, the following can be easily proved.

Theorem 4.10. *Let $R_{r_1,q} = [0, r_1] \times [0, q]$. If $b \leq m_2(\infty)$ then the set $R^{\#}_{r_1,q}$ is positively invariant by the nonlinear semigroup $\tilde{T}(t), t \geq 0$ and bounded in Y for any $q \geq 0$ and $r_1 \geq r_c$.*

Given a subset C of the phase space Y^+, C is said to attract a point $w_0 \in Y^+$ if for any ε there exists $t_0 > 0$ such that for $t \geq t_0$, $\tilde{T}(t)w_0$ belongs to the $\varepsilon-$ neighbourhood of C.

Theorem 4.11. *Let us assume the hypotheses of Theorem 4.9 or Theorem 4.10.*

i) Let $R_1 = [0, r_c] \times \{0\}$. If $b \leq m_2(\infty)$ then $R_1^\#$ attracts every point of Y^+.

ii) Let $R_2 = [0, r_b] \times \{0\}$ where $m_2(r_b) = b$. If $m_2(\infty) < b \leq m_2(r_c)$ then $R_2^\#$ attracts every point of Y^+.

iii) Finally, Let $R_3 := B_{r_c, q(r_c)}$ defined in Theorem 4.9. If $m_2(r_c) < b < m_2(0)$ then $R_3^\#$ attracts every point of Y^+.

Proof. Let $w_0 \in Y^+$ and let $(r(t), (p+v)(t)), t \geq 0$ be the curve in the first quadrant of $\mathbb{R}^2$ defined by the solution of (2.6) with initial condition w_0. Notice that w_0 belongs to some $B^\#_{r_1,q}$ or $R^\#_{r_1,q}$ defined in the previous theorems. Therefore, this curve is bounded for all $t \geq 0$ and hence its $\omega-$limit set ω, i.e., the set of limit points of sequences $(r(t_n), (p+v)(t_n))$ with $t_n \to \infty$, is non-empty.

Now notice that the proofs of Theorem 4.10 and Theorem 4.9 give respectively that a point x in the interior of the first quadrant and belonging to the boundary of $R_{r_1,q}$ in the case i) and to the boundary of $B_{r_1,q}$ in the cases ii) and iii) cannot belong to ω. Indeed, let us assume that $x \in \omega$, i.e. that any neighbourhood of x contains points of the curve $(r(t), (p+v)(t))$. From the proof of Theorem 4.9 and by continuity of the right hand side of (4.18) it follows that there exists a neighbourhood of x so small that any curve $(r(t), (p+v)(t))$ entering it goes into the positively invariant region and cannot visit this neighbourhood again, which is a contradiction.

On the other hand, let us show that a point not belonging to $R_i, i = 1, 2, 3$ does belong to the boundary of some of these invariant regions.

In the case i) a point $(r_0, q_0) \notin R_1$ belongs to the boundary of R_{r_1,q_0} for $r_1 = \max(r_0, r_c)$.

Let us consider the remaining cases. Notice that in the cases ii) and iii), points $(r_0, 0)$ with $r_0 > \max(r_b, r_c)$ belong to the boundary of $B_{r_0,q}$ whenever $q > q(r_0)$. Notice also that $q(r_1)$ is bounded on bounded subintervals of $[\max(r_b, r_c), \infty)$.

In the case ii) $(r_b \geq r_c)$, a point (r_0, q_0) with $r_0 > r_b$ and $q_0 > 0$ belongs to the boundary of $B_{r_1,q}$ for r_1 sufficiently close to r_0 in order that $q := q_0 \frac{r_1 - r_b}{r_1 - r_0}$ is larger than $q(r_1)$. A point (r_0, q_0) with $r_0 \leq r_b$ and $q_0 > 0$ belongs to the boundary of B_{r_1,q_0} for r_1 sufficiently close to r_b in order that $q(r_1) < 0$.

In the case iii) $(r_b < r_c)$, (r_0, q_0) with $r_0 \leq r_b$ and $q_0 > q(r_c)$ belongs to the boundary of B_{r_c,q_0}; $(r_0, q_0) \notin R_3$ and $r_b < r_0 < r_c$ belongs to the boundary of $B_{r_c,q}$ where

$q := q_0 \frac{r_c - r_b}{r_c - r_0}$ and finally, (r_0, q_0) with $r \geq r_c$ and $q_0 > 0$ belongs to the boundary of $B_{r_1,q}$ for r_1 sufficiently close to r_0 in order that $q := q_0 \frac{r_1 - r_b}{r_1 - r_0}$ is larger than $q(r_1)$.

The previous results show that, in the three cases, the regions R_1, R_2 and R_3 respectively, contain ω. In the case i) (respectively ii) notice that any $\varepsilon-$neighbourhood E of R_1 (respectively R_2) contains a region of the form $R_{r_1,q}$ (respectively $B_{r_1,q}$) which in its turn contains R_1 (respectively R_2) and hence ω. So any curve $(r(t), (p+v)(t))$ eventually enters E and does not leave it. Therefore the corresponding solution $(u(t), v(t), r(t))$ enters $E^{\#}$ which is the $\varepsilon-$neighbourhood of $R_1^{\#}$ (respectively $R_2^{\#}$) and does not leave it. In the third case, as $\omega \neq \varnothing$, any curve $(r(t), (p+v)(t))$ goes into the positively invariant region R_3 and hence, the corresponding solution $(u(t), v(t), r(t))$ enters $R_3^{\#}$ and does not leave it. So in the three cases $R_i^{\#}, i = 1, 2, 3$ attracts every point in Y^+. $\square$

A semigroup is said to be point dissipative if there exists a bounded set that attracts every point of the phase space (see [39, Sect. 3.4]). $\tilde{T}(t)$ is dissipative. Moreover, $\tilde{T}(t)$ is compact for $t \geq l$ by Theorem 4.8 and so, [39, Theorem 3.4.8] gives the following theorem on the existence of a global attractor, i.e., a subset of Y^+ which is a maximal compact invariant set attracting each bounded set contained in Y^+.

Theorem 4.12. *Let us assume the hypotheses of Theorem 4.11. For the nonlinear semigroup $\tilde{T}(t), t \geq 0$ there exists a compact global attractor if $b < m_2(0)$. It is contained in the bounded sets $R_1^{\#}, R_2^{\#}$ or $R_3^{\#}$ depending on the conditions of Theorem 4.11.*

Theorem 4.2 and Theorem 4.12 immediately yield the following

Corollary 4.2. *Under the hypotheses of Theorem 4.9, if $b < m_2(r_c)$, the compact global attractor of $\tilde{T}(t)$ in Y^+ reduces to the set including the equilibrium points $(0, 0, 0)$ and $(0, 0, r_c)$ and the segment joining them. Moreover any solution with $r_0 > 0$ tends to $(0, 0, r_c)$.*

Chapter 5

Small Perturbations of the Age-Dependent Model: Coexistence Equilibrium Point and its Stability

5.1 Introduction

In the second chapter, we have, assuming that $m_2(r) = m_1(r) + constant$, completely determined the dynamics of Problem (2.1) if $g(r) = r$. We have exploited its special form,

$$x' = Ax - m(r)x,$$

to make a reduction to a nonautonomous two dimensional ordinary differential system, which, in turn is studied analyzing its limit system by classical methods (using Markus's theorems, see Subsect 3.3.3). In Chap. 4, without assuming the previous condition, we have not been able to determine the asymptotic behaviour of Problem (2.1). But, we have proved that the coexistence equilibrium point (u_e, v_e, r_e), whenever it exists, turns out to be always asymptotically stable if $m_1'(r_e) = m_2'(r_e)$ (see Theorem 4.6). This is a (local) generalization of the results of Chap. 3 where the coexistence equilibrium solution is shown to be a global attractor if $g(r) = r$. Also, assuming $L(u, v) := \int_0^l u(a)\, da + v$, we proved that the coexistence equilibrium solution (u_e, v_e, r_e) is asymptotically stable if we perturb slightly the death rate of the juveniles $m_1(r)$ or the death rate of the adults $m_2(r)$ or both by functions depending on level resources (see Theorem 4.7).

The subject of this chapter originates in the study, without assuming that $L(u, v) := \int_0^l u(a)\, da + v$, of the stability of the coexistence equilibrium point of Problem (2.1) perturbing the death rate of the juveniles, $m_1(r) = m_2(r) + constant$, by a function $\varepsilon(a, r)$ depending on the age and the amount of the resources. We note that this will give a generalization of the results of Theorem 4.7. Finally, we give a condition about the norm of the function $\varepsilon(a, r)$ in order the coexistence equilibrium point of the perturbed problem is asymptotically stable. We emphasize that the aim has been to obtain a "computable" condition depending only on the parameter functions of the problem (see the main result of this chapter, Theorem 5.3) and not just giving a statement like "for any sufficiently small ε". We note that it is perhaps possible to go further in the sense of proving, using results of [77], that the coexistence equilibrium point of Problem (2.1) is a global attractor for some ε, obviously, forgetting any attempt to compute how small must ε be. We also notice that applying [77], Magal and Webb proved in [57] for a different model with a similar algebraic structure that there is a unique globally asymptotically stable attractor for a sufficiently small perturbation.

In this chapter, we let $m(r) := m_1(r)$ and we assume that $m_2(r) = m(r) + \nu$ where ν is a constant. We also let $(w_e, r_e) := (u_e, v_e, r_e)$. Recall that $u_e = bv_e e^{-m(r_e)a}$, $v_e = h^{-1}(g(r_e))/L(be^{-m(r_e)a}, 1)$ and r_e satisfies $(m(r_e) + \nu)e^{m(r_e)l} = b$. Also notice that $w_e = v_e\varphi$ where $\varphi(a) := (be^{-\lambda^* a}, 1)$ is an eigenvector of the operator A associated to the dominant real eigenvalue $\lambda^* = m(r_e)$.

5.2 Preliminary

Firstly, we state the following proposition which will allow to prove that the coexistence equilibrium point (u_e, v_e, r_e) of the initial value problem (2.1) is hyperbolic.

Proposition 5.1. *The dominant real eigenvalue λ^* of the operator A is simple.*

Proof. Let (u,v) be an eigenvector of A associated to λ^*, so

$$\begin{cases} -u' &= \lambda^* u, \\ -\nu v + u(l) &= \lambda^* v, \\ u(0) &= bv. \end{cases}$$

The first equation and the boundary condition of this system imply that $u(a) = bve^{-\lambda^* a}$. On the other hand, from the second equation we have $be^{-\lambda^* l} - \nu - \lambda^* = 0$ if $v \neq 0$, i.e., if also $u \not\equiv 0$. So, it follows that $(u(a), v) = v\varphi(a)$. In other words, $span\{\varphi\}$ is the kernel of the operator $(A - \lambda^*)$. Now, let us prove that the kernel of the operator $(A-\lambda^*)^2$ is equal to $span\{\varphi\}$. Let $(u_1, v_1) \in D(A)$ such that $(A-\lambda^*)^2(u_1, v_1) = 0$ and $(u_1, v_1) \notin span\{\varphi\}$. Therefore, there exists a complex number $\lambda \neq 0$ such that $(A-\lambda^*)(u_1, v_1) = \lambda\varphi$, i.e.,

$$\begin{cases} -u_1' - \lambda^* u_1 &= \lambda b e^{-\lambda^* a}, \\ -\nu v_1 + u_1(l) - \lambda^* v_1 &= \lambda, \\ u_1(0) &= bv_1. \end{cases} \tag{5.1}$$

The first and the third equations are equivalent to having $u_1(a) = be^{-\lambda^* a}(v_1 - \lambda a)$. The second equation implies that $-\nu v_1 + b(v_1 - \lambda l)e^{-\lambda^* l} - \lambda^* v_1 = \lambda$, i.e., $(be^{-\lambda^* l} - \lambda^* - \nu)v_1 = \lambda(1 + ble^{-\lambda^* l})$ since $v_1 \neq 0$. This is impossible because the term of the left hand side of this last equality is equal to 0 but the term of the right hand side is different from 0. Hence, the proof of the statement of this proposition . $\square$

The following result will be used when we deal with the stability of the coexistence equilibrium point after the perturbation of the age-dependent model (2.1) in the forthcoming subsection.

Theorem 5.1. *Let $m_2(r) = m_1(r) + \nu$. Then the coexistence equilibrium point (u_e, v_e, r_e) of the initial value problem (2.1) is hyperbolic.*

Proof. Firstly, recall that we denote that $m(r) = m_1(r)$. Now, let λ be an eigenvalue of the linear part of Problem (2.1) at the coexistence equilibrium point (w_e, r_e) and let (u, v, r) be its eigenvector. So, from Subsect. 4.2.1 we have the following system

$$\begin{cases} Aw - m(r_e)w - m'(r_e)rw_e &= \lambda w, \\ -r_e h'(Lw_e)Lw + g'(r_e)r_e r &= \lambda r, \end{cases} \tag{5.2}$$

where $(u,v) = w$. Hence,

$$(A - m(r_e) - \lambda)w = m'(r_e)rw_e. \tag{5.3}$$

If $m(r_e)+\lambda \in \sigma(A)$ then $Re\lambda \leq 0$ because $\lambda^* = m(r_e)$ is a dominant real eigenvalue of the operator A. Let us assume that $Re\lambda = 0$, so $\lambda = 0$. From (5.3) it follows that $(A-m(r_e))w = m'(r_e)rw_e$. Thus, w belongs to the kernel of the operator $(A-m(r_e))^2$. On the other hand, w does not belong to the kernel of the operator $A - m(r_e)$. This contradicts the fact that λ^* is a simple eigenvalue of A. Then, in this case $Re\lambda < 0$.

If $m(r_e) + \lambda \in \rho(A)$ then $(A - m(r_e) - \lambda)^{-1}$ exists. So, using Equation (5.3) and the second equation of System (5.2), we obtain

$$-r_e h'(Lw_e)L(A - m(r_e) - \lambda)^{-1} m'(r_e)w_e + g'(r_e)r_e = \lambda.$$

Since $(A - m(r_e) - \lambda)^{-1}w_e = -w_e/\lambda$, it is clear that

$$\frac{m'(r_e)r_e}{\lambda} h'(Lw_e)Lw_e + g'(r_e)r_e - \lambda = 0$$

which implies that

$$\lambda^2 - g'(r_e)r_e\lambda - m'(r_e)r_e h'(Lw_e)Lw_e = 0.$$

The solutions of this quadratic equation are

$$\lambda = \frac{g'(re)r_e \pm \sqrt{(g'(r_e)r_e)^2 + 4m'(r_e)r_e h'(Lw_e)Lw_e}}{2}.$$

In the right hand of this equality, the first term is always negative and, since $m'(r_e) < 0$, its absolute value is bigger than the second term, so $Re\lambda < 0$.

