Appendix A

Proofs

A.1 Proof of Proposition 52

It is clear that there are no situations where a positive measure \( \sigma \) of independent buyers or sellers have incentive to deviate and to join the cartel on their respective side of the market. Indeed, outsiders obtain a higher payoff than insiders provided that the cartel is active. Hence the only relevant deviations consist in situations in which a positive measure \( \sigma \) of cartel members wish to defect from the cartel.

Consider first market outcomes with symmetric trade. These outcomes can arise for all levels of market frictions provided that the fraction of independent traders are such that \( \sigma \frac{1}{2} \) and \( \frac{1}{2} \). In particular, assume that the fractions of independent buyers and sellers are such that

\[
\max f^{-1}_{\bar{q}}, \bar{q} = \frac{3}{4} \cdot \min f^{-1}_{-2}; sg = \min f^{-1}_{-2}; \bar{q}; sg
\]

whereby equilibrium strategies are such that \( (\bar{q}; \frac{3}{4}) = (\bar{q}; \bar{q}) \) with

\[
\frac{3}{4} \bar{q} = \frac{2(\sigma + 2^*)}{\left(1 + 3(\sigma + 2^*)\right)} \cdot \bar{q} \cdot \min f^{-1}_{-2}; sg
\]

Suppose further that, at a given market outcome, a strictly positive measure \( \sigma \) of the members of buyers’ cartel leave the cartel. When such a defection occurs, the best reply function of buyers’ cartel shifts towards the right, due to an increase in the fraction of independent buyers (the latter changes from \( \sigma \) to \( \sigma + \sigma^+ \) as a consequence of the defection). If the equilibrium measure of active traders \( q \) is such that

\[
\frac{3}{4} q^+ = \frac{2(\sigma + 2^*)}{\left(1 + 3(\sigma + 2^*)\right)} \cdot q \cdot \min f^{-1}_{-2}; sg
\]

then we claim that leaving the cartel is beneficial. Indeed, after the defection, the buyers’ cartel continues to set its measure of active traders equal to \( \sigma + 2^+; \bar{q} \) = \( q \), which yields per capita payoffs

\[
\frac{3}{4} (q, q) = \frac{3}{8} (q, q) = \frac{(2^+)}{2}
\]
to outsiders (and to defecting cartel members). Prior to the defection, the individual payoff to cartel members is

$$\frac{B^{q_0} (q_0)}{(1, \theta)} = \frac{(q_0, \theta^*(2, \theta^*))}{2(1, \theta)}$$

and it is immediate to check that $$\frac{B^{q_0} (q_0)}{(1, \theta)} < \frac{1}{B} (q, q)$$. However, if $$q$$ is such that

$$\frac{3q}{q} = \frac{2(2, \theta^*)}{(1 + \theta^*(2, \theta^*))} \cdot q < \frac{2(\theta^*+\theta^*)(2, \theta^*)}{(1 + \theta^*(\theta^*+\theta^*)(2, \theta^*))} = \frac{3q}{q}$$

then the buyers’ cartel breaks down completely as a consequence of the defection, and it plays \( \theta^*, \frac{q}{q} \) = 1: In this situation, defecting buyers would receive a payoff equal to

$$\frac{3q}{q} (1, q) = \frac{q(3, \theta^*)}{4(1, \theta^*)}$$

each, and a defection would not be proﬁtable if $$\frac{B^{q_0} (1, \theta)}{(1, \theta)} \cdot \frac{1}{B} (1, q)$$; or else if $$\frac{3q}{q} \cdot q < 1$$; which is precisely the case at hand. Conversely, it remains proﬁtable for members of sellers’ cartel to leave their cartel, and consequently a defection of \( n \) members does not induce sellers’ cartel to modify the chosen measure $$q$$ of active sellers, being $\frac{3q}{q} \cdot \frac{1}{q}$.

In general, when \( \theta^* \), \( \theta \) and \( \frac{3q}{q} \), \( \frac{3q}{q} \) both cartels are stable if and only if

$$\frac{3q}{q} = \frac{2(2, \theta^*)}{(1 + \theta^*(2, \theta^*))} = \frac{2(\theta^*+\theta^*)(2, \theta^*)}{(1 + \theta^*(\theta^*+\theta^*)(2, \theta^*))} = \frac{3q}{q} = \frac{3q}{q}$$

which implies

$$\theta^* = \frac{\frac{3q}{q}}{1 + \theta^*(2, \theta^*)}$$

Notice that, in order for both cartels to be active (i.e. in order for $$\frac{3q}{q} < q$$ to be true), it must be that \( \theta^* < \theta^* \) and \( \frac{3q}{q} < \frac{3q}{q} \); where both $$0 < \frac{3q}{q} < 1$$ and $$0 < \theta^* < 1$$ hold. A symmetric stable cartel configuration thus exists when \( \theta^* \), \( \theta^* \), \( \theta \) and \( \frac{3q}{q} \), \( \frac{3q}{q} \).

