Strategic Incentives in Multivalued Social Choice Processes

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A mi familia y a Laura.

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Chapter 1

Introduction

Elections can be modeled as two stages decision processes. In a first stage, a set of candidates decide whether they run the fray or they withdraw their candidacy. Later, a voting rule is used to select among the running candidates based on voters' preferences. The objective of this thesis is the analysis of strategic incentives of the participants in election, both voters and candidates.

For an election represent an optimal compromise of the conflicting interests of the individuals who form a society, the public decision must be based on their preferences. However, these preferences are usually private of the individuals. A voting rule seeks to elicit the true preferences of each individuals, for if the voters have incentives to misrepresent their preferences the public decision will not be based on the right information. In particular, the social choice may be far from efficient with respect the true preferences. So we impose a normative property on the voting rule: an individual cannot benefit by announcing a false preference independent of the preferences of the other members of the groups. In other terms, for each voter it must be a dominant strategy to report her true preferences and the social choice will be based on the true preferences of the voters. This property is called non-manipulability or *strategy-proofness.*

Depending on the voting rule, it is possible that some candidate prefers to quit her candidacy in order to affect the social choice in her own advantage. If it is the case, the set of feasible alternatives could not be considered as exogenous and independent of any individual decision. This fact would complicate any attempt of positive analysis. Thus, it seems desirable that a voting rule does not provide incentives to the candidates to withdraw their candidacy unilater-

ally. In other words, staying as a candidate when the remaining candidates stay should always be a Nash equilibrium strategy for all candidates. A voting rule that meets these requirements is called *candidate stable*.

This thesis focuses on elections in which the participants face the possibility that the social choice outcome may consist of more than one alternative. Although it is natural to think that an election results in the selection of a unique candidate, in many public decision environments this assumption is very restrictive, since it excludes the possibility of ties among alternatives. It may be the case that voters' preferences do not provide enough information to lead to a resolute choice. For instance, think of a society with three individuals who have to select from three candidates and voters' preferences present the classical voting paradox. In this situation, a tie among the three candidates would be natural. We could interpret a non-resolute voting rule as a first screening device which narrows the social agenda to a smaller set of alternatives. The individuals consider this first selection as the social outcome since they are not aware of the way in which the final decision is to be made.

The crucial point of the analysis of strategic incentives of the participants in multivalued rules is how the individuals compare the social outcome when these outcomes are no longer single alternatives but sets of alternatives. The individuals must be endowed with preferences over sets of alternatives. These preferences are going to be restricted. They must be consistent not only with the preferences over single alternatives, but also they should depend on the interpretation attached to the notion of sets and on how the individuals think that the ties are going to be resolved. Many different domains of preferences can be proposed. For instance, individuals can compare sets according to their expected utility. Alternatively, they could rely on extreme attitudes towards risk. Once the preferences over sets of the individuals are specified, it is possible to define the game that the participants in an election play. Then, the multivalued counterparts to the strategy-proofness and candidate stability can be formulated and the results can be compared.

The first part of this thesis studies the incentives of the candidates in multivalued voting rules. Dutta, Jackson and Le Breton [17] have analyzed the implications of candidate stability for single-valued voting rules. When candidates are not allowed to vote, they have shown that

only rules that select the best preferred candidate of an arbitrary voter, a dictator, are candidate stable and unanimous. Their work is extended here in two complementary directions. Chapter 2 considers rules that select a set of candidates for each configuration of the agenda and each voters' preference profile. The results depend crucially on the way in which candidates are assumed to compare sets of candidates. If candidates compare sets according to the axioms of Expected Utility Theory and Bayesian Updating from some prior assessment, the results of Dutta, Jackson and Le Breton [17] basically hold. Nevertheless, if candidates' preferences over sets are consistent with extreme attitudes towards risk, candidate stability becomes less stringent and positive results arise.

In Chapter 3, we propose an alternative framework. We consider voting rules that select a lottery on the set of running candidates for each agenda and each profile of voters' preferences. This framework would encompass voting rules in which chance plays an explicit role. It would also comprehend situations in which candidates can assess different subjective probabilities to each candidate to be the final winner of the election. Assuming that candidates are expected utility maximizers and are not allowed to vote it is shown that, generically, only random combinations of dictatorial rules are candidate stable and unanimous. However, the results depend dramatically on the size of the society. For instance, when there are only three initial candidates, sub-additive distributions of the veto power are also admitted.

Finally, we investigate the possibility of constructing strategy-proof multivalued voting rules. In single-valued environments, provided that there are at least three alternatives and voters' preferences are unrestricted, the celebrated Gibbard-Satterthwaite Theorem states that only dictatorial rules are strategy-proof (Gibbard [21], Satterthwaite [37].) In the last years several papers tried to extend this result to multivalued environments. Among them, Duggan and Schwartz [16] propose a weak definition of strategy-proofness and some apparently mild regularity conditions, yet they obtain a dictatorship result. In Chapter 4, we analyze the Duggan and Schwartz's framework and highlight the role of one of the regularity conditions: *Residual Resoluteness*. This property compels the social choice to be single-valued when the preferences of the voters are very similar. We show that this condition is very compelling for small societies and, in fact, it is responsible of Duggan and Schwartz's pessimistic result. However, it is proved that

only rules that centralize the power of decision in an arbitrary group of voters are strategy-proof and include all single alternatives in the range. Moreover, this is also the case even when the voters are allowed to report their preferences over sets of candidates. Thus, this work provides further evidence on the strength and robustness of the Gibbard-Satterthwaite Theorem.

Each chapter is self-contained and thus, the chapters can be read independently.

Chapter 2

Candidate Stability and Voting Correspondences

2.1 Introduction

The Social Choice literature has addressed the study of strategic manipulation of collective choice processes. Much work has been devoted to analyze the incentives of the voters to misrepresent their preferences. But there are other ways in which the participants in an election can affect social choices. The set of candidates (or possible alternatives) is often taken as exogenous and independent of any individual decision. Nevertheless, to enter an election implies an explicit decision by the candidates, and candidates can influence the result of the choice procedure by leaving the fray, even when they have no chance of winning. For instance, we can think of a presidential election in which there are three candidates, one of them, conservative, with the support of 40% of the voters, and two of them, leftist, with the support of 35%, and 25%, respectively. If the voting rule is plurality and voters cast their ballot according to their true preferences, the conservative candidate wins, but if one of the leftist candidates quits, the other leftist candidate will be the one chosen. If the leftist candidates prefer a leftist president to a conservative one, they will have incentives to exit the poll and transfer their votes to the other leftist candidate.

Elections can be modeled as two stage processes. In a first stage, a set of candidates decide

whether they run the election or they leave the ballot. Later on, voters decide the social outcome according to a voting mechanism that takes into account their preferences over the running candidates. In this paper, we will focus on the incentives at the first stage and analyze the possibility of constructing voting mechanisms such that a candidate never prefers to exit the election unilaterally rather than to run the fray. In other terms, staying as a candidate when the rest of the candidates stay is always a Nash Equilibrium strategy for all the candidates. This condition is called "candidate stability". Only if candidate stability holds we can consider the set of feasible candidates as exogenous.

The analysis of candidates' strategic concerns is a relatively new topic in the Social Choice literature. Osborne and Slivinski [31] and Besley and Coate [10] inaugurated the study of strategic candidacy in representative democracy models. They focused on the number of candidates and the pattern of entry of candidates in elections ruled by plurality. Instead, we will follow the normative analysis of the subject proposed by Dutta, Jackson and Le Breton [17],[1] who introduced candidate stability condition in single-valued elections. Their main result is that any single-valued unanimous and candidate stable voting procedure must be dictatorial if candidates are not allowed to vote. In the case of voting candidates, they also show that an impossibility result holds for a large family of voting rules.

In this work, we focus on the case in which there are not voting candidates and generalize DJL's analysis to multivalued environments. Therefore, we model elections as voting correspondences. For each set of candidates at stake and each preference profile over candidates in the ballot of voters, a voting correspondence chooses a set of candidates. We can interpret this voting correspondence as the first screening device which narrows the social agenda to a smaller set of candidates. Candidates do know this first selection but they are not aware of the final resolution of the social choice. Therefore, candidates consider this set as the result of their stay-quit decision.

Another possible interpretation fits environments in which the voting mechanism played by the voters selects a single candidate but it admits multiple equilibria. We can think of situations in which the candidates know the set of equilibria of the voting mechanism, but they do not

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[1]Henceforth DJL.

have enough information to deduce which are the strategies actually played by the voters and the equilibrium that eventually arises. In these cases, the candidates cannot apply any backward induction argument to focus on a unique equilibrium of the voting stage. Nevertheless, they may consider the whole set of possible equilibria as the result of the social choice.

When the result of the voting correspondence is multivalued, the study of the candidates' strategic concerns becomes problematic. Individuals are naturally endowed with preferences over candidates, but such preferences do not contain enough information to compare sets of candidates. Thus, we have to provide candidates with preferences over sets of candidates, but these preferences are going to be naturally restricted in two ways. First, they must respect candidates' original preferences over singleton sets. Moreover, they should be consistent with the interpretation attached to the concept of sets of candidates. In this work we propose different restricted domains of preferences over sets that fit different interpretations of the resolution of the social choice. For instance, we can suppose that candidates compare sets according to their expected utility. Alternatively, they could rely on some criterion consistent with extreme attitudes towards risk, and their preferences may be consistent with leximin, maximin or maximax criteria.[2] Indeed, the implications of candidate stability will depend crucially on the choice of the specific domain of preferences over sets.

We prove that if candidates compare sets according to their expected utility conditional on some prior probability assessment, only dictatorial rules are candidate stable and unanimous, whereas if the prior assessments are restricted to be uniform, also rules that select the best candidates for two fixed agents satisfy both conditions. On the other hand, positive results are obtained if candidates' preferences over sets are consistent with extreme attitudes towards risk.

Besides the work by DJL on single-valued candidate stable voting procedures, there are Ehlers and Weymark [18] and Eraslan and McLennan [19] who investigate candidate stability in multivalued environments. These papers and ours are independent but complementary con-

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[2]These domains of preferences over sets have been applied to the study of strategy-proofness of social choice correspondences in Pattanaik [32] and [33], Gärdenfors [20], Barberà, Dutta and Sen [6], and Campbell and Kelly [11] among others. We refer the reader to Barberà, Bossert and Pattanaik [5] for a recent survey on ranking sets of objects.

tributions extending the DJL's model to set valued elections. Both [18] and [19] propose the same condition of candidate stability, but they do not model explicitly the role of preferences of candidates over sets of candidates. Hence, they restrict attention to only one of the possible scenarios considered here. In [18] an alternative direct proof of the main theorem of DJL is provided. It is also shown that the arguments in their proof can be applied to multivalued environments to obtain that only dictatorial rules are candidate stable and unanimous. On the other hand, in [19] DJL's framework is also extended to allow the voters to report indifferences. They prove that only serial dictatorships satisfy candidate stability and unanimity.

A companion paper, Rodríguez-Álvarez [36],[3] proposes a different approach to the analysis of multivalued elections. In [36] elections are modeled as probabilistic voting procedures, that is, rules that for each agenda and each profile of preferences select a lottery on the set of candidates. In this stochastic environment, the implications of candidates stability depend crucially on the size of the set of candidates. Not surprisingly, random combinations of dictatorial rules are always unanimous and candidate stable. Nevertheless, many other possibilities can be devised. For instance, when there are only three candidates a rule assigning equal probability to the set of efficient candidates at stake is unanimous and candidate stable.

We can mention the work of Sönmez [41] who studies the incentives of the agents to affect the social agenda in a multivalued economic environment: centralized two sided matching markets. Specifically, he focuses on the incentives of an hospital to destroy positions available to medical interns and he shows that the solution used to match students and hospitals in the US is manipulable in this way. The main difference with our analysis relies on his interpretation of multiple outcomes. A hospital may be interested in hiring more than one medical intern. So, the students matched to each hospital are compatible and there is no need to solve possible ties. This implies that the domain of preferences over sets of students employed by hospitals does not fit the preferences used by candidates to compare sets of candidates.

Finally, we refer to an interesting paper by Berga, Bergantiños, Massó and Neme [9], who also analyze the incentives of the participants in a multivalued election to withdraw. They study the problem of a society choosing a subset of new members from a finite set of candidates. They

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[3]See Chapter 3.

consider explicitly the possibility that the founding members of this society may want to leave the society if they do not like the resulting new society. It is shown that only unanimity rule is non-manipulable and stable, no founder prefers to quit rather than to stay in the club. In contrast to candidate stability, their stability condition is not strategic. In their framework, if considering exiting, an agent does not take into account the effect of her withdrawal on the social outcome. Instead, she considers to withdraw because she may prefer to stay alone rather than to belong to the resulting new society.

The remainder of the paper is structured as follows. We devote Section 2.2 to outline the setting and provide definitions and notation, and Section 2.3 to develop the implications of candidate stability when candidates are expected utility maximizers and are not allowed to vote. In Section 2.4 we refer to the possibilities that arise when the preferences of the candidates over sets are consistent with extreme attitudes towards risk. Finally, in Section 2.5 we discuss the possible overlap between candidates and voters. In Appendix 1 we include the proofs of the theorems. In Appendix 2 we prove some intermediate results.

2.2   Definitions

2.2.1   Candidates, Voters and Preferences

Voters and Candidates

Let $\mathcal{N}$ be a society formed by a finite set of voters $\mathcal{V}$, and a finite set of candidates $\mathcal{C}$, $\mathcal{N} = \mathcal{C} \cup \mathcal{V}$. We consider the case in which $\#\mathcal{C} \geq 3$ since the strategic concerns with $\#\mathcal{C} \leq 2$ are trivial.[4] We focus on the case in which there is no overlap between the sets of voters and candidates, $\mathcal{C} \cap \mathcal{V} = \{\varnothing\}$. It could be an approximation to the situation in which the set of candidates is relatively small with respect to the set of voters, and hence, their capability to influence the social choice is negligible. Moreover, in this scenario, we can isolate the incentives of the candidates to participate in an election, regardless of their concerns as voters.[5]

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[4]$\#A$ refers to the cardinality of the set A.

[5]When candidates can be voters, the analysis of stability becomes more complicated because candidates' preferences are assumed to favor their own election. Then, stronger conditions are required and their interpretation is not clear in terms of candidates' incentives. This case is left to Section 2.5.

Preferences over Candidates

Let $\mathcal{P}$ be the set of all linear orders on $\mathcal{C} \cup \{\varnothing\}$, where the empty set refers to the situation in which no candidate is elected.[6] Each individual $i \in \mathcal{N}$ is endowed with a linear order $P_i \in \mathcal{P}$. Let $\mathcal{P}^i \subset \mathcal{P}$ be the set of preferences over candidates admissible for individual i. We assume that for all i, any candidate is preferred to the empty set. Preferences of voters over candidates are no longer restricted, but each candidate considers herself as the best alternative. That is, aP_ab for all $a, b \in \mathcal{C}$, $P_a \in \mathcal{P}^a$. We denote by $P \in \mathcal{P}^{\mathcal{V}}$ a voters' preference profile.

Let $2^{\mathcal{C}} \backslash \{\varnothing\}$ be the set of all non-empty finite subsets of $\mathcal{C}$. For each $A \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, $P \mid_A$ denotes the restriction of P to the set A. By *top* (A, P_i) and *bottom* (A, P_i) we mean, respectively, the best and worst element of A according to the order $P_i \in \mathcal{P}$.

Abusing notation, for any set $I \subseteq \mathcal{N}$, $\mathcal{P}^I$, is defined as the set of admissible preferences profiles for I. Then, P_I refers to the restriction of the profile P to the members of I, .

Preferences over Sets of Candidates

As we have already noted, we are interested in the incentives of candidates to withdraw from multivalued elections. Then, in order to compare social outcomes, candidates must be endowed with preferences over sets of candidates.

Let $\mathcal{D}$ be the set of all complete and reflexive binary relations on $2^{\mathcal{C}} \backslash \{\varnothing\}$. We denote by $\succsim \in \mathcal{D}$ an arbitrary preference relation over sets of candidates. Although we assume strict preferences over alternatives, we do not rule out the possibility of indifference among sets. Hence for each $\succsim \in \mathcal{D}$, $\succ$ refers to the strict component of $\succsim$, while $\sim$ refers to the indifference term that is immediately defined from $\succ$ in the standard manner.