Finally, let $m(r_e) + \lambda \in \rho(A)$ such that λ does not satisfy the last quadratic equation. Let us now prove that λ belongs to the resolvent set of the operator defined by the left side of (5.2) which will be denoted by $\tilde{S}$. In order to do this, let $(f,s) \in Y$ and let us look for $(w,r) \in D(\tilde{S})$ such that $(\tilde{S} - \lambda)(w,r) = (f,s)$. In other words,

$$\begin{cases} (A - m(r_e) - \lambda)w - m'(r_e)rw_e & = & f, \\ -r_e h'(Lw_e)Lw + (g'(r_e)r_e - \lambda)r & = & s. \end{cases} \tag{5.4}$$

The first equation of this system implies that $w = (A - m(r_e) - \lambda)^{-1} f - m'(r_e) r w_e/\lambda$. From the second equation, this yields that

$$\left(\frac{m'(r_e)r_e}{\lambda} h'(Lw_e)Lw_e + g'(r_e)r_e - \lambda\right) r = s.$$

Since $\lambda^2 - g'(r_e)r_e\lambda - m'(r_e)r_e h'(Lw_e)Lw_e \neq 0$, it follows that

$$r = \frac{\lambda s}{\lambda^2 - g'(r_e)r_e\lambda - m'(r_e)r_e h'(Lw_e)Lw_e}.$$

This completes the proof of Theorem 5.1. $\square$

5.3 Perturbation of the death rate of the juveniles

In this section we perturb the death rate of the young in (2.1) with a function $\varepsilon(a,r)$. Thus, Problem (2.1) becomes

$$\left\{\begin{array}{rcl} \binom{u}{v}' & = & A\binom{u}{v} - m(r)\binom{u}{v} - \binom{\varepsilon(a,r)u}{0}, \\ r' & = & (g(r) - h(L(u,v)))\, r, \\ u(a,0) & = & u_0(a), \quad v(0) = v_0, \quad r(0) = r_0. \end{array}\right. \tag{5.5}$$

Notice that we define ε as a function from $Z := [0,l] \times \mathbb{R}_+$ into $\mathbb{R}$ such that, in order to make biological sense, $m(r) + \varepsilon(a,r)$ is non-negative and decreasing as a function of the resources. On the other hand, let $||\varepsilon||_Z := \sup_{r\in \mathrm{R}_+} ||\varepsilon(.,r)||_{L^1[0,l]}$ and $||\varepsilon||_{C^1} := ||\varepsilon||_Z + ||\frac{\partial\varepsilon}{\partial r}||_Z$.

Finally, we note that this section is devoted to study the local stability of the coexistence equilibrium point of (5.5).

5.3.1 Existence of the coexistence equilibrium point

From perturbation theory, the local stability of the coexistence equilibrium point of (5.5) depends on the norm of ε. We start by giving conditions that $||\varepsilon||_Z$ have to satisfy in order to ensure the existence of a nontrivial steady state of (5.5).

Theorem 5.2. *Let*

$$||\varepsilon||_Z \leq \min\left(\ln \frac{b}{m(r_c)+\nu} - lm(r_c), lm(0) - \ln \frac{b}{m(0)+\nu}\right) := M. \tag{5.6}$$

Then there exists a unique coexistence equilibrium point $(u_\varepsilon, v_\varepsilon, r_\varepsilon) =: (w_\varepsilon, r_\varepsilon)$ *of 5.5. Moreover,*

$$u_\varepsilon(a) = bv_\varepsilon e^{-m(r_\varepsilon)a} e^{-\int_0^a \varepsilon(a', r_\varepsilon)da'},$$
$$v_\varepsilon = \frac{h^{-1}(g(r_\varepsilon))}{L(be^{-m(r_\varepsilon)a}e^{-\int_0^a \varepsilon(a, r_\varepsilon)da}, 1)}$$

and

$$(m(r_\varepsilon) + \nu)e^{m(r_\varepsilon)l}e^{\int_0^l \varepsilon(a, r_\varepsilon)da} = b \tag{5.7}$$

Finally, $(w_\varepsilon, r_\varepsilon) \longrightarrow (w_e, r_e)$ *as* $\varepsilon \longrightarrow 0$.

Proof. First, notice that the coexistence equilibrium point of Problem (5.5) satisfies the following system

$$\begin{cases} u'(a) + (m(r) + \varepsilon(a, r))u(a) &= 0, \\ u(l) - (m(r) + \nu)v &= 0, \\ g(r) - h(L(u, v)) &= 0, \\ u(0) - bv &= 0, \end{cases} \tag{5.8}$$

with $u \not\equiv 0, v \neq 0$ and $r \neq 0$. From the first and the last equations in (5.8) we have $u(a) = bve^{-m(r)a}e^{-\int_0^a \varepsilon(a', r)da'}$. On the other hand, the second equation reduces to $(m(r) + \nu)e^{m(r)l}e^{\int_0^l \varepsilon(a', r)da'} = b$. Notice that the left hand side term of this equality is decreasing as function of r. Then, there exists $r_\varepsilon \in (0, r_c)$ satisfying this equality if only if we have

$$(m(r_c) + \nu)e^{m(r_c)l}e^{\int_0^l \varepsilon(a', r_c)da'} < b < (m(0) + \nu)e^{m(0)l}e^{\int_0^l \varepsilon(a', 0)da'}.$$

This is equivalent to

$$\begin{aligned} \textstyle\int_0^l \varepsilon(a', r_c)da' &< \ln \tfrac{b}{m(r_c)+\nu} - lm(r_c) \\ -\textstyle\int_0^l \varepsilon(a', 0)da' &< lm(0) - \ln \tfrac{b}{m(0)+\nu} \end{aligned} \tag{5.9}$$

Notice that the fact that $(m(r_e) + \nu)e^{m(r_e)l} = b$ implies that $\ln \frac{b}{m(r_c)+\nu} - lm(r_c) > 0$ and $lm(0) - \ln \frac{b}{m(0)+\nu} > 0$). Thus, (5.9) is ensured by the assumption (5.6). That is, there exists a unique $r_\varepsilon \in (0, r_c)$ such that $(m(r_\varepsilon) + \nu)e^{m(r_\varepsilon)l}e^{\int_0^a \varepsilon(a', r_\varepsilon)da'} = b$. On the other hand, as h is an increasing unbounded function vanishing at 0 then there exists a unique v_ε such that $h(v_\varepsilon L(be^{-m(r_\varepsilon)a}, 1)) = g(r_\varepsilon)$, and this gives $u_\varepsilon(a) = bv_\varepsilon e^{-m(r_\varepsilon)a}e^{-\int_0^a \varepsilon(a', r_\varepsilon)da'}$. Finally, the second result is straightforward. $\square$

5.3.2 Local stability of the coexistence equilibrium point

The goal of this subsection originates in the study of the local stability of the nontrivial steady state of the perturbed problem (5.5). Let us start computing the linearization of Problem (5.5) at the nontrivial steady state $(w_\varepsilon, r_\varepsilon)$.

Firstly, let $(u, v) = w_\varepsilon + w$ with $w = (\tilde{u}, \tilde{v})$ and $r = r_\varepsilon + s$. Using the Taylor expansion it follows

$$\begin{cases} \binom{\tilde{u}}{\tilde{v}}' &= A\binom{\tilde{u}}{\tilde{v}} + A\binom{u_\varepsilon}{v_\varepsilon} - (m(r_\varepsilon) + m'(r_\varepsilon)s + \ldots)\binom{\tilde{u}+u_\varepsilon}{\tilde{v}+v_\varepsilon} \\ &\quad -\binom{(\varepsilon(a,r_\varepsilon)+\frac{\partial\varepsilon(a,r_\varepsilon)}{\partial r}s+\ldots)(\tilde{u}+u_\varepsilon)}{0}, \\ s' &= (g(r_\varepsilon) + g'(r_\varepsilon)s + \ldots - h(L(u_\varepsilon, v_\varepsilon)) - h'(L(u_\varepsilon, v_\varepsilon))L(\tilde{u}, \tilde{v}) - \ldots)(r_\varepsilon + s). \end{cases}$$

Dropping higher order terms, one obtains the following linear system

$$\begin{cases} \binom{\tilde{u}}{\tilde{v}}' &= A\binom{\tilde{u}}{\tilde{v}} - m(r_\varepsilon)\binom{\tilde{u}}{\tilde{v}} - m'(r_\varepsilon)s\binom{u_\varepsilon}{v_\varepsilon} - \binom{\varepsilon(a,r_\varepsilon)\tilde{u}}{0} - \binom{\frac{\partial\varepsilon(a,r_\varepsilon)}{\partial r}su_\varepsilon}{0}, \\ s' &= g'(r_\varepsilon)r_\varepsilon s - r_\varepsilon h'(L(u_\varepsilon, v_\varepsilon))L(\tilde{u}, \tilde{v}). \end{cases}$$

Now, setting $J_\varepsilon(w, s) := -\binom{\varepsilon(a,r_\varepsilon)\tilde{u}}{0} - \binom{(s\frac{\partial\varepsilon}{\partial r}(a,r_\varepsilon)u_\varepsilon}{0}$, this linear system can be written as

$$\begin{cases} \begin{pmatrix} w \\ s \end{pmatrix}' &= \begin{pmatrix} Aw - m(r_\varepsilon)w - m'(r_\varepsilon)sw_\varepsilon + J_\varepsilon(w,s) \\ -r_\varepsilon h'(Lw_\varepsilon)Lw + g'(r_\varepsilon)r_\varepsilon s \end{pmatrix} \\ &= \begin{pmatrix} Aw - m(r_e)w - m'(r_e)sw_e \\ -r_e h'(Lw_e)Lw + g'(r_e)r_e s \end{pmatrix} \\ &\quad + \begin{pmatrix} (m(r_e) - m(r_\varepsilon))w + (m'(r_e)w_e - m'(r_\varepsilon)w_\varepsilon)s + J_\varepsilon(w,s) \\ (r_e h'(Lw_e) - r_\varepsilon h'(Lw_\varepsilon))Lw - (g'(r_e)r_e - g'(r_\varepsilon)r_\varepsilon)s \end{pmatrix} \end{cases}$$

Let $S(w, s)$ denote the linearization of (5.5) at its coexistence equilibrium point $(u_\varepsilon, v_\varepsilon, r_\varepsilon)$. We note that $S(w, s) = T(w, s) + B(w, s)$ where

$$T(w, s) := \begin{pmatrix} Aw - m(r_e)w - m'(r_e)sw_e \\ -r_e h'(Lw_e)Lw + g'(r_e)r_e s \end{pmatrix}$$

is the linearization of Problem (2.1) at its equilibrium point (w_e, r_e) and

$$B(w,s) := \begin{pmatrix} (m(r_e) - m(r_\varepsilon))w + (m'(r_e)w_e - m'(r_\varepsilon)w_\varepsilon)s + J_\varepsilon(w,s) \\ (r_e h'(Lw_e) - r_\varepsilon h'(Lw_\varepsilon))Lw - (g'(r_e)r_e - g'(r_\varepsilon)r_\varepsilon)s \end{pmatrix}.$$

Now we give the following lemmas which allow to prove the main result of this chapter. From now on, let $A_0 := A - \lambda^* I$.

Lemma 5.1. *If the real part of λ is non-negative then*

$$\left\|\lambda(A_0 - \lambda)^{-1}\right\| \leq \left(|\lambda| l + \frac{1 + bl}{K}\right)$$

and for any element $(f, \alpha) \in X$, we have

$$\left|(L\varphi)(A_0 - \lambda)^{-1}\binom{f}{\alpha} - \left(L(A_0 - \lambda)^{-1}\binom{f}{\alpha}\right)\varphi\right|_X \leq \frac{l(2K + bl)}{K}||L||\ |\varphi|_X |(f,\alpha)|_X$$

where the non-negative constant K depends on $be^{-\lambda^* l}$, and we recall that $\varphi(a) := (be^{-\lambda^* a}, 1)$.

