Chapter 3

Candidate Stability and Probabilistic Voting Procedures

3.1 Introduction

Social decision procedures can be described as two stages processes. In the first stage a set of possible candidates decide whether they enter the election or they leave the fray. In the second stage, once the agenda is formed, the social choice is taken from the voters' preferences. It seems desirable that the voting procedure does not provide incentives to the candidates to strategically affect the social decision by withdrawing their candidacy. Only if this is the case, we can consider the set of feasible alternatives as stable and exogenous and independent of any individual decision.

In a recent paper Dutta, Jackson and Le Breton [17][1] have inaugurated the normative analysis of candidates' incentives to withdraw from an election by introducing a condition on the voting rules called Candidate Stability. Given a set of initial candidates, a voting rule is candidate stable if a candidate never prefers to leave the ballot unilaterally rather than to stay. They prove that only dictatorial rules are candidate stable, unanimous and single-valued when the candidates are not allowed to vote (Theorem 1, DJL.) If there are voting candidates the study becomes problematic, but they obtain an impossibility result for a large class of single-valued

[1] Henceforth DJL.

voting rules (Theorems 2 and 3, DJL.)

A drawback of DJL is that only resolute decision processes are taken into account. Although, the single-valuedness assumption is reasonable, in many social choice environments it can be very restrictive. For instance, think of a society with three voters who have to choose from three candidates and assume that voters' preferences are symmetrical among voters and lead to the voting paradox. In that case, no anonymous and neutral argument can be applied to select a single alternative and a tie would be natural. When ties are allowed, we can think of the voting procedure as a first screening device which narrows the social choice from an initial agenda and the individuals are not aware of how the final decision is to be made.

Another possible interpretation, which fits nicely our framework, is to suppose that in the second stage voters play a voting mechanism that admits several equilibria. Candidates know which are the possible equilibria and the final outcomes for each profile of voters' preferences, but they cannot use any backward induction argument to focus on a specific equilibrium. It could be the case that they do not have enough information to know the strategies actually played by the voters and the equilibrium that eventually arises. In this situation the candidates may assess different probabilities to each candidate to be the final winner of the election.

In this chapter, we depart from DJL framework in two ways. Firstly, we model elections as probabilistic voting procedures. We consider rules that for each configuration of the agenda and each voters' preference profile select a lottery on the set of running candidates and this chosen lottery does not depend on the voters' preferences over the candidates who are not at stake. In addition, we also focus on the stability of any possible set of running candidates and not only on the stability of a given agenda. Therefore, we consider a new stability condition stronger than candidate stability, **exit stability**. A voting rule is exit stable if for any set of candidates at stake, a candidate can never be better off by leaving the fray unilaterally.

In this article, it is assumed that candidates cannot vote and their preferences over lotteries on the set of candidates consistent with the Expected Utility Theory. In Lemma 3.1 we show **exit stability** is equivalent to a regularity condition: whenever a candidate drops the election the probability of any of the remaining running candidates cannot diminish. This condition was proposed by Pattanaik and Peleg [34] for the study of the distribution of power under probabilistic voting procedures. They proved that any efficient and regular probabilistic voting procedure is a probabilistic combination of dictatorial rules (Theorem 4.14, [34].) In our Theorem 3.1, we show that the efficiency requirement can be weakened to unanimity. Therefore, only probabilistic random dictatorships satisfy unanimity and exit stability.

The arguments in the proofs of [34] cannot be applied to obtain the characterization of candidate stable probabilistic voting procedures. Regularity condition applies to all possible configurations of the agenda, while candidate stability only focuses on the relation of the social choice under the full agenda and the selection with the agendas in which only one candidate withdraws. Nevertheless, we see that our characterizations are in line with the results and examples proposed in [34]. The characterization depends crucially on the number of candidates at stake. When there are at least four initial candidates, a candidate stable and unanimous probabilistic voting procedure must be a convex combination of dictatorial rules whenever a candidate leaves the election. Nevertheless, this does not imply that it should also be the case when no candidate quits (Theorem 3.2.) Finally, if the initial agenda contains only three candidates, we show that the decision power will be concentrated in the hands of an arbitrary group of voters, but the distribution of the veto power is not necessarily additive. (Theorem 3.3.)

Closely related to this article are the works of Ehlers and Weymark [18], Eraslan and McLennan [19] and Rodríguez-Álvarez [35].[2] They study the implications of candidate stability in voting rules which for each configuration of the agenda and each preference profile choose a set of candidates.[3] In [18] and [19] a strong version of candidate stability for correspondences is introduced, and it is shown that only dictatorial rules satisfy it together with unanimity. Moreover, in [19], the analysis is strengthened by allowing voters to express weak preferences it is shown that only serially dictatorial rules are candidate stable and unanimous. On the other hand, in [35] the incentives of the candidates to quit an election are explicitly modeled by endowing them with preferences over sets of candidates. These preferences over sets are naturally restricted in

[2]See the previous Chapter 2.

[3]We address the interested reader to DJL and [35] for further references on the topic of endogenous agenda formation and strategic candidacy.

order to match the interpretation attached to the notion of sets of candidates. In [35] we propose several domains of preferences over sets. For instance, these preferences may be consistent with Expected Utility Theory and Bayesian updating from some prior assessment.[4] If these prior assessments are unrestricted, candidate stability is equivalent to the condition introduced by [18] and [19]. Nevertheless, if the candidates are constrained to assess even-chance lotteries on the set of elected candidates, candidate stability is less compelling, but only rules that select the best candidates of two arbitrarily fixed voters are admitted besides dictatorial ones. When the candidates compare sets consistently with extreme attitudes towards risk (like for instance *leximin, maximin, maximax* criteria) candidate stability becomes even less stringent and new possibilities arise.

The key point in the proofs of the theorems of this work relies on the relation between candidate (exit) stable probabilistic voting procedures and leximin candidate stable voting correspondences. By exploring this relation we can show that any unanimous and exit stable probabilistic voting procedure only assigns a positive probability to efficient candidates. This allows us to apply the results in [34] and also obtain the characterization for candidate stable rules.

The remainder of the paper proceeds as follows. In Section 3.2, we introduce the set up and notation, while in Section 3.3 we present some examples and the characterization theorems. We devote Section 3.4 to the proofs of the theorems. In Section 3.5 we conclude by discussing the case of voting candidates and the role played by unanimity.

3.2 Definitions and Notation

3.2.1 Voters, Candidates and Preferences

Let $\mathcal{N}$ be a society formed by a finite set of voters $\mathcal{V}$, and an infinite set of candidates $\mathcal{C}$, $\mathcal{N} = \mathcal{C} \cup \mathcal{V}$. We focus on the case in which there is no overlap between the sets of voters and candidates ($\mathcal{C} \cap \mathcal{V} = \{\varnothing\}$.) In this scenario, we can isolate the incentives of the candidates to

[4] Barberà, Dutta and Sen [6] analyze this class of preferences over sets in the context of strategy-proof social choice correspondences.

participate in an election, regardless of their concerns as voters.[5]

Let $2^{\mathcal{C}}$ denote the set of all finite subsets of $\mathcal{C}$. We call $A \in 2^{\mathcal{C}}$ an agenda. We assume that only finite agendas are feasible for this society. The whole set of potential candidates cannot run the election simultaneously.

Each individual $i \in \mathcal{N}$ is endowed with a linear order on $\mathcal{C} \cup \{\varnothing\}$,[6] where the empty set refers to the situation in which no candidate is elected. We assume that for all i any candidate is preferred to the empty set. We denote by P_i a preference order of individual i, and by $\mathcal{P}^i$ the set of admissible preferences over candidates for individual i. For any $A \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ and P_i, $top\,(A, P_i)$ refers to the best element of A according to the preference order P_i. Preferences of voters over candidates are unrestricted, but each candidate considers herself as the best alternative, that is, for all $a \in \mathcal{C}$, and for all $P_a \in \mathcal{P}^a$, $a = top\,(\mathcal{C}, P_a)$. We denote by $P \in \mathcal{P}^{\mathcal{V}}$ a voters preference profile. For each $A \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, $P\mid_A$ denotes the restriction of P to the set A. Abusing notation, for any set $I \subseteq \mathcal{N}$, P^I refers to the restriction of the profile P to the members of I, $\mathcal{P}^I$, is defined as the set of admissible preferences profiles for I. For any set of voters I, we denote by $-I$ the set of voters $\mathcal{V} \backslash I$. Finally, let $\#B$ stand for the cardinality of the set B.