The preferences over sets of candidates are restricted. They must satisfy the restriction on their preferences over single candidates and our interpretation of sets of candidates. According to the different interpretations in the resolution of the social choice we present several environments which define different domains of admissible preferences over sets. We will defer the presentation

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[6]A linear order is a complete, antisymmetric and transitive binary relation. For each $P_i \in \mathcal{P}$, the associated weak preference relation is denoted by R_i.

of the different domains until later sections. At this point, we introduce some notation.

For any $a \in \mathcal{C}$, the domain $\mathcal{D}_a^{\mathcal{E}} \subset \mathcal{D}$ denotes the set of admissible preferences over sets of candidates for candidate a in the environment $\mathcal{E}$.

Equivalently, we can define $\mathcal{D}_a^{\mathcal{E}}$ via the admissible preferences over single candidates. Given the environment $\mathcal{E}$, for any $P_a \in \mathcal{P}^a$, let $\succsim^{\mathcal{E}}(P_a) \subset \mathcal{D}$ be the set of preferences over sets consistent with P_a in the environment $\mathcal{E}$. Then $\mathcal{D}_a^{\mathcal{E}}$ could be defined in the following way:

$$\mathcal{D}_a^{\mathcal{E}} = \{\succsim \in \mathcal{D} \text{ s.t. there is } P_a \in \mathcal{P}^a \text{ with } \succsim \in \succsim^{\mathcal{E}}(P_a)\}.$$

2.2.2 Voting Correspondences

The object of interest of this article is an aggregation rule, a voting correspondence v, that for each configuration of the ballot and each preference profile selects a set of candidates at contest.

Definition 2.1. *A voting correspondence is a mapping* $v : 2^{\mathcal{C}} \backslash \{\varnothing\} \times \mathcal{P}^{\mathcal{V}} \to 2^{\mathcal{C}} \backslash \{\varnothing\}$ *such that for all* $A \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ *and for all* $P \in \mathcal{P}^{\mathcal{V}}$:

i) $v(A, P) \subseteq A$.

ii) $v(A, P) = v(A, P')$ *for all* $P' \in \mathcal{P}^{\mathcal{V}}$ *such that,* $P \mid_A = P' \mid_A$.

Item i) refers to the fact that candidates cannot be selected if they are not at stake.

In item ii) we assume Arrow's Independence of Irrelevant Alternatives, only the preferences of voters over the candidates at stake matter.

Notice that a voting correspondence is a collection of social choice correspondences, one for each configuration of the agenda. A voting correspondence is more general than a voting procedure as defined in DJL, since a voting correspondence allows for multivalued choices.

2.2.3 Unanimity

In this work we abstract from the mechanism used in the voting stage. We only assume unanimity, which is uncontroversial. When all voters agree on who is the best candidate, this is elected.

Definition 2.2. *The voting correspondence v satisfies unanimity if for any $B \in 2^{\mathcal{C}} \setminus \{\varnothing\}$ and any $b \in B$, $b = top(B, P_i)$ for all $i \in \mathcal{V}$, implies $v(B, P) = b$.*

2.2.4 Candidate Stability

The candidate stability condition captures the lack of incentives of candidates to affect the social outcome by leaving the electoral fray. The feasible set of alternatives can be considered independent of individual decisions and exogenously fixed, only if a candidate never has incentives to withdraw her candidacy.

We say a voting correspondence v is candidate stable if and only if for all $a \in \mathcal{C}$ and for all $P \in \mathcal{P}^{\mathcal{V}}$, a does not prefer $v(\mathcal{C} \setminus \{a\}, P)$ to $v(\mathcal{C}, P)$. Equivalently, a voting correspondence is candidate stable if and only if the entry of all candidates always constitutes a Nash equilibrium strategy for all candidates.

Candidate stability is not a condition on the entry-exit game form, but on the family of games which are generated from the entry-exit game form once the preferences of agents over social choice outcomes are specified. Since the result of the stay-quit decision by the candidates is not necessarily a singleton, these preferences cannot be trivially derived from candidates' preferences over candidates. That is why candidates have to be endowed with some criterion in order to extend their preferences from alternatives to sets of alternatives. In order for the Nash equilibrium requirements not to be trivially fulfilled, we have assumed that preferences over sets of candidates are complete.

The choice of the domain restriction on candidates' preferences over sets will become crucial for our analysis since the different domains lead to different implications of candidate stability. We now include a generic definition of candidate stability in an arbitrary environment $\mathcal{E}$.

Definition 2.3. *A voting correspondence v is candidate stable in the environment $\mathcal{E}$ (or $\mathcal{E}$ candidate stable) if and only if for all $a \in \mathcal{C}$, for all $P \in \mathcal{P}^{\mathcal{V}}$ and for all $\succsim_a \in \mathcal{D}_a^{\mathcal{E}}$, $v(\mathcal{C}, P) \succsim_a v(\mathcal{C} \setminus \{a\}, P)$.*

Notice that we do not introduce stability conditions regarding the incentives of non-running candidates to enter the election. While it seems necessary to introduce restrictions on the

incentives to enter the ballot of the candidates who are not in the agenda, it is a debatable issue to define a stability condition that applies to them. Thus, in this paper we focus our attention on the strategic incentives to leave the ballot.

When defining candidate stability, we only refer to the stability of the whole set of candidates $\mathcal{C}$. That is, we investigate the necessary conditions for the set $\mathcal{C}$ to be a stable agenda. With such definition of stability it should not be surprising that the implications of candidate stability depend on the composition of the set of initial candidates.

We can provide different definitions of stability while maintaining the same spirit of candidate stability. For instance, we could investigate rules for which any possible agenda were stable. That is, candidates should never have incentives to leave the fray independently of the configuration of the ballot. With this definition of stability, we would introduce the logic of candidate stability at any possible agenda $A \in 2^{\mathcal{C}} \backslash \{\varnothing\}$. Evidently, such condition would be a stronger one than candidate stability, but we would obtain the same characterization results (up to minimal modifications in the statements.) Although this new stability condition may facilitate the proofs and the presentation of the results, we follow DJL formulation of the stability condition, which allows us to state stronger results.

2.3   Candidate Stability and Conditional Expected Utility Maximizers

We begin the formal analysis by introducing two domains of preferences over sets that match scenarios in which the candidates are expected utility maximizers. These domains of preferences were proposed in [6] for the study of strategy-proof social choice correspondences. In these environments, candidates are assumed to interpret the possible social choice outcomes as the basis of lotteries and to have von Neumann-Morgenstern preferences over lotteries. If a set is chosen, candidates attribute some probabilities of choice to the elements of this. The proposed scenarios differ in the description of the way individuals will assess the probabilities associated to sets. In the first one, *Conditional Expected Utility Consistent Preferences*(or simply, BDS1), each candidate is assumed to subjectively assess one probability distribution over $\mathcal{C}$ and to evaluate each

subset according to its conditional expected utility. In the second scenario, *Conditional Expected Utility Consistent with Even Chance Probabilities Preferences*(BDS2), the initial distribution of probability is restricted to be uniform, all candidates associate each set with an even chance lottery over all its components. We now state this formally.

A utility function is a mapping $u : \mathcal{C} \longrightarrow \mathbb{R}$. We say that u fits preference ordering $P \in \mathcal{P}$ or represents P if for any $a, b \in \mathcal{C}$, aPb iff $u(a) > u(b)$. An individual probability assessment λ_i is a function $\lambda_i : \mathcal{C} \longrightarrow (0, 1)$, such that $\sum\limits_{a \in \mathcal{C}} \lambda_i(a) = 1$. We rule out degenerate lotteries as probability assessments since those are not consistent with our interpretation of sets of candidates as the possible results of the social choice process.

Definition 2.4 (Conditional Expected Utility Consistent; BDS1). *The ordering $\succsim \in \mathcal{D}$ is a Conditional Expected Utility Consistent ordering admissible for candidate $a \in \mathcal{C}$, we write $\succsim \in \mathcal{D}_a^1$, if and only if there is $P_a \in \mathcal{P}^a$, utility function u_a representing P_a along with a prior assessment λ_a such that for all $X, Y \in 2^{\mathcal{C}} \setminus \{\varnothing\}$:*

$$X \succsim Y \Leftrightarrow \sum_{x \in X} \frac{\lambda_a(x) u_a(x)}{\left(\sum_{x \in X} \lambda_a(x)\right)} \geq \sum_{y \in Y} \frac{\lambda_a(y) u_a(y)}{\left(\sum_{y \in Y} \lambda_a(y)\right)}.$$

Definition 2.5 (Conditional Expected Utility Consistent with Equal Probabilities; BDS2). *The ordering $\succsim \in \mathcal{D}$ is a Conditional Expected Utility Consistent with Equal Probabilities ordering admissible for candidate $a \in \mathcal{C}$, $\succsim \in \mathcal{D}_a^2$, if and only if there is $P_a \in \mathcal{P}^a$ and a utility function u_a fitting P_a such that for all $X, Y \in 2^{\mathcal{C}} \setminus \{\varnothing\}$:*

$$X \succsim Y \Leftrightarrow \frac{1}{\#X} \sum_{x \in X} u_a(x) \geq \frac{1}{\#Y} \sum_{y \in Y} u_a(y).$$

Remark 2.1. *BDS2 preferences are a special case of BDS1 preferences, where prior assessments are restricted to be uniform. $\mathcal{D}_a^2 \subset \mathcal{D}_a^1$.*

Before proceeding to state the restrictions that candidate stability implies in both environments, we begin with a lemma on the existence of various admissible preference orderings in $\mathcal{D}_a^1$ and $\mathcal{D}_a^2$. Its proof appears in Appendix 2.

Lemma 2.1. *Let $a \in \mathcal{C}$ and $X, Y \in 2^{\mathcal{C}} \setminus \{\varnothing\}$:*

i) *If $a \notin (X \cup Y) \Rightarrow$ there exists a $\succsim \in \mathcal{D}_a^2$, such that $X \succ Y$.*

ii) *If X, Y are such that $X = (Y \cup \{a\})$ and $a \notin Y, \Rightarrow$ for all $\succsim \in \mathcal{D}_a^1$, $X \succ Y$.*

iii) *If $a \in Y$, $Y \neq (X \cup \{a\})$, and $\#Y \neq 1 \Rightarrow$ there is a $\succsim \in \mathcal{D}_a^1$, such that $X \succ Y$.*

iv) *If there is $b \in Y$ such that $(X \backslash \{a\}) = (Y \backslash \{b\}) \Rightarrow$ for all $\succsim \in \mathcal{D}_a^2$, $X \succ Y$.*

v) *If $a \in Y$, $\#Y \neq 1$ and for all $b \in (X \cup \{\varnothing\})$, $(X \backslash \{b\}) \neq (Y \backslash \{a\}) \Rightarrow$ there is a $\succsim \in \mathcal{D}_a^2$, such that $X \succ_a^2 Y$.*

Lemma 2.1 presents the main features of the admissible preferences over sets in both BDS domains. Items i), iii) and v) imply that the preferences of the candidates over a significant amount of sets are not restricted. Moreover, item iv) says that the BDS2 criterion generates responsive preferences: for any two sets that differ in only one candidate, the set containing the more desirable candidate is always preferred.

Now, we can state the implications of candidate stability in these environments. In single-valued elections, as a result of the self-preference of the candidates, DJL have proved that the incentives to withdraw only affect the non-elected candidates. In multivalued voting processes this fact no longer holds, as we see in the following lemmata.

Lemma 2.2. *A voting correspondence v is BDS1 candidate stable if and only if for all $a \in \mathcal{C}$, $P \in \mathcal{P}^{\mathcal{V}}$:*

i) *if $a \in v(\mathcal{C}, P)$ and $v(\mathcal{C}, P) \neq \{a\}, \Rightarrow v(\mathcal{C}, P) = (v(\mathcal{C} \backslash \{a\}, P) \cup \{a\})$ and,*

ii) *if $a \notin v(\mathcal{C}, P)$, $v(\mathcal{C}, P) = v(\mathcal{C} \backslash \{a\}, P)$.*

Proof. We start by proving the sufficiency side. Consider an arbitrary candidate $a \in \mathcal{C}$ and an arbitrary profile of voters preferences $P \in \mathcal{P}^{\mathcal{V}}$. Assume first that $v(\mathcal{C}, P) = \{a\}$. By the self preference of the candidates, for all $\succsim \in \mathcal{D}_a^1$, $v(\mathcal{C}, P) \succsim v(\mathcal{C} \backslash \{a\}, P)$, and the candidate a has no incentive to leave the fray. Suppose now that $a \in v(\mathcal{C}, P), v(\mathcal{C}, P) \neq \{a\}$ and $(v(\mathcal{C}, P) = v(\mathcal{C} \backslash \{a\}, P) \cup \{a\})$, by item ii) of Lemma 2.1, we know that for all $\succsim \in \mathcal{D}_a^1$, $v(\mathcal{C}, P) \succ v(\mathcal{C} \backslash \{a\}, P)$. Finally, if $a \notin v(\mathcal{C}, P)$ and $v(\mathcal{C}, P) = v(\mathcal{C} \backslash \{a\}, P)$, then the candidate a's withdrawal does not affect the social choice so she is not better off after leaving the fray. This case exhaust the possibilities, and as the choice of P was arbitrary, sufficiency holds.

Now we check the necessity. First, let us prove i). Assume to the contrary v is BDS1 candidate stable but there are $a \in \mathcal{C}, P \in \mathcal{P}^{\mathcal{V}}$ such that $a \in v(\mathcal{C}, P), v(\mathcal{C}, P) \neq \{a\}$ but $v(\mathcal{C}, P) \neq (v(\mathcal{C}\backslash\{a\}, P) \cup \{a\})$. By item iii) of Lemma 2.1, we know there is $\succsim \in \mathcal{D}_a^1$ such that $v(\mathcal{C}\backslash\{a\}, P) \succ v(\mathcal{C}, P)$, which contradicts BDS1 candidate stability. In order to check the necessity of ii), assume again that v is BDS1 candidate stable but there are $a \in \mathcal{C}, P \in \mathcal{P}^{\mathcal{V}}$ such that $a \notin v(\mathcal{C}, P)$, but $v(\mathcal{C}, P) \neq v(\mathcal{C}\backslash\{a\}, P)$. Notice that, $a \notin (v(\mathcal{C}, P) \cup v(\mathcal{C}\backslash\{a\}, P))$, hence by item i) of Lemma 2.1 and Remark 2.1, there is $\succsim \in \mathcal{D}_a^1$ with $v(\mathcal{C}\backslash\{a\}, P) \succ v(\mathcal{C}, P)$, again contradicting BDS1 candidate stability.

∎

Lemma 2.2 shows that in the domain of BDS1 preferences, if an elected candidate quits the ballot, then the result of the social choice after her withdrawal must be the election of the remaining elected candidates. Moreover, the withdrawal of a non-elected candidates must have no effect on the social choice. Lemma 2.2 provides an expected utility rationale for the strong candidate stability condition proposed in [18] and [19]. Only when the candidates cannot vote and compare sets according to Conditional Expected Utility Consistent preferences, the strong candidate stability condition introduced in these works is equivalent to a candidate stability condition derived from candidates preferences over the possible social choice outcomes.

Lemma 2.3. *A voting correspondence v is BDS2 candidate stable if and only if for all $a \in \mathcal{C}$ and $P \in \mathcal{P}^{\mathcal{V}}$:*

   i) *If $a \in v(\mathcal{C}, P)$ and $v(\mathcal{C}, P) \neq \{a\}$ either:*

   a) $v(\mathcal{C}, P) = v(\mathcal{C}\backslash\{a\}, P) \cup \{a\}$ *or,*

   b) *there is $b \in \mathcal{C}\backslash\{a\}$ such that $v(\mathcal{C}\backslash\{a\}, P) = (v(\mathcal{C}, P)\backslash\{a\}) \cup \{b\}$.*

   ii) *If $a \notin v(\mathcal{C}, P)$, $v(\mathcal{C}, P) = v(\mathcal{C}\backslash\{a\}, P)$.*

Proof. In order to prove sufficiency, we only have to check item $i, b)$ since, as for all $a \in \mathcal{C}$ $\mathcal{D}_a^2 \subset \mathcal{D}_a^1$, the arguments in Lemma 2.2 apply directly to the remaining cases. Hence, consider $P \in \mathcal{P}^{\mathcal{V}}$ and $a \in v(\mathcal{C}, P)$ such that there is another candidate $b \in \mathcal{C}$ satisfying $v(\mathcal{C}\backslash\{a\}, P) = (v(\mathcal{C}, P)\backslash\{a\} \cup \{b\}$. By the self preference of the candidates, and by the responsiveness of the

preferences in $\mathcal{D}_a^2$, items ii) and iv) of Lemma 2.1, we have that for all $\succsim \in \mathcal{D}_a^2$, $v(\mathcal{C}, P) \succ v(\mathcal{C}\backslash\{a\}, P)$. So, BDS2 candidate stability holds.