Secondly, consider market outcomes with asymmetric trade. Recall that these outcomes, where at most one cartel is active, are attained only when market frictions are not high.

Suppose that the strategy pair \( b(s); s \) is played. In this situation, condition (3.7) fails and the necessary and suﬃcient conditions for \( b(s); s \) to be an equilibrium is that max \( f_{\frac{3q}{q}}/3g \cdot s \); whereby \( b(s) \cdot s \); and \( b(s) < \min f_{\frac{3q}{q}}/3g \). Take $$s$$ as the quantity offered by the sellers and consider a defection from the buyers’ cartel.

Suppose, for the time being, that max \( f_{\frac{3q}{q}}/3g = \frac{3q}{q} \). After the deviation, the measure of independent buyers becomes \( \theta \) and the cartel’s best reply function shifts slightly towards the right. Such a defection has three possible consequences, depending on the magnitude of \( \theta \) and \( \theta \). (i) The buyers’ cartel continues to respond to the total quantity \( s \) offered by sellers demanding \( \frac{3q}{q} (s) = \frac{b^{\theta^*+\theta^*}}{b^{\theta^*+\theta^*}} (s) \), where

$$b^{\theta^*+\theta^*} (s) = \frac{s(2, \theta^*)}{g(1, \theta^*)} \frac{p(1, \theta^*)}{b^{\theta^*+\theta^*}} (b^{\theta^*+\theta^*})(s)$$

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and the equilibrium of the quantity-setting game is \( b^{(b^{+};1)}(s);s \): This occurs when \( \& < 0 \); or equivalently when \( \frac{3}{2} b^{+} < s \); and when \( b^{(b^{+};1)}(s) < \min f^{-1}_{1} - 3g \). We claim that, when no cartel is active on the supply side, it is pro\( \cdot \)table for a measure \( " > 0 \) of members of the buyers’ cartel to defect. Observe that the per capita utility of outsiders after the defection is equal to

\[
\frac{3}{8} b^{(b^{+};1)}(s);s = \frac{\sigma (s_{1}^{2} + (s_{1}^{*} + s_{1}^{*})^{2})}{4(s_{1} - (s_{1}^{*}))};
\]

(A.1)

whereas the per capita utility that cartel members receive prior to the defection is

\[
\frac{b(\frac{b^{(b^{+};1)}(s);s}{4} + g)}{4(s_{1} - (s_{1}^{*}))};
\]

(A.2)

Furthermore note that the inequality \( \frac{3}{8} b^{(b^{+};1)}(s);s > \frac{b(\frac{b^{(b^{+};1)}(s);s}{4} + g)}{4(s_{1} - (s_{1}^{*}))} \) always holds and that the expression for \( \frac{3}{8} b^{(b^{+};1)}(s);s \) is decreasing in \( " \). Thus, for \( " \) small enough, expression (A.1) is strictly greater than (A.2). (ii) After the defection, the buyers’ cartel demands \( b^{(b^{+};1)}(s) \) but no equilibrium of the quantity-setting game exists. This is the case when \( \min f^{-1}_{1} - 3g < s \) and \( \min f^{-1}_{1} - 3g \cdot b^{(b^{+};1)}(s) < s \). This situation could then be discarded. (iii) The buyers’ cartel sets the measure of active traders \( s_{1}^{*} = s \) and the equilibrium of the game \( G^{(b^{+};1)}(s) = (s; s) \): Then it must be that \( \& > \min f^{-1}_{1} - 3g \) or that \( \frac{3}{8} b^{+} \cdot s \); and that \( \min f^{-1}_{1} - 3g > s \): When this deviation occurs, outsiders have individual pay\( \cdot \)s equal to

\[
\frac{3}{8} (s; s) = \frac{1}{8} (2; 0);
\]

cartel members have individual pay\( \cdot \)s given by (A.2) before the defection, and \( \frac{b(\frac{b^{(b^{+};1)}(s);s}{4} + g)}{4(s_{1} - (s_{1}^{*}))} > \frac{3}{8} (s; s) \). Therefore, if the measures of independent buyers and sellers are such that \( \& = \& \) and \( \frac{3}{8} > \max f_{1/2}; 3g \) respectively, the pro\( \cdot \)le \( \& \) is "\( - \)stable.