Preferences over Lotteries.

Let $\mathcal{L}$ be the set of lotteries on the set $\mathcal{C}$. That is:

$$\mathcal{L} = \left\{ \lambda \in \mathbb{R}_+^{\#\mathcal{C}} \text{ such that for all } a \in \mathcal{C}, \lambda(a) \geq 0 \text{ and } \sum_{a \in \mathcal{C}} \lambda(a) = 1 \right\}$$

Candidates are endowed with complete, reflexive and transitive preferences on $\mathcal{L} \cup \{\varnothing\}$ and we assume that they are expected utility maximizers. Again, we assume that any candidate always prefers any $\lambda \in \mathcal{L}$ to the empty set. A utility function is a mapping $u_i : \mathcal{C} \to \mathbb{R}$. A utility function fits the preference ordering $P_i \in \mathcal{P}$ if and only if for any $a, b \in \mathcal{C}$, $u_i\,(a) > u_i(b)$ iff aP_ib. Then, given two lotteries $\lambda, \lambda' \in \mathcal{L}$, a candidate $a \in \mathcal{C}$ with preferences over candidates P_a and

[5]When candidates can be voters, the analysis of stability becomes more complicated, because candidates' preferences are assumed to favor their own election. We postpone the discussion of this interesting case to the concluding section.

[6]A linear order is a complete, antisymmetric and transitive binary relation.

consistent utility function u_a prefers the lottery λ to the lottery λ' for a, if and only if:

$$\sum_{b\in\mathcal{C}} \lambda(b)u_a(b) > \sum_{b'\in\mathcal{C}} \lambda'(b')u_a\left(b'\right).$$

Once defined the strict component of the candidates preferences over lotteries, the weak component is defined in the usual way.

3.2.2 Probabilistic Voting Procedures

The object of interest of this article is an aggregation rule, a probabilistic voting procedure p, that selects for each configuration of the ballot and each preference profile a lottery on the set of the candidates.

Definition 3.1. *A probabilistic voting procedure is a mapping* $p : 2^{\mathcal{C}} \times \mathcal{P}^{\mathcal{V}} \rightarrow \mathcal{L} \cup \{\varnothing\}$ *such that for all* $A \in 2^{\mathcal{C}}, a \in \mathcal{C}$ *and* $P \in \mathcal{P}^{\mathcal{V}}$*:*

i) $p\left(a, A, P\right) = 0$*, if* $a \notin A$*, and* $p(A, P) = \{\varnothing\}$ *if and only if* $A = \{\varnothing\}$*,*

ii) $p\left(A, P\right) = p\left(A, P'\right)$ *for all* $P' \in \mathcal{P}^{\mathcal{V}}$ *such that,* $P\mid_A = P'\mid_A$*,*

where $p\left(a, A, P\right)$ *denotes the probability assigned to candidate* a *at profile* P *when the agenda is conformed by the candidates in* A*.*

Item $i)$ states that a candidate cannot be selected if she is not at stake. Moreover, whenever a candidate runs the election, the election always results in somebody selected.

Finally, $ii)$ is in the spirit of Arrow's Independence of Irrelevant Alternatives. Namely, only the preferences of voters over the candidates who eventually run the election are relevant.

Our definition of probabilistic voting procedures is less general than the one proposed in [34], since we embed Independence of Irrelevant Alternatives in the definition. A probabilistic voting procedure is a generalization of a single-valued voting procedure proposed in DJL. A single-valued voting procedure is restricted to select degenerate lotteries that assign probability one to a unique candidate. A probabilistic voting procedure is also more precise than a voting correspondence as defined in [18], [19], and [35], since it assigns a specific probability distribution to each preference profile and not only a set of possible outcomes. Thus, it is more flexible than a voting correspondence to meet voters' preferences. Notice that a probabilistic voting procedure may assign different probabilities to the candidates for different preference profiles even when the set of selected candidates do not change. Finally, a probabilistic voting procedure is a family of decision schemes as defined and analyzed in [22], one for each configuration of the agenda.

3.2.3 Unanimity and Efficiency

In this work, we care about probabilistic voting procedures which satisfy a minimal responsiveness condition. When all voters agree on the best candidate, she is uniquely selected.

Definition 3.2. *A probabilistic voting procedure p is unanimous if and only if for all $c \in \mathcal{C}$, $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ and for all $P \in \mathcal{P}^{\mathcal{V}}$ such that $c = top\,(C, P_i)$ for all $i \in \mathcal{V}$, $p\,(c, C, P) = 1$.*

For all $A \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, for all $I \subset \mathcal{V}$ and for all $P_I \in \mathcal{P}^I$, $Pareto\,(A, P_I) = \{a \in A$, such that there is no $b \in A$, bP_ia for all $i \in I\}$.

Definition 3.3. *A probabilistic voting procedure p is (ex-post) Pareto efficient if and only if for all $c \in \mathcal{C}$, $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ and $P \in \mathcal{P}^{\mathcal{V}}$, $c \notin Pareto(C, P)$ implies $p(c, C, P) = 0$.*

Evidently, unanimity is less stringent than ex-ante and ex-post Pareto efficiency. Unanimity does not rule out the possibility that Pareto dominated candidates receive a positive probability. We discuss the consequences of its relaxation in Subsection 5.3.2.

3.2.4 Exit Stability and Candidate Stability

In this paper we are interested in designing elections for which the agenda can be considered exogenous and independent of the preferences of voters. In order to consider an agenda $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ as exogenous it will be necessary that:

- The candidates who are in the ballot ($c \in C$) never have incentives to leave the fray.
- The candidates who are not at stake ($b \in \mathcal{C} \backslash C$) never improve by entering the election.

Although it seems uncontroversial to introduce a stability condition for candidates who are in the ballot (they should not benefit by quitting), it is not clear which would be a correct statement of stability regarding the incentives of the non-running candidates. To ask for rules that never provide incentives to outsiders to enter would be too compelling and it would lead immediately to impossibility results. Hence, we only focus on the incentives of the running candidates to withdraw. By doing so, we accommodate circumstances in which the candidates are free to leave the election once they are at stake, but they cannot enter in the fray by themselves. Indeed, we will see that our non-exit conditions are rather powerful and they will reduce considerably the incentives of outsider candidates to run the fray.

We present two parallel stability conditions regarding strategic withdrawal of the candidates. Candidate stability is defined as in DJL. Given a fixed agenda, candidate stability focuses on the lack of candidates' incentives to withdraw unilaterally from this agenda. On the other hand, Exit Stability condition captures the same idea, but applying it to any possible agenda. We say a probabilistic voting procedure p is exit stable if and only if a candidate never benefits by withdrawing her candidacy, independently of the remaining candidates who stay in the fray.

Both exit stability and candidate stability imply that the set of candidates can be treated as exogenous (since no candidate will have incentives to leave the poll.) Nevertheless, under exit stability the results are independent of the set of candidates at stake, while the initial agenda is crucial for candidate stability. For instance, by the self-preference of candidates, candidate stability is empty of content for agendas with only two candidates. No candidate can prefer to drop from the ballot because it implies that the remaining candidate will be the sure winner of the election. In fact, we will see that the characterization of the family of candidate stable and unanimous probabilistic voting procedures depends crucially on the size of the initial set of candidates. We provide now the formal definitions:

Definition 3.4. *A probabilistic voting procedure p is **exit stable** if and only if for all $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, $a \in C$, for all $P_a \in \mathcal{P}^a$, for all utility function u_a consistent with P_a and all $P \in \mathcal{P}^{\mathcal{V}}$:*

$$\sum_{b \in C} p\left(b, C, P\right) u_a\left(b\right) \geq \sum_{b' \in C \backslash \{a\}} p(b', C\backslash\{a\}, P) u_a(b').$$

Naturally, we do not include in the definition the empty set since no candidate can withdraw from an agenda in which no candidate runs the election. We finish this section with the definition of candidate stability.