Now, we prove the necessity. Assume to the contrary that v is BDS2 candidate stable but it does not satisfy item i). Hence, there are $a \in v(\mathcal{C}, P)$ and $P \in \mathcal{P}^{\mathcal{V}}$ such that there is no candidate $b \in (v(\mathcal{C}\backslash\{a\}, P) \cup \{\varnothing\})$, such that $v(\mathcal{C}\backslash\{a\}, P) = (v(\mathcal{C}, P)\backslash\{a\}) \cup \{b\}$. Applying item v) of Lemma 2.1, we know there is $\succsim \in \mathcal{D}_a^2$ such that $v(\mathcal{C}\backslash\{a\}, P) \succ v(\mathcal{C}, P)$, which contradicts BDS2 candidate stability. Finally, the arguments in the proof of necessity of Lemma 2.2 can also be applied in the BDS2 environment in order to show the necessity of item ii).

■

Even though BDS2 restricts considerably the preferences of the candidates, BDS2 candidate stability is not much weaker than BDS1 candidate stability. Basically, the only new possibility that arises with respect to the BDS1 environment is that an elected candidate does not have incentives to leave the fray if her withdrawal results in the selection of another candidate.

In the following Theorem 2.1 and Theorem 2.2, we show that candidate stability and unanimity introduce rather compelling restrictions in both BDS scenarios. Almost any unanimous voting correspondence is subject to the incentives of the candidates to leave the ballot in environments in which sets of candidates are compared according to the postulates of the Expected Utility Theory. Before we state the characterization results, we need to introduce some notation.

A voting correspondence v is dictatorial if there exists a voter $i \in \mathcal{V}$ such that $v(A, P) = top(A, P_i)$ for all $P \in \mathcal{P}^{\mathcal{V}}$ and all agendas $A \in \{\mathcal{C}, \{\mathcal{C}\backslash\{a\}\}_{a \in \mathcal{C}}\}$. Analogously, a voting correspondence v is bidictatorial if there are two voters $i, j \in \mathcal{V}$ such that $v(A, P) = top(A, P_i) \cup top(A, P_j)$ for all $P \in \mathcal{P}^{\mathcal{V}}$ and all agendas $A \in \{\mathcal{C}, \{\mathcal{C}\backslash\{a\}\}_{a \in \mathcal{C}}\}$.[7] Finally, for any $I \subseteq \mathcal{V}$, $P \in \mathcal{P}^{\mathcal{V}}$ and $A \in 2^{\mathcal{C}}\backslash\{\varnothing\}$, we define $Pareto(A, P_I) = \{a \in A,$ s.t. for no $b \in A,$ $bP_i a$ for all $i \in I\}$.

Theorem 2.1. *A voting correspondence v is unanimous and BDS1 candidate stable if and only if v is dictatorial.*

Theorem 2.1 is equivalent to the Theorem in [19] when candidates cannot vote and voters are not allowed to express weak preferences. The result in DJL's Theorem 1 also applies to

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[7]This restricted definition of dictatorial rules is due to the fact that we only care about the stability of the whole set of candidates.

multivalued environments if the candidates preferences over sets of candidates are not severely restricted.

Theorem 2.2. *Let v be a unanimous voting correspondence.*

i) *If $\#\mathcal{C} \geq 4$; v is candidate stable according to BDS2 if and only if v is dictatorial or bidictatorial.*

ii) *If $\#\mathcal{C} = 3$; v is candidate stable according to BDS2 if and only if there is a set of voters $S \subseteq \mathcal{V}$, such that for all $A \in \{\mathcal{C}, \{\mathcal{C}\backslash\{a\}\}_{a\in\mathcal{C}}\}$ and for all $P \in \mathcal{P}^{\mathcal{V}}$:*

a) *$v(A, P) \subseteq Pareto(A, P_S)$ and,*

b) *for any $a, b \in \mathcal{C}$, $v(A, P) \neq \{a\}$ if there is $i \in S$ such that $bP_i a$.*

In the light of Lemma 2.1 and Lemma 2.3, the results of Theorem 2.2 are not surprising. In general, only rules that choose the best candidates of two arbitrarily fixed voters are allowed apart from dictatorial ones. However, when the initial set of candidates is fairly small another possibilities arise. In this case the self preference of the candidates and the responsiveness of the preferences reduce considerably the set of admissible preferences and candidates' incentives to withdraw.

The proofs of Theorem 2.1 and Theorem 2.2 are in Appendix 1. The proofs rely on the arguments of a more general result, Theorem 2.4, that appears in Section 2.5. Theorem 2.4 allows for an overlap between the candidates' and voters' sets at the cost of employing stability notions that are not directly derived from the candidates' incentives to withdraw the election. We provide now a sketch of the proofs. We analyze first the election when voters preferences are restricted to prefer three arbitrary candidates to the rest and voters also agree in their preferences over the remaining candidates. In this restricted environment, from a voting correspondence we can define an auxiliary group decision rule. In both conditional expected utility scenarios, candidate stability and unanimity imply that this ancillary group decision rule always generates quasitransitive orderings on $\mathcal{C}$,[8] and satisfies Unrestricted Domain, Pareto Efficiency and Arrow's Independence of Irrelevant Alternatives. Hence, we can apply a version of Arrow's Theorem

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[8]A binary relation is quasitransitive if its strict component is transitive.

stated by Mas-Colell and Sonnenschein [28] to obtain that there is a group of voters holding veto power over v. The result can be extended to arbitrary profiles of preferences. Moreover, in the BDS1 scenario, it is immediate to see that only one voter may hold veto power and then the dictatorship result is obtained.[9]

Finally, in the BDS2 environment, we need at least four candidates to get the bidictatorship result. When, there are only three candidates, many rules satisfy the requirements of unanimity BDS2 and candidate stability. We can mention the correspondences choosing the top candidates of an arbitrarily large fixed group of voters, or the Pareto correspondence. Nevertheless, many other rules fulfill the conditions of Theorem 2.2.

Example 2.1. *Let $\mathcal{V} = \{1, ..., \#\mathcal{V}\}$, and $\#\mathcal{C} = 3$. For any $P \in \mathcal{P}^{\mathcal{V}}$ and $A \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ define by $k^*(A, P) = \min\{i \in \mathcal{V}, \ s.t. \ top(A, P_i) \neq top(A, P_1)\}$. Construct the voting correspondence $\tilde{v}$ in the following way, for all $A \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, and for all $P \in \mathcal{P}^{\mathcal{V}}$, $\tilde{v}(A, P) = top(A, P_1) \cup top\left(A, P_{k^*(A,P)}\right)$. Notice that $\tilde{v}$ satisfies ii,a) ii,b) of Theorem 2.2, hence, $\tilde{v}$ is unanimous and candidate stable according to BDS2.*

Finally, we have to remark that, apart from the case in which only three candidates are at stake, Theorem 2.1 and Theorem 2.2 are in the line of the results in [6] for strategy-proof correspondences. Nevertheless, we want to highlight that strategy-proofness and candidate stability are very different and independent conditions. Basically, non-manipulability via preferences is a monotonicity condition on the preferences of the voters expressed in a fixed agenda framework, whereas candidate stability is defined in variable agenda settings, and implies consistency conditions in the choice at different ballots. The similarity of the results is due to the relation of both frameworks with some version of the Arrow's Theorem. In the following section we will show that the differences are not exclusive to the BDS2 domain.

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[9]The theorems admit other proofs. For instance, following [18], we can show that if a voting correspondence is candidate stable according BDS1 or BDS2 and unanimous, it is Pareto Efficient when the agenda contains at least $\#\mathcal{C} - 1$ candidates. Then, from the results of Denicolò [14] on non-binary choice, we can develop the arguments that lead us to complete the proof. Denicolò's theorems are not concerned with candidate stability, but they make use of consistency conditions that can be derived from both BDS versions of candidate stability.

2.4 Discussion on the Domain of Preferences over Sets

The results stated in the previous section depend dramatically on how candidates compare the sets of candidates. Both BDS scenarios are plausible when candidates can make use of plenty of information and construct preferences over sets in a somehow sophisticated way. Moreover, in both BDS scenarios the domains of admissible preferences over sets of candidates are very large. For each preference relation over candidates there are infinite utility functions representing it, and many different orderings over $2^{\mathcal{C}} \backslash \{\varnothing\}$ can be devised from each preference ordering over candidates. Therefore, as we have seen, many possibilities of profitable withdrawal arise. In this section, we assume that candidates cannot assess any probability distribution on the set of elected candidates. In such situations, candidates may have extreme attitudes towards risk, consistent with the leximin, maximin or maximax preferences.[10] They may also be reluctant to express strict preference among sets which are not clearly better or worse than other. In this case the candidates may use other criteria to construct their preferences over sets like, for example, the Gärdenfors preferences.

Candidate Stability for Leximiners

Leximin preferences reflect the way in which sophisticated pessimistic individuals construct their preferences over sets. When comparing two sets, a leximiner first compares the worst alternatives in each set. If they are the same, leximiners care about the next to the worst ones, and if they are again the same, leximiners follow the procedure in the same fashion. We express this formally:

Definition 2.6 (Leximin Preferences). *Let $a \in \mathcal{C}$, $P_a \in \mathcal{P}^a$ and $X \in 2^{\mathcal{C}} \backslash \{\varnothing\}$. First define $X^1(P_a) = bottom(X, P_a)$. Then proceed inductively. Once $X^t(P_a)$ is defined for any $t \geq 0$ define $X^{t+1}(P_a) = \{\varnothing\}$ if $X = \cup_{i=1}^{t} X^i(P_a)$ and $X^{t+1}(P_a) = bottom([X \backslash \cup_{i=1}^{t} X^i(P_a)], P_a)$ otherwise.*

The ordering $\succsim \in \mathcal{D}$ is the leximin preference over sets extending the preference ordering over candidates P_a, $\succsim = \succsim^{lex}(P_a)$, if and only if for all $X, Y \in 2^{\mathcal{C}} \backslash \{\varnothing\}$:

$$X \succ Y \leftrightarrow X^{t'}(P_a) P_a Y^{t'}(P_a),$$

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[10]These preference domains were first proposed in Pattanaik [32] and [33].

where t' is the smallest integer t such that $X^t(P_a) \neq Y^{t'}(P_a)$.

Then, the domain of admissible leximin preferences can be defined by:

$$\mathcal{D}_a^{lex} = \{\succsim \in \mathcal{D}, \text{ such that there is } P_a \in \mathcal{P}^a \text{ with } \succsim = \succsim^{lex}(P_a)\}.$$

Notice that for each $P_a \in \mathcal{P}^a$, there is only one leximin preference over sets ($\# \succsim^{lex}(P_a) = 1$.) Moreover, as it is shown in [33], leximin preferences are incompatible with conditional expected utility maximization. These two facts may seem sufficient to conjecture that the opportunities of profitable withdrawal must be considerably reduced in the leximin scenario. The following lemma shows that this is not true.

Lemma 2.4. *A voting procedure v is leximin candidate stable if and only if for all $a \in \mathcal{C}$, $P \in \mathcal{P}^{\mathcal{V}}$:*

i) *(No Harm)* *If $a \in v(\mathcal{C}, P)$, $v(\mathcal{C}, P) \subseteq (v(\mathcal{C}\backslash\{a\}, P) \cup \{a\})$.*

ii) *(Insignificance)* *If $a \notin v(\mathcal{C}, P)$, $v(\mathcal{C}, P) = v(\mathcal{C}\backslash\{a\}, P)$.*

Proof. Again, we start by proving sufficiency. Consider an arbitrary $P \in \mathcal{P}^{\mathcal{V}}$ and $a \in \mathcal{C}$ such that $a \in v(\mathcal{C}, P)$, then $v(\mathcal{C}, P) \subseteq (v(\mathcal{C}\backslash\{a\}, P) \cup \{a\})$. In order to simplify the notation, denote the sets $v(\mathcal{C}, P) = X$ and $v(\mathcal{C}\backslash\{a\}, P) = Y$. Consider an arbitrary $P_a \in \mathcal{P}^a$ and find the smallest integer t' such that $X^{t'}(P_a) \neq Y^{t'}(P_a)$. Notice that $X^{t'}(P_a) \in Y$, and thus for some integer $t'' > t'$, $X^{t'}(P_a) = Y^{t''}(P_a)$. This implies that $X^{t'}(P_a)P_a Y^{t'}(P_a)$, and $v(\mathcal{C}, P) \succ^{lex}(P_a)v(\mathcal{C}\backslash\{a\}, P)$. As the choice of P_a was arbitrary, this implies that for all $\succsim \in \mathcal{D}_a^{lex}$, $v(\mathcal{C}, P) \succ v(\mathcal{C}\backslash\{a\}, P)$. Then, elected candidates do not have incentives to quit. Now suppose $a \notin v(\mathcal{C}, P)$, $v(\mathcal{C}, P) = v(\mathcal{C}\backslash\{a\}, P)$. Since a's withdrawal has no effect on the social choice, she can not improve by leaving. This concludes the proof of sufficiency, since the choice of P was arbitrary.

Now we prove necessity. First, let us prove item i). Assume v is leximin candidate stable but there are $a, b \in \mathcal{C}$, $P \in \mathcal{P}^{\mathcal{V}}$ such that $a, b \in v(\mathcal{C}, P)$ but $b \notin v(\mathcal{C}\backslash\{a\}, P)$. Find a preference ordering $P_a' \in \mathcal{P}^a$ such that $b = bottom(\mathcal{C}, P_a')$. But this implies that $v(\mathcal{C}\backslash\{a\}, P) \succ^{lex}(P_a')v(\mathcal{C}, P)$, which contradicts leximin candidate stability. On the other hand, assume v is candidate stable, but ii) does not hold. Hence, there are $a \in \mathcal{C}$ and $P \in \mathcal{P}^{\mathcal{V}}$, such that $a \notin v(\mathcal{C}, P)$ but $v(\mathcal{C}, P) \neq v(\mathcal{C}\backslash\{a\}, P)$. Now find $P_a'' \in \mathcal{P}^a$ such that for all $b \in (v(\mathcal{C}\backslash\{a\}, P) \backslash v(\mathcal{C}, P))$, $c \in$

$(v\left(\mathcal{C},P\right)\cap v\left(\mathcal{C}\backslash\{a\},P\right))$ and $d\in\left(v\left(\mathcal{C},P\right)\backslash v\left(\mathcal{C}\backslash\{a\},P\right)\right),$ $bP_a''cP_a''d.$ But then, $v\left(\mathcal{C}\backslash\{a\},P\right)\succ^{lex}$ $(P_a'')v\left(\mathcal{C},P\right),$ which again contradicts leximin candidate stability.

∎

Lemma 2.4 shows that leximin preferences also introduce strong restrictions on candidate stable voting correspondences. The implications are very similar to those derived for the conditional expected utility scenarios. Namely, the withdrawal of an elected candidate cannot harm the remaining elected candidates (we could dub this property as *no harm*), while the exit of a non-elected candidate cannot affect the final outcome (*insignificance.*) Albeit the leximin domain reduces the strategic incentives to withdraw from the ballot, only rules endowing some voters with veto power satisfy unanimity, no harm, and insignificance.