Proof. Let $(u, v) \in D(A)$ such that $(A_0 - \lambda)^{-1}(f, \alpha) = (u, v)$, i.e.,

$$\begin{cases} -u' - (\lambda^* + \lambda)u &= f, \\ u(l) - (\nu + \lambda^* + \lambda)v &= \alpha, \\ u(0) &= bv. \end{cases}$$

Using the variation of constants formula, it follows that the first and the last equations of this system are equivalent to having

$$u(a) = bve^{-(\lambda^* + \lambda)a} - \int_0^a e^{-(\lambda^* + \lambda)(a-s)} f(s) ds. \tag{5.10}$$

After, if $be^{-(\lambda^* + \lambda)l} - (\nu + \lambda^* + \lambda) \neq 0$ then the second equation yields that

$$v = \frac{\alpha + \int_0^l e^{-(\lambda^* + \lambda)(l-s)} f(s) ds}{be^{-(\lambda^* + \lambda)l} - (\nu + \lambda^* + \lambda)}.$$

We note that the resolvent operator of A_0 is not defined at 0 since $be^{-\lambda^* l} - (\nu + \lambda^*) = 0$. For any strictly positive real number γ let us consider the complex function, for

$\lambda \geq 0$, $\psi_\gamma(\lambda) := \gamma(e^{-\lambda l} - 1)/\lambda$. Firstly, notice that this function tends to $-\gamma l$ as λ goes to 0 and tends to 0 as $|\lambda|$ goes to infinity (from now we define $\psi_\gamma(0) = -\gamma l$). So, in particular, there exists $R > 0$ such that $|\psi_\gamma(\lambda)| < 1/2$ whenever $|\lambda| > R$. Moreover, the mean-value theorem implies that $|(e^{-\lambda l} - 1)/\lambda| \leq l$. Therefore, the function $\psi_\gamma(\lambda)$ transforms the half plane $\{\lambda \in \mathbb{C}; Re\lambda \geq 0\}$ into a compact set which does not contain 1 for any $\gamma > 0$. Indeed, for $\lambda \neq 0$, $\psi_\gamma(\lambda) = 1$ is equivalent to the equation $e^{-\lambda l} = 1+\frac{\lambda}{\gamma}$ which does not have a non-vanishing solution with non-negative real part because the modulus of the left hand side is less or equal to 1 whereas the modulus of the right hand side is strictly larger than 1. Therefore, there exists a positive real number $K \neq 0$ such that

$$\left|\frac{\lambda}{be^{-(\lambda^*+\lambda)l} - (\nu + \lambda^* + \lambda)}\right| = \left|\frac{1}{be^{-\lambda^* l}\left(\frac{e^{-\lambda l}-1}{\lambda}\right) - 1}\right| \leq \frac{1}{K}.$$

Then we have

$$|\lambda v| \leq \frac{|\alpha| + ||f||_{L^1[0,l]}}{K} \tag{5.11}$$

and

$$||\lambda u||_{L^1[0,l]} \leq bl\frac{|\alpha| + ||f||_{L^1[0,l]}}{K} + |\lambda| l ||f||_{L^1[0,l]}.$$

Therefore,

$$\begin{array}{rcl} \left|\lambda(A_0 - \lambda)^{-1}\binom{f}{\alpha}\right|_X & \leq & \frac{1+bl}{K}|\alpha| + \left(|\lambda|l + \frac{1+bl}{K}\right)||f||_{L^1[0,l]} \\ & \leq & \left(|\lambda|l + \frac{1+bl}{K}\right)|(f,\alpha)|_X \end{array}$$

which implies that

$$\left\|\lambda(A_0 - \lambda)^{-1}\right\| \leq \left(|\lambda|l + \frac{1+bl}{K}\right).$$

For the second statement, let us add and subtract $bve^{-\lambda^* a}$ in the right hand term of (5.10). This allow to write the following

$$\binom{u}{v}(a) = v\varphi(a) + \binom{bve^{-\lambda^* a}\left(e^{-\lambda a} - 1\right) - \int_0^a e^{-(\lambda^*+\lambda)(a-s)}f(s)ds}{0}.$$

Thus, it follows that

$$\begin{array}{rl} & (L\varphi)(A_0 - \lambda)^{-1}\binom{f}{\alpha} - \left(L(A_0 - \lambda)^{-1}\binom{f}{\alpha}\right)\varphi \\ = & L\varphi\begin{pmatrix} bve^{-\lambda^* a}\left(e^{-\lambda a} - 1\right) - \int_0^a e^{-(\lambda^*+\lambda)(a-s)}f(s)ds \\ 0 \end{pmatrix} \\ & -L\left(bve^{-\lambda^* a}\left(e^{-\lambda a} - 1\right) - \int_0^a e^{-(\lambda^*+\lambda)(a-s)}f(s)ds, 0\right)\varphi. \end{array}$$

Finally, using (5.11) and the fact that $|(e^{-\lambda l}-1)/\lambda| \leq l$ we obtain

$$\begin{array}{rl} & \left|(L\varphi)(A_0-\lambda)^{-1}\binom{f}{\alpha} - \left(L(A_0-\lambda)^{-1}\binom{f}{\alpha}\right)\varphi\right|_X \\ \leq & 2\left(bl^2\frac{|\alpha|+||f||_{L^1[0,l]}}{2K} + l||f||_{L^1[0,l]}\right)||L||\ |\varphi|_X \\ = & 2\left(\frac{bl^2}{2K}|\alpha| + \frac{l(2K+bl)}{2K}||f||_{L[0,l]}\right)||L||\ |\varphi|_X \\ \leq & \frac{l(2K+bl)}{K}||L||\ |\varphi|_X\ |(f,\alpha)|_X. \end{array}$$

Lemma 5.2. *Let λ be a complex number such that its real part is non-negative. Then, the following holds*

$$\begin{array}{rl} \|(T-\lambda)^{-1}\| \leq & \max\left(\frac{|\lambda|+c\,|\varphi|_X}{|\lambda(\lambda+d)+ckL\varphi|}, \frac{|\lambda|lK+1+bl}{|\lambda(\lambda+d)+ckL\varphi|K}(|\lambda+d|+k\,||L||)\right. \\ & \left. + \frac{ckl(2K+bl)}{|\lambda(\lambda+d)+ckL\varphi|K}||L||\,|\varphi|_X\right), \end{array}$$

where $c := -m'(r_e)ve$, $d := -r_e g'(r_e)$ and $k := r_e h'(veL\varphi)$ are three strictly positive constants.

Proof. Let $(w,s) = (T-\lambda)^{-1}(f,\alpha)$. That is,

$$\left\{\begin{array}{rcl} Aw - \lambda^* w - vem'(r_e)s\varphi - \lambda w & = & f, \\ r_e g'(r_e)s - r_e h'(veL\varphi)Lw - \lambda s & = & \alpha, \end{array}\right.$$

which is equivalent to having

$$\left\{\begin{array}{rcl} (A_0-\lambda)w + cs\varphi & = & f, \\ -ds - kLw - \lambda s & = & \alpha, \end{array}\right. \tag{5.12}$$

where $A_0 := A-\lambda^*$. As the functions m and g are strictly decreasing, h is an increasing function, L is a linear positive form and ve is positive, so c, d and k are non-negative constants. Thus, $Re\lambda \geq 0$ implies that $\lambda(\lambda+d)+ckL\varphi \neq 0$. Therefore, using the second equation of (5.12), $s = -\frac{\alpha+kLw}{\lambda+d}$, which implies using the first equation of this system that

$$w = (A_0-\lambda)^{-1}f + c\frac{\alpha+kLw}{\lambda+d}(A_0-\lambda)^{-1}\varphi. \tag{5.13}$$

After, using the fact that $(A_0-\lambda)^{-1}\varphi = -\varphi/\lambda$, it follows that

$$Lw = \frac{\lambda(\lambda+d)}{\lambda(\lambda+d)+ckL\varphi}\left(L(A_0-\lambda)^{-1}f - \frac{c\alpha}{\lambda(\lambda+d)}L\varphi\right). \tag{5.14}$$

Substituting now Lw with this expression in (5.13), we have

$$\begin{aligned} w &= (A_0-\lambda)^{-1}f - \tfrac{c}{\lambda(\lambda+d)}\left(\alpha + \tfrac{k\left(\lambda(\lambda+d)L(A_0-\lambda)^{-1}f - c\alpha L\varphi\right)}{\lambda(\lambda+d)+ckL\varphi}\right)\varphi \\ &= (A_0-\lambda)^{-1}f - c\tfrac{\alpha+kL(A_0-\lambda)^{-1}f}{\lambda(\lambda+d)+ckL\varphi}\varphi \\ &= \tfrac{1}{\lambda(\lambda+d)+ckL\varphi}\lambda(\lambda+d)(A_0-\lambda)^{-1}f + \tfrac{c}{\lambda(\lambda+d)+ckL\varphi}\left(kL\varphi(A_0-\lambda)^{-1}f\right. \\ &\quad \left. - k\left(L(A_0-\lambda)^{-1}f\right)\varphi - \alpha\varphi\right). \end{aligned}$$

Using also (5.14), it follows that

$$\begin{aligned} s &= \tfrac{-1}{\lambda+d}\left(\alpha + k\tfrac{\lambda(\lambda+d)L(A_0-\lambda)^{-1}f - c\alpha L\varphi}{\lambda(\lambda+d)+ckL\varphi}\right) \\ &= -\lambda\tfrac{\alpha+kL(A_0-\lambda)^{-1}f}{\lambda(\lambda+d)+ckL\varphi}. \end{aligned}$$

On the other hand, we obtain, from Lemma 5.1,

$$\begin{aligned} |w|_X &\le \tfrac{1}{|\lambda(\lambda+d)+ckL\varphi|K}|\lambda+d|\left(|\lambda|lK+1+bl\right)|f|_X \\ &\quad + \left|\tfrac{ck}{\lambda(\lambda+d)+ckL\varphi}\right|\left(\tfrac{l(2K+bl)}{K}||L||\,|\varphi|_X\right)|f|_X + \tfrac{c\,|\varphi|_X}{|\lambda(\lambda+d)+ckL\varphi|}|\alpha| \end{aligned}$$

and

$$\begin{aligned} |s| &\le \tfrac{|\lambda|\ |\alpha| + k\left(|\lambda|l + \frac{1+bl}{K}\right)||L||\ |f|_X}{|\lambda(\lambda+d)+ckL\varphi|} \\ &= \tfrac{|\lambda|}{|\lambda(\lambda+d)+ckL\varphi|}|\alpha| + \tfrac{k}{|\lambda(\lambda+d)+ckL\varphi|K}\left(|\lambda|lK+1+bl\right)||L||\ |f|_X. \end{aligned}$$

That is,

$$\begin{aligned} |(w,s)|_Y &\le \tfrac{|\lambda|+c\,|\varphi|_X}{|\lambda(\lambda+d)+ckL\varphi|}|\alpha| + \tfrac{|\lambda|lK+1+bl}{|\lambda(\lambda+d)+ckL\varphi|K}(|\lambda+d|+k\,||L||)|f|_X \\ &\quad + \tfrac{ckl(2K+bl)}{|\lambda(\lambda+d)+ckL\varphi|K}||L||\,|\varphi|_X|f|_X. \end{aligned}$$

Hence,

$$\begin{aligned} \|(T-\lambda)^{-1}\| &\le \max\left(\tfrac{|\lambda|+c\,|\varphi|_X}{|\lambda(\lambda+d)+ckL\varphi|}, \tfrac{|\lambda|lK+1+bl}{|\lambda(\lambda+d)+ckL\varphi|K}(|\lambda+d|+k\,||L||)\right. \\ &\qquad \left. + \tfrac{ckl(2K+bl)}{|\lambda(\lambda+d)+ckL\varphi|K}||L||\,|\varphi|_X\right). \quad \square \end{aligned}$$

Now, from the result of the last lemma, we can conclude the following

Corollary 5.1. *Let λ be a complex number such that its real part is non negative. Then, we have*

$$\begin{aligned} \|(T-\lambda)^{-1}\| &\le \max\left(\tfrac{cd|\varphi|_X+\sqrt{\tilde{K}(\beta',d)}}{d\sqrt{\tilde{K}(\beta',d)}}\quad , \quad \tfrac{(1+bl)\left(d^2+k||L||d+\sqrt{\tilde{K}(\beta',d)}\right)+clkd(2K+bl)||L||\,|\varphi|_X}{dK\sqrt{\tilde{K}(\beta',d)}}\right. \\ &\quad \left. + \tfrac{lk||L||}{d} + l\tilde{C}(\beta',d)\right) =: M_1 \end{aligned}$$

where c, d and k are defined in Lemma 5.2, $\beta' := ckL\varphi$,

$$\tilde{K}(\beta', d) := \begin{cases} \beta'^2 & if \quad \beta' \leq \frac{d^2}{2} \\ \frac{d^2}{4}(4\beta' - d^2) & if \quad \beta' \geq \frac{d^2}{2} \end{cases}$$

and

$$\tilde{C}(\beta', d) := \sqrt{\frac{1}{1 - 4\beta' \frac{\sqrt{\beta'^2 + 2\beta' d^2}}{\left(\beta' + \sqrt{\beta'^2 + 2\beta' d^2}\right)\left(\beta' + 2d^2 + \sqrt{\beta'^2 + 2\beta' d^2}\right)}}}$$

Proof. Let $\lambda \in \mathbb{C}$ such that $Re\lambda \geq 0$, so from Lemma 5.2 we have

$$||(T - \lambda)^{-1}|| \leq \max(I(\lambda), J(\lambda))$$

where

$$I(\lambda) := \frac{(|\lambda| + c|\varphi|_X)}{|\lambda(\lambda + d) + ckL\varphi|}$$

and

$$J(\lambda) := \frac{(|\lambda| lK + 1 + bl)\,(|\lambda + d| + k\,||L||) + ckl(2K + bl)||L||\,|\varphi|_X}{|\lambda(\lambda + d) + ckL\varphi| K}.$$

Let now $\psi_1(\lambda) := |\lambda|/|\lambda(\lambda + d) + ckL\varphi|$ and $\psi_2(\lambda) := c|\varphi|_X/|\lambda(\lambda + d) + ckL\varphi|$. So $I(\lambda) = \psi_1(\lambda) + \psi_2(\lambda)$.