Definition 3.5. *Given the agenda $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, a probabilistic voting procedure is candidate stable at C if to stay in the fray is always a Nash equilibrium strategy for all the candidates in C. That is, for all $a \in C$, for all $P_a \in \mathcal{P}^a$, for all utility function u_a consistent with P_a and all $P \in \mathcal{P}^{\mathcal{V}}$, it holds that:*

$$\sum_{b \in C} p\left(b, C, P\right) u_a\left(b\right) \geq \sum_{b' \in C \backslash \{a\}} p(b', C \backslash \{a\}, P) u_a(b').$$

3.3 Implications of Exit Stability and Candidate Stability in Probabilistic Environments

As it is mentioned before, in this paper we assume that the candidates cannot vote. Hence, although exit stability and candidate stability are not conditions regarding the preferences of the candidates, these are not an input of a probabilistic voting procedure. The preferences of the candidates introduce restrictions on the social choice relating the outcome when all the candidates are at stake to the outcome when a unique candidate drops the ballot. We start this section by providing a crucial lemma that introduces the main implications of exit stability in our probabilistic environment. Namely, a probabilistic voting procedure is exit stable if and only if whenever a candidate withdraws from the poll, then no other candidate reduces her probability of being finally elected.

Lemma 3.1. *A probabilistic voting procedure p is exit stable if and only if for all $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ for all $a, b \in C$, and for all $P \in \mathcal{P}^{\mathcal{V}}$ it holds that $p\left(b, \mathcal{C} \backslash \{a\}, P\right) \geq p\left(b, \mathcal{C}, P\right)$.*

Proof. Assume "ad contrarium" that the probabilistic voting procedure p is exit stable but there exist $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, $a, b \in C$, and $P \in \mathcal{P}^{\mathcal{V}}$ such that $p\left(b, C \backslash \{a\}, P\right) < p\left(b, C, P\right)$. Consider $P_a \in \mathcal{P}^a$, such that for any $c \in C \backslash \{b\}$, $cP_a b$. Notice first, that by $i)$ in the definition of probabilistic voting procedure, for any $d \in (\mathcal{C} \backslash C)$, $p(d, C, P) = 0$. We can find a utility function u_a fitting P_a with $u_a\left(b\right)$ small enough to get $\sum_{c \in C \backslash \{a\}} p\left(c, C \backslash \{a\}, P\right) u_a\left(c\right) > \sum_{c' \in C} p\left(c', C, P\right) u_a\left(c'\right)$, which contradicts exit stability.

On the other hand, assume that for all $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, $a, b \in C$, and for all $P \in \mathcal{P}^{\mathcal{V}}$ it holds that $p\left(b, \mathcal{C} \backslash \{a\}, P\right) \geq p\left(b, \mathcal{C}, P\right)$. Consider $P \in \mathcal{P}^{\mathcal{V}}$, $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ and $a \in C$ such that for all

$b \in C\backslash\{a\}$, $p(b, C\backslash\{a\}, P) = p(b, C, P)$, then $p(C, P) = p(C\backslash\{a\}, P)$, and trivially a cannot better off withdrawing. Finally, consider $C \in 2^{\mathcal{C}}\backslash\{\varnothing\}$, $P \in \mathcal{P}^{\mathcal{V}}$ and $a \in C$ such that there are some $b \in C\backslash\{a\}$ with $p(b, C\backslash\{a\}, P) > p(b, C, P)$ then it holds that,

$$\sum_{b \in C\backslash\{a\}} p(b, C\backslash\{a\}, P) - p(b, C, P) = p(a, C, P) > 0.$$

And as for all P_a, all u_a, and all $b \in \mathcal{C}\backslash\{a\}$, $u_a(a) > u_a(b)$,

$$p(a, C, P)\, u_a(a) > \sum_{b \in C\backslash\{a\}} ((p(b, C\backslash\{a\}, P) - p(b, C, P))\, u_a(b)),$$

and then also,

$$\sum_{b \in C} p(b, C, P)\, u_a(b) > \sum_{b' \in C\backslash\{a\}} p(b', C\backslash\{a\}, P)\, u_a(b'),$$

and candidate stability holds.

■

Remark 3.1. *A probabilistic voting procedure p is exit stable if and only if for all* $P \in \mathcal{P}^{\mathcal{V}}$ *and* $C, C' \in 2^{\mathcal{C}}\{\varnothing\}$ *with* $C \subseteq C'$, *for all* $c \in C$, $p(c, C, P) \geq p(c, C', P)$.

The previous remark just implies that any exit stable probabilistic voting procedure satisfies regularity as defined by Pattanaik and Peleg. (Definition 3.8, [34].)

By using the same arguments for a specific agenda C, we can provide the following lemma characterizing the consequences of candidate stability in probabilistic environments.

Lemma 3.2. *Given the agenda* $C \in 2^{\mathcal{C}}\backslash\{\varnothing\}$, *a probabilistic voting procedure p is canidate stable at* C *if and only if for all* $a, b \in C$, *and for all* $P \in \mathcal{P}^{\mathcal{V}}$ *it holds that* $p(b, C\backslash\{a\}, P) \geq p(b, C, P)$.

Remark 3.2. *If p is candidate stable at agenda* C, *then for all* $P \in \mathcal{P}^{\mathcal{V}}$, *and* $b \in C$ *such that* $p(b, C, P) = 0$, $p(C, P) = p(C\backslash\{b\}, P)$.

Proof. Assume the contrary, then there are $P \in \mathcal{P}^{\mathcal{V}}, b \in C$ such that $p(b, C, P) = 0$, but $p(C, P) \neq p(C\backslash b, P)$. In this case, it must be the case that there is $a \in C$ such that $p(a, C, P) > p(a, C\backslash\{b\}, P)$, which contradicts candidate stability at agenda C.

■

Notice that candidate stability is empty of content for agendas with less than three candidates. At agendas with only one candidate at stake, this candidate is always elected with certainty. By the restrictions on candidates preferences, this outcome is strictly preferred to the result of a withdrawal, the empty set. On the same fashion, when candidate quits an agenda with only two candidates, she cannot be better off. By leaving the fray, she gets the other candidate elected (which cannot be better than the social choice when both candidates run the election.)

DJL have proved that only dictatorial rules are candidate stable and unanimous in deterministic environments. To check that they are also exit stable is immediate. We can expect that probabilistic combinations of dictatorial rules, *random dictatorships* are also exit stable. According to a *random dictatorship* a group of vetoers have the possibility of becoming a dictator. The vetoers are asked for their best candidate, they introduce a given number of ballots with the name of their preferred candidate in a hat, and then a ballot is drawn at random. Hence, a candidate has a positive probability of being elected if is the best candidate for some vetoer and the probability of being elected is the proportion of ballots that have her name. We state this formally.

Definition 3.6 (*Dictatorship*). *A probabilistic voting procedure d_i is* ***dictatorial*** *if and only if there is a voter $i \in \mathcal{V}$, such that for all $a \in \mathcal{C}$, for all $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, and for all $P \in \mathcal{P}^{\mathcal{V}}$,*

$$d_i(a, C, P) = \begin{cases} 1 & \text{if } a = top(C, P_i), \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.7 (*Random Dictatorship*). *The probabilistic voting procedure p is a* ***random dictatorship*** *if and only if it is a probabilistic combination of dictatorial rules. That is, there is a group voters $S \subseteq \mathcal{V}$, called the vetoers, a set of weights $\{\alpha_i\}_{i \in S}$, ($\alpha_i > 0$ for all $i \in S$ and $\sum_{i \in S} \alpha_i = 1$) and a set of dictatorial probabilistic voting procedures $\{d_i\}_{i \in S}$ such that for all $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, $P \in \mathcal{P}^{\mathcal{V}}$, $p(C, P) = \sum_{i \in S} \alpha_i d_i(C, P)$.*

It is easy to check that any random dictatorship is unanimous and candidate stable. Notice that if a candidate withdraws then the ballots that did not have her name remain with the same name, while those which had her name are changed to another candidate. Hence, no other candidate's support is reduced, and no candidate can improve by leaving the fray.