Theorem 2.3. **i)** *If a voting correspondence v is unanimous and leximin candidate stable then there is a subset of voters $S\subseteq\mathcal{V}$, such that for all $A\in\{\mathcal{C},\{\mathcal{C}\backslash\{a\}\}_{a\in\mathcal{C}}\}$ and for all $P\in\mathcal{P}^{\mathcal{V}}$:*

   a) $v\left(A,P\right)\subseteq Pareto\left(A,P_S\right),$

   b) for any $a\in\mathcal{C}$, $v(A,P)\neq\{a\}$ if there are $i\in S$ and $b\in A$ with $bP_ia,$

   c) for any $a\in\mathcal{C}$, $\cup_{i\in S}top(\mathcal{C}\backslash\{a\},P_i)\subseteq v(\mathcal{C}\backslash\{a\},P).$

ii) *Let v be such that for all $A\in 2^{\mathcal{C}}\backslash\{\varnothing\}$ and all $P\in\mathcal{P}^{\mathcal{V}}$, $v(A,P)=Pareto(A,P)$, then v is unanimous and candidate stable according to the leximin extension.*

Proof. Item i) is proved in the Appendix 1. Hence, we prove item ii). It is clear that Pareto correspondences are unanimous. On the other hand, consider an arbitrary profile of voters preferences $P\in\mathcal{P}^{\mathcal{V}}$ and some $a\in\mathcal{C}$. If $a\in Pareto\left(\mathcal{C},P_S\right)$ for all $b\in Pareto(\mathcal{C},P)$, also $b\in Pareto(\mathcal{C}\backslash\{a\},P)$, since if b is not dominated by any alternative in $\mathcal{C}$, it cannot be dominated by any alternative in $\mathcal{C}\backslash\{a\}$. Then, for all $a\in\mathcal{C}$, $Pareto(\mathcal{C},P)\subseteq(Pareto(\mathcal{C}\backslash\{a\},P)\cup\{a\})$. Moreover, those candidates who were dominated when a was at stake, were dominated by some candidate in $\mathcal{C}\backslash\{a\}$. Notice also that the candidates that are not Pareto dominated when all candidates are at stake are not dominated by any alternative in $\mathcal{C}\backslash\{a\}$. Hence, whenever $a\notin v(\mathcal{C},P)$, $Pareto\left(\mathcal{C},P_S\right)=Pareto\left(\mathcal{C}\backslash\{a\},P_S\right).$ This suffices to prove that the Pareto correspondence satisfies the requirements of Lemma 2.4.

∎

The reader can check that the conditions in item i) of Theorem 2.3 are also sufficient when there are only three candidates. We remark that, besides Pareto correspondences, many other rules satisfy the requirements of Theorem 2.3 independently of the size of the set of candidates. Moreover, since any unanimous and leximin candidate stable voting correspondence is efficient, Theorem 2.3 can be viewed as a positive result: many interesting voting correspondences do not provide incentives for candidates to withdraw if they use the leximin extension criterion.[11]

The remainder of this section is devoted to presenting the possibilities that arise when the domains of candidates preferences over sets are even more restricted. In particular, their preferences could be consistent with maximin, maximax or Gärdenfors preferences. In these scenarios, candidate stability is not very stringent, and many interesting rules are not affected by the incentives of the candidates to quit the election. Nevertheless, the arguments we use in the proofs of the previous theorems do not apply to these new environments. The crucial point is that it is not possible to construct the auxiliary group decision rules and to employ the classical results on rationalizability of the choice functions. Therefore, instead of looking for characterization results our strategy will be the following. First, we derive the specific implications of candidate stability in each context. Then we will provide some relevant examples of voting correspondences that are candidate stable in these new environments.

Candidate Stability for Maximiners

The maximin preferences can be interpreted as the behavior of extreme risk averters who only care about the worst alternative in a set.

Definition 2.7. *The ordering* $\succsim \in \mathcal{D}$ *is the maximin ordering over sets extending the preference ordering over candidates* $P_a \in \mathcal{P}^a$, $\succsim = \succsim^{min}(P_a)$, *if and only if for all* $X, Y \in 2^{\mathcal{C}} \backslash \{\varnothing\}$,

$$X \succsim Y \leftrightarrow bottom\,(X, P_a)\,R_a bottom\,(Y, P_a)\,.$$

The domain $\mathcal{D}_a^{min}$ *contains all the maximin orderings admissible for candidate a.*

The implications of candidates stability when candidates are maximiners are much weaker than under leximin preferences. Notice that leximin preferences apply the maximin logic of comparisons of the worst elements in each set iteratively. Thus, whenever a maximiner finds profitable to quit an election, she would also improve in the case she were a leximiner. In Proposition 2.1, we specify the constrains that candidate stability introduces in the maximin scenario. We also present the family of core correspondences which includes Pareto correspondences and satisfies unanimity and maximin candidate stability.

The Pareto correspondence is an example of a core correspondence. A candidate b is not selected by the Pareto correspondence if there is another candidate a such that all the voters prefer a to b. In this case, we say that the candidate a blocks the selection of the candidate b since all the voters prefer a to b and they have the power to enforce the social outcome to be a. Hence, we can say that when a society uses the Pareto correspondence as a voting correspondence, only the whole society has blocking power.

The distribution of the blocking power implicit in the definition of the Pareto correspondence is rather restrictive. We can think of a Parliament or a society in which the agreement of two thirds of its members is enough to impose an alternative (or any other qualified majority rule.) Core Correspondences allow for general distributions of the blocking power. In order to describe these possibilities, we introduce the definition of Effectivity functions and core correspondences.

An *Effectivity Function E* is a mapping from the set of non-empty sets of voters, $2^{\mathcal{V}}\backslash\{\varnothing\}$, to $2^{\mathcal{C}}\backslash\{\varnothing\}$ such that (i) for any $A \in 2^{\mathcal{C}}\backslash\{\varnothing\}$, $A \in E(\mathcal{V})$; and (ii), for any $S \in 2^{\mathcal{V}}\backslash\varnothing$, $\mathcal{C} \in E(S)$. Given the effectivity function E, we say that the coalition $S \subseteq \mathcal{V}$ has the power to force the social outcome to belong to the set $A \subseteq \mathcal{C}$ iff $A \in E(S)$.

The coalition $S \subseteq \mathcal{V}$ blocks candidate b at profile $P \in \mathcal{P}^{\mathcal{V}}$ if there is another candidate $a \in \mathcal{C}$ such that for all $i \in S$, aP_ib, and $a \in E(S)$, we say a dominates b at profile P. A coalition blocks a candidate if they coalition has the effective power to force the social outcome to be another candidate, and all the members of the coalition prefer this available candidate to the first one. For any $A \in 2^{\mathcal{C}}\backslash\{\varnothing\}$ and $P \in \mathcal{P}^{\mathcal{V}}$, the *Core correspondence* associated to the effectivity function E, C_E, selects the set of candidates in A which are not dominated by any other candidate in A at profile P.

Proposition 2.1. i) *A voting correspondence v is maximin candidate stable if and only if for all $a \in \mathcal{C}$ and for all $P \in \mathcal{P}^\mathcal{V}$, $v(\mathcal{C}, P) \subseteq (v(\mathcal{C} \backslash \{a\}, P) \cup \{a\})$.*

ii) *Given an effectivity function E, if for all $A \in \{\mathcal{C}, \{\mathcal{C} \backslash \{a\}\}_{a \in \mathcal{C}}\}$ and all $P \in \mathcal{P}^\mathcal{V}$ its associated Core correspondence, C_E, is non-empty, then C_E is unanimous and maximin candidate stable.*

Proof. i) In order to prove sufficiency, consider an arbitrary candidate $a \in \mathcal{C}$ and an arbitrary preference profile $P \in \mathcal{P}^\mathcal{V}$. Notice that $v(\mathcal{C}, P) \subseteq (v(\mathcal{C} \backslash \{a\}, P) \cup \{a\})$ implies that for all $P_a \in \mathcal{P}^a$, $bottom(v(\mathcal{C}, P, P_a) \in v(\mathcal{C} \backslash \{a\}, P), P_a)$, and also $bottom(v(\mathcal{C}, P), P_a) R_a bottom(v(\mathcal{C} \backslash \{a\}, P), P_a)$. As the choice of a and P was arbitrary this suffices to prove that no maximiner candidate has incentive to leave the ballot.

On the other hand, in order to prove necessity assume v is maximin candidate stable but there are $a, b \in \mathcal{C}$, $P \in \mathcal{P}^\mathcal{V}$, such that $b \in v(\mathcal{C}, P)$, $b \notin v(\mathcal{C} \backslash \{a\}, P)$. Take a preference $P_a \in \mathcal{P}^a$ such that $b = bottom(\mathcal{C}, P)$, then we can find $\succsim \in \mathcal{D}_a^{min}$ such that $v(\mathcal{C} \backslash \{a\}, P) \succ v(\mathcal{C}, P)$ contradicting maximin candidate stability.

ii) First notice that the core correspondences are unanimous, since all the sets $A \in 2^\mathcal{C} \backslash \{\varnothing\}$ belong to $E(\mathcal{V})$. To check maximin candidate unanimity is also immediate. Just notice that if a candidate a is not dominated when all the candidates are at stake, she cannot be dominated when another candidate leaves the fray. Then for all $a \in \mathcal{C}$ and all $P \in \mathcal{P}^\mathcal{V}$, $C_E(\mathcal{C}, P) \subseteq C_E(\mathcal{C} \backslash \{a\}, P)$.

∎

Item i) of Proposition 2.1 proves that maximin candidate stability only prevents the voting correspondence from shrinking the chosen set when a candidate withdraws. In item ii), we show that as maximin preferences reduce the candidates' incentives to withdraw, general distributions of the blocking power are admitted. The following example provides a situation in which we can obtain a non-empty core correspondence from majority comparisons of candidates.

Example 2.2. *Consider a society with four voters and three candidates. For any $A \in 2^\mathcal{C} \backslash \{\varnothing\}$ and any $P \in \mathcal{P}^\mathcal{V}$, $v(A, P) = \{a \in A$ s.t. for no $b \in A$ there are three voters preferring b to $a\}$. That is, v selects the candidates that are not dominated according to the strict majority relation. The reader can check that v satisfies item i) of Proposition 2.1, and hence it is maximin*

candidate stable.

Candidate Stability and Gärdenfors Preferences

The Gärdenfors preferences match scenarios in which the candidates are not extreme risk averters, but they are very reluctant to express strict preference between sets unless one set clearly improves the other. These preferences can be suitable when the candidates have very little information on the resolution of the social choice.

Definition 2.8. *The ordering $\succsim \in \mathcal{D}$ is the Gärdenfors ordering over sets extending the preference ordering over candidates P_a, $\succsim = \succsim^G (P_a)$, if and only if for all $X, Y \in 2^{\mathcal{C}} \backslash \{\varnothing\}$,*

$$X \succsim Y \leftrightarrow \text{ for all } b \in X \backslash Y, c \in X \cap Y \text{ and } d \in Y \backslash X, bP_a cP_a d.$$

The domain $\mathcal{D}_a^G$ contains all the Gärdenfors orderings admissible for candidate a.

Gärdenfors preferences can be equivalently defined in the following way. A candidate a with preferences over candidates P_a, prefers the set X to the set Y, if for some u_a fitting P_a, for all possible prior assessments the conditional expected utility obtained from the set X is always higher than the conditional expected utility that the set Y yields.[12] Gärdenfors preferences fit uncertain environments in which any distribution of probabilities among alternatives is plausible and candidates are reluctant to use a specific one.

It is clear that the Gärdenfors preferences are weak, in the sense that it is difficult to obtain strict preference over sets. This makes the incentives to quit of the voters to be very scarce. In fact, we will see that even rules based on majority comparison of the candidates are Gärdenfors candidate stable.

Let $\mathcal{B}$ denote the set of complete binary relations on $\mathcal{C}$. For any $P \in \mathcal{P}^{\mathcal{V}}$, the strict majority relation, $\bar{M}(P)$ is defined as follows. For any $a, b \in \mathcal{C}$, a beats b, $a\bar{M}_P b$, if and only if $\#\{i \in \mathcal{V}$ s.t. $aP_i b\} > \#\{j \in \mathcal{V}$ s.t. $bP_i a\}$. The weak component of the majority relation, $M(P)$, is defined in the usual way. For any $A \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ and for all $P \in \mathcal{P}^{\mathcal{V}}$, the *top cycle* of the majority relation $tc(A, P)$ is defined as the minimal (w.r.t. inclusion) set on A such that all its elements

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[12]A proof of this statement can be found in Ching and Zhou [12]. They make use of degenerate lotteries but a similar argument applies to our scenario.

defeat all the candidates who belong to $A \backslash tc(A, P)$ by majority comparisons.[13] The top cycle correspondence is never empty and satisfies interesting properties. For instance, it is a singleton if and only if there is a candidate who beats all the remaining ones by majority comparison, that is the top cycle equals the Condorcet winner whenever the latter exists.

Proposition 2.2. *i) A voting correspondence v is Gärdenfors candidate stable if and only if for all $P \in \mathcal{P}^{\mathcal{V}}$ $a \notin v(\mathcal{C}, P) \Rightarrow v(\mathcal{C}, P) = v(\mathcal{C} \backslash \{a\}, P)$.*

ii) The majority Top Cycle Correspondence is unanimous and Gärdenfors candidate stable.

Proof. i) In order to prove sufficiency, consider an arbitrary preference profile $P \in \mathcal{P}^{\mathcal{V}}$ and a candidate $a \in \mathcal{C}$. Notice first that if $a \in v(\mathcal{C}, P)$, the self preference of the candidates implies that $v(\mathcal{C}, P) \succsim v(\mathcal{C} \backslash \{a\}, P)$ for all $\succsim \in \mathcal{D}_a^G$. On the other hand, if $a \notin v(\mathcal{C}, P)$ she cannot improve by withdrawing, since she cannot change the outcome of the election.

Finally, let us check necessity. Assume there is $P \in \mathcal{P}^{\mathcal{V}}$, such that a non-elected candidate $a \notin v(\mathcal{C}, P)$ can affect the result of social choice and $v(\mathcal{C}, P) \neq v(\mathcal{C} \backslash \{a\}, P)$. Then, as $a \notin v(\mathcal{C}, P) \cup v(\mathcal{C} \backslash \{a\}, P)$, we can find $P_a \in \mathcal{P}^a$ such that for all $b \in v(\mathcal{C} \backslash \{a\}, P) \backslash v(\mathcal{C}, P)$, $c \in v(\mathcal{C}, P) \cup v(\mathcal{C} \backslash \{a\}, P)$ and $d \in v(\mathcal{C}, P) \backslash v(\mathcal{C} \backslash \{a\}, P)$, $b P_a c P_a d$. But this implies that there is $\succsim \in \mathcal{D}_a^G$ such that $v(\mathcal{C} \backslash \{a\}, P) \succ v(\mathcal{C}, P)$, which contradicts Gärdenfors candidate stability.

ii) Unanimity of the top cycle correspondence is clear. To prove Gärdenfors candidate stability is also immediate. If a candidate $a \notin tc(\mathcal{C}, P)$ leaves the stake then all candidates within $tc(\mathcal{C}, P)$ beat all candidates outside it. Since $tc(\mathcal{C}, P)$ is minimal for $\mathcal{C}$, it is also minimal for $\mathcal{C} \backslash \{a\}$. Thus, for all $P \in \mathcal{P}^{\mathcal{V}}$ and for all $a \notin tc(\mathcal{C}, P)$, $tc(\mathcal{C}, P) = tc(\mathcal{C} \backslash \{a\}, P)$.

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In item i) of Proposition 2.2, we have proved that under Gärdenfors preferences the incentives to leave the ballot only affect the non-elected candidates. In fact, candidate stability reduces to insignificance of the non-elected candidates and it is equivalent to the DJL's formulation of strong candidate stability in single-valued elections. Nevertheless the implications of insignificance are dramatically different in multivalued environments. In the single-valued case, it is equivalent to

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[13]Analogously, we could define the Top Cycle of any binary relation $B \in \mathcal{B}$ in the same fashion. The results in the item ii) of Proposition 2.2 are also valid for the Top Cycle of any binary relation $B \in \mathcal{B}$.

the Arrow's Choice Axiom and it implies the transitive rationalizability of the voting procedure. However, as item ii) shows, its implications are much weaker when applied to multivalued environments.

Candidate Stability for Maximaxers

Finally, maximax extension is obviously defined as the way in which extremely optimistic people compare sets of candidates.