Firstly as $Re\lambda > 0$,

$$\psi_1(\lambda) = \frac{1}{|\lambda + d + ckL\varphi/\lambda|} \leq \frac{1}{d}.$$

Moreover, since $\psi_2(\lambda)$ tends to 0 as $|\lambda|$ goes to infinity, then, by the maximum modulus theorem we can affirm that the function ψ_2 attains its maximum on the imaginary axis. Now, for any $\alpha \in \mathbb{R}$, $\psi_2(\alpha i) = c|\varphi|_X/\sqrt{(\alpha^2 - \beta')^2 + \alpha^2 d^2}$. After, a simple computation shows that $\tilde{K}(\beta', d)$ is the minimum value of the real function $\phi_1(\alpha) := (\alpha^2 - \beta')^2 + \alpha^2 d^2$. So $|\psi_2(\lambda)| \leq c|\varphi|_X/\sqrt{\tilde{K}(\beta', d)}$. Therefore,

$$I(\lambda) \leq \frac{1}{d} + \frac{c|\varphi|_X}{\sqrt{\tilde{K}(\beta', d)}}.$$

On the other hand, we have

$$\begin{aligned} J(\lambda) &= \frac{((1+bl)+cl(2K+bl)|\varphi|_X)\,k||L||}{|\lambda(\lambda+d)+ckL\varphi|K} + \frac{|\lambda|lk||L||}{|\lambda(\lambda+d)+ckL\varphi|} \\ &+ \frac{(1+bl)|\lambda+d|}{|\lambda(\lambda+d)+ckL\varphi|K} + \frac{|\lambda||\lambda+d|l}{|\lambda(\lambda+d)+ckL\varphi|}. \end{aligned}$$

Using the previous results, it follows immediately that

$$\frac{((1+bl)+cl(2K+bl)|\varphi|_X)\,k||L||}{|\lambda(\lambda+d)+ckL\varphi|K} \le \frac{((1+bl)+cl(2K+bl)|\varphi|_X)\,k||L||}{K\sqrt{\tilde{K}(\beta',d)}},$$

$$\frac{|\lambda|lk||L||}{|\lambda(\lambda+d)+ckL\varphi|} \le \frac{lk||L||}{d}$$

and

$$\begin{aligned} \frac{(1+bl)|\lambda+d|}{|\lambda(\lambda+d)+ckL\varphi|K} &\le \frac{(1+bl)(|\lambda|+d)}{|\lambda(\lambda+d)+ckL\varphi|K} \\ &\le \frac{(1+bl)\left(d^2+\sqrt{\tilde{K}(\beta',d)}\right)}{dK\sqrt{\tilde{K}(\beta',d)}}. \end{aligned}$$

Now let $l\psi_3(\lambda)$ denote the fourth term of the right hand side of the previous equality. As $\psi_3(\lambda) \longrightarrow 1$ as $|\lambda| \longrightarrow \infty$, by the maximum modulus theorem again, the function ψ_3 has a maximum on the imaginary axis (because it takes values larger than 1 on it). We have $\psi_3(0) = 0$ and for any real number $\alpha \neq 0$

$$\begin{aligned} \psi_3(\alpha i) &= \sqrt{\frac{\alpha^4+\alpha^2d^2}{(\alpha^2-\beta')^2+\alpha^2d^2}} = \sqrt{\frac{\alpha^4+\alpha^2d^2}{\alpha^4-2\alpha^2\beta'+\beta'^2+\alpha^2d^2}} \\ &= \sqrt{\frac{1}{(1+\beta'\frac{\beta'-2\alpha^2}{\alpha^2(\alpha^2+d^2)}}} \le \sqrt{\frac{1}{(1+\beta'\min_{x\in\mathbb{R}_+}\phi_2(x)}} \end{aligned}$$

where, for any $x > 0, \phi_2(x) := (\beta'-2x)/x(x+d^2)$ (notice $\psi_3(\alpha i) > 1$ for $\alpha > \sqrt{\beta'/2}$). A simple computation shows also that $(\beta' + \sqrt{\beta'^2 + 2\beta' d^2})/2$ is the point of the minimum of the function ϕ_2, that is $\psi_3(\lambda) \le \tilde{C}(\beta', d)$. Finally, we obtain

$$\begin{aligned} J(\lambda) \le\ & \frac{(1+bl)}{dK\sqrt{\tilde{K}(\beta',d)}}\left(d(d+k||L||)+\sqrt{\tilde{K}(\beta',d)}\right) + \frac{ckl(2K+bl)||L|||\varphi|_X}{K\sqrt{\tilde{K}(\beta',d)}} \\ &+ \frac{lk||L||}{d} + l\tilde{C}(\beta',d). \quad \square \end{aligned}$$

Lemma 5.3. *Let (u_e, v_e, r_e) and $(u_\varepsilon, v_\varepsilon, r_\varepsilon)$ respectively be the two nontrivial equilibrium points of the problem (2.1) and the perturbed problem (5.5). Then, we have*

$$|r_e - r_\varepsilon| \leq \frac{(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1}||\varepsilon||_Z.$$

Furthermore if, for some real number M_2,

$$\|\varepsilon\|_Z \leq M_2 < \frac{2c_1 L(be^{-m(r_e)a},1)}{||L||bl\left(2c_1 + c_3 l m(r_e) e^{(m(0)+m(r_e)l}\right)}$$

then

$$\begin{aligned}||(u_e,v_e)-(u_\varepsilon,v_\varepsilon)||_Y \quad \leq \quad & \Bigg[\Bigg(\frac{h^{-1}(g(0))||L||bl\left(2c_1+c_3 l(m(r_e)+\nu)e^{(m(0)+m(r_e))l}\right)}{2c_1 L(be^{-m(r_e)a},1)-||L||M_2 bl\left(2c_1+c_3 l(m(r_e)+\nu)e^{(m(0)+m(r_e))l}\right)} \\ & + \frac{c_2(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1}\Bigg)(1+bl) \\ & + \frac{1}{2}h^{-1}(g(0))bl\left(2c_1 + c_3 l(m(r_e)+\nu)e^{(m(0)+\,m(r_e))l}\right)\Bigg] \\ & \frac{1}{L(be^{-m(r_e)a},1)}||\varepsilon||_Z =: \tilde{M}_2||\varepsilon||_Z.\end{aligned}$$

where $c_1 := \min_{r\in[0,r_c]}|m'(r)|$, $c_2 := \max_{r\in[0,r_c]}|g(r)/h'(h^{-1}g(r))|$ and $c_3 := \max_{r\in[0,r_c]}|m'(r)|$.

Proof. Using the fact that $(m(r_e)+\nu)e^{m(r_e)l} = b$ and (5.7), it follows

$$\left(\tilde{f}(m(r_e)) - \tilde{f}(m(r_\varepsilon))\right)e^{\int_0^l \varepsilon(a,r_\varepsilon)da} + \tilde{f}(m(r_e))\left(1 - e^{\int_0^l \varepsilon(a,r_\varepsilon)da}\right) = 0$$

where $\tilde{f}$ is a real function such that $\tilde{f}(x) := (x+\nu)e^{xl}$. Now, from the mean-value theorem it is clear that there exists $\xi \in (m(r_c), m(0))$ such that

$$\tilde{f}'(\xi)|m(r_e) - m(r_\varepsilon)| = \tilde{f}(m(r_e))e^{m(0)l}\left|e^{-m(0)l} - e^{-\int_0^l(m(0)+\varepsilon(a,r_\varepsilon))da}\right|.$$

the mean-value theorem again, it follows that there exists $\eta \in (0, r_c)$ and ξ' between $-m(0)l$ and $-\int_0^l(m(0)+\varepsilon(a,r_\varepsilon))da$ such that

$$\tilde{f}'(\xi)|m'(\eta)||r_e - r_\varepsilon| \leq \tilde{f}(m(r_e))e^{m(0)l}e^{\xi'}\int_0^l |\varepsilon(a,r_\varepsilon)|da.$$

Since $m(\infty) > -\nu$ so $\xi > -\nu$. This implies that $\tilde{f}'(\xi) = (1 + l(\xi+\nu)e^{\xi l} \geq 1$. On the other hand, it is clear that $\xi' < 0$. Therefore

$$|r_e - r_\varepsilon| \leq \frac{(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1}||\varepsilon||_Z.$$

Let now

$$I'_{e,\varepsilon} := \left|\frac{1}{L(be^{-m(r_e)a},1)} - \frac{1}{L(be^{-m(r_\varepsilon)a}e^{-\int_0^a \varepsilon(a',r_\varepsilon)},1)}\right|$$

and

$$J'_{e,\varepsilon} := \frac{|h^{-1}g(r_e) - h^{-1}g(r_\varepsilon)|}{L(be^{-m(r_e)a},1)}.$$

Then, we have

$$|v_e - v_\varepsilon| \leq h^{-1}(g(r_\varepsilon))I'_{e,\varepsilon} + J'_{e,\varepsilon}.$$

On the other hand, using the first statement and the mean-value theorem it follows that

$$\begin{array}{rcl} |J'_{e,\varepsilon}| & \leq & \frac{1}{L(be^{-m(r_e)a},1)}\max_{r\in[0,r_c]}\left|\frac{g'(r)}{h'(h^{-1}(g(r)))}\right||r_e - r_\varepsilon| \\ & \leq & \frac{c_2(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1 L(be^{-m(r_e)a},1)}||\varepsilon||_Z \end{array}$$

and

$$\begin{array}{rcl} ||be^{-m(r_e)a} - be^{-m(r_\varepsilon)a}e^{-\int_0^a \varepsilon(a',r_\varepsilon)da'}||_{L^1[0,l]} & \leq & b\int_0^l \Big|m(r_e)a - m(r_\varepsilon)a \\ & & -\int_0^a \varepsilon(a',r_\varepsilon)da'\Big|\, da \\ & \leq & \frac{bl^2}{2}\max_{r\in[0,r_c]}|m'(r)||r_e - r_\varepsilon| + bl||\varepsilon||_Z \\ & \leq & bl\left(1 + \frac{c_3 l(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{2c_1}\right)||\varepsilon||_Z. \end{array}$$

Let us assume now that there exists a constant M_2 such that

$$\|\varepsilon\|_Z \leq M_2 < \frac{2c_1 L(be^{-m(r_e)a},1)}{||L||bl\left(2c_1 + c_3 l(m(r_e)+\nu)e^{(m(0)+m(r_e)l}\right)}.$$

Then,

$$\begin{array}{rcl} I'_{e,\varepsilon} & = & \frac{\left|L(be^{-m(r_e)a},1) - L(be^{-m(r_\varepsilon)a}e^{-\int_0^a \varepsilon(a',r_\varepsilon)da'},1)\right|}{L(be^{-m(r_e)a},1)L(be^{-m(r_\varepsilon a}e^{-\int_0^a \varepsilon(a',r_\varepsilon)da'},1)} \\ & = & \left|\frac{L(be^{-m(r_e)a},1) - L(be^{-m(r_\varepsilon)a}e^{-\int_0^a \varepsilon(a',r_\varepsilon)da'},1)}{L(be^{-m(r_e)a},1)\Big(L(be^{-m(r_e)a},1) + L\Big(\Big(be^{-m(r_\varepsilon)a}e^{-\int_0^a \varepsilon(a',r_\varepsilon)da'},1\Big) - \Big(be^{-m(r_e)a},1\Big)\Big)\Big)}\right| \\ & \leq & \frac{||L||bl\Big(2c_1 + c_3 l(m(r_e)+\nu)e^{(m(0)+m(r_e)l}\Big)||\varepsilon||_Z}{L(be^{-m(r_e)a},1)\Big(2c_1 L(be^{-m(r_e)a},1) - ||L||M_2 bl\Big(2c_1 + c_3 l(m(r_e)+\nu)e^{(m(0)+m(r_e))l}\Big)\Big)}. \end{array}$$

This yields that

$$
\begin{aligned}
|v_e - v_\varepsilon| \quad \leq \quad & \Bigg(\tfrac{h^{-1}(g(0))||L||bl\left(2c_1+c_3 l(m(r_e)+\nu)e^{(m(0)+m(r_e))l}\right)}{L(be^{-m(r_e)a},1)\left(2c_1 L(be^{-m(r_e)a},1)-||L||M_2 bl\left(2c_1+c_3 l(m(r_e)+\nu)e^{(m(0)+m(r_e))l}\right)\right)} \\
& + \tfrac{c_2(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1 L(be^{-m(r_e)a},1)}\Bigg) ||\varepsilon||_Z
\end{aligned}
$$

and

$$
\begin{aligned}
|u_e - u_\varepsilon|_{L^1[0,l]} \quad = \quad & \Big\| bv_e \left(e^{-m(r_e)a} - e^{-m(r_\varepsilon)a} e^{-\int_0^a \varepsilon(a',r_\varepsilon)da'}\right) \\
& + b(v_e - v_\varepsilon)e^{-\int_0^a (m(r_\varepsilon)+\varepsilon(a',r_\varepsilon))da'} \Big\|_{L^1[0,l]} \\
\leq \quad & \tfrac{h^{-1}(g(0))bl}{2c_1 L(be^{-m(r_e)a},1)} \left(2c_1 + c_3 l(m(r_e)+\nu)e^{(m(0)+m(r_e))l}\right) ||\varepsilon||_Z \\
& + \Bigg(\tfrac{h^{-1}(g(0))||L||b^2 l^2\left(2c_1+c_3 l(m(r_e)+\nu)e^{(m(0)+m(r_e))l}\right)}{L(be^{-m(r_e)a},1)\left(2c_1 L(be^{-m(r_e)a},1)-||L||M_2 bl\left(2c_1+c_3 l(m(r_e)+\nu)e^{(m(0)+m(r_e))l}\right)\right)} \\
& + bl \tfrac{c_2(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1 L(be^{-m(r_e)a},1)}\Bigg) ||\varepsilon||_Z .
\end{aligned}
$$

Finally, it follows that

$$
\begin{aligned}
||(u_e,v_e)-(u_\varepsilon,v_\varepsilon)||_Y \quad \leq \quad & \Bigg[\Bigg(\tfrac{h^{-1}(g(0))||L||bl\left(2c_1+c_3 l(m(r_e)+\nu)e^{(m(0)+m(r_e))l}\right)}{2c_1 L(be^{-m(r_e)a},1)-||L||M_2 bl\left(2c_1+c_3 l(m(r_e)+\nu)e^{(m(0)+m(r_e))l}\right)} \\
& + \frac{c_2(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1}\Bigg)(1+bl) \\
& + \frac{1}{2}h^{-1}(g(0))bl\left(2c_1 + c_3 l(m(r_e)+\nu)e^{(m(0)+\ m(r_e))l}\right)\Bigg] \\
& \frac{1}{L(be^{-m(r_e)a},1)}||\varepsilon||_Z . \quad \square
\end{aligned}
$$

Lemma 5.4. *If there exists a constant M_2 such that*

$$
\|\varepsilon\|_Z \leq M_2 < \frac{2c_1 L(be^{-m(r_e)a},1)}{||L||bl\left(2c_1 + c_3 l(m(r_e)+\nu)e^{(m(0)+m(r_e))l}\right)},
$$

then

$$
\|B(w,s)\|_Y \leq M_3 \max\left(||\varepsilon||_\infty, ||\varepsilon||_{C^1}\right) ||(w,s)||_Y
$$

where c_1, c_2, c_3 are all defined in the last Lemma,

$$
\begin{aligned}
M_3 \quad := \quad & \Bigg(1 + \tfrac{c_3(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1} + \tfrac{c_5(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1}||w_e||_X \\
& + \tilde{M}_2 c_3 + \tfrac{2c_1 bh^{-1}(g(0))}{2c_1 L(be^{-m(r_e)a},1)-||L||M_2 bl\left(2c_1+c_3 l(m(r_e)+\nu)e^{(m(0)l+m(r_e)l}\right)} \\
& + \tfrac{h'(Lw_e)(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1}||L|| + \tilde{M}_2 r_c c_6 ||L||^2 \\
& + \tfrac{c_7(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1}\Bigg),
\end{aligned}
$$

$c_4 := \min_{r\in[0,r_c]} L(be^{-m(r)a},1), c_5 := \max_{r\in[0,r_c]} |m''(r)|, c_6 := ||h''||_\infty$ *and*
$c_7 := \max_{r\in[0,r_c]} |g'(r)+rg''(r)|$.