In fact, just by rephrasing Theorem 4.14 in [34], we know that only probabilistic combinations of dictatorial rules are exit stable and efficient. Our Theorem 3.1 extends that result by relaxing efficiency to unanimity.

Theorem 3.1. *A probabilistic voting procedure is unanimous and exit stable if and only if it is a random dictatorship.*

The proof of Theorem 3.1 appears in the next section, we provide now a sketch of the proof. In order to prove that unanimity and exit stability imply efficiency, we exploit the results of Theorem 3 in [35] on candidate stable voting correspondences. (Theorem 2.3 in the previous chapter.) Given a unanimous and exit stable probabilistic voting procedure p, we can construct an auxiliary voting correspondence v_p in the following way, for any $a \in \mathcal{C}$, for all $A \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, and for all $P \in \mathcal{P}^{\mathcal{V}}$, $a \in v_p(A, P)$ iff $p(a, A, P) > 0$. It is easy to check that if p is unanimous and exit stable, for any agenda $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, v_p is also unanimous and candidate stable at C when the voters compare sets according to the leximin extension criterion. By Theorem 3 in [35], v_p must be efficient and, moreover, there exists a group of voters holding veto power. Once efficiency is proved, the result follows immediately from Theorem 4.14 in [34], since exit stability is equivalent to regularity.[7]

At the light of Theorem 3.1 we can evaluate the incentives of an outsider candidate to enter a unanimous and exit stable election. It is clear that only the candidates who are going to become the best candidate at stake for some dictator can affect the social outcome. Moreover, the effect of their entry will be that they will become elected (with some positive probability.) Therefore, unanimous and exit stable probabilistic voting procedures do not provide incentives to the entry of *frivolous* candidates. A candidate without any possibility of being elected does not have incentives to enter the fray.

In order to analyze the implications of candidate stability, we cannot address directly to the results in [34]. As candidate stability focuses on the stability of a given agenda, we cannot

[7] We present the results of [34] and [35] in the following section as Propositions 3.1 and 3.2.

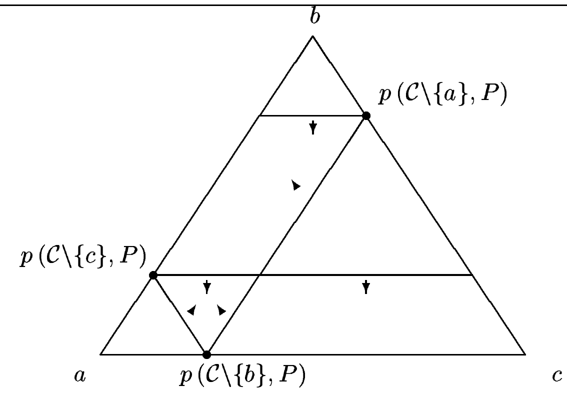

Figure 3.1: Example 3.1: $p(\mathcal{C}, P)$

apply the full power of regularity. In fact, candidate stability and unanimity no longer compel the election to be driven by a random dictatorship. The following example shows the new possibilities that arise when we focus on the stability of fixed agendas.

Example 3.1. *Let $\mathcal{V} = \{1, 2\}$, $\mathcal{C} = \{a, b, c\}$. Let the probabilistic voting procedure p be a random dictatorship whenever only two candidates are at stake. That is for all $d \in \mathcal{C}$, for all $P \in \mathcal{P}^{\mathcal{V}}$, w.l.o.g. there is $\alpha \in \left(\frac{1}{2}, 1\right)$, such that $p(\mathcal{C}\backslash\{d\}, P) = \alpha d_1 (\mathcal{C}\backslash\{d\}, P) + (1-\alpha) d_2 (\mathcal{C}\backslash\{d\}, P)$. Let $P \in \mathcal{P}^{\mathcal{V}}$ be such that aP_1bP_1c , cP_2bP_2a, Figure 1 shows that candidate stability does not imply that $p(\mathcal{C}, P) = \alpha d_1 (\mathcal{C}, P) + (1-\alpha) d_2 (\mathcal{C}, P)$.*

Nevertheless, if voter 2's preferences are $bP_2'cP_2'a$, then $p(\mathcal{C}, P') = p(\mathcal{C}\backslash\{c\}, P)$, as it is shown in Figure 2.

Before following with the analysis of candidate stability and finite fixed agendas, we introduce now a bit of notation.

Let p be a probabilistic voting procedure and $C \in 2^{\mathcal{C}}\backslash\{\varnothing\}$ a finite agenda. For any $P \in \mathcal{P}^{\mathcal{V}}$ and for any $a \in C$, define

$$L_p(a, C, P) = \{\lambda \in \mathcal{L}, \text{ such that for all } b \in C\backslash\{a\}, \ \lambda(b) \leq p(b, C\backslash\{a\}, P)\}.$$

The set $L_p(a, C, P)$ contains all the lotteries that are admissible for $p(C, P)$ given the selection of p when the candidate a withdraws. At this point, we can present a rephrasal of Lemma 3.2 in terms of the sets of admissible lotteries. The probabilistic voting procedure p is candidate

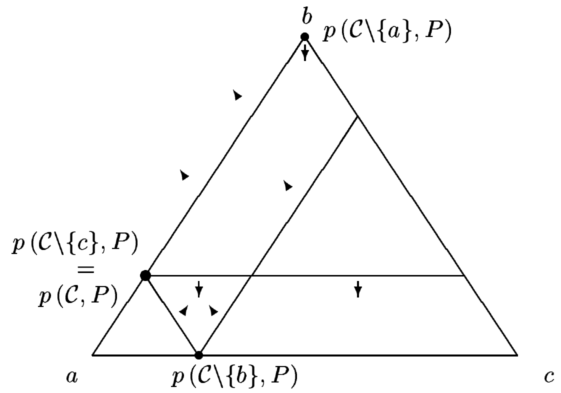

Figure 3.2: Example 3.1: $p(\mathcal{C}, P')$

stable at agenda C if and only if $p(\mathcal{C}, P) \in \bigcap_{b\in\mathcal{C}} L_p(b, C, P)$.

The following Theorem 3.2 characterizes the family of candidate stable and unanimous probabilistic voting procedures when there are at least four candidates. As we have seen in the previous example, candidate stability and unanimity allow for a certain flexibility in the social choice when all the candidates are at stake, it is only at the cost of distributing the decision power additively among the vetoers when a candidate leaves the ballot.

Theorem 3.2. *Let $C \in 2^{\mathcal{C}}\backslash\{\varnothing\}$ be a finite agenda containing at least four alternatives. A probabilistic voting procedure p is unanimous and candidate stable at C if and only if there is a group of vetoers $S \subseteq \mathcal{V}$, a set of weights $\{\alpha_i\}_{i\in S}$, ($\alpha_i > 0$ for all $i \in S$ and $\sum_{i\in S} \alpha_i = 1$), and a set of dictatorial probabilistic voting procedures $\{d_i\}_{i\in S}$, such that for all $P \in \mathcal{P}^{\mathcal{V}}$,*

$$p(C\backslash\{a\}, P) = \sum_{i\in S} \alpha_i d_i (C\backslash\{a\}, P) \text{ for all } a \in C,$$

$$p(C, P) \in \bigcap_{b\in C} L_p(b, C, P).$$

Theorem 3.2 is parallel to the Theorem 4.11 in [34]. Nevertheless, we have already mentioned it cannot be derived directly from it. The crucial point is that candidate stability focuses on the stability of the full agenda, while the results in [34] apply regularity to all possible agenda. Instead, this result is directly derived from the results in [35] on candidate stable voting correspondences.

In the following lemma we specify the restrictions that $\bigcap_{b\in C} L_p(b,C,P)$ satisfies whenever a probabilistic voting procedure is a combination of dictatorial rules when a candidate leaves the election.