Definition 2.9. *The ordering $\succsim \in \mathcal{D}$ is the maximax ordering over sets extending the preference ordering over candidates P_a, $\succsim = \succsim^{max}(P_a)$, if and only if for all $X, Y \in 2^{\mathcal{C}} \backslash \{\varnothing\}$,*

$$X \succsim Y \leftrightarrow top\left(X, P_a\right) R_a top\left(Y, P_a\right).$$

The domain $\mathcal{D}_a^{max}$ contains all the maximax orderings admissible for candidate a.

Although the maximax and Gärdendfors domains are independent, the maximax version of candidate stability is even weaker than the Gärdenfors' one. We directly introduce another example of correspondence that fits the requirements of maximax candidate stability, the Uncovered Set.[14] Let $\#\mathcal{V}$ be odd. We say a is covered by b according to the majority relation $M(P)$ if $b\bar{M}(P)a$ and for all c such that $a\bar{M}(P)c$, also $b\bar{M}_Pc$. For all $A \subseteq \mathcal{C}$, and for all $P \in \mathcal{P}^{\mathcal{V}}$ define the Uncovered Set of the majority relation $uc\left(A, P\right)$, as the set of candidates in A who are not covered by any other candidate in A.

Proposition 2.3.   *i) A voting correspondence v is candidate stable according to the maximax extension if and only if for all $P \in \mathcal{P}^{\mathcal{V}}$ for all $a \notin v(\mathcal{C}, P)$; $v(\mathcal{C}\backslash\{a\}, P) \subseteq v(\mathcal{C}, P)$.*

ii) If $\#\mathcal{V}$ is odd, the Uncovered Set of the majority relation is unanimous and maximax candidate stable.

Proof. i) In order to prove sufficiency, consider an arbitrary profile of voters preferences $P \in \mathcal{P}^{\mathcal{V}}$. Notice first that an elected candidate never has incentives to leave the fray, since for all $P_a \in \mathcal{P}^a$, $a = top(v(\mathcal{C}, P)P_a)P_a top(v(\mathcal{C}\backslash\{a\}, P), P_a)$. Moreover, if for all $a \notin v(\mathcal{C}, P)$,

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[14]See Miller [29] and Chapter 9 in Moulin [30].

$v(\mathcal{C}\backslash\{a\}, P) \subseteq v(\mathcal{C}, P)$, $top\left(v(\mathcal{C}\backslash\{a\}, P), P_a\right) \subseteq v(\mathcal{C}, P)$. This implies that for all $\succsim \in \mathcal{D}_a^{max}$, $v(\mathcal{C}, P) \succsim v(\mathcal{C}\backslash\{a\}, P)$. Hence, sufficiency holds.

Necessity also follows immediately. Assume ad contrarium that v is maximax candidate stable but there are $P \in \mathcal{P}^{\mathcal{V}}$ and $a \notin v(\mathcal{C}, P)$ such that there is another candidate $b \in v\left(\mathcal{C}\backslash\{a\}, P\right)\backslash v\left(\mathcal{C}, P\right)$. Take $P_a \in \mathcal{P}^a$, such that $b = top\left(\mathcal{C}\backslash a, P_a\right)$, as $a \notin v(\mathcal{C}, P)$. This suffices to ensure the existence of a maximax preference over sets $\succsim \in \mathcal{D}_a^{max}$ such that $v(\mathcal{C}\backslash\{a\}, P) \succ v(\mathcal{C}, P)$, which contradicts candidate stability.

ii) Notice that the uncovered set is unanimous since at unanimous profiles there is a candidate that beats all the remaining candidates . Assume the uncovered is not maximax candidate stable. Then, there are a preference profile $P \in \mathcal{P}^{\mathcal{V}}$, and two candidates $a, b \in \mathcal{C}$ such that $a \notin uc\left(\mathcal{C}, P\right)$ and $b \in uc\left(\mathcal{C}\backslash\{a\}, P\right)$, but $b \notin uc\left(\mathcal{C}, P\right)$. Notice that if b is not covered by any $c \in \mathcal{C}\backslash\{a\}$ but it is covered in $\mathcal{C}$, then b is covered by a. By the transitivity of the covering relation and as $a \notin uc\left(\mathcal{C}, P\right)$, b also should be covered by some candidate in $\mathcal{C}\backslash a$, a contradiction. This implies that the uncovered set satisfies the conditions of item i) and it is always maximax candidate stable.

■

Maximax candidate stability is slightly weaker than Gärdenfors' candidate stability. Hence, any Gärdenfors candidate stable voting correspondence is also maximax candidate stable, while the converse is not true. In fact, it is easy to see that the uncovered set of the majority relation is not candidate stable under Gärdenfors preferences.

Example 2.3. *Let* $\#\mathcal{V} = 3$, $\mathcal{C} = \{a, b, c, d, e\}$. *Consider the preference profile* $P \in \mathcal{P}^{\mathcal{V}}$ *such that* $aP_1eP_1bP_1cP_1d$; $bP_2cP_2dP_2aP_2e$ *and* $cP_3eP_3dP_3aP_3b$, *then* $UC\left(\mathcal{C}, P\right) = \{a, b, c, e\}$, *whereas* $UC\left(\mathcal{C}\backslash d, P\right) = \{a, b, c\}$. *If the candidate* $d's$ *preferences over candidates are* $dP_jaP_jbP_jcP_je$ *and she uses the Gärdenfors criterion to compare sets of candidates, d has an incentive to quit.*

In order to close this section a few remarks are in order. First, we have seen that Condorcet consistent rules as the Top Cycle and the Uncovered Set satisfy maximax candidate stability. However, we have not mentioned the effects of the incentives to withdraw of the candidates for other prominent class of voting rules, the family of scoring methods. Under scoring methods,

each voter assigns points to each candidate, linearly increasing with her ranking in the voter's preferences. The candidates who accumulate more points are finally elected. It is not difficult to check that scoring rules are not candidate stable neither under the maximin domain nor under maximax preferences.

Finally note that a voting procedure v is maximin and Gärdenfors (or maximax) candidate stable if and only if v is candidate stable according to the leximin extension. This final remark implies that a core correspondence (that is unanimous and maximin candidate stable) is maximax candidate stable only if it is the Pareto correspondence.[15]

2.5   Overlap between Voters and Candidates

We have already discussed that our assumption on non-voting candidates is suitable in many voting situations and can be a good approximation to large elections. However, the possibility of being at the same time a voter and a candidate should not be ruled out, and it deserves interest on its own. Unfortunately, it is not trivial to extend the analysis of strategic incentives of the candidates to this environment.

A first difficulty on the analysis of environments with voting candidates is derived from the restriction on the self preference of the candidates. If a candidate can vote the voting correspondence no longer operates in an universal domain. Hence, we cannot use directly any version of the Arrow's Theorem. Moreover, unanimity becomes empty of content. Whenever there are at least two voting candidates, no unanimous profile exists. This problem can be overcome by introducing stronger versions of unanimity. For instance, we say a voting correspondence is strongly unanimous if whenever there is a running candidate who is the best candidate at stake for all the voters who are not candidates, and the second best for all the voting candidates, the voting correspondence chooses this candidate. Formal definition follows.

We say $B \subseteq \mathcal{C}$ is a restricted top-set of profile P if for all $i \in \mathcal{V}$, for all $b \in B$, $c \in (\mathcal{C} \backslash \{i\}) \backslash B$,

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[15]Demange [13] shows that many core correspondences, those generated from convex effectivity functions, are not manipulable according to the maximax. This family of core correspondences strictly includes Pareto correspondences. Hence, we cannot state that candidate stability is generically weaker than strategy-proofness, as many maximax strategy-proof social choice correspondences are not maximax candidate stable.

$bP_i c$. We write $P \in \mathcal{P}^{\mathcal{V}}(B)$.

Definition 2.10. *The voting correspondence v is strongly unanimous iff for all $P \in \mathcal{P}^{\mathcal{V}}$, and all $B \subseteq \mathcal{C}$, whenever there is $b \in B$, such that b is a restricted top-set of P then $v(B, P) = b$.*

The above problems put in jeopardy our normative analysis but are not overwhelming. However, the most important difficulty arises when defining the implications of candidate stability. If candidates can vote, our Lemmata 2.2, 2.3, 2.4 and the items i) of Propositions 2.1, 2.2, 2.3 do not apply, since the social choice depends on the preferences of the withdrawing candidates. Notice that the society could use *punishment strategies* to eliminate the incentives of the candidates to leave. For instance, we can think of a voting correspondence which selects the worst candidate according to the preferences of the withdrawing candidate. Of course, such choices may be incompatible with strong unanimity, but other interesting examples can be devised.

Example 2.4. *Consider a society in which all the voters are also candidates, $\mathcal{C} = \mathcal{V}$. Define the restricted Pareto correspondence in the following way. For all $A \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ and all $P \in \mathcal{P}^{\mathcal{V}}$:*

$$Pareto^*(A, P) = \{a \in A \text{ s.t. for no } b \in A, bP_i a, \text{ for all } i \in (\mathcal{V} \backslash \{a\})\}.$$

Now define the voting correspondence v^* in the following way. For all $P \in \mathcal{P}^{\mathcal{V}}$ and all $a \in \mathcal{C}$,

$$v^*(\mathcal{C}, P) = Pareto^*(\mathcal{C}, P), \text{ and}$$

$$v^*(\mathcal{C} \backslash \{a\}, P) = bottom(Pareto^*(\mathcal{C} \backslash \{a\}, P), P_a).$$

It is immediate to check that v^* is candidate stable with respect to the domains of preferences we have presented. Notice that $Pareto^*(\mathcal{C}, P) \subseteq Pareto^*(\mathcal{C} \backslash \{a\}, P)$. Hence for any P_a and for all $b \in Pareto^*(\mathcal{C}, P)$, $bR_a v(\mathcal{C} \backslash \{a\}, P)$. Therefore, the outcome after the withdrawal cannot improve the result when all the candidates run the election. Moreover, v is also strongly unanimous. However, the use of the punishment could imply that the social choice may be against the interest of almost all the members of the society when $\#\mathcal{V}$ is very large.

Another possible approach consists of introducing some candidate stability condition without modeling explicitly candidates' incentives to abandon the election. This possibility is proposed in [18] and [19], where a strong candidate stability condition is provided. We can weaken their

condition and ask for voting correspondences satisfying "no harm" of the withdrawal of an elected candidate and "insignificance" of non-elected candidates.[16] In this case we can obtain a result similar to the item i) of Theorem 2.3.

Theorem 2.4. *If a voting correspondence v satisfies strong unanimity, no harm and insignificance, then there is a set of voters* $S \subseteq (\mathcal{V} \backslash \mathcal{C})$ *(who are not candidates) such that, for all* $P \in \mathcal{P}^{\mathcal{V}}$:

a) $v(A, P) \subseteq Pareto(A, P_S),$

b) *for any* $a \in \mathcal{C}$, $v(A, P) \neq \{a\}$ *if there are* $i \in S$ *and* $b \in A$ *with* $bP_ia,$

c) *for any* $a \in \mathcal{C}$, $\cup_{i \in S} top(\mathcal{C} \backslash \{a\}, P_i) \subseteq v(\mathcal{C} \backslash \{a\}, P).$

The above theorem has a chief drawback. By imposing an "ad hoc" stability condition, it does not consider the real incentives of the candidates to withdraw. As the general analysis of the voting candidates case seems out of reach, DJL have proposed the study of specific classes of voting rules in single-valued environments. In their Theorem 2, they have proved that candidate stability is incompatible with a strong version of unanimity (m-2 unanimity) for a large family of voting procedures satisfying a restricted version of monotonicity (top-pair monotonicity) [17] The reader can check that the arguments in the proof of Theorem 2 in DJL can be applied to multivalued environments when candidates preferences over sets belong to the conditional expected utility and leximin domains with minimal modifications in the proofs.

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[16]See Lemma 2.4

[17]A voting rule satisfies $m - 2$ unanimity if whenever there are a group of $\#\mathcal{V} - 2$ voters who prefer a candidate above the remaining ones, that candidate is elected. A voting procedure is top-pair monotonic if i) whenever there are two candidates who are a top-set, then the remaining candidates are not chosen, and ii) an increase of the support of a candidate never harms.

2.6 Appendices

2.6.1 Appendix 1: Proofs of the Theorems

The proofs below do not appear in the same order as the results. This economizes the exposition, as some results that appear later in the paper (namely, Theorem 2.4) are used in the proofs of ones stated earlier (Theorems 2.1, 2.2 and 2.3.) We have preferred to highlight the less general theorems regarding the non-overlap between voters and candidates' sets because they admit much better interpretation in terms of candidates incentives.

We have already mentioned that the main theorem exploits the relation between candidate stable voting correspondences and rationalizable group decision rules. This allows us to employ at a crucial step of the proof a version of the Arrow's Theorem by Mas-Colell and Sonnenschein [28]. It is stated in Proposition 2.5 below.

In order to prove the main theorems we need to introduce some notations and definitions.

A *group decision rule* F is a function mapping profiles of voters preferences on binary relations on C, $F : \mathcal{P}^{\mathcal{V}} \longrightarrow \mathcal{B}$. A group decision rule is *quasitransitive* iff for all $P \in \mathcal{P}^{\mathcal{V}}$, $F(P)$ is a quasiorder. A group decision rule is *unanimous* if for any $a, b \in \mathcal{C}$ for all profiles $P \in \mathcal{P}^{\mathcal{V}}$ such that $aP_i b$ for all $i \in \mathcal{V}$, $aF(P)b$ but not $bF(P)a$. A group decision rule satisfies *Arrow's Independence of Irrelevant Alternatives (AIIA)* iff for all $P, P' \in \mathcal{P}^{\mathcal{V}}$ such that for all $i \in \mathcal{V}$, $(aP_i b$ if and only if $aP_i' b)$, $aF(P)b$ implies $aF(P')b$. Finally a group decision rule is *oligarchical* iff there is a set of voters $S \subseteq \mathcal{V}$ such that for all $P \in \mathcal{P}^{\mathcal{V}}$ $aF(P)b$ if and only if there is $i \in S$ with $aP_i b$.

Notice that the object of interest of our work are not group decision rules but voting correspondences. Nevertheless, for all $P \in \mathcal{P}^{\mathcal{V}}$, we can define the binary relation $R_P^v \in \mathcal{B}$ in the following way, for any $a, b \in C$, $aR_P^v b$ iff and only if $a \in v(\{a, b\}, P)$.[18] Now, for any or any voting correspondence v, we can define an associated group decision rule F^v in such a way that for all $P \in \mathcal{P}^{\mathcal{V}}$, $F^v(P) = R_P^v$.

Finally we include a pair of regularity conditions on voting correspondences. These conditions will become useful to establish the properties of the properties of the group decision rule

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[18]The strict component of R_P^v, $\bar{R}_P^v$ is defined in the usual way.

associated to a candidate stable voting correspondences. [19]

Condition 2.1 (Chernoff). *A voting correspondence v satisfies Chernoff iff for all $P \in \mathcal{P}^{\mathcal{V}}$, for all $A, B \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, $v(A \cup B, P) \subseteq v(A, P) \cup B$.*

Condition 2.2 (Aizerman). *A voting correspondence v satisfies Aizerman iff for all $P \in \mathcal{P}^{V}$ $A, B \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, $v(B, P) \subseteq A \subseteq B$ implies $v(A, P) \subseteq v(B, P)$.*

Proposition 2.4 (Sen [39]). *A voting correspondence v satisfies Chernoff and Aizerman if and only if R_P^v is a quasiorder for all $P \in \mathcal{P}^{\mathcal{V}}$.*

Proposition 2.5 (Mas-Colell and Sonnenschein [28]). *A group decision rule F satisfies quasitransitivity, AIIA, unanimity if and only if F is oligarchical.*

Proof of Theorem 2.4.

The proof of Theorem 2.4 proceeds through a series of lemmata. From now on, we assume that v is a strongly unanimous voting correspondence satisfying no harm and insignificance.

We say B is a restricted top set of P at profile $\bar{P}$, if B is a restricted top-set of P and $P \mid_{\mathcal{C} \backslash B} = \bar{P} \mid_{\mathcal{C} \backslash B}$, we denote the set of profiles for which B is a top-set at profile $\bar{P}$ as $\mathcal{P}^{\mathcal{V}}(B, \bar{P})$.