Proof. Using the fact that $m(r_\varepsilon)+\varepsilon(a,r_\varepsilon)>0$ for any $a\in[0,l]$ and the proof of Lemma 5.3 we have

$$\begin{aligned}\|u_\varepsilon\|_\infty &\le bv_\varepsilon \le \frac{bh^{-1}(g(0))}{L(be^{-m(r_e)a},1)+L\Big(\big(be^{-m(r_\varepsilon)a}e^{-\int_0^a \varepsilon(a',r_\varepsilon)da'},1\big)-\big(be^{-m(r_e)a},1\big)\Big)}\\ &\le \frac{2c_1bh^{-1}(g(0))}{2c_1L(be^{-m(r_e)a},1)-||L||M_2bl\big(2c_1+c_3l(m(r_e)+\nu)e^{(m(0)+m(r_e))l}\big)}.\end{aligned}$$

This implies

$$\begin{aligned}||J_\varepsilon(w,s)||_Y \le\ & ||\varepsilon||_\infty||w||_X\\ &+\frac{2c_1bh^{-1}(g(0))}{2c_1L(be^{-m(r_e)a},1)-||L||M_2bl\big(2c_1+c_3l(m(r_e)+\nu)e^{(m(0)+m(r_e))l}\big)}||\tfrac{\partial\varepsilon}{\partial r}||_Z|s|.\end{aligned}$$

Therefore, we have

$$\begin{aligned}||B(w,s)||_Y \le\ & c_3|r_e-r_\varepsilon|||w||_X+||w_e(m'(r_e)-m'(r_\varepsilon))+(w_e-w_\varepsilon)m'(r_\varepsilon)||_X\,|s|\\ &+||\varepsilon||_\infty||w||_X+c_7|r_e-r_\varepsilon|\,|s|\\ &+\frac{2c_1bh^{-1}(g(0))}{2c_1L(be^{-m(r_e)a},1)-||L||M_2bl\big(2c_1+c_3l(m(r_e)+\nu)e^{(m(0)l+m(r_e)l}\big)}||\tfrac{\partial\varepsilon}{\partial r}||_Z|s|\\ &+|h'(Lw_e)(r_e-r_\varepsilon)+r_\varepsilon(h'(Lw_e)-h'(Lw_\varepsilon))|\,||L||\,||w||_X\\ \le\ & \frac{c_3(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1}||\varepsilon||_Z||w||_X+\frac{c_5(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1}||w_e||_X||\varepsilon||_Z|s|\\ &+\tilde{M}_2c_3||\varepsilon||_Z|s|+||\varepsilon||_\infty||w||_X\\ &+\frac{2c_1bh^{-1}(g(0))}{2c_1L(be^{-m(r_e)a},1)-||L||M_2bl\big(2c_1+c_3l(m(r_e)+\nu)e^{(m(0)l+m(r_e)l}\big)}||\tfrac{\partial\varepsilon}{\partial r}||_Z|s|\\ &+\frac{h'(Lw_e)(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1}||\varepsilon||_Z||L||\,||w||_X\\ &+\tilde{M}_2r_cc_6||L||^2\,||\varepsilon||_Z||w||_X\\ &+\frac{c_7(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1}||\varepsilon||_Z|s|\\ \le\ & \Big(1+\frac{c_3(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1}+\frac{c_5(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1}||w_e||_X\\ &+\tilde{M}_2c_3+\frac{2c_1bh^{-1}(g(0))}{2c_1L(be^{-m(r_e)a},1)-||L||M_2bl\big(2c_1+c_3l(m(r_e)+\nu)e^{(m(0)l+m(r_e)l}\big)}\\ &+\frac{h'(Lw_e)(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1}||L||+\tilde{M}_2r_cc_6||L||^2\\ &+\frac{c_7(m(r_e)+\nu)e^{(m(r_e)+m(0))l}}{c_1}\Big)\max\left(||\varepsilon||_\infty,||\varepsilon||_{C^1}\right)||(w,s)||_Y. \quad \square\end{aligned}$$

Finally, we state the most important result of this chapter. The lemmas given above allow us to prove it using Theorem 4.4, (Theorem 3.17 in [48]).

Theorem 5.3. *If*

$$\max\left(||\varepsilon||_{\infty}, ||\varepsilon||_{C^1}\right) \leq \min\left(M, M_2, 1/M_1M_3\right)$$

where M, M_1, M_2 and M_3 were respectively given in Theorem 5.2, Corollary 5.1, Lemma 5.3 and Lemma 5.4, then the coexistence equilibrium point $(u_\varepsilon, v_\varepsilon, r_\varepsilon)$ of Problem (5.5) is asymptotically stable.

Proof. Firstly, the assumption $\|\varepsilon\|_{C^1} \leq M$ guarantees the existence of the coexistence equilibrium point $(u_\varepsilon, v_\varepsilon, r_\varepsilon)$. Let $\lambda \in \mathbb{C}$ such that $Re\lambda \geq 0$. On the other hand, recall that the operator T is the linear part of Problem (2.1) at its coexistence equilibrium point (u_e, v_e, r_e). Recall also that this equilibrium is asymptotically stable and hyperbolic. So, $\lambda \in \rho(T)$. Now, notice that S denote the linear part of Problem (5.5) at its coexistence equilibrium point $(u_\varepsilon, v_\varepsilon, r_\varepsilon)$. Let us recall also that $S = T + B$. As the operator A is the infinitesimal generator of a semigroup and $T-A$ is bounded, then the operator T generates a semigroup too. Thus, the operator T is closed. Moreover, since the operator B is bounded, $D(T) \subset D(B)$. On the other hand, from Corollary 5.1 and Lemma 5.4 we have respectively $||(T-\lambda)^{-1}||_Y \leq M_1$ and $\|B(w,s)\|_Y \leq M_3 \max\left(||\varepsilon||_{\infty}, ||\varepsilon||_{C^1}\right) ||(w,s)||_Y$. Furthermore, the assumptions of Theorem 5.3 yield that $M_3 \max\left(||\varepsilon||_{\infty}, ||\varepsilon||_{C^1}\right) ||(T-\lambda)^{-1}||_Y \leq M_1M_3 \max\left(||\varepsilon||_{\infty}, ||\varepsilon||_{C^1}\right) < 1$. Finally, using Theorem 4.4, we obtain $\lambda \in \rho(S)$ which completes the proof of the statement of this theorem.

Chapter 6

A numerical implicit method for the Initial Value Problem

6.1 Introduction

Unfortunately, in many structured population models, a rather general formulation hinders the global analysis of the solutions. It is even complicated to determine the stability of an equilibrium solution for the model in its most general form. This occurs too in our case when (2.1) is undertaken without the assumption of equality of the death rates, i.e. without assuming $m_1(r) \equiv m_2(r) + constant$. Thus, it seems that the analysis of these models must rely heavily on numerical solution techniques.

Until the nineties, numerical methods have rarely been applied to solving partial differential equations of age-structured population models. Afterwards, several numerical methods for the analysis of the physiologically structured population models have been developed in order to solve these problems, and they have also been decisive in the study of the qualitative behaviour of these classes of dynamical systems. Moreover, they enabled the detailed analysis of concrete cases arising in applications. The importance of numerical methods for examination of realistically formulated models has firstly been demonstrated in important studies of the AIDS epidemic (see [46] and [47]). In [31], Goudriaan indicates in Sect. 6 the utility of applying the escalator boxcar train method to demographic age structured problems. Neither of these studies include density dependence in the demographic rates. After, de Roos introduced in [70] the escalator boxcar train method, which is a semidiscrete method, to approximate some momenta of the original density function with biological meaning. In [78], Sulsky examines some numerical schemes for the general age-structured population model, including the possibility of density dependence. Later on, in [49] the authors aimed at deriving similar methods for a special class of deterministic infinite dimensional dynamical systems describing physiologically structured populations, with the study of equilibria and their stability as their primary goals. Recently, extensive studies of physiologically structured population dynamics using numerical methods were undertaken and can be found in several papers. For example, Angulo and López-Marcos [2] considered numerical schemes for size-structured population equations based on Runge-Kutta methods, and Ackleh and Ito [1] developed an implicit finite difference scheme for an nonlinear size-structured population model that had been studied analytically by Calsina and Saldaña in [18].

Fortunately, entirely new numerical techniques applied to structured population models need not to be developed since all these methods are based on ideas borrowed from numerical schemes considered earlier in problems involving the flow of compressible fluid (see e.g [21] and [53]). The common link is that the governing equation is a hyperbolic conservation law. Sulsky [78] points out that two aspects of structured population models are not met in compressible fluid flow, the functional dependence of the birth and death rates on the total population size, and the integral boundary condition associated with the birth process. Thus, basic explicit schemes for compressible fluids have an implicit component when applied to structured population models. An adaptation of standard methods for compressible flow to age-structured population models is presented in [78].

The most straightforward numerical method can be applied if the population is age-structured. In [78], Sulsky used an explicit Eulerian method with a fixed grid in order to compute the density. In general, $\triangle t \leq \triangle a$, where $\triangle t$ is the time step and $\triangle a$ is the stepsize in age, is the stability restriction condition for age-structured models. This condition is more restrictive for mass-structured models (see [79]).

The numerical method examined here is an implicit method that is presented in [79] for a mass-structured model in order to eliminate the restriction imposed by the explicit scheme. This class of implicit method adaptable to population models has been presented by Warming and Beam [85] for hyperbolic conservation laws. Finally, note that an implicit method requires in general more computational work per timestep than an explicit method; however, the use of a much larger timestep usually offsets the extra work (see [79]).

6.2 Linear implicit scheme

6.2.1 The setting

The main goal of this section is to adapt an implicit method, which is presented in [79], to the numerical study of the asymptotic behaviour of the solutions in a neighborhood of the coexistence equilibrium point (u_e, v_e, r_e) of (2.1). This work requires to assume in the forthcoming somehow special forms for the operator L and for the real functions m_1, m_2, g and h. First, setting $L(u,v) := c_1 \int_0^l u(a)\ da + c_2 v$ where c_1 and c_2 are positive constants. Notice that $L(u,v)$ measures now the relative weight of the consumers in terms of predation pressure and it is equal to the total population of predators when $c_1 = c_2 = 1$. On the other hand, setting $g(r) = c_3(1 - \frac{r}{r_c})$ where $c_3 > 0$. With this choice and when the functional response of the consumer $h(L(u,v))$ equals 0, the third equation of Problem (2.1) will be a logistic equation where c_3 is the intrinsic growth rate and r_c is the carrying capacity of the amount of the available resources. Finally, we will take $h(x) = x$ as identity real function and for $i = 1, 2, m_i(r) = m_i(\infty) + \frac{m_i(0) - m_i(\infty)}{1 + k_i r}$ where $k_i > 0$.

For simplicity, the plane (a,t) is discretized by choosing a uniform stepsize in age $\Delta a = a_{j+1} - a_j, j = 0, 1, ..., J$ and a uniform time step $\Delta t = t_{n+1} - t_n, n = 0, 1, 2, ...$ Respectively, we also define by u^n, v^n, r^n and, for $i = 1, 2$, m_i^n the difference approximations $u(a, t_n), v(t_n), r(t_n)$ and $m_i(r(t_n))$ and we let $\Delta u = u^{n+1} - u^n, \Delta v = v^{n+1} - v^n$ and $\Delta r = r^{n+1} - r^n$.

We discretize the age-structured population model (2.1) using the following second order temporal difference approximation

$$\left\{ \begin{array}{lcl} u^{n+1} & = & u^n + \frac{\Delta t}{2}\left[(u_t)^n + (u_t)^{n+1}\right] + O(\Delta t^3), \\ v^{n+1} & = & v^n + \frac{\Delta t}{2}\left[(v')^n + (v')^{n+1}\right] + O(\Delta t^3), \\ r^{n+1} & = & r^n + \frac{\Delta t}{2}\left[(r')^n + (r')^{n+1}\right] + O(\Delta t^3). \end{array} \right. \tag{6.1}$$

After, replacing the time derivatives in the last system using the initial value problem (2.1), we have

$$\left\{ \begin{array}{rcl} u^{n+1} & = & u^n - \frac{\Delta t}{2}\left[\frac{\partial}{\partial a}u^n + \frac{\partial}{\partial a}u^{n+1} + m_1^n u^n + m_1^{n+1}u^{n+1}\right] + O(\Delta t^3), \\ v^{n+1} & = & v^n - \frac{\Delta t}{2}\left[-u^n(l) - u^{n+1}(l) + m_2^n v^n + m_2^{n+1} v^{n+1}\right] + O(\Delta t^3), \\ r^{n+1} & = & r^n + \frac{\Delta t}{2}\left[g(r^n)r^n + g(r^{n+1})r^{n+1}\right. \\ & & \left. -h(L(u^n, v^n))r^n - h(L(u^{n+1}, v^{n+1}))r^{n+1}\right] + (\Delta t^3), \\ u(0, t_{n+1}) & = & u(0, t_n) + b\Delta v. \end{array} \right. \tag{6.2}$$

Next, using the Taylor series expansion, it follows that

$$\left\{\begin{array}{lcl} u^{n+1} & = & u^n + (u^{n+1} - u^n) \\ & \simeq & u^n + \Delta t \frac{\partial u^n}{\partial t}, \\ v^{n+1} & = & v^n + (v^{n+1} - v^n) \\ & \simeq & v^n + \Delta t \frac{\partial v^n}{\partial t}, \\ r^{n+1} & = & r^n + (r^{n+1} - r^n) \\ & \simeq & r^n + \Delta t \frac{\partial r^n}{\partial t}, \end{array}\right. \tag{6.3}$$

and for $i = 1, 2$

$$\begin{array}{rcl} m_i^{n+1} & \simeq & m_i^n + \Delta r \frac{dm_i^n}{dr}, \\ g(r^{n+1}) & \simeq & g(r^n) + \Delta r g'(r^n), \\ h(L(u^{n+1}, v^{n+1})) & \simeq & h(L(u^n, v^n)) + L(\Delta u, \Delta v) h'(L(u^n, v^n)). \end{array} \tag{6.4}$$