Lemma 3.3. *Let p be a probabilistic voting procedure candidate stable at agenda C. Then, for all $P \in \mathcal{P}^{\mathcal{V}}$:*

i) $\sum_{i\in S} \alpha_i d_i(C,P) \in \bigcap_{b\in C} L_p(b,C,P)$.

ii) *If $a, a' \in \bigcup_{i\in S} top(C,P_i)$, there is no $\lambda \in \bigcap_{b\in C} L_p(b,C,P)$ such that $\lambda(a) = \lambda(a') = 0$.*

iii) $\sum_{i\in S} \alpha_i d_i(C,P) \neq \bigcap_{b\in C} L_p(b,C,P)$ *if and only if there is $a \in C$, such that for all $b \in C\backslash\{a\}$, there is $i \in S$ with $a = top(C\backslash\{b\},P_i)$ and $a \neq top(C,P_i)$.*

Proof. i) Notice that $\sum_{i\in S} \alpha_i d_i(\mathcal{C},P) \in \mathcal{L}$, and therefore for all $a,b \in \mathcal{C}$, $\sum_{i\in S} \alpha_i d_i(a,\mathcal{C},P) \leq \sum_{i\in S} \alpha_i d_i(a,\mathcal{C}\backslash\{b\},P)$.

ii) Notice that for all $P \in \mathcal{P}^{\mathcal{V}}$ and $a \in \bigcup_{i\in S} top(\mathcal{C},P_i)$, if there is $\lambda \in L_p(a,C,P)$ with $\lambda(a) = 0$, then $\lambda = \sum_{i\in S} \alpha_i d_i(\mathcal{C}\backslash\{a\},P)$. Then for all $a' \in \left(\left(\bigcup_{i\in S} top(\mathcal{C},P_i)\right)\backslash\{a\}\right)$, $\lambda(a') = \sum_{i\in S} \alpha_i d_i(a',\mathcal{C}\backslash\{a\},P) > 0$. Then as $L_p(a,P)$ is a convex subset of the $\#\mathcal{C}-1$ simplex there is no $\lambda' \in L_p(a,P)$ such that $\lambda'(a) = \lambda'(a') = 0$.

iii) Assume that for $P \in \mathcal{P}^{\mathcal{V}}$ there is not such candidate a, then for all $a \in \mathcal{C}$ there is $b \in \mathcal{C}\backslash\{a\}$ such that $\{i \in S,\ a = top(\mathcal{C}\backslash\{b\},P_i)\} = \{i \in S,\ a = top(\mathcal{C},P_i)\}$. Hence, for all $a \in \mathcal{C}$ we get $\sum_{i\in S} \alpha_i d_i(a,\mathcal{C},P) = \min_{d\in\mathcal{C}\backslash\{a\}} \sum_{i\in S} \alpha_i d_i(a,\mathcal{C}\backslash d,P)$. For any $\lambda \in \bigcap_{b\in\mathcal{C}} L_p(b,C,P)$, we have that for all $a \in \mathcal{C}$, $\lambda(a) \leq \sum_{i\in S} \alpha_i d_i(a,\mathcal{C},P)$. As $\sum_{i\in S} \alpha_i d_i(a,\mathcal{C},P) \in \mathcal{L}$, it must be the case that for all $a \in \mathcal{C}$, $\lambda(a) = \sum_{i\in S} \alpha_i d_i(a,\mathcal{C},P)$, and then $\bigcap_{b\in\mathcal{C}} L_p(b,C,P) = \sum_{i\in S} \alpha_i d_i(\mathcal{C},P)$.

Conversely, take $P \in \mathcal{P}^{\mathcal{V}}$ such that for some $a \in \mathcal{C}$ for any $b \in (\mathcal{C}\backslash\{a\})$, there is $i \in S$ with $a = top(\mathcal{C}\backslash\{b\},P_i)$, whereas $a \neq top(\mathcal{C},P_i)$. Notice first that

$$\sum_{i\in S} \alpha_i d_i(a,C,P_i) < \min_{d\in C} \left\{ \sum_{i\in S} \alpha_i d_i(a,C\backslash\{d\},P) \right\}.$$

This implies there is $\lambda \in \mathcal{L}$ such that

$$\sum_{i\in S} \alpha_i d_i(a,C,P_i) < \lambda(a) \leq \min_{d\in C} \left\{ \sum_{i\in S} \alpha_i d_i(a,C\backslash\{d\},P) \right\},$$

while for all $b \in \mathcal{C} \backslash \{a\}$,

$$\lambda(b) \leq \sum_{i \in S} \alpha_i d_i \left(b, \mathcal{C}, P\right) \leq \min_{d \in \mathcal{C}} \left\{ \sum_{i \in S} \alpha_i d_i \left(b, \mathcal{C} \backslash d, P\right) \right\},$$

and therefore $\lambda \in \bigcap_{b \in \mathcal{C}} L_p(b, P)$ but $\lambda \neq \sum_{i \in S} \alpha_i d_i(\mathcal{C}, P)$.

■

In item *i)* we show that the conditions of Theorem 3.2 define properly probabilistic voting procedures since a random dictatorship always satisfies them.

Item *ii)* implies that only one of the best preferred candidates according to the vetoers preferences can receive null probability when no candidate leaves the agenda C.

Finally, item *iii)* characterizes the preference profiles in which a candidate stable and unanimous probabilistic voting procedure may be different than a random dictatorship. Only at preference profiles in which there is a candidate who becomes the best preferred candidate among those remaining in the fray for some vetoer whoever is the other candidate who leaves the fray, the choice when all the candidates run the election is different than the lottery that the random dictatorship mechanism will yield. In those cases, this candidate will have more possibility of being elected if any other candidate withdraws and then she can receive more probability than the assigned by the *random dictatorship* lottery when all the candidates run the election.

Notice that whenever the number of candidates exceeds the number of vetoers in two, our characterization collapse to *random dictatorships*, since there is no preference profile in which a candidate always increases her support when any other candidate leaves the ballot.[8] Nevertheless, this case corresponds to situations in which the set of vetoers is relatively small with respect to the set of candidates and hence the power of decision is very concentrated in a few voters.

Just as a matter of clarification we present now an example of a *modified random dictatorship.* It resembles Example 5.6 in [34].

Example 3.2. *Let* $C = \{a, b, c, d\}$, *and* $\mathcal{V} = \{1, 2, 3, 4\}$; $S = \{1, 2, 3\}$ *and* $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$. *Let* $\mathcal{P}^* \subset \mathcal{P}^{\mathcal{V}}$ *denote the set of preference profiles such that each vetoer has different top candidate*

[8]This fact is also noted in [34] for regular probabilistic voting procedures.

but the remaining candidate is the second best for all the vetoers and the best for voter 4. *Then, we define the probabilistic voting procedure* p^* *in such a way that for all* $A \in \{C, \{C\backslash\{a\}\}_{a\in C}\}$ *and all* $P \in \mathcal{P}^{\mathcal{V}}$*:*

$$p^*(A,P) = \begin{cases} \left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right) & \text{if } P \in \mathcal{P}^* \text{ and } A = C, \\ \sum_{i\in S} \alpha_i d_i(A,P) & \text{otherwise.} \end{cases}$$

It is easy to check that p^* *satisfies the conditions of the Theorem* 3.2. *For this election method, the set of admissible lotteries for* p^* *is not a singleton if and only if all the vetoers report different top candidates and the remaining candidate is the second best for all the vetoers. We have tailored* p^* *in such a way that the preferences of the voter without veto power are also relevant for the social choice.*

Theorem 3.2 does not cover the case of agendas with only three candidates. In fact, we need to consider agendas with at least four candidates at several steps of the proof to obtain the result. The following example shows that there exist unanimous and candidate stable probabilistic voting procedures that are not *modified random dictatorships* when only three candidates are at stake.

Example 3.3. *Let* $C = \{a,b,c\}$. *Construct now a probabilistic voting procedure* p' *such that for all* $a \in C$*, for all* $A \in 2^C\backslash\{\varnothing\}$*, and for all* $P \in \mathcal{P}^{\mathcal{V}}$*,:*

$$p'(a,A,P) = \begin{cases} \frac{1}{\#Pareto(A,P)} & \text{if } a \in Pareto(A,P), \\ 0 & \text{otherwise.} \end{cases}$$

p' *is unanimous and candidate stable. Nevertheless, it is not candidate stable for any agenda with more than three candidates.*

In [35], we provide a characterization of the family of unanimous candidate stable voting correspondences when candidates are expected utility maximizers and consider that the ties are solved using even chance probabilities. [9] In this environment, a voting correspondence defines immediately a candidate stable probabilistic voting procedure. Roughly speaking, almost any oligarchical rule satisfy the requirements. The previous example is just an example. Notice that

[9]See the previous Chapter, Theorem 2.2.

p' is not a *random dictatorship* even when a candidate leaves the election, since the distribution of the veto power is not additive. Every group of voters has the same capacity to impose their most preferred candidate when only two are at stake. In the next theorem, we see that any monotonic and sub-additive distribution of the veto power suffices to reconcile unanimity and candidate stability.