Lemma 2.5. *Consider any arbitrary set of three candidates $\{a, b, c\} \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ and an arbitrary preference profile $\bar{P} \in P^{V}$. Then for all $P \in \mathcal{P}^{\mathcal{V}}(\{a, b, c\}, \bar{P})$ $v(\mathcal{C}, P) \subseteq \{a, b, c\}$.*

Proof. Pick any arbitrary $\{a, b, c\} \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ and $\bar{P} \in \mathcal{P}^{\mathcal{V}}$. Consider a preference profile $P \in \mathcal{P}^{\mathcal{V}}(\{a, b, c\}, \bar{P})$ such that $\{a, b\}$ is a restricted top-set. Notice that either $a \notin v(\mathcal{C}, P)$, or $b \notin v(\mathcal{C}, P)$, or $\{a, b\} \subseteq v(\mathcal{C}, P)$. Assume first, $b \notin v(\mathcal{C}, P')$, then by insignificance $v(\mathcal{C}, P') = v(\mathcal{C} \backslash \{b\}, P')$, and by strong unanimity $v(\mathcal{C} \backslash \{b\}, P') = \{a\}$, and also $v(\mathcal{C}, P') = \{a\}$. If $\{a, b\} \subseteq v(\mathcal{C}, P)$ by strong unanimity $v(\mathcal{C} \backslash \{a\}, P') = \{b\}$, and $v(\mathcal{C} \backslash \{b\}, P') = \{a\}$. By no harm, we know $v(\mathcal{C}, P) \subseteq (v(\mathcal{C} \backslash \{a\}, P) \cup \{b\})$, and therefore $v(\mathcal{C}, P) = \{a, b\}$. As the choice of a and b was arbitrary, it follows that if $\{x, y\} \subset \{a, b, c\}$ is a restricted top-set of $P \in \mathcal{P}^{\mathcal{V}}(\{a, b, c\}, \bar{P})$, then $v(\mathcal{C}, P) \subseteq \{x, y\}$.

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[19]Excellent compendia of the choice theory literature can be found in Sen [39] and [40], Moulin [30], and Aizerman and Aleskerov [1].

Now, take an arbitrary $P' \in \mathcal{P}^{\mathcal{V}}\left(\{a,b,c\},\bar{P}\right)$. Suppose there is some $x \notin \{a,b,c\}$; $x \in v\left(\mathcal{C},P\right)$. Consider now $\hat{P} \in \mathcal{P}^{\mathcal{V}}(a,b,\bar{P}$ such that $P'\mid_{\mathcal{C}\backslash\{c\}} = \hat{P}\mid_{\mathcal{C}\backslash\{c\}}$. By $ii)$ in the definition of voting correspondence, it should be the case that $v\left(\mathcal{C}\backslash\{c\},P'\right) = v\left(\mathcal{C}\backslash\{c\},\hat{P}\right)$. By the arguments in the previous paragraph, $v\left(\mathcal{C},\hat{P}\right) \subseteq \{a,b\}$, and by insignificance, $v\left(\mathcal{C},\hat{P}\right) = v\left(\mathcal{C}\backslash\{c\},\hat{P}\right)$. Hence, $v\left(\mathcal{C}\backslash\{c\},P'\right) \subseteq \{a,b\}$, and applying no harm we obtain that, $v\left(\mathcal{C},P'\right) \subseteq \{a,b,c\}$, contrary to our supposition.

Moreover, from similar arguments, it follows that also for any $d \in \mathcal{C}$, $v\left(\mathcal{C}\backslash\{d\},P\right) \subseteq \{a,b,c\}\backslash\{d\}$.

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At this point, for any arbitrary triple of candidates $\{a,b,c\} \in 2^{\mathcal{C}}\backslash\{\varnothing\}$ and preference profile $\bar{P} \in \mathcal{P}^{\mathcal{V}}$, we can define an ancillary restricted voting correspondence

$$\hat{v}_{\left(\{a,b,c\},\bar{P}\right)} : 2^{\{a,b,c\}}\backslash\{\varnothing\} \times \mathcal{P}^{\mathcal{V}}\left(\{a,b,c\},\bar{P}\right) \longrightarrow 2^{\{a,b,c\}}\backslash\{\varnothing\},$$

in such a way that for all $P \in \mathcal{P}^{\mathcal{V}}\left(\{a,b,c\},\bar{P}\right)$ and for all $x \in \{a,b,c\}$; $\hat{v}_{\left(\{a,b,c\},\bar{P}\right)}\left(\{a,b,c\},P\right) = v\left(\mathcal{C},P\right)$, and $\hat{v}_{\left(\{a,b,c\},\bar{P}\right)}\left(\{a,b,c\}\backslash\{x\},P\right) = v\left(\mathcal{C}\backslash\{x\},P\right)$. Then, $\hat{v}_{\left(\{a,b,c\},\bar{P}\right)}$ is a well defined choice function in the restricted domain $\mathcal{P}^{\mathcal{V}}\left(\{a,b,c\},\bar{P}\right)$ and satisfies strong unanimity,no harm and insignificance since v also does. Moreover as $\hat{v}_{\left(\{a,b,c\},\bar{P}\right)}$ is always well defined, for any set of three candidates and any preference profile we can define the group decision rule $F^{\hat{v}(\{a,b,c\},\bar{P})}$, that maps voters preference profile over the candidates $\{a,b,c\}$ to binary relations on $\{a,b,c\}$.

Lemma 2.6. *For any* $\{a,b,c\} \in 2^{\mathcal{C}}\backslash\{\varnothing\}$ *and* $\bar{P} \in \mathcal{P}^{\mathcal{V}}$, $F^{\hat{v}(\{a,b,c\},\bar{P})}$ *is quasitransitive.*

Now, we are going to focus on the implications of no harm and insignificance in an arbitrary restricted domain $\mathcal{P}^{\mathcal{V}}(\{a,b,c\},\bar{P})$. In order to simplify the notation, from now on we refer to $\hat{v}_{\left(\{a,b,c\},\bar{P}\right)}$ simply by $\hat{v}$, and also $F^{\hat{v}(\{a,b,c\},\bar{P})}$, by $F^{\hat{v}}$.

Proof. In order to prove this Lemma we have to check that for all $P \in \mathcal{P}^{\mathcal{V}}\left(\{a,b,c\},\bar{P}\right)$, $\hat{v}$ satisfies Chernoff and Aizerman conditions. Then the result holds directly from the application of Proposition 2.4. Hence, consider arbitrary $P \in \mathcal{P}^{\mathcal{V}}\left(\{a,b,c\},\bar{P}\right)$ and $A,B \in 2^{\{a,b,c\}}\backslash\{\varnothing\}$ and let us check the different possibilities.

"Chernoff". Notice first that Chernoff holds immediately from the definition of voting correspondence in the following cases: if A and B are singletons, if either A or B are $\{a, b, c, \}$, and if $\#(A \cup B) = 2$. So suppose $(A \cup B) = \{a, b, c\}$ and either A or B is a singleton (say B). By no harm, $\hat{v}(A \cup B, P) \subseteq \hat{v}(A \backslash B, P) \cup B$, and immediately we obtain $\hat{v}(A \cup B, P) \subseteq \hat{v}(A, P) \cup B$. On the other hand, from the definition of voting correspondence we also have that, $\hat{v}(A \cup B, P) \subseteq \hat{v}(B, P) \cup A$, and we have that Chernoff holds. Thus, the only remaining possibility is that A and B are both duples and $(A \cup B) = \{a, b, c\}$. In this case, there are candidates $d = (B \backslash A)$ and $e = (A \backslash B)$. Notice that no harm and insignificance imply that $\hat{v}(A \cup B, P) \subseteq (\hat{v}(A \cup B \backslash \{d\}, P) \cup \{d\})$. Finally as $(A \cup B \backslash \{d\}) = A$, and $\{d\} \subset B$, the previous expression also implies that $\hat{v}(A \cup B) \subseteq (\hat{v}(A, P) \cup B)$. Repeating the argument for the withdrawal of the candidate e, we have that $\hat{v}$ satisfies Chernoff.

"Aizerman". Aizerman holds immediately in the cases in which the set A is a singleton or the whole set of candidates $\{a, b, c\}$. Hence we only have to study the case in which A is a duple. Without lost of generality, assume $A = \{a, b\}$, $B = \{a, b, c\}$. If $\hat{v}(\{a, b, c\}, P) \subseteq \{a, b\}$, insignificance implies that $\hat{v}(\{a, b\}, P) = \hat{v}(\{a, b, c\}, P)$. Thus we have that for all A, B, $\hat{v}(B, P) \subseteq A \subseteq B$, implies $\hat{v}(A, P) \subseteq \hat{v}(B, P)$.

As the choice of $P \in \mathcal{P}^V(\{a, b, c\}, \bar{P})$ was arbitrary, we can apply Proposition 2.4 to state that for all $P \in \mathcal{P}^V(\{a, b, c\}, \bar{P})$, $R_P^{\hat{v}}$ is a quasiorder, and then it follows from its very definition that the group decision rule $F^{\hat{v}}$ is quasitransitive.

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Lemma 2.7. *For any $\{a, b, c\} \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ and any $\bar{P} \in \mathcal{P}^V$ the group decision rule $F^{\hat{v}(\{a,b,c\}, \bar{P})}$ is oligarchical with vetoers $S \subseteq (\mathcal{V} \backslash \{a, b, c\})$.*

Proof. Consider an arbitrary profile $\bar{P} \in \mathcal{P}^V$ and a restricted profile of the candidates $\{a, b, c\}$ such that, $P_{\{a,b,c\}}^* \in \mathcal{P}^{\{a,b,c\}}(\{a, b, c\}, \bar{P})$ and $aP_a^* b P_a^* c$; $b P_b^* c P_b^* a$, and $c P_c^* a P_c^* b$. Now define an auxiliary group decision rule F^* on the preferences of the voters who are not in $\{a, b, c\}$ in the following way, for all $P_{(\mathcal{V} \backslash \{a,b,c\})} \in \mathcal{P}^{\mathcal{V} \backslash \{a,b,c\}}(\{a, b, c\}, \bar{P})$, $F^*(P_{(\mathcal{V} \backslash \{a,b,c\})}) = F^{\hat{v}}((P_{(\mathcal{V} \backslash \{a,b,c\})}), P_{\{a,b,c\}}^*)$. Construct now the preference profile $P^1 \in \mathcal{P}^V(\{a, b, c\}, \bar{P})$ in the following way, for all $i \in (\mathcal{V} \backslash \{a, b, c\})$, P_i^1 is such that $a P_i^1 b P_i^1 c$, while $P_{\{a,b,c\}}^1 = P_{\{a,b,c\}}^*$. By strong unanimity $v(\mathcal{C} \backslash \{c\}, P^1) =$

$\{a\}$, and $v(\mathcal{C}\backslash\{a\}, P^1) = \{b\}$, and as $F^{\hat{v}(\{a,b,c\},\bar{P})}$ is quasitransitive, it must be the case that also $v(\mathcal{C}\backslash\{b\}, P^1) = \{a\}$. Then, by the definition of $F^{\hat{v}}$, $F^{\hat{v}}(P^1) = P_i^1$ for any $i \in (\mathcal{V}\backslash i)$.

Consider now, the profile $P^2 \in \mathcal{P}^{\mathcal{V}}(\{a,b,c\}, \bar{P}$ such that for all $i \in (\mathcal{V}\backslash\{a,b,c\})$, $bP_i^2 c P_i^2 a$, and $P_{\{a,b,c\}}^2 = P_{\{a,b,c\}}^*$. Applying the same arguments than we use for P^1 we have that $F^{\hat{v}}(P^2) = P_i^2$ for some $i \in (\mathcal{V}\backslash\{a,b,,c\})$. Analogously, for the profile $P^3 \in \mathcal{P}^{\mathcal{V}}(\{a,b,c\}, \bar{P}$ such that for all $i \in (\mathcal{V}\backslash\{a,b,c\})$, $cP_i^3 a P_i^3 b$, and $P_{\{a,b,c\}}^3 = P_{\{a,b,c\}}^*$ we obtain $F^{\hat{v}}(P^3) = P_i^3$, for some $i \in (\mathcal{V}\backslash\{a,b,c\})$.

At this point, consider an arbitrary profile of preferences for the voters $\tilde{P} \in \mathcal{P}^{\mathcal{V}}(\{a,b,c\}, \bar{P})$ such that for all $i \in (\mathcal{V}\backslash\{a,b,c\})$, $a\tilde{P}_i b$, while $\tilde{P}_{\{a,b,c\}} = P_{\{a,b,c\}}^*$. Notice that $\tilde{P}\mid_{\mathcal{C}\backslash\{c\}} = P^1 \mid_{\mathcal{C}\backslash\{c\}}$, then by $ii)$ in the definition of voting correspondence, $v\left(\mathcal{C}\backslash\{c\}, \tilde{P}\right) = \{a\}$. By the definition of the group decision rule F^* this implies that $aF^*\left(\tilde{P}_{(\mathcal{V}\backslash\{a,b,c\})}\right)b$, and not $bF^*\left(\tilde{P}_{(\mathcal{V}\backslash\{a,b,c\})}\right)a$. We can use the same argument for any profile in which a candidate is unanimoulsy preferred to another one by all the voters in $(\mathcal{V}\backslash\{a,b,c,\}$ to we conclude that F^* satisfies unanimity.

We have seen that F^* satisfies unanimity on $\mathcal{P}^{\mathcal{V}\backslash\{a,b,c\}}(\{a,b,c\}, \bar{P})$, unrestricted domain, since the preferences of the relevant voters over $\{a,b,c\}$ are completely free, Arrow's Independence of Irrelevant Alternatives by ii) in the definition of voting correspondence and quasitransitivity by the quasitransitivity of $F^{\hat{v}}$. Now we can apply Proposition 2.5 to state that F^* is oligarchical. Moreover, since the preferences of a, b and c are fixed, the vetoers must belong to $(\mathcal{V}\backslash\{a,b,c\})$.

Now, we have to extend this partial result to all possible configuration of the preferences of candidates a, b and c. If a, b and c are not voters, we are already done, since their preferences do not affect the voting correspondence, so we suppose that $\{a,b,c\}$ are also voters. Consider now $P_{\{a,b,c\}}^{**} \in P^{\{a,b,c\}}(\{a,b,c\}, \bar{P})$ such that $aP_a^{**}cP_a^{**}b$, and $P_{\{b,c\}}^{**} = P_{\{b,c\}}^*$. Define now the auxiliary group decision rule F^{**} in the same fashion that F^*, that is for any $P_{(\mathcal{V}\backslash\{a,b,c\})} \in \mathcal{P}^{\mathcal{V}\backslash\{a,b,c\}}(\{a,b,c\}, \bar{P})$, $F^{**}\left(P_{(\mathcal{V}\backslash\{a,b,c\})}\right) = F^{\hat{v}}\left(P_{(\mathcal{V}\backslash\{a,b,c\})}, P_{\{a,b,c\}}^{**}\right)$. Consider now the profile $P^{3'} \in \mathcal{P}^{\mathcal{V}}(\{a,b,c\}, \bar{P})$, such that for all $j \in \mathcal{V}\backslash a$, $P_j^{3'} = P_j^3$ and $P_a^{3'} = P_a^{**}$. Notice that $P^{3'}\mid_{\mathcal{C}\backslash\{b\}} = P^3 \mid_{\mathcal{C}\backslash\{b\}}$ and $P^{3'}\mid_{\mathcal{C}\backslash\{c\}} = P^3 \mid_{\mathcal{C}\backslash\{c\}}$. By ii) of the definition of a voting correspondence, $v\left(\mathcal{C}\backslash\{b\}, P^{3'}\right) = \{c\}$, $v\left(\mathcal{C}\backslash\{c\}, P^{3'}\right) = \{a\}$ and by the quasitransitivity of $F^{\hat{v}}$, we obtain that $v\left(\mathcal{C}\backslash\{a\}, P^{3'}\right) = \{c\}$, and then, $F^{**}\left(P_{(\mathcal{V}\backslash\{a,b,c\})}^{3'}\right) = P_i^3$, for some $i \in (\mathcal{V}\backslash\{a,b,c\})$.