When instead \( \max f_{1/2}; 3g = 1/2 \) and a defection from buyers’ cartel occurs, the following cases have to be considered. (i) The buyers’ cartel contiguously to respond to \( s \) setting \( s_{1}^{*} = b^{(b^{+};1)}(s) \) and the outcome of the quantity-setting game is still \( b^{(b^{+};1)}(s);s \) after the defection. If \( \frac{3}{8} b^{+} < s \); which implies \( b^{(b^{+};1)}(s) < s \); and \( b^{(b^{+};1)}(s) < \min f^{-1}_{1} - 3g \); then the deviation is pro\( \cdot \)table. (ii) The buyers’ cartel plays \( b^{(b^{+};1)}(s) \); but no Nash equilibrium exists for \( b^{(b^{+};1)}(s) \), \( \min f^{-1}_{1} - 3g \) or \( b^{(b^{+};1)}(s) > s \). (iii) When \( \frac{3}{8} b^{+} \), \( \& \); or equivalently \( \& = \& \) and \( \frac{3}{8} > \max f_{1/2}; 3g \): the defection from the cartel triggers the response \( \& = b^{+}(s) \); and when the case the equilibrium outcome is \( (1; s) \) and the deviating members are not better off. Then the pro\( \cdot \)le \( \& b^{(b^{+};1)}(s);s \) is an "\( - \)stable market outcome for \( \& = \& \) and \( \frac{3}{8} > \max f_{1/2}; 3g \):

Finally, consider market outcomes of the form \( (b^{+};1; b^{(1)}(s)) \): In order for such pro\( \cdot \)le to be an "\( - \)stable market outcome, it must be the case that the strategy pair \( (1; b^{(1)}(s)) \) be a Nash equilibrium of the quantity-setting game \( G^{(b^{+};1)} \). Recall that \( b^{(1)}(s) < s \) if and only if

\[
\frac{\sigma(2^{*}; 2^{*})(s^{*})^{2}}{(1/2); 3g} = \frac{3}{8}.
\]
Therefore, \( \text{pro.le} (\frac{1}{2}; 1; \mathbf{b}(1)) \); where \( @ \), \( @_i \) is "-stable only if the additional condition \( \frac{1}{2} \) is satisfied, which is the case if and only if \( s > s_* \); where \( s_* \) solves \( \frac{1}{2} = \frac{\pi}{2} \) and is such that

\[
s_* = \frac{(4_1^*) + 2(5_1) + (1_1^*)^2}{2(5_1)^2} ;
\]

with \( s_* < 1 \) being true for all \( ^* > 0 \). Finally, the \( \text{pro.le} (\frac{1}{2}; 1; \mathbf{b}(1)) \) represents an "-stable market outcome only if \( \frac{1}{2} < \frac{\pi}{2} \) and \( \pi > \pi \).

A.2 Proof of Theorem 76

Consider 1st case (i); where \( b_+ > s_H \). Then the Sorting, the One-Sided Pooling and the Pooling QEI might all exist and the middleman’s pro.ts corresponding to these three quasi-equilibria must be compared. In particular, the payo. associated to the Sorting QEI is \( \frac{3}{2} \) in expression (4.41), the bene.ts that the intermediary obtains at the One-Sided Pooling QEI are either \( \frac{1}{2} \) in (4.45) or \( \frac{1}{2} \) in (4.47), and finally the pro.ts corresponding to the Pooling QEI are given by \( \frac{1}{2} \) in (4.49).

Suppose that \( ^* \); in which case no One-Sided Pooling Equilibrium exists and the only candidates for a SEI reduce to the Sorting or the Pooling QEI. Now, one has that

\[
\frac{3}{2} > \frac{1}{2} (\quad b_+ < \frac{\mu}{1(1+\mu)^2} s_H \quad) \cdot \frac{\pi}{2} ;
\]

where \( s_H < \frac{\pi}{2} \): Observe that when condition (A.3) is satisfied, then also (4.42) holds. Then it is more pro..table for the middleman to serve only high surplus agents rather than serving all potential traders when the bid and ask prices posted in the former case are such that \( P_b < s_H \) and \( P_a > b_+ \): Conversely, if serving only the high surplus agent implies setting too narrow a bid-ask spread, namely \( (P_a \cdot P_b) < (b_+ \cdot s_L) \); then it is surely more pro..table for the middleman to serve the whole market and let low surplus agents be just indi. erent between trading or not.