Theorem 3.3. *Assume the agenda C contains only three candidates. A probabilistic voting procedure is candidate stable and unanimous at C if and only is there is a group of voters $S \subseteq \mathcal{V}$, a set of weights $\{\alpha_T\}_{T \subseteq S}$ such that:*

$$p\,(a, C\backslash\{b\}, P) = \alpha_T \leftrightarrow T = \{j \in S,\ a = top\,(C\backslash\{b\}, P_j)\},$$
$$p\,(C, P) \in \bigcap_{b \in C} L_p\,(b, C, P)\,,$$

for all $P \in \mathcal{P}^{\mathcal{V}}$ and $a, b \in \mathcal{C}$; furthermore,

i) $\alpha_{\{\varnothing\}} = 0$ *and for all $T \in S$, $\alpha_T = 1 - \alpha_{S\backslash T}$,*

ii) (monotonicity) *for all $T, T' \subseteq S$, $T \subseteq T'$; $\alpha_T \leq \alpha_{T'}$,*

iii) (sub-additivity) *for all disjoint $T, T' \subset S$, $\alpha_T + \alpha_{T'} \geq \alpha_{(T \cup T')}$.*

Although Theorem 3.2 does not cover the case of three candidates, the main arguments in its proof can be applied to get the result at agendas with only three candidates. On the other hand, when there are only three initial candidates, a candidate stable probabilistic voting procedure is indeed regular and we can also address to [34]'s Remark 4.15 to prove that sub-additive distributions of the veto power among the vetoers are compatible with candidate stability and unanimity. In the next section we provide the complete proof of Theorem 3.3 from the arguments in [35].

It may seem surprising that the implications of candidate stability depend so much on the number of candidates at stake. The differences between the results of Theorem 3.2 and Theorem 3.3 are due to the fact that when there are only three candidates, and one of them withdraws, the choice of the lottery on the remaining candidates is a binary choice. Thus, we can expect that the results when $\#C = 3$ are in the line of those in [7] and [27] on stochastic social preferences (or probabilistic binary choice.) As it is remarked in [34],[10] when there are three candidates, a candidate stable probabilistic voting procedure is rationalizable. That is, for every profile of individual orderings one can specify a probability distribution r over linear orderings of candidates, such that given a feasible agenda A, the probability of a candidate a being chosen by the voters from A, is the sum of the probabilities assigned by r to all those linear orderings $P' \in \mathcal{P}$ where $a = top(A, P')$. Nevertheless, when there are at least four candidates, regularity no longer implies rationalizability. But in this case, candidate stability forces the distribution of the veto power to be additive.

3.4 Proofs of the Theorems

As we have already mentioned the proof of Theorem 3.1 relies on the results in [34]. Theorem 3.3 can also be proved by using the results in [34] but Theorem 3.2 cannot be proved with the same arguments, since candidate stability is not equivalent to regularity. Hence, we exploit the relation between exit (candidate) stable probabilistic voting procedures and leximin candidate stable voting correspondences. The crucial argument relies on the application of the important results of [35] and [34] to our environment. These results are expressed as Proposition 3.1 and Proposition 3.2 below. But before we need the following definitions.

A *voting correspondence* v is a mapping $v : 2^{\mathcal{C}} \backslash \{\varnothing\} \times \mathcal{P}^{\mathcal{V}} \to 2^{\mathcal{C}} \backslash \{\varnothing\}$ such that:

i) For all $A \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, $v(A, P) \subseteq A$.

ii) $v(A, P) = v(A, P')$ for all $P' \in \mathcal{P}^{\mathcal{V}}$ such that, $P \mid_A = P' \mid_A$.

A voting correspondence v is unanimous if whenever there is $b \in B \subseteq \mathcal{C}$ and $P \in \mathcal{P}^{\mathcal{V}}$ such that $top(B, P_i) = b$ for all $i \in \mathcal{V}$, then $v(B, P) = b$. Finally, a voting correspondence v is leximin candidate stable if for all $P \in \mathcal{P}^{\mathcal{V}}$ and $a \in \mathcal{C}$, $v(\mathcal{C}, P) \subseteq v(\mathcal{C} \backslash \{a\}, P) \cup \{a\}$ if $a \in v(\mathcal{C}, P)$ (*no harm*), and $v(\mathcal{C}, P) = v(\mathcal{C} \backslash \{a\}, P)$ if $a \notin v(\mathcal{C}, P)$ (*insignificance*.)

Proposition 3.1 (Theorem 3, Rodríguez-Álvarez [35]). *A voting correspondence v is unanimous and leximin candidate stable at agenda $C \in 2^{C} \backslash \{\varnothing\}$ if and only if there is a set*

[10]See Remarks 3.12 ; 4.15 and Lemma 3.13, [34].

of voters $S \subseteq \mathcal{V}$, called the vetoers, such that for all $a \in C$, $A \in \{C, C \backslash \{a\}\}_{a \in C}$, $v(A,P) \subseteq Pareto(A, P_S)$ and moreover if $v(A,P) \neq \{a\}$ if there is $b \in A$, $i \in S$ such that $bP_i a$..

Proposition 3.2 (Theorem 4.14, Pattanaik and Peleg [34]). *If there exists $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ with $\#C \geq \#\mathcal{V} + 2$ then a probabilistic voting procedure p satisfies regularity and efficiency if and only if p is a random dictatorship.*

For any probabilistic voting procedure p we can define an ancillary voting correspondence v_p in the following way, for any $a \in \mathcal{C}$, for all $A \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, and for all $P \in \mathcal{P}^{\mathcal{V}}$, $a \in v_p(A, P)$ if and only if $p(a, A, P) > 0$. By $i)$ and $ii)$ of the definition of probabilistic voting procedures v_p is well defined as a voting correspondence.

Proof of Theorem 3.1.
Sufficiency is clear, so we focus on necessity. The crucial point in the proof of necessity is to check that unanimity and exit stability imply efficiency. This is proved through the following lemma that relates stable probabilistic procedures and leximin candidate stable voting correspondences.

Lemma 3.4. *If p is a unanimous and exit stable voting procedure, then its associated voting correspondence v_p is unanimous and leximin candidate stable at any agenda $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$.*

Proof. The unanimity of p implies v_p also is unanimous. Hence, we check now leximin candidate stability. As p is exit stable, for any $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ $a, b \in C$, and $P \in \mathcal{P}^{\mathcal{V}}$; $p(a, C \backslash \{b\}, P) \geq p(a, C, P)$. Take an arbitrary agenda C, and a preference profile P and find $b \in C$ such that $p(b, C, P) > 0$. By exit stability, we know that for all $a \in C \backslash \{b\}$ such that $p(a, C, P) > 0$, also $p(a, C \backslash \{b\}, P) > 0$ and by the definition of v_p, $a \in v_p(C, P)$ and $a \in v_p(C \backslash \{b\}, P)$. As the choice of C and P was arbitrary, for all $b \in v_p(C, P)$, $v(C, P) \subseteq v_p(C \backslash \{b\}, P) \cup \{b\}$.

Finally, for an arbitrary agenda C and an arbitrary preference profile P, find $b' \in C$ such that $p(b', C, P) = 0$, by Remark 3.2 we know that $p(C, P) = p(C \backslash \{b'\}, P)$. Again as the choice of C and P was arbitrary, for all $b' \notin v_p(\mathcal{C}, P)$, $v_p(\mathcal{C}, P) = v_p(\mathcal{C} \backslash \{b'\}, P)$, and we get the desired conclusion.