Consider finally the profile $P^4 \in \mathcal{P}^{\mathcal{V}}\left(\{a,b,c\}, \bar{P}\right)$, such that for all $i \in (\mathcal{V}\backslash\{a,b,c\})$, $bP_i^4 a P_i^4 c$, while $P_{\{a,b,c\}}^4 = P_{\{a,b,c\}}^{**}$. Notice that, $P^4 \mid_{\mathcal{C}\backslash\{c\}} = P^2 \mid_{\mathcal{C}\backslash\{c\}}$, and $P^4 \mid_{\mathcal{C}\backslash\{b\}} = P^1 \mid_{\mathcal{C}\backslash\{b\}}$. Following the same arguments of the previous paragraph, we can see that $F^{**}(P_{(\mathcal{V}\backslash\{a,b,c\})}^4) = P_i^4$ for some $i \in (\mathcal{V}\backslash\{a,b,c\})$. Then repeating the same reasoning, it is easy to prove that F^{**} is unanimous. Therefore, by Proposition 2.5, as F^{**} is a quasitransitive group decision rule defined in a universal domain and satisfying unanimity and AIIA, we know it is oligarchical.

At this point, we have to prove that the same voters hold veto power for F^* and for F^{**}. Assume the contrary, and w.l.o.g. suppose there is $i \in (\mathcal{V}\backslash\{a,b,c\})$, holding veto power over F^{**}, but not over F^*. Consider the restricted profile of preferences $\hat{P}_{(\mathcal{V}\backslash\{a,b,c\})} \in \mathcal{P}^{\mathcal{V}\backslash\{a,b,c\}}\left(\{a,b,c\}, \bar{P}\right)$ such that for all $j \in \mathcal{V}\backslash\{a,b,c,i\}$, $a\hat{P}_j b \hat{P}_j c$, while $b\hat{P}_i a \hat{P}_i c$. The voter i is not a vetoer for F^* but he is for F^{**}, we have that:

$$v\left(\mathcal{C}\backslash\{c\}, \left(\hat{P}_{(\mathcal{V}\backslash\{a,b,c\})}, P_{\{a,b,c\}}^*\right)\right) = \{a\},$$
$$b \in v\left(\mathcal{C}\backslash\{c\}, \left(\hat{P}_{(\mathcal{V}\backslash\{a,b,c\})}, P_{\{a,b,c\}}^{**}\right)\right).$$

But this is a contradiction with ii) in the definition of the voting correspondence, since we have that $\left(\hat{P}_{(\mathcal{V}\backslash\{a,b,c\})}, P_{\{a,b,c\}}^*\right) \mid_{\mathcal{C}\backslash\{c\}} = \left(\hat{P}_{(\mathcal{V}\backslash\{a,b,c\})}, P_{\{a,b,c\}}^{**}\right) \mid_{\mathcal{C}\backslash\{c\}}$.

We can apply the same arguments iteratively for all possible $P_{\{a,b,c\}} \in \mathcal{P}^{\{a,b,c\}}\left(\{a,b,c\}, \bar{P}\right)$. Thus we can conclude that the group decision rule $F^{\hat{v}}$ is oligarchical and, of course, the vetoers S belong to $(\mathcal{V}\backslash\{a,b,c\})$. Moreover, as the choice of $\{a,b,c\}$ and P was arbitrary, for any triple of candidates $\{a,b,c\} \in 2^{\mathcal{C}}\backslash\{\varnothing\}$ and any preference profile $P \in \mathcal{P}^{\mathcal{V}}$, $F^{\hat{v}(\{a,b,c\},P)}$ is oligarchical, and the candidates {a,b,c} cannot be vetoers.

∎

Consider now arbitrary $\{a,b,c\} \in 2^{\mathcal{C}}\backslash\{\varnothing\}$ and $P \in \mathcal{P}^{\mathcal{V}}$, we will see now which are the consequences of the previous lemma over the original voting correspondence v. Let S denote the oligarchy for $F^{\hat{v}}$, we will see they have veto power for v. First we prove the necessity of item a) of Theorem 2.4 in this restricted domain. Notice that, item a) holds for all the agendas in which a candidate leaves the fray by the very definition of $F^{\hat{v}}$ and S. If there is a candidate $d \notin Pareto\left(\mathcal{C}, P_S\right)$, there is $e \in \{a,b,c\}$, with $eP_i d$ for all $i \in S$. Notice now that for $f = (\{a,b,c\}\backslash\{d,e\})$, $v\left(\mathcal{C}\backslash\{f\}, P\right) = \{e\}$, since S have veto power for $F^{\hat{v}}$, therefore by

insignificance and no harm it must be the case that $v(\mathcal{C}, P) \subseteq \{e\} \cup \{f\}$, and then $d \notin v(\mathcal{C}, P)$. In order to prove b), notice that it holds from the definition of $F^{\hat{v}}$ whenever a candidate withdraws. Hence, we only have to check that a candidate cannot be the unique choice of the society unless she unanimously preferred by the members of S to the remaining ones. Assume without loss of generality, there is a non-unanimous profile $P'_S \in \mathcal{P}^S(\{a, b, c\}, \bar{P})$, such that $v(\mathcal{C}, P') = \{a\}$ but there is $i \in S$ such that $bP'_i a$. As i is a vetoer for $F^{\hat{v}}$, $b \in v(\mathcal{C}\backslash\{c\}, P')$. But insignificance implies that $v(\mathcal{C}, P') = v(\mathcal{C}\backslash\{c\}, P')$, which contradicts our assumption. Finally, notice that as $F^{\hat{v}}$ is oligarchical *item c*) also holds trivially in the domain $\mathcal{P}^{\mathcal{V}}(\{a, b, c\}, \bar{P})$.

This concludes the proof in the case in which there are only three candidates $\{a, b, c\}$, since $\mathcal{P}^{\mathcal{V}} = \mathcal{P}^{\mathcal{V}}(\{a, b, c\}, \bar{P})$. We will prove the result for arbitrary sizes of the set of candidates using an induction argument. Before, we present the induction step we introduce a pair of lemmata that will allow us to state the induction step properly.

Lemma 2.8. *Consider an arbitrary set of three candidates* $\{a, b, c\} \in 2^{\mathcal{C}}\backslash\{\varnothing\}$, v *is oligarchical in* $\mathcal{P}^{\mathcal{V}}(\{a, b, c\})$.

Proof. First, we extend our previous result to those profiles for which $\{a, b, c\}$ is top-set but are not uniform for the rest of alternatives. Take $\bar{P}$, $\tilde{P}$ such that $\{a, b, c\}$ is a top-set for both profiles, but there is $d \in \mathcal{C}\backslash\{a, b, c\}$, $\bar{P} \neq \tilde{P}$; but $\bar{P}\mid_{\mathcal{C}\backslash\{d\}} = \tilde{P}\mid_{\mathcal{C}\backslash\{d\}}$. We know, by the previous lemma, there are oligarchies for each profile of preferences $F^{\hat{v}(\{a,b,c\},\bar{P})}$ and $F^{\hat{v}(\{a,b,c\},\tilde{P})}$ are oligarchical, and let us denote their respective oligarchies by $\bar{S}$ and $\tilde{S}$. Suppose to the contrary $\bar{S} \neq \tilde{S}$, then there is some voter $j \in \bar{S}$, $j \notin \tilde{S}$. Consider now the profiles $P \in \mathcal{P}V(\{a, b, c\}, \bar{P})$ and $P' \in \mathcal{P}^{\mathcal{V}}(\{a, b, c\}, \tilde{P})$ such that $P\mid_{\mathcal{C}\backslash\{d\}} = P'\mid_{\mathcal{C}\backslash\{d\}}$ and for all $i \in \mathcal{V}\backslash j$, $aP_i bP_i c$, while $bP_j aP_j c$. Then, $v(\mathcal{C}, P) = \{a\}$, while $b \in v(\mathcal{C}, P')$. Then, by no harm and insignificance, $v(\mathcal{C}\backslash\{d\}, P) = \{a\}$, while $b \in v(\mathcal{C}\backslash\{d\}, P')$. But $P\mid_{\mathcal{C}\backslash\{d\}} = P'\mid_{\mathcal{C}\backslash\{d\}}$, and ii) in the definition of voting correspondence implies $v(\mathcal{C}\backslash\{d\}, P) = v(\mathcal{C}\backslash\{d\}, P')$, and contradicts our supposition about $\bar{S} \neq \tilde{S}$. We can repeat the argument as many times as necessary to prove the statement for any arbitrary $P \in \mathcal{P}^{\mathcal{V}}$, which leads to the desired result. ■

Lemma 2.9. *For any* $\{a, b, c\} \in 2^{\mathcal{C}}\backslash\{\varnothing\}$, v *is oligarchical on* $\mathcal{P}^{\mathcal{V}}(\{a, b, c\})$ *(with the same vetoers* $S \subseteq (\mathcal{V}\backslash\mathcal{C})$ *for each of these domains.*

Proof. It is enough to consider some $\{a, b, d\}$ distinct from $\{a, b, c\}$ and show that the same veto-ers "dictate" on $\mathcal{P}^{\mathcal{V}}(\{a, b, c\})$ and $\mathcal{P}^{\mathcal{V}}(\{a, b, d\})$. By the previous lemma, we know there are an oligarchy S which dictates on $\mathcal{P}^{\mathcal{V}}(\{a, b, c\})$ and an oligarchy S' which dictates on $\mathcal{P}^{\mathcal{V}}(\{a, b, d\})$. Suppose to the contrary, $S \neq S'$, and $j \in (S' \backslash S)$. Consider the profile $\hat{P} \in \mathcal{P}^{\mathcal{V}}(\{a, b, c, d\})$ such that $a = top(\mathcal{C}, \hat{P}_i)$ for all $i \in (\mathcal{V} \backslash \{j\})$ but $b = top\left(\mathcal{C}, \hat{P}_j\right)$, $a = top\left(\mathcal{C} \backslash \{b\}, \hat{P}_j\right)$ and $d = bottom\left(\{a, b, c, d\}, \hat{P}_i\right)$ for all $i \in \mathcal{V}$. Then, as $\hat{P} \in \mathcal{P}^{\mathcal{V}}(\{a, b, c\})$, $v\left(\mathcal{C}, \hat{P}\right) = \{a\}$, and by insignificance we know also that $v\left(\mathcal{C} \backslash \{c\}, \hat{P}\right) = \{a\}$. Now, consider the profile P^*, such that c and d are switched, hence $\hat{P}\mid_{\mathcal{C}\backslash\{c\}} = P^* \mid_{\mathcal{C}\backslash\{c\}}$, whereas $P^* \in \mathcal{P}^{\mathcal{V}}(\{a, b, d\})$. As $j \in S'$ $b \in v(\mathcal{C}, P^*)$, and by insignificance and no harm also $b \in v(\mathcal{C} \backslash \{c\}, P^*)$. However, this contradicts *ii*) in the definition of voting correspondence, since $\hat{P}\mid_{\mathcal{C}\backslash\{c\}} = P^* \mid_{\mathcal{C}\backslash\{c\}}$. We can repeat the argument for any set of three candidates to obtain the desired result.

∎

Note that as the set of vetoers is the same for any $\{a, b, c\}$ and the candidates cannot be vetoers, this implies that if $\mathcal{V} = \mathcal{C}$ no strongly unanimous voting correspondence satisfies no harm and insignificance.

At this point we are already at condition to extend our result to arbitrary sizes of the top-set candidates.

Induction Step There is $m \geq 3$ such that for all $B \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ with $\#B \leq m$, then for all $P \in \mathcal{P}^{\mathcal{V}}(B)$ it holds that there is a set of voters $S \subseteq (\mathcal{V} \backslash \mathcal{C})$ such that for all $A \in \{\mathcal{C}, \{\mathcal{C} \backslash \{a\}\}_{a \in \mathcal{C}}\}$:

a) $v(A, P) \subseteq Pareto(A, P_S)$.

b) $v(A, P) \neq a$ if there is $b \in A$, $i \in S$ with $bP_i a$.

Now we will prove that the result is also true for $\#B = m + 1$.

Proof. Consider an arbitrary $B \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, with $\#B = m + 1$, and an arbitrary $P \in \mathcal{P}^{\mathcal{V}}(B)$.

We start proving *a*) for the case in which all the candidates run the election. Let $d \notin Pareto(\mathcal{C}, P_S)$, as $\#B \geq 3$ there is $f \in B$ such that $d \notin Pareto(\mathcal{C} \backslash \{f\}, P)$. Find now, $P' \in P(\mathcal{C} \backslash \{f\})$ with $P\mid_{\mathcal{C}\backslash\{f\}} = P' \mid_{\mathcal{C}\backslash\{f\}}$. Evidently, $\#(B \backslash \{f\}) = m$, then by the induction

hypotheses $v\left(\mathcal{C}\backslash\{f\}, P'\right) \subseteq Pareto\left(\mathcal{C}\backslash\{f\}, P'_S\right) = Pareto\left(\mathcal{C}\backslash\{f\}, P_S\right)$. By ii) in the definition of voting correspondence $v\left(\mathcal{C}\backslash\{f\}, P\right) = v\left(\mathcal{C}\backslash\{f\}, P'\right)$, therefore by no harm, we have that:

$$v\left(\mathcal{C}, P\right) \subseteq v\left(\mathcal{C}\backslash\{f\}, P\right) \cup \{f\} \subseteq Pareto\left(\mathcal{C}\backslash\{f\}, P\right) \cup \{f\},$$

which suffices to prove that Pareto dominated (according to the preferences of the vetoers) candidates are not elected at P when all the candidates are at stake.

Now we check that the result also holds when a candidate a withdraws. Assume first $a \notin v\left(\mathcal{C}, P\right)$. Notice that if $d \notin Pareto(\mathcal{C}\backslash\{a\}, P_S)$, $d \notin Pareto\left(\mathcal{C}, P_S\right)$ and by the previous paragraph, we know that $d \notin v\left(\mathcal{C}, P\right)$. But by insignificance $v\left(\mathcal{C}, P\right) = v\left(\mathcal{C}\backslash\{a\}, P\right)$, and no Pareto dominated candidate is elected. Finally in the case in which $a \in v\left(\mathcal{C}, P\right)$, the argument applied in the previous paragraph can be replicated to prove the desired result (notice that if a candidate a is elected, a must belong to B and evidently $\#\left(B\backslash\{a\}\right) = m$.)

We prove b) using similar arguments. Consider the election at profile P when al the candidates are at stake and assume to the contrary that there are $a, b \in \mathcal{C}$ such that $v\left(\mathcal{C}, P\right) = a$ but bP_ja for some $j \in S$. For any $f \in \left(B\backslash\{a\}\right)$, by insignificance as $f \notin v\left(\mathcal{C}, P\right)$; $v\left(\mathcal{C}\backslash\{f\}, P\right) = \{a\}$. Now pick $P'' \in \mathcal{P}^{\mathcal{V}}\left(B\backslash\{f\}\right)$, with $P\mid_{\mathcal{C}\backslash\{f\}} = P''\mid_{\mathcal{C}\backslash\{f\}}$. As $\#\left(B\backslash\{f\}\right) = m$, and bP''_ja, by the induction hypotheses, $v\left(\mathcal{C}\backslash\{f\}, P''\right) \neq \{a\}$, which leads to a contradiction since by ii) in the definition of voting correspondence $v(\mathcal{C}\backslash\{f\}, P) = v\left(\mathcal{C}\backslash\{f\}, P''\right)$, and then our assumption cannot be correct.

Finally, we can replicate the arguments in the proof of item a) to prove b) when a candidate withdraws. If some candidate $e \notin v\left(\mathcal{C}, P\right)$ quits, then $v\left(\mathcal{C}, P\right) = v\left(\mathcal{C}\backslash\{e\}, P\right)$, but in the previous paragraph we have already proved that the election at P cannot be a singleton if a is not unanimously preferred for the vetoers. If an elected candidate leaves the ballot, the induction hypotheses applies directly to obtain the result.