Substituting the right hand sides of (6.3), (6.4) for $u^{n+1}, v^{n+1}, m_i^{n+1}, h(L(u^{n+1}, v^{n+1})), r^{n+1}$ and $g(r^{n+1})$ yields

$$\left\{\begin{array}{rcl} m_1^{n+1} u^{n+1} & \simeq & m_1^n u^n + u^n \Delta r \frac{dm_1^n}{dr} + \Delta u m_1^n, \\ m_2^{n+1} v^{n+1} & \simeq & m_2^n v^n + v^n \Delta r \frac{dm_2^n}{dr} + \Delta v m_2^n, \\ g(r^{n+1}) r^{n+1} & \simeq & g(r^n) r^n + r^n g'(r^n) \Delta r + g(r^n) \Delta r, \\ h(L(u^{n+1}, v^{n+1})) r^{n+1} & \simeq & r^n h(L(u^n, v^n)) + r^n L(\Delta u, \Delta v) h'(L(u^n, v^n)) \\ & & + h(L(u^n, v^n)) \Delta r. \end{array}\right. \tag{6.5}$$

Substituting the right hand side of (6.5) for the corresponding left side in 6.2), we obtain the following system linear in $\Delta u, \Delta v$ and Δr

$$\left\{\begin{array}{lcl} \Delta u & = & -\frac{\Delta t}{2} \left(\frac{\partial u^n}{\partial a} + \frac{\partial u^{n+1}}{\partial a} + 2 m_1^n u^n + u^n m_1'^n \Delta r + m_1^n \Delta u \right) + O(\Delta t^3), \\ \Delta v & = & -\frac{\Delta t}{2} \left(-u^n(l) - u^{n+1}(l) + 2 m_2^n v^n + v^n m_2'^n \Delta r + m_2^n \Delta v \right) + O(\Delta t^3), \\ \Delta r & = & \frac{\Delta t}{2} \left(2 g(r^n) r^n + r^n g'(r^n) \Delta r + g(r^n) \Delta r - 2 h(L(u^n, v^n)) r^n \right. \\ & & \left. - r^n L(\Delta u, \Delta v) h'(L(u^n, v^n)) - h(L(u^n, v^n)) \Delta r \right) + O(\Delta t^3) \end{array}\right. \tag{6.6}$$

where $m_i'^n = m_i'(r^n)$, for $i = 1, 2$. Notice also that $u(l) = u(a_J)$. So, System (6.6) can be written as

$$\left\{\begin{array}{lcl} \Delta u & = & -\frac{\Delta t}{2} \left(2 \frac{\partial u^n}{\partial a} + \frac{\partial \Delta u}{\partial a} + 2 m_1^n u^n + u^n m_1'^n \Delta r + m_1^n \Delta u \right) + O(\Delta t^3), \\ \Delta v & = & -\frac{\Delta t}{2} \left(-2 u^n(l) - \Delta u(l) + 2 m_2^n v^n + v^n m_2'^n \Delta r + m_2^n \Delta v \right) + O(\Delta t^3), \\ \Delta r & = & \frac{\Delta t}{2} \left(2 g(r^n) r^n + \left[r^n g'(r^n) + g(r^n) - h(L(u^n, v^n)) \right] \Delta r \right. \\ & & \left. - 2 r^n h(L(u^n, v^n)) - r^n h'(L(u^n, v^n)) L(\Delta u, \Delta v) \right) + O(\Delta t^3), \\ \Delta u(0) & = & b \Delta v. \end{array}\right. \tag{6.7}$$

Finally, passing the terms $\Delta u, \Delta v$ and Δr to the right hand side in (6.7), it follows

$$\left\{\begin{array}{rcl}
\left(1+\frac{\Delta t}{2}m_1^n\right)\Delta u+\frac{\Delta t}{2}\frac{\partial \Delta u}{\partial a}+\frac{\Delta t}{2}u^n m_1'^n\Delta r & = & -\Delta t\left(m_1^n u^n+\frac{\partial u^n}{\partial a}\right)\\
 & & +O(\Delta t^3),\\
\left(1+\frac{\Delta t}{2}m_2^n\right)\Delta v-\frac{\Delta t}{2}\Delta u(l)+\frac{\Delta t}{2}v^n m_2'^n\Delta r & = & \Delta t\left(u^n(l)-m_2^n v^n\right)\\
 & & +O(\Delta t^3),\\
\left(1+\frac{\Delta t}{2}\left[-r^n g'(r^n)-g(r^n)+h(L(u^n,v^n))\right]\right)\Delta r & & \\
+\frac{\Delta t}{2}r^n h'(L(u^n,v^n))L(\Delta u,\Delta v) & = & \Delta t r^n\left(g(r^n)\right.\\
 & & \left.-h(L(u^n,v^n))\right)+O(\Delta t^3),\\
\Delta u(0)-b\Delta v & = & 0.
\end{array}\right. \tag{6.8}$$

In this system, partial derivatives of the density $u(a,t)$ with respect to a can be expressed as a difference quotient plus a truncation error, and the integrals can also be approximated by numerical quadrature formulas. So, the symbol u^n will be evaluated at grid points $a_j, j=0,1,2,...J$ and it will be replaced by $u_j^n=u(a_j,t_n)$. Thus, the numerical algorithm becomes for the unknowns $\Delta u_j=u_j^{n+1}-u_j^n$, $\Delta v=v^{n+1}-v^n$ and $\Delta r=r^{n+1}-r^n$.

Using central difference approximations for the derivatives with respect to a, we have for $1\le j\le J$,

$$\begin{array}{rcl}
\frac{\partial \Delta u_j}{\partial a} & = & \frac{\Delta u_{j+1}-\Delta u_{j-1}}{2\Delta a},\\
\frac{\partial u_j}{\partial a} & = & \frac{u_{j+1}-u_{j-1}}{2\Delta a}.
\end{array}$$

and

$$\begin{array}{rcl}
\frac{\partial \Delta u_J}{\partial a} & = & \frac{\Delta u_J-\Delta u_{J-1}}{\Delta a},\\
\frac{\partial u_J}{\partial a} & = & \frac{u_J-u_{J-1}}{\Delta a}.
\end{array}$$

Using also trapezoidal rule, one obtains

$$\begin{array}{rcl}
L(\Delta u,\Delta v) & = & c_1\int_0^l \Delta u\,da+c_2\Delta v=c_1\Delta a\left[\frac{\Delta u_0}{2}+\sum_{j=1}^{J-1}\Delta u_j+\frac{\Delta u_J}{2}\right]+c_2\Delta v,\\
L(u^n,v^n) & = & c_1\int_0^l u^n\,da+c_2 v=c_1\Delta a\left[\frac{u_0^n}{2}+\sum_{j=1}^{J-1}u_j^n+\frac{u_J^n}{2}\right]+c_2 v.
\end{array}$$

Now, with the unknowns ordered $\vec{u}:=(\Delta v,\Delta u_0,\Delta u_1,\Delta u_2,...,\Delta u_J,\Delta r)^T$, (6.8) is equivalently written, taken $h(x)=x$ from now on, as the following system of linear equation on the grid points $(a_j,t_n), j=0,1,2,...,J$ and $n=0,1,2,...$

$$MAT\vec{u}=\vec{b} \tag{6.9}$$

where

$$\vec{b} := \begin{pmatrix} \Delta t(u(l) - m_2 v) \\ 0 \\ \Delta t\left(m_1 u_1 + \frac{u_2 - u_0}{2\Delta a}\right) \\ \vdots \\ \Delta t\left(m_1 u_{i-2} + \frac{u_{i-1} - u_{i-3}}{2\Delta a}\right) \\ \vdots \\ \Delta t\left(m_1 u_J + \frac{u_J - u_{J-1}}{\Delta a}\right) \\ \Delta t r(g - L(u, v)) \end{pmatrix}$$

$$MAT := \begin{pmatrix} 1+\alpha_2 & & & & & & & & -\frac{\Delta t}{2} & \alpha_3 \\ -b & 1 & & & & & & & & \\ & -\beta_1 & 1+\alpha_1 & \beta_1 & & & & & & \gamma_1 \\ & & -\beta_1 & 1+\alpha_1 & \beta_1 & & & & & \gamma_2 \\ & & & . & . & . & & & & . \\ & & & & . & . & . & & & . \\ & & & & & . & . & & . & . \\ & & & & & -\beta_1 & 1+\alpha_1 & & \beta_1 & \gamma_{J-1} \\ & & & & & & -2\beta_1 & & 1+\alpha_1+2\beta_1 & \gamma_J \\ \beta_2 & \beta_3 & 2\beta_3 & . & . & . & 2\beta_3 & & \beta_3 & 1+\alpha_4 \end{pmatrix}$$

where entries not shown are zero, $\alpha_1 := \frac{\Delta t}{2} m_1, \alpha_2 := \frac{\Delta t m_2}{2}, \alpha_3 := \frac{\Delta t v m_2'}{2}, \alpha_4 := \frac{\Delta t}{2}(-rg' - g + L(u, v)), \beta_1 := \frac{\Delta t}{4\Delta a}, \beta_2 := \frac{\Delta t c_2 r}{2}, \beta_3 := \frac{\Delta t \Delta a c_1 r}{4}$ and for $i = 1, 2, ..., J$, $\gamma_i := \frac{\Delta t u_i m_1'}{2}$

The $(J+3) \times (J+3)$ matrix MAT is essentially tridiagonal and has almost the same form as the matrix obtained by Sulsky [79] when solving a mass-structured population model using this linear implicit method. The matrix MAT is simpler since it has a unique fully nonzero row which arises from the dependence of Δr on all the terms Δu and Δv. Moreover, in the last column the second entry is the unique vanishing entry. This is a consequence of the dependence of the death rates of the predator populations on the amount of the resources. In [79], the last nonzero rows arise from the dependence of ΔM and $\Delta \rho_0$ on all the values of $\Delta \rho$ where ΔM is the stepsize in total biomass and $\Delta \rho$ is stepsize in the population density function. Notice also that in [79], the last nonzero column results from the dependence of the demographic rates on the total biomass.

Sulsky [79] recommends the use of the Sherman-Morrison-Woodbury formula (see [38]) in order to solve this type of systems exploiting their tridiagonal structured. In our case, the matrix MAT can be expressed as $MAT = I - \overline{MAT}$ where I is the $(J+3)\times(J+3)$ identity real matrix and

$$\overline{MAT} := -\begin{pmatrix}
\alpha_2 & & & & & & & -\frac{\Delta t}{2} & \alpha_3 \\
-b & & & & & & & & \\
 & -\beta_1 & \alpha_1 & \beta_1 & & & & & \gamma_1 \\
 & & -\beta_1 & \alpha_1 & \beta_1 & & & & \gamma_2 \\
 & & & . & . & . & & & . \\
 & & & & . & . & . & & . \\
 & & & & & . & . & . & . \\
 & & & & & -\beta_1 & \alpha_1 & \beta_1 & \gamma_{J-1} \\
 & & & & & & -2\beta_1 & \alpha_1+2\beta_1 & \gamma_J \\
\beta_2 & \beta_3 & 2\beta_3 & . & . & . & 2\beta_3 & \beta_3 & \alpha_4
\end{pmatrix}$$

where the entries not shown are zero. We can take Δt and Δa in order that a matrix norm of $\overline{MAT}$ is strictly less than 1. This ensures that the matrix MAT is invertible and we can use the Gaussian elimination to solve the linear system (6.8). At the end of this chapter, there is a Fortran program which will be used to determine the dynamical behaviour of the solutions of Problem (2.1).

For simplicity, in the forthcoming, let $l = 1$, $c_1 = c_2 = 1$ as in Chap. 4, and the initial juvenile density $u_0(a) = bve^{-0.2a}$. We will also take $J = 97$, i.e., $\Delta a = 1/97$ and MAT will be a 100×100 matrix.

6.2.2 Two examples

First, we note that the numerical experiments made using this implicit linear scheme show that in a wide range of value of the parameters defined at the beginning of Subsection 6.2.1 the coexistence equilibrium point (u_e, v_e, r_e) of (2.1) is stable. However, in agreement with Section 4.2, we found some examples where this equilibrium loses its stability via Hopf bifurcations changing the dearth rates $m_1(r)$ or $m_2(r)$.

On the other hand, we note the following: when we take the parameters of the initial value problem (2.1) such that the curve $\Phi_{\alpha,M_1,M_2}(\omega) = (M_1'(\omega), M_2'(\omega)), \omega > 0$, which has been defined in Subsect. 4.2.2, starts in the fourth quadrant the stability of the equilibrium (u_e, v_e, r_e) depends more on the death rate of the juveniles $m_2(r)$. But, while the curve $\Phi_{\alpha,M_1,M_2}(\omega)$ originates in the second quadrant the stability depends more on the death rate of the adults $m_1(r)$.

Next, we study two numerical examples where (u_e, v_e, r_e) loses its stability. In the first one, we fix all the parameters of (2.1) except $m_2(r)$, varying k_2. In the second one, we change only k_1, i.e., we vary $m_1(r)$.

First example

In this case, we take $b = 1$. So, according to Figure 4.1, the curve $\Phi_{\alpha,M_1,M_2}(\omega) = (M_1'(\omega), M_2'(\omega)), \omega > 0$ starts in the fourth quadrant (as in Subsect. 4.2.1, we define $M_i := m_i(r_e)$ and $M_i' := m_i'(r_e)$ for $i = 1, 2$). In order to study the loss of stability of the coexistence equilibrium point (u_e, v_e, r_e) via Hopf bifurcation, we choose to take k_2 as a variable and we fix the other parameters, setting $k_1 = 2, m_1(0) = m_2(0) = 1, m_1(\infty) = 0.2, m_2(\infty) = 0.1, c_3 = 1.5$ and the carrying capacity of the amount of the resources, r_c, to 10.

The numerical results obtained using the implicit linear method show that there exists a constant $\tilde{k_2} \simeq 70$ such that if $k_2 < \tilde{k_2}$ then (u_e, v_e, r_e) is stable. It loses its stability if $k_2 > \tilde{k_2}$. Thus, we study five cases, three before bifurcation and two after the loss of stability of (u_e, v_e, r_e). In the forthcoming, N denotes the number of the iterations made by the numerical implicit linear program and p_e is the number of the consumer population steady states.

Case 1: $k_2 = 1$

The numerical computations, made with Maple, show in this case that $p_e = 1.380671722$, $v_e = .7740098896$, $\beta = r_e = .7955310957, M_1 = .5087536852, M_2 = .6012444519$, $M_1' = -.2383220953, M_2' = -.2791622228$ and $\alpha = -.1193296644$ where α and β have been defined in Subsect. 4.2.1.

The following figure represents the consumer population number of the adults as function of the amount of a available resources. That is, the projection of the curve $(u(t), v(t), r(t))$ on the v-r plane.