When \( ^* < \frac{\mu}{2} \) and \( s_H \cdot b_+ < \frac{\pi}{2} \); then an OSP quasi-equilibrium also exists. The middleman’s payo. in this case is \( \frac{1}{2} \) which is such that

\[
\frac{1}{2} < \frac{1}{2} (\quad b_+ > \frac{\mu}{1(1+\mu)(s_H \cdot s_L)} \quad) \cdot \frac{\pi}{2} ;
\]

Moreover the inequality \( \frac{1}{2} < \frac{1}{2} \) always holds for \( ^* < \frac{1}{2} ; \) otherwise for \( \frac{1}{2} < \frac{\mu}{2} \) it is satisfied if and only if

\[
b_+ > \frac{(1(1+\mu)(s_H \cdot s_L))}{\frac{\pi}{2}} \quad \frac{\pi}{2} \quad > \quad s_H ;
\]

where \( \frac{\pi}{2} \) is true whenever \( ^* < \frac{1}{2} \). Further observe that \( \frac{\pi}{2} > \frac{\pi}{2} \) if and only if

\[
\pi > \frac{1(1+\mu)^2}{2\mu(1+\mu)^2} \quad \frac{\pi}{2} ;
\]

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where $\circ_1 < \circ_2 < \frac{1}{2}$ and that the chain of inequalities $t_{L}^{\text{SA}} > t_{P}^{\text{AOSP}} > t_{L}^{\text{SAOSP}}$ holds if and only if

$$\circ < \frac{1 + p}{1 + 2\mu^2 + 4\mu^2 + 2\mu} \cdot \circ_3;$$

where $\frac{1}{2} < \circ_3 < \circ_0$: Therefore if $\circ_1 < \circ < \frac{1}{2}$ then the OSP quasi-equilibrium is always dominated in terms of the benefits accruing to the middleman, which are maximal at a Sorting QEI if $b < b_{L}^{\text{A}}$ and otherwise it is maximal at a Pooling QEI.

If instead $\circ$, $\circ_0$, then the middleman receives payoffs $\frac{1}{2}^{\text{OSP}}$ at a OSP quasi-equilibrium, where $\frac{1}{2}^{\text{OSP}} > \frac{1}{2}$ always holds and where

$$\frac{1}{2}^{\text{OSP}} > \frac{1}{2} \quad \text{if} \quad s_{H} < b_{L} < b_{L}^{\text{OSP}} < \frac{(1+\mu)s_{H} - s_{L}^{(1+\mu)s_{L}}}{(2+\mu)(2+\mu)} \cdot \frac{1}{2}^{\text{OSP}}.$$

Examine now case (ii) in which $b_{L} < s_{H}$: In this event the pooling quasi-equilibrium does not exist, and the One-Sided Pooling and the Sorting QEI only hold under certain conditions. OSP quasi-equilibria always exist for $\circ > \circ_1$: In particular, $\frac{1}{2}^{\text{OSP}}$, as given by (4.48) is the relevant payoff for $\circ$, $\circ_0$ or $\circ < \circ_0$ and $b_{L} < b_{L}^{\text{OSP}}$; whereas $\frac{1}{2}^{\text{OSP}}$ is relevant when $b_{L} > b_{L}^{\text{OSP}}$ and $\circ_1 < \circ < \circ_0$: Sorting quasi-equilibria yield payoffs $\frac{1}{2}$, whose expression is (4.43), and exist for any $\circ$ and $b_{L} > b_{L}^{\text{OSP}}$.

Note that $b_{L} > b_{L}^{\text{OSP}}$ if and only if $\circ < \frac{1}{1+\mu}$; $\circ_4$, where $\circ_4 < \circ_0$ always holds and where $\circ_4 > \circ_1$ if and only if $\mu < \frac{1}{2}$: Suppose that $\circ = \min f^*_{2} + \circ_4 g_{2}$, then for $b_{L}^{\text{OSP}} < b_{L} < b_{L}^{\text{OSP}}$ no equilibrium with intermediation exists. For $b_{L} < b_{L}^{\text{OSP}}$; the unique SEI is the one corresponding to the OSP quasi-equilibrium yielding benefits $\frac{1}{2}^{\text{OSP}}$; and for $b_{L} > b_{L}^{\text{OSP}}$ the unique SEI corresponds to the Sorting quasi-equilibrium. Conversely, for $\circ > \min f^*_{2} + \circ_4 g_{2}$ an equilibrium with intermediation always exists. Note that $\frac{1}{2}^{\text{OSP}} > \frac{1}{2}$ always holds and $\frac{1}{2}^{\text{OSP}} > \frac{1}{2}$ is true if and only if

$$\frac{1}{2}^{\text{OSP}} > \frac{1}{2} \quad \text{if} \quad b_{L} < < 2^{\circ} (s_{H} + s_{L}) + s_{L} \cdot \frac{1}{2}^{\text{OSP}};$$

where $\frac{1}{2}^{\text{OSP}} > \frac{1}{2}$ always holds and $\frac{1}{2}^{\text{OSP}} > \frac{1}{2}$ is true whenever $\circ > \min f^*_{2} + \circ_4 g_{2}$. Hence the One-Sided Pooling quasi-equilibrium is always dominating for $b_{L} < b_{L}^{\text{OSP}}$ and the Sorting quasi-equilibrium dominates only if $b_{L} < b_{L}^{\text{OSP}}$ and $\circ < \circ_0$. 
Bibliography