■

By Proposition 3.1, we know that for any $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ there is a group of voters $S \subseteq \mathcal{V}$ holding veto power for v_p, and therefore for all $P \in \mathcal{P}^{\mathcal{V}}$, $v_p(C, P) \subseteq Pareto(C, P_S)$, which implies that p is efficient.

Once we know that a unanimous and exit stable probabilistic voting procedure is indeed efficient, the result follows immediately. Notice that since $\mathcal{C}$ is an infinite set, we can always find an agenda C such that $\#C \geq \#\mathcal{V}+2$. Moreover, we have seen that an exit stable probabilistic voting procedure satisfies regularity. Therefore, applying Proposition 3.2, we can see that any unanimous and exit stable probabilistic voting procedure is a random dictatorship. ■

Proof of Theorem 3.2.
We start with sufficiency. By Lemma 3.3, we know that a probabilistic voting procedure satisfying the requirements of Theorem 3.2 is well defined. In order to check unanimity, notice that a random dictatorship is unanimous, and by *ii*) of Lemma 3.3, p may differ from a random dictatorship only at non-unanimous profiles. On the other hand, it is not difficult to see that the conditions are enough to get candidate stable at C, since for all $P \in \mathcal{P}^{\mathcal{V}}$ $p(C, P) \in \bigcap_{b \in C} L_p(b, C, P)$.

So from now on we assume that p is candidate stable at the agenda C, with $\#C \geq 4$. From the arguments of Lemma 3.4, we know there is a set of voters S, such that for all $A \in \{C, \{C \backslash \{a\}\}_{a \in C}\}$, $b \in C$ and $P \in \mathcal{P}^{\mathcal{V}}$, $p(b, C, P) > 0$ only if $b \in Pareto(C, P_S)$.

Claim 3.1. *There is a set of weights $\{\alpha_T\}_{T \subseteq S}$ with $\alpha_T > 0$, and $\alpha_T + \alpha_{(S \backslash T)} = 1$ for all $T \subseteq S$ such that for all $a, b \in C$, $P \in \mathcal{P}^{\mathcal{V}}$ with $Pareto(C, P_S) = \{a, b\}$, it holds that:*

$$p(a, C, P) = \alpha_T \longleftrightarrow T = \{i \in S \text{ such that } aP_i b\}.$$

Proof. Take a pair of candidates $a, b \in C$ and find $P \in \mathcal{P}^{\mathcal{V}}$, $T \subsetneq S$ such that $a = top(C, P_i)$, $b = top(C \backslash \{a\}, P_i)$ for all $i \in T$ while $b = top(C, P_j)$, $a = top(C \backslash \{b\}, P_j)$ for all $j \in (S \backslash T)$. Let us denote $p(a, C, P) = \alpha$ and as $Pareto(C, P_S) = \{a, b\}$, $p(b, C, P) = (1 - \alpha)$. The proof proceeds by a series of steps.

Step 1. First, we are going to prove that if we fix the preferences of the voters not in S, whenever two candidates are the only efficient candidates (according to the preferences of the vetoers) the probability of each candidate to be the winner only depends on the group of vetoers who support each efficient candidate.

Consider $P' \in \mathcal{P}^{\mathcal{V}}$, such that $P_S \mid_{\{a,b\}} = P'_S \mid_{\{a,b\}}$, $Pareto\,(C, P_S) = \{a, b\}$ while $P_{-S} = P'_{-s}$. Take now the profile of preferences $\tilde{P} \in \mathcal{P}^{\mathcal{V}}$ such that a vetoer $i \in S$ changes the order in which she compares two contiguous candidates with respect to her initial preferences. That is, there is $i \in S$, $d, e \in C$ ($\{d, e\} \neq \{a, b\}$) such that $P_{-i} = \tilde{P}_{-i}$; $P_i \mid_{C\setminus\{d\}} = \tilde{P}_i \mid_{C\setminus\{d\}}$, $P_i \mid_{C\setminus\{e\}} = \tilde{P}_i \mid_{C\setminus\{e\}}$ and dP_ie while $e\tilde{P}_id$. Notice that at least one of the candidates, say e, $e \notin \{a, b\}$. Moreover, $e \notin Pareto\,(C, P_S)$, $e \notin Pareto\left(C, \tilde{P}_S\right)$. By Remark 3.2, $p\,(C, P) = p\,(C\setminus\{e\}, P)$, and also $p\left(C, \tilde{P}\right) = p\left(C\setminus\{e\}, \tilde{P}\right)$. Finally, applying $ii)$ of the definition of probabilistic voting correspondence $p\,(C, P) = p\left(C, \tilde{P}\right)$. Repeating the argument with one such a change at the preferences of one vetoer as many time as necessary we get the desired result, that is $p\,(a, C, P) = p\,(a, C, P') = \alpha$.

Step 2. Now, we prove that for all profiles $P'' \in \mathcal{P}^{\mathcal{V}}$ such that there two candidates such that $Pareto\,(C, P_S) = \{d, e\}$ and all the vetoers in T prefer d to e while the remaining vetoers prefer e to d, we have that $p\,(d, C, P'') = \alpha$.

Consider a profile $\hat{P} \in \mathcal{P}^{\mathcal{V}}$ such that $a = top\left(C, \hat{P}_i\right)$, $b = top\left(C\setminus\{a\}, \hat{P}_i\right)$ for all $i \in T$ while $b = top\left(C, \hat{P}_j\right)$, $a = top\left(C\setminus\{b\}, \hat{P}_j\right)$ for all $j \in (S\setminus T)$ and $c = top\left(C\setminus\{a, b\}, \hat{P}_{i'}\right)$ for all $i' \in S$. By the arguments on the previous paragraphs, we know that $p\left(a, C, \hat{P}\right) = \alpha$, $p\left(b, C, \hat{P}\right) = (1 - \alpha)$, since $Pareto\left(C, \hat{P}_S\right) = \{a, b\}$ and $P \mid_{\{a,b\}} = \hat{P} \mid_{\{a,b\}}$. Take now the preference profile $P^* \in \mathcal{P}^{\mathcal{V}}$ such that $\hat{P}_k = P^*_k$ for all $k \notin S$ while the voters in S the positions of the candidates b and c change their positions, that is $aP^*_icP^*_ib$ for all $i \in T$ and $cP^*_jbP^*_ja$ for all $j \in (S\setminus T)$, and $P^*_S \mid_{C\setminus\{b\}} = \hat{P}_S \mid_{C\setminus\{b\}}$ and also $P^*_S \mid_{C\setminus\{c\}} = \hat{P}_S \mid_{C\setminus\{c\}}$. As $c \notin Pareto\left(\mathcal{C}, \hat{P}_S\right)$, $p\left(c, \mathcal{C}, \hat{P}\right) = 0$. By Remark 3.2, we have that $p\left(C, \hat{P}\right) = p\left(C\setminus\{c\}, \hat{P}\right)$ and by $ii)$ in the definition of probabilistic voting procedure, $p\left(a, C\setminus\{c\}, \hat{P}\right) = p\,(a, C\setminus\{c\}, P^*) = \alpha$. Finally, by candidate stability $p\,(a, \mathcal{C}, P^*) \leq \alpha$. We now check that $p\,(a, \mathcal{C}, P') = \alpha$. Assume "ad contrarium" that $p\,(a, C, P^*) < \alpha$. Given that $b \notin Pareto\,(C, P^*_S)$, from efficiency we obtain $p\,(b, C, P^*) = 0$ and by Remark 3.2, candidate stability implies $p\,(a, C, P^*) = p\,(a, C\setminus\{b\}, P^*) < \alpha$. Notice now that by $ii)$ in the definition of probabilistic voting procedure $p\,(a, C\setminus\{b\}, P^*) = p\left(a, C\setminus\{b\}, \hat{P}\right)$. This is a contradiction with candidate stability, since $p\left(a, C, \hat{P}\right) = \alpha > p(a, C\setminus\{b\}, \hat{P})$. Therefore, we must have that $p\,(a, C, P^*) = \alpha$, $p\,(c, C, P^*) = (1 - \alpha)$. Repeating the reasoning as many times as necessary we obtain the desired result.