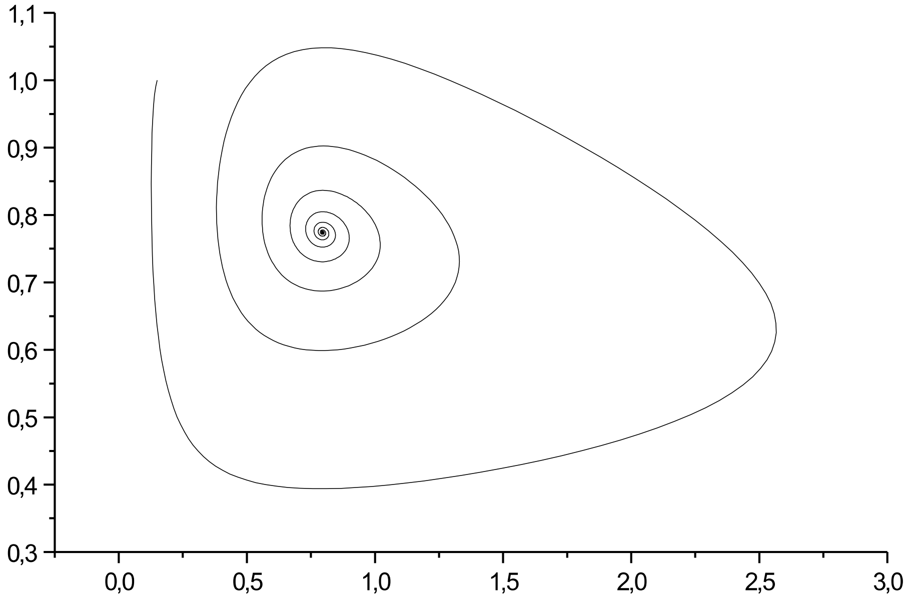

Figure 6.1: The curve (r, v) for $k_2 = 1, v_0 = 1, r_0 = 0.15, \Delta t = 0.1$ and $N = 3000$

Figure 6.2 gives the curve (r, p) of the amount of the resources and the total predator population.

Figure 6.3 shows that (M_1', M_2') lies in the stability region which implies that the coexistence equilibrium solution (u_e, v_e, r_e) is stable for $k_2 = 1$. This is enlightened by Figure 6.1 and Figure 6.2. On the other hand, solving the characteristic equation 4.12, using Maple, the eigenvalue of the linearization at this equilibrium with the greatest real part is $\lambda = -.05947590834 + .5350234716i$.

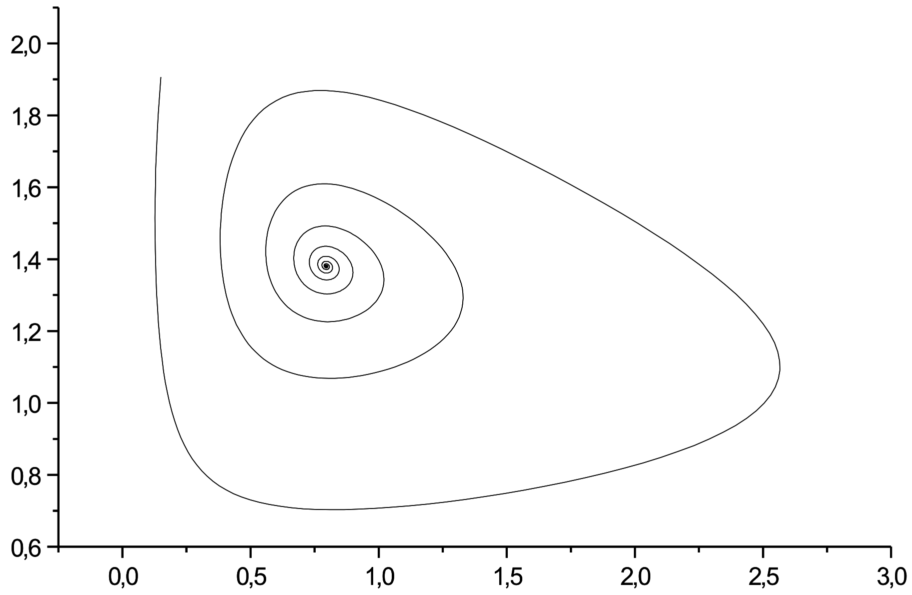

Figure 6.2: The curve (r, p) for $k_2 = 1, v_0 = 1, r_0 = 0.15, \Delta t = 0.1$ and $N = 3000$

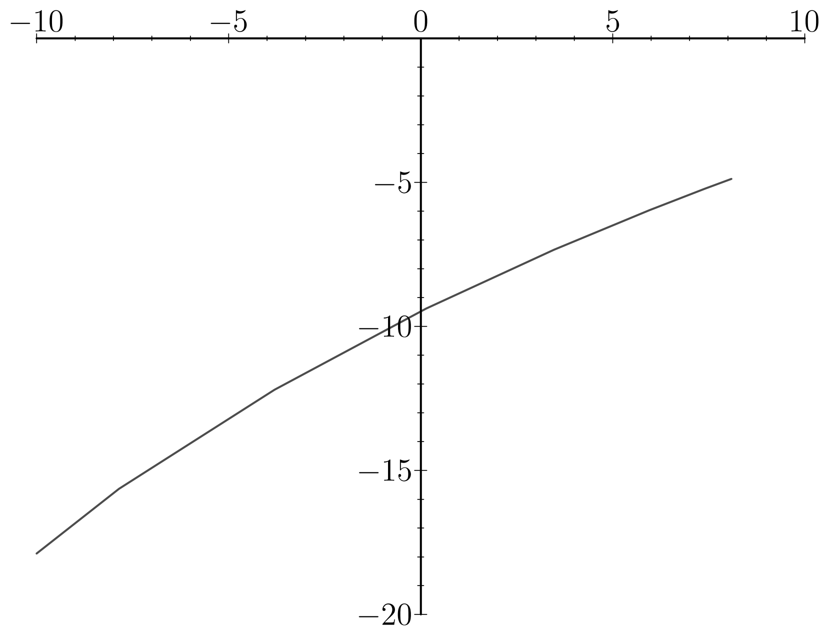

Figure 6.3: The curve $\Phi_{\alpha,M_1,M_2}(\omega)$ for $k_2 = 1$

Case 2: $k_2 = 30$

Using the numerical mathematical computations, made with Maple again, we obtain $p_e = 1.490181641$, $v_e = .8991570060$, $\beta = r_e = .06548445491, M_1 = .9073580829$, $M_2 = .4035890657$, $M_1' = -1.250888644, M_2' = -3.072210693$ and $\alpha = -.009822668237$.

We note that in Figure 6.6 (M_1', M_2') belongs to the stability region. This is

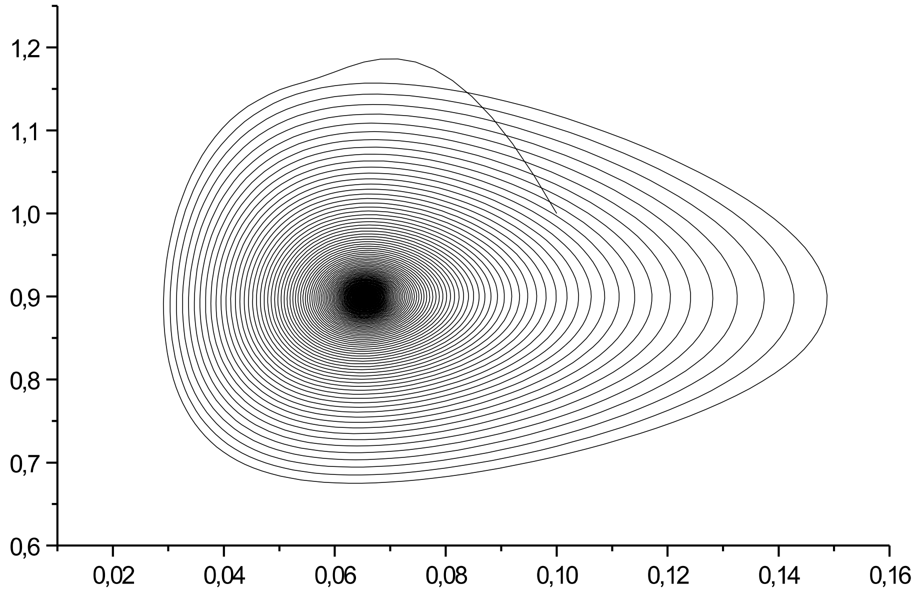

Figure 6.4: The curve (r, v) for $k_2 = 30, v_0 = 1, r_0 = 0.1, \Delta t = 0.1$ and $N = 20000$

asserted by Figure 6.4 and Figure 6.5. Moreover, the eigenvalue of the linearization at this equilibrium with the dominant real part is $\lambda = -.003317333795 + .4986702481i$. Therefore, the coexistence equilibrium solution (u_e, v_e, r_e) is stable for $k_2 = 30$.

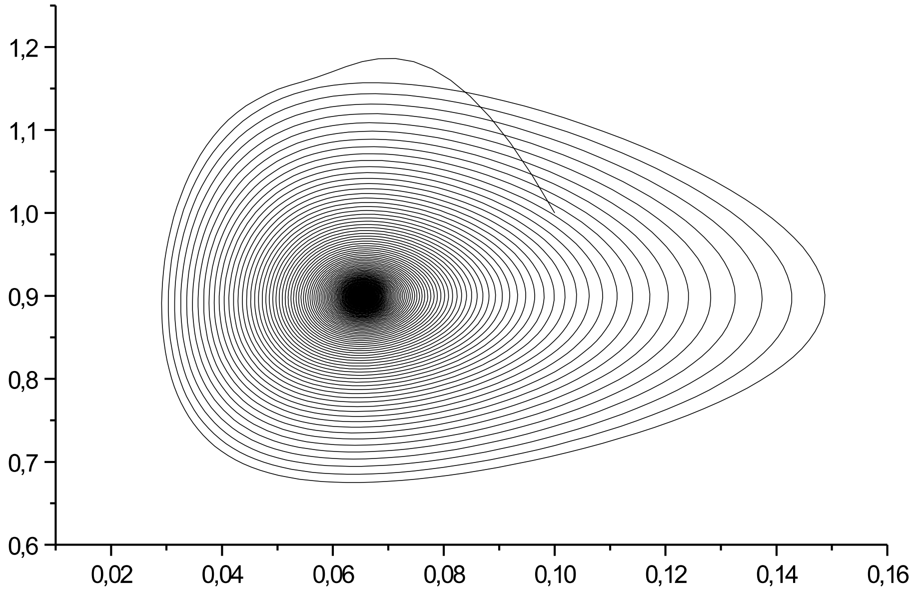

Figure 6.5: The curve (r, p) for $k_2 = 30, v_0 = 1, r_0 = 0.1, \Delta t = 0.1$ and $N = 20000$

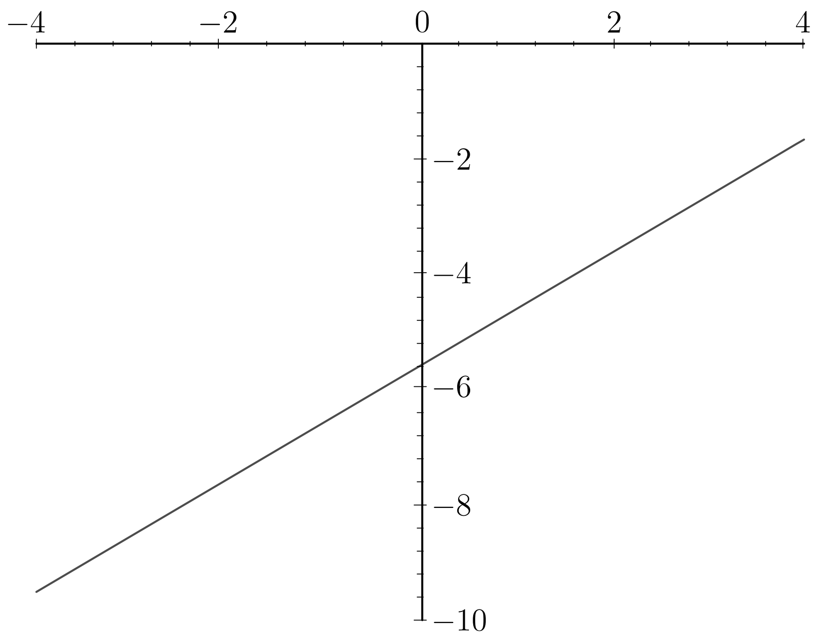

Figure 6.6: The curve $\Phi_{\alpha,M_1,M_2}(\omega)$ for $k_2 = 30$

Case 3: $k_2 = 60$

In this case we obtain, from the numerical computations, made with Maple, that $p_e = 1.494688983$, $v_e = .9078833898$, $\beta = r_e = .03543787546$, $M_1 = .9470521200$, $M_2 = .3878827716$, $M_1' = -1.395217175$, $M_2' = -5.525099344$ and $\alpha = -.005315681319$. This allows to plot the Figure 6.9 which gives the curve $\Phi_{\alpha,M_1,M_2}(\omega), \omega \geq 0$.

It is clear that Figure 6.7, Figure 6.8 and Figure 6.9 reflect the fact that

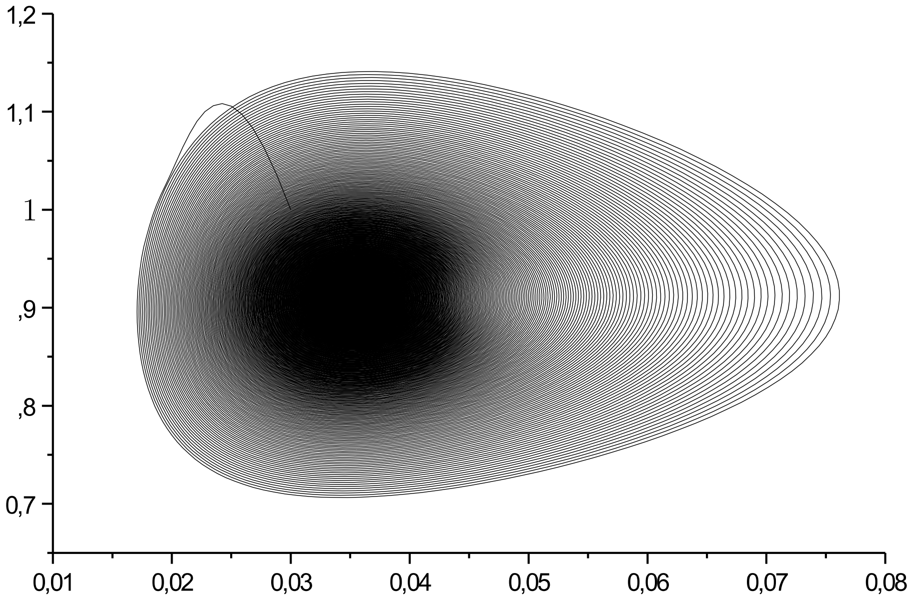

Figure 6.7: Curve (r, v) for $k_2 = 60, v_0 = 1, r_0 = 0.03, \Delta t = 0.1$ and $N = 100000$

the coexistence equilibrium solution (u_e, v_e, r_e) is still stable for $k_2 = 60$. Indeed, the eigenvalue of the linearization at this equilibrium with the greatest real part is $\lambda = -.0005902348483 + .4811459107i$.