∎

In order to close the proof of Theorem 2.4 we only have to show that the best candidates of the vetoers when another candidate withdraws are elected. Take $a \in \mathcal{C}$ and $P \in \mathcal{P}^{\mathcal{V}}$. Let $b = top\left(\mathcal{C}\backslash\{a\}, P_j\right)$ for $j \in S$. Consider now $P^* \in \mathcal{P}^{\mathcal{V}}$, such that $P\mid_{\mathcal{C}\backslash\{a\}} = P^*\mid_{\mathcal{C}\backslash\{a\}}$, $a = top\left(\mathcal{C}, P^*_i\right)$ for all $i \in \left(\mathcal{V}\backslash\{j\}\right)$, while $a = top\left(\mathcal{C}\backslash\{b\}, P^*\right)$. By the arguments in the previous paragraphs,

$v\left(\mathcal{C}, P^{*}\right)=\{a, b\}$, and by no harm $b \in v\left(\mathcal{C} \backslash\{a\}, P^{*}\right)$. Finally, by ii) in the definition of voting correspondence $v\left(\mathcal{C} \backslash\{a\}, P\right)=v\left(\mathcal{C} \backslash\{a\}, P^{*}\right)$, which yields the desired result, since the choice of P, b and j was arbitrary.

∎

Proof of Theorem 2.1.

It is evident that dictatorial rules are unanimous and BDS1 candidate stable so we focus on necessity.

Notice first that strong unanimity and unanimity are equivalent when candidates cannot vote. Moreover, by Lemma 2.2, we know that in this case, BDS1 candidate stability implies no harm and insignificance. Therefore, by the results in Theorem 2.4 ,there is a set of voters, S, with veto power over v. We prove now that there can only be one such a voter, namely a dictator.

Assume to the contrary $\#S \geq 2$ and consider the preference profile $P \in \mathcal{P}^{\mathcal{V}}$ such that for some candidates $a, b, c \in \mathcal{C}$, aP_ibP_ic for some $i \in S$, while bP_jcP_ja for all $j \in(\mathcal{V} \backslash\{i\})$. By Theorem 2.4, we have that $v\left(\mathcal{C}, P\right)=\{a, b\}$ whereas $v\left(\mathcal{C} \backslash\{b\}, P\right)=\{a, c\}$, which is a contradiction with item i) of Lemma 2.2 and v being BDS1 candidate stable. Hence, a unanimous BDS1 candidate stable voting correspondence only admits one vetoer, and therefore, it is dictatorial.

∎

Proof of Theorem 2.2.

It is clear that dictatorial and bidictatorial voting correspondences are BDS2 candidate stable and unanimous.

Let us check sufficiency when there are only three candidates. Notice first that the voting correspondence selects just a candidate only at unanimous profiles (for the vetoers.) In this case, the elected candidate never improves by quitting while the non-elected ones do not affect the social choice (the profile remains unanimous after the withdrawal of a non-elected candidate.) When $\#(v\left(\mathcal{C}, P\right))=2$, both elected candidates are not Pareto dominated, therefore if the non-elected candidate leaves, items a) and b) imply that the social choice does not change. On the other hand, if a chosen candidate withdraws, the outcome will become either the other

selected candidate or the other two candidates are elected. In either case, the withdrawal is not profitable. Finally, if $v(\mathcal{C}, P) = \mathcal{C}$, as no candidate is Pareto dominated, for any $a \in \mathcal{C}$, $v(\mathcal{C}\backslash\{a\}, P) = (\mathcal{C}\backslash\{a\})$ and no candidate is better off after quitting.

In order to prove necessity, notice that strong unanimity and unanimity are equivalent when candidates do not vote, and BDS2 candidate stable voting correspondence satisfy no harm and insignificance when candidates are not allowed to vote. Therefore, from Theorem 2.4, we know that any candidate stable under BDS2 and unanimous voting correspondence is oligarchical. We have just seen that the conditions of Theorem 2.4 are also sufficient for the case in which there are only three candidates. We will prove immediately that when there are at least four candidates, there cannot be more than two vetoers.

Assume to the contrary that a voting correspondence v is BDS2 candidate stable and unanimous, but there are at least voters holding veto power. Consider $\{a, b, c\} \in 2^{\mathcal{C}}\backslash\{\varnothing\}$, $P' \in \mathcal{P}^{\mathcal{V}}(\{a, b, c\})$, $i, j \in S$ such that $a = top(C, P'_i)$, $b = top\left(\mathcal{C}, P'_j\right)$ and $c = top(C, P'_k)$ for all $k \in (\mathcal{V}\backslash\{i, j\})$. By Theorem 2.4, $v(\mathcal{C}, P') = \{a, b, c\}$, and also for all $d \in (\mathcal{V}\backslash\{a, b, c\})$, $v(\mathcal{C}\backslash\{d\}, P') = \{a, b, c\}$. Pick now $d' \in (\mathcal{V}\backslash\{a, b, c\})$, $\hat{P} \in \mathcal{P}^{\mathcal{V}}$ such that $\hat{P}\mid_{\mathcal{C}\backslash\{d'\}} = P'\mid_{\mathcal{C}\backslash\{d'\}}$ and $d' = top\left(\mathcal{C}\backslash\{a\}, \hat{P}_i\right)$, $d' = top\left(\mathcal{C}, \hat{P}_{k'}\right)$ for all $k' \in (\mathcal{V}\backslash\{i\})$. Then, by Theorem 2.4 $v\left(\mathcal{C}, \hat{P}\right) = \{a, d\}$, while by ii) in the definition of voting correspondence $v\left(\mathcal{C}\backslash\{d\}, \hat{P}\right) = \{a, b, c\}$, but this contradicts item i) of Lemma 2.3, thus $\#S \leq 2$.

Finally, we have to prove that a candidate is elected only if she is the best preferred candidate for one of the vetoers $S = \{i, j\}$. Assume to the contrary there is $P^* \in \mathcal{P}^{\mathcal{V}}$ such that $a = top(C, P^*_i)$, while $b = top\left(\mathcal{C}, P^*_j\right)$ but there is $c \in (\mathcal{C}\backslash\{a, b\})$, $c \in v(\mathcal{C}, P^*)$. Then for all $d \in (\mathcal{C}\backslash\{a, b, c\})$, $\{a, b, c\} \in v(\mathcal{C}\backslash\{d\}, P^*)$. Pick now $\tilde{P} \in \mathcal{P}^{\mathcal{V}}$, $\tilde{P}\mid_{\mathcal{C}\backslash\{d\}} = P^*\mid_{\mathcal{C}\backslash\{d\}}$, such that $d = top\left(\mathcal{C}\backslash\{a\}, \tilde{P}_i\right)$, $d = top\left(\mathcal{C}, \tilde{P}_j\right)$. By Theorem 2.4, $v\left(\mathcal{C}, \tilde{P}\right) = \{a, d\}$, while by ii) of the definition of voting correspondence, $\{a, b, c\} \subseteq v\left(\mathcal{C}\backslash\{d\}, \tilde{P}\right)$, but this contradicts item i) of Lemma 2.3, hence only the best preferred candidates of the vetoers can be elected when all the candidates run the election. A similar argument applies to the agendas in which a candidate withdraws, which concludes the proof.

■

Proof of Theorem 2.3.

Again notice that strong unanimity and unanimity are equivalent when candidates cannot vote. Moreover, in Lemma 2.4 we have proved that leximin candidate stability is equivalent to no harm and insignificance both together when the candidates are not voters. Then the result is a direct application of Theorem 2.4. $\blacksquare$

2.6.2 Appendix 2: Proof of Lemma 2.1

We introduce a final piece of notation which becomes quite useful in the proof of Lemma 2.1. For any $a \in \mathcal{C}$, $P_a \in \mathcal{P}^a$ and any integer $n = 1, ..., \#\mathcal{C}$, $x(n, P_a)$ refers to the candidate in the n-th position according to the preference ordering P_a.

Proof of Lemma 2.1.

Item i) Notice that $a \notin (X \cup Y)$, then we can find $P_a \in \mathcal{P}^a$ such that for all $b \in (X \backslash Y)$, $c \in (X \cap Y)$ and $d \in (Y \backslash X)$, $bP_a cP_a d$. Then, for an arbitrary u_a fitting P_a, we have that

$$\frac{1}{\#X} \sum_{x \in X} u_a(x) > \frac{1}{\#Y} \sum_{y \in Y} u_a(y).$$

This is enough to show there is $\succsim_a^2 \in \mathcal{D}_a^2$, such that $X \succ_a^2 Y$.

Item ii) Assume to the contrary there is $\succsim \in \mathcal{D}_a^1$ such that $Y \succsim X$. In this case there should exist $P_a \in \mathcal{P}^a$, u_a fitting P_a and a prior probability assessment λ such that:

$$\sum_{y \in Y} \frac{\lambda(y) u_a(y)}{\left(\sum_{y \in Y} \lambda(y)\right)} \geq \frac{\lambda(a) u_a(a)}{\lambda(a) + \sum_{y \in Y} \lambda(y)} + \sum_{y \in Y} \frac{\lambda(y) u_a(y)}{\left(\lambda(a) + \sum_{y \in Y} \lambda(y)\right)}. \tag{2.1}$$

Nevertheless, (2.1) implies that $\sum_{y \in Y} \frac{\lambda(y) u_a(y)}{\left(\sum_{y \in Y} \lambda(y)\right)} \geq u_a(a)$ which is a contradiction with u_a being consistent with some P_a. Therefore our assumption was false, and for all $a \in \mathcal{C}$, $X, Y \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ such that $X = (Y \cup \{a\})$, $\succsim \in \mathcal{D}_a^1$, $X \succ Y$.

Item iii) Let $X, Y \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, be such that $\#Y > 1$, $a \in Y$, $Y \neq (X \cup \{a\})$. We can study two cases separately:

Case iii-a) There is $b \in (Y \backslash \{a\})$, $b \notin X$.

Case iii-b) There is some $c \in X$, $c \notin Y$.

Case iii-a). Consider $b \in (Y \backslash \{a\})$, $b \notin X$ and find $P_a \in \mathcal{P}^a$ with $b = bottom(\mathcal{C}, P_a)$, a prior assessment λ such that for some $\varepsilon > 0$, $\lambda(b) = 1 - \varepsilon$, $\lambda(x) = \frac{\varepsilon}{\#(\mathcal{C}-1)}$ for all $x \in (\mathcal{C} \backslash \{b\})$. Construct now a utility function u_a fitting P_a in such a way that $u_a(x(n, P_a)) = 1 - \frac{(n-1)}{\#\mathcal{C}} \varepsilon$ for $n = 1, ..., \#\mathcal{C} - 1$, while $u_a(x(\#\mathcal{C}, P_a)) = u_a(b) = 0$.

Notice that, for some $\varepsilon > 0$ small enough we obtain

$$\sum_{x \in X} \frac{\lambda(x) u_a(x)}{\left(\sum_{x \in X} \lambda(x)\right)} > \sum_{y \in Y} \frac{\lambda(y) u_a(y)}{\left(\sum_{y \in Y} \lambda(y)\right)},$$

since the left hand side converges to 1 as ε goes to 0, while the right hand side converges to 0 as ε decreases. This suffices to prove the existence of the desired ordering over sets in the case $iii - a)$.

Case iii-b). Assume now there is $c \in X$, $c \notin Y$. Find $P'_a \in \mathcal{P}^a$ such that $c = top(\mathcal{C} \backslash a, P_a)$, prior assessment λ' such that for some $\varepsilon' > 0$, $\lambda'(c) = 1 - \varepsilon'$ while $\lambda'(x) = \frac{\varepsilon'}{\#(\mathcal{C}-1)}$ for all $x \in (\mathcal{C} \backslash \{c\})$. Finally, construct a utility function representing P'_a, u'_a in such a way that $u'_a(a) = 1$, $u'_a(c) = 1 - \varepsilon'$, and $u'_a(x(n, P_a)) = \frac{\varepsilon'}{n}$ for all $n = 3, ..., \#\mathcal{C}$.

For some $\varepsilon' > 0$ small enough we get again that

$$\sum_{x \in X} \frac{\lambda'(x) u'_a(x)}{\left(\sum_{x \in X} \lambda'(x)\right)} > \sum_{y \in Y} \frac{\lambda'(y) u'_a(y)}{\left(\sum_{y \in Y} \lambda'(y)\right)},$$

since the left hand side converges to 1, while the right hand side converges to $\frac{1}{\#Y}$, which is smaller to 1 since we assumed $\#Y > 1$. Therefore, using P'_a, u'_a and λ' we obtain the result for the case $iii - b)$, which concludes the proof.

Item iv) We only have to check the case in which for some $b \in Y$, $(X \backslash \{a\}) = (Y \backslash \{b\})$. Consider an arbitrary $P_a \in \mathcal{P}^a$. Notice that $a P_a b$, and moreover $\#X = \#Y$, and $(X \backslash \{a\}) =$

$(Y \backslash \{b\}) = (X \cap Y)$. Then for all utility function u_a representing P_a, it holds that

$$\frac{1}{\#X} \left(u_a(a) + \sum_{x \in X \cap Y} u_a(x) \right) > \frac{1}{\#Y} \left(u_a(b) + \sum_{x \in X \cap Y} u_a(x) \right).$$

As the choice of P_a was arbitrary, this suffices to prove that for all BDS2 orderings $\succsim \in \mathcal{D}_a^2$, $X \succ Y$.

Item v) Let $a \in \mathcal{C}$, $X, Y \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ be such that $a \in Y$, $\#Y = 1$ and for any $b \in (Y \cup \{\varnothing\})$, $(X \backslash \{a\}) \neq (Y \backslash \{b\})$. We have to study two cases:

Case v-a) There is $c \in (Y \backslash \{a\})$, $c \notin X$.

Case v-b) There is a set of candidates $\{b_1, ..., b_k\}$ such that either $X = (Y \cup \{b_1, ..., b_k\})$, or
$X = (Y \cup \{b_1, ..., b_k\}) \backslash \{a\}$ (in this case $k > 1$.)

Case v-a). Find $\hat{P}_a \in \mathcal{P}^a$ such that $c = bottom\left(\mathcal{C}, \hat{P}_a\right)$. Construct now a utility function fitting $\hat{P}_a$ such that

$$\hat{u}_a(c) < \left(\frac{\#Y}{\#X} \sum_{x \in X} \hat{u}_a(x) - \sum_{y \in Y} \hat{u}_a(y) \right).$$

We can find such a $\hat{u}_a$ since c is the worst candidate according to the preference order $\hat{P}_a$. This suffices to prove the case $v - a$).

Case v-b). Find $P_a^* \in \mathcal{P}^a$ such that for all $c \in \mathcal{C} \backslash (\{b_1, ..., b_n\} \cup a)$ and for all $l = 1, ..., k$ $b_l P_a^* c$. Construct u_a^* in such a way that for some $\varepsilon > 0$ small enough $u_a^*(x(n, P_a^*)) = 1 - \frac{(n-1)}{\#\mathcal{C}} \varepsilon$ for all $n = 1, ..., k+1$, while $u_a^*(x(n, P_a^*)) = \frac{\varepsilon}{n}$, otherwise.

If $X = (Y \cup \{b_1, ..., b_k\})$, for some ε sufficiently small we obtain that $\frac{1}{\#X} \left(\sum_{x \in X} u_a^*(x) \right)$ is arbitrarily close to $\frac{k+1}{\#Y+k}$, while $\frac{1}{\#Y} \left(\sum_{y \in Y} u_a^*(y) \right)$ is arbitrarily close to $\frac{1}{\#Y}$. Notice that $\frac{k+1}{\#Y+k} > \frac{1}{\#Y}$ if $\#Y > 1$, and there exists $\succsim \in \mathcal{D}_a^2$, such that $X \succ Y$.

Finally if $X = (Y \cup \{b_1, ..., b_k\}) \backslash \{a\}$ (with $k > 1$) we can find ε small enough to get $\frac{1}{\#X} \left(\sum_{x \in X} u_a^*(x) \right)$ arbitrarily close to $\frac{k}{\#Y+k-1}$, while $\frac{1}{\#Y} \left(\sum_{y \in Y} u_a^*(y) \right)$ is again arbitrarily close to $\frac{1}{\#Y}$. In this case notice that $\frac{k}{\#Y+k-1} > \frac{1}{\#Y}$, since $\#Y$ and k are both greater than 1, and the desired conclusion holds.

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