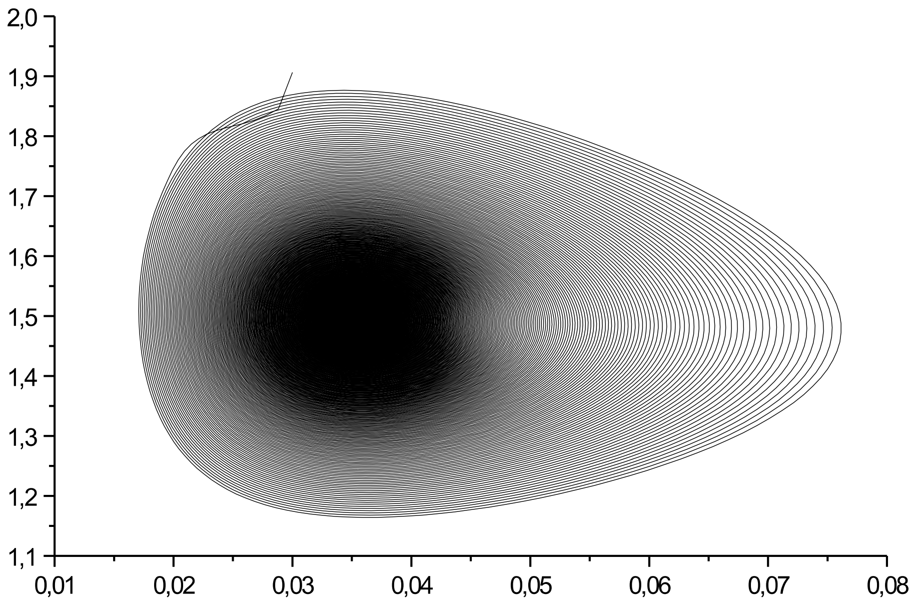

Figure 6.8: Curve (r, p) for $k_2 = 60, v_0 = 1, r_0 = 0.03, \Delta t = 0.1$ and $N = 100000$

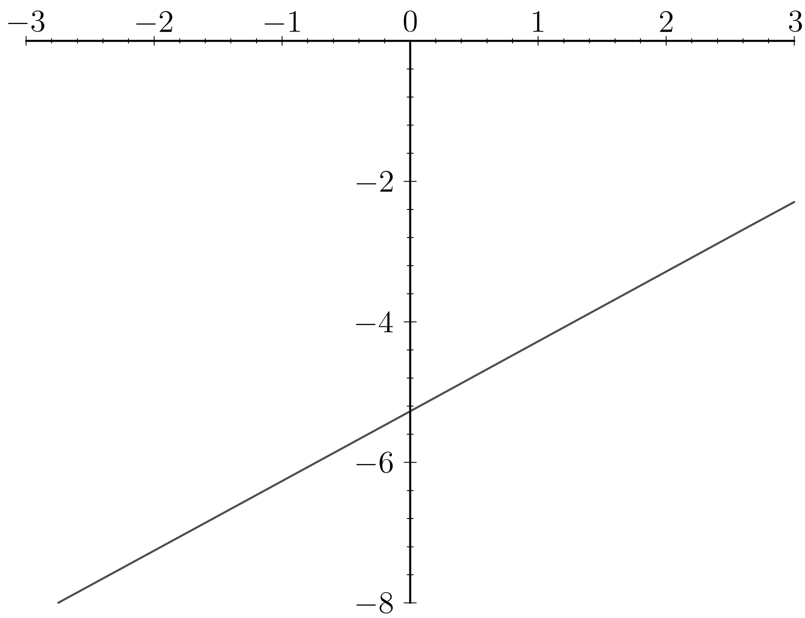

Figure 6.9: The curve $\Phi_{\alpha,M_1,M_2}(\omega)$ for $k_2 = 60$

Case 4: $k_2 = 80$

Using again the numerical computations, it follows that $p_e = 1.495924887, v_e = .9103895452$, $\beta = r_e = .02719922755, M_1 = .9587264532, M_2 = .3833808287, M_1' = -1.439164577$, $M_2' = -7.138195023$ and $\alpha = -.004079884133$. In Figure 6.12, We give the curve $\Phi_{\alpha,M_1,M_2}(\omega), \omega \geq 0$ which shows the stability and instability regions.

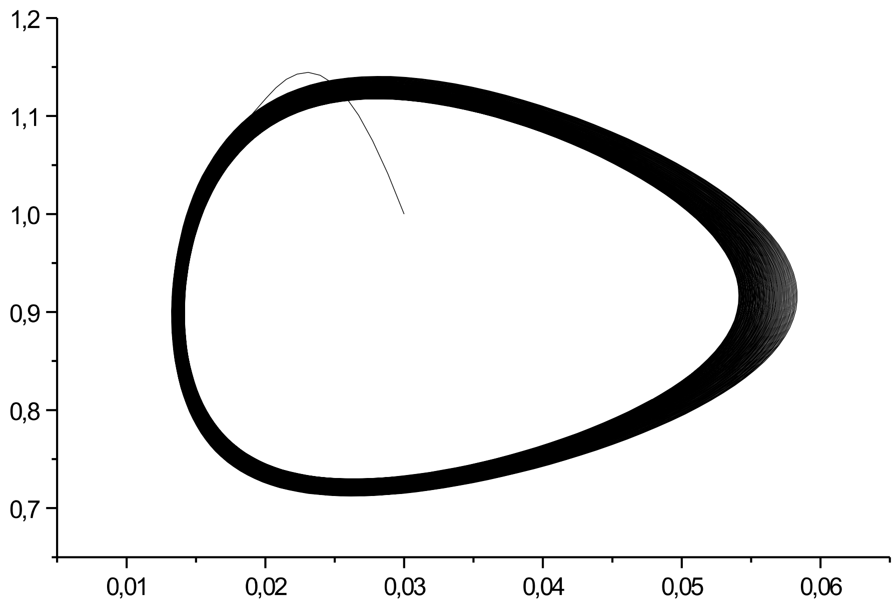

Figure 6.10: The curve (r, v) for $k_2 = 80, v_0 = 1, r_0 = 0.03, \Delta t = 0.1$ and $N = 10000$

Figure 6.12 shows that (M_1', M_2') belongs to the instability region. That is, the coexistence equilibrium point (u_e, v_e, r_e) is unstable when $k_2 = 80$. This is also asserted by Figure 6.10 and Figure 6.11. Indeed, the real part of the dominant eigenvalue of the linearization at this equilibrium point is $\lambda = .0001845920204 + .4755582779i$. Moreover, these figures show that the loss of stability of (u_e, v_e, r_e) generated a Hopf bifurcation of a periodic solution. This orbit encloses (u_e, v_e, r_e) and it is stable.

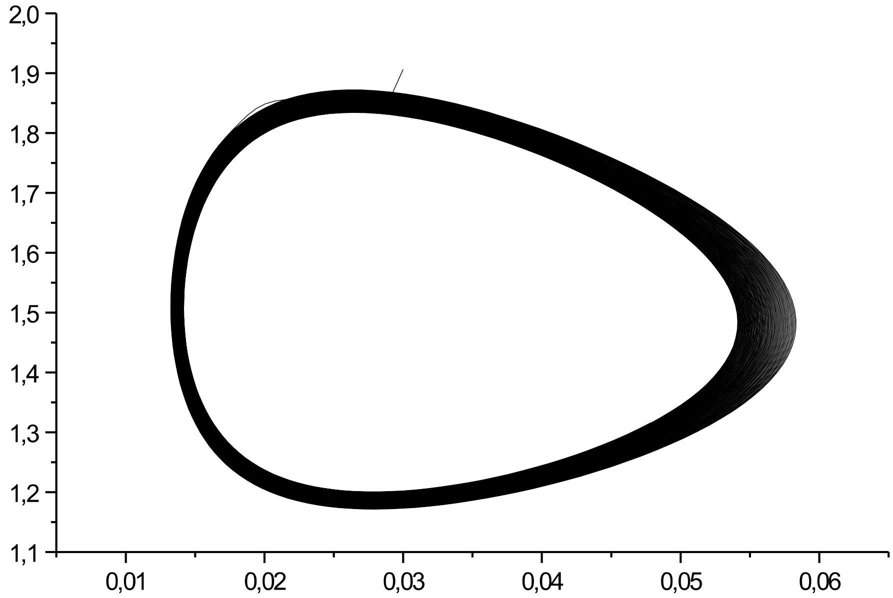

Figure 6.11: The curve (r, p) for $k_2 = 80, v_0 = 1, r_0 = 0.03, \Delta t = 0.1$ and $N = 10000$

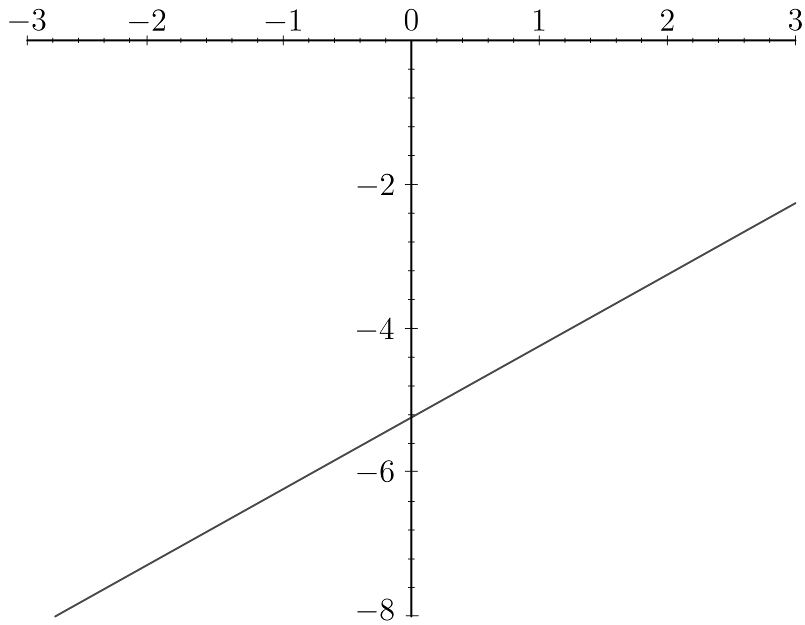

Figure 6.12: The curve $\Phi_{\alpha,M_1,M_2}(\omega)$ for $k_2 = 80$

Case 5: $k_2 = 150$

Finally, we study the case $k_2 = 150$ in order to see what can occur when taking k_2 far from $\tilde{k}_2$ (recall that $\tilde{k}_2$ is the value corresponding to the Hopf bifurcations). In this case the numerical mathematical computations show that $p_e = 1.497750752, v_e = .9141890444$, $\beta = r_e = .01502782964, M_1 = .9766570602, M_2 = .3765678408, M_1' - 1.507990473$, $M_2' = -12.74829508$ and $\alpha = -.002254174446e$. This allow to draw the curve $\Phi_{\alpha,M_1,M_2}(\omega), \omega \geq 0$ which shows the stability and instability regions.

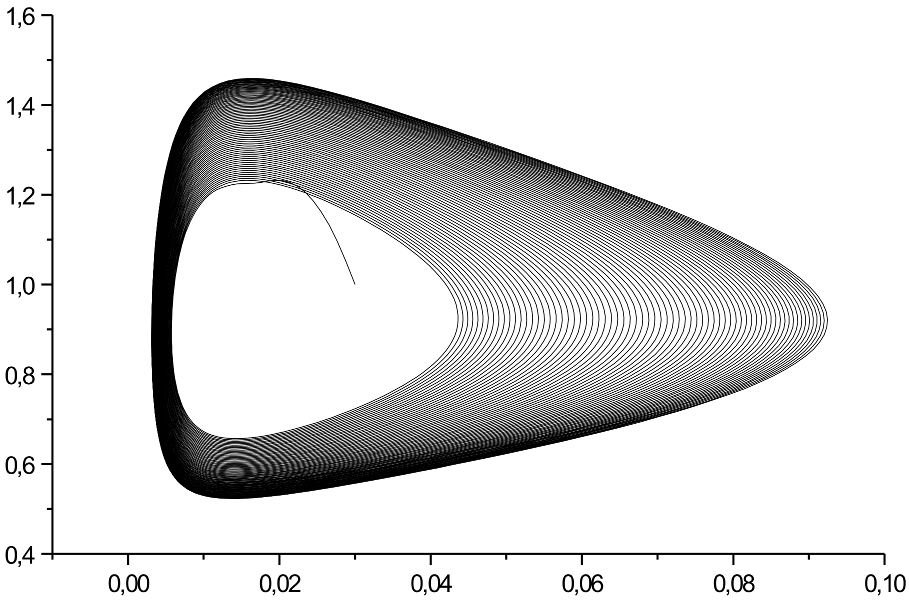

Figure 6.13: The curve (r, v) for $k_2 = 150, v_0 = 1, r_0 = 0.03, \Delta t = 0.1$ and $N = 10000$

Figure 6.13, Figure 6.14 and Figure 6.15 enlighten that the coexistence equilibrium point (u_e, v_e, r_e) is unstable when $k_2 = 150$. Indeed, the real part of the dominant eigenvalue of the linearization at this equilibrium point is $\lambda = .001354915787 + .4665854392i$. Notice that the loss of stability of (u_e, v_e, r_e) happened via Hopf bifurcation. This generates a stable periodic solution enclosing (u_e, v_e, r_e).