Step 3. Finally, we prove that the probability of a being elected does not depend on the preferences of the voters who are not oligarchs. That is, for all $P' \in \mathcal{P}^{\mathcal{V}}$ such that $P_S = P'_S$, $p(a, C, P') = \alpha$ (and therefore also $p(b, C, P') = (1-\alpha)$.)

Consider a profile of preferences $\bar{P} \in \mathcal{P}^{\mathcal{V}}$, such that there is a voter $k \notin S$ that switches the order in which she compares two continuous candidates (without lost of generality $dP_k e$, $e\bar{P}_k d$, $P_k |_{C\backslash\{d\}} = \bar{P}_k |_{C\backslash\{d\}}$ and $P_k |_{C\backslash\{e\}} = \bar{P}_k |_{C\backslash\{e\}}$ and $P_{-k} = \bar{P}_{-k}$.) If $\{d, e\} \neq \{a, b\}$, either d or e (or both) do not receive positive probability neither at profile P nor at profile P^*. We can assume without lost of generality that $d \notin Pareto\left(C, \bar{P}_S\right)$ and $p(d, C, P) = p\left(d, C, \bar{P}\right) = 0$. By the joint application of Remark 3.2 and *ii)* in the definition of probabilistic voting procedure $p(C, P) = p(C\backslash\{d\}, P) = p(C\backslash\{d\}, P^*) = p(C, P^*)$, and the change of k's preferences has no effect on the choice. So assume now that $\{d, e\} = \{a, b\}$. By the arguments in the previous paragraph and *ii)* in the definition of probabilistic voting procedure, we know that $p\left(a, C\backslash\{b\}, \bar{P}\right) = \alpha$. Notice that also $P' |_{C\backslash\{b\}} = \left(P'_S, P^*_{-S}\right) |_{C\backslash\{b\}}$, and since $b \notin Pareto(C, P'_S)$, applying again *ii)* of the definition of probabilistic voting procedure, we get $p\left(b, \mathcal{C}, \left(P'_S, P^*_{-S}\right)\right) = 0$. And finally, from Remark 1 we obtain that $p(C, P^*) = p\left(C\backslash\{b\}, \left(P'_S, P^*_{-S}\right)\right)$, that is $p\left(a, C, \left(P'_S, P^*_{-S}\right)\right) = \alpha$, $p\left(c, C, \left(P'_S, P^*_{-S}\right)\right) = (1-\alpha)$. Repeating the arguments with one such a switch in the preferences of one voter we obtain the result for arbitrary preference profiles of the voters without veto power.

It is easy to see that the three steps lead to the desired result.

■

Claim 3.2. *For all* $T, T' \subset S$ *with* $T \cap T' = \{\varnothing\}$, $\alpha_T + \alpha_{T'} = \alpha_{(T \cup T')}$.

Proof. Pick two arbitrary disjoints subsets of the set of vetoers, $T, T' \subset S$, $T \cap T' = \{\varnothing\}$. Consider $P \in \mathcal{P}^{\mathcal{V}}$, $a, b, c, d \in C$ such that $aP_i bP_i cP_i dP_i e$ for all $i \in T$, $bP_j cP_j aP_j dP_j e$ for all $j \in T'$, $cP_k aP_k bP_k dP_k e$ for all $k \in (S\backslash T)$, and for all $e \in (C\backslash\{a, b, c, d\})$. Construct now $P^1 \in \mathcal{P}^{\mathcal{V}}$ such that $P |_{C\backslash\{d\}} = P^1 |_{C\backslash\{d\}}$ and $d = top\left(C\backslash\{a\}, P^1_i\right)$ for all $i \in T$, $d = top\left(C, P^1_{i'}\right)$ for all $i' \in (S\backslash T)$. By the arguments of the previous claim, $p\left(a, C, P^1\right) = \alpha_T$. Analogously construct the profile $P^2 \in \mathcal{P}^{\mathcal{V}}$ such that $P |_{C\backslash\{d\}} = P^2 |_{C\backslash\{d\}}$ and $d = top\left(C\backslash\{b\}, P^2_j\right)$ for all $j \in T'$, $d = top\left(C, P^2_{j'}\right)$ for all $j' \in (S\backslash T')$, and we get, $p\left(b, C, P^2\right) = \alpha_{T'}$. In the same fashion take the profile $P^3 \in \mathcal{P}^{\mathcal{V}}$ such that $P |_{C\backslash\{d\}} = P^3 |_{C\backslash\{d\}}$ and $d = top\left(C\backslash\{c\}, P^3_k\right)$ for all

$k \in S\backslash(T \cup T')$, $d = top\left(C, P^3_{k'}\right)$ for all $k' \in (T \cup T')$ and, $p\left(c, C, P^3\right) = \alpha_{S\backslash(T\cup T')}$.

Now, notice that $P\mid_{C\backslash\{d\}} = P^1\mid_{C\backslash\{d\}} = P^2\mid_{C\backslash\{d\}} = P^3\mid_{C\backslash\{d\}}$, therefore by candidate stability, *ii*) in the definition of probabilistic voting procedure and efficiency

$$\begin{aligned} p(a, C, P) &= p\left(a, C\backslash\{d\}, P^1\right) \geq \alpha_T, \\ p(b, C, P) &= p\left(b, C\backslash\{d\}, P^2\right) \geq \alpha_{T'}, \\ p(c, C, P) &= p\left(c, C\backslash\{d\}, P^3\right) \geq \alpha_{S\backslash(T\cup T')}; \end{aligned}$$

while, also by efficiency, $p(e, C, P) = 0$ for all $e \notin \{a, b, c\}$. At this point, we will show that the inequalities must be binding.

Assume to the contrary and without lost of generality that $p(a, C, P) > \alpha_T$. Construct now the profile $P^* \in \mathcal{P}^{\mathcal{V}}$ such that $P\mid_{C\backslash\{b\}} = P^*\mid_{C\backslash\{b\}}$ and such that for all $e \in \{a, b\}$, $l \in S$, $eP^*_l b$, and therefore $Pareto(C, P^*) = \{a, c\}$. By the previous claim we have that $p(a, C, P^*) = \alpha_T$, and by Remark 3.2, $p(a, C\backslash\{b\}, P^*) = \alpha_T$. But this is a contradiction since by candidate stability $p(a, C\backslash\{b\}, P) > \alpha_T$ whereas by ii) in the definition of probabilistic voting procedure $p(C\backslash\{b\}, P) = p(C\backslash\{b\}, P^*)$.

We can repeat the argument with candidates b and c to prove that $p(a, C, P) = \alpha_T$, $p(b, C, P) = \alpha_{T'}$ and $p(c, C, P) = \alpha_{S\backslash(T\cup T')}$. Moreover, by efficiency:

$$\alpha_T + \alpha_{T'} + \alpha_{S\backslash(T\cup T')} = 1. \tag{*}$$

Finally, note that as $\alpha_{S\backslash(T\cup T')} = \left(1 - \alpha_{(T\cup T')}\right)$, (*) implies $\alpha_T + \alpha_{T'} = \alpha_{(T\cup T')}$.

■

In order to close the proof of the theorem, we only have to prove that for any $a, b \in C$, $P \in \mathcal{P}^{\mathcal{V}}$, $p(a, C\backslash\{b\}, P) > 0$ only if there is $i \in S$ with $a = top(C\backslash\{b\}, P_i)$, and moreover $p(a, C\backslash\{b\}, P) = \alpha_T$ where $T = \{i \in S, a = top(C\backslash\{b\}, P_i)\}$.

Consider $P \in \mathcal{P}^{\mathcal{V}}$, such that there are $a, b \in C$, $T \subset S$, $T = \{i \in S, a = top(C\backslash\{b\}, P_i)\}$. Construct now the profile $P' \in \mathcal{P}^{\mathcal{V}}$ such that $P\mid_{C\backslash\{b\}} = P'\mid_{C\backslash\{b\}}$ with $b = top(C\backslash\{a\}, P'_i)$ for all $i \in T$ for all $j \in (S\backslash T)$, while $b = top\left(C, P'_j\right)$. Then we have that $p(a, C, P') = \alpha_T$, and therefore, by ii) in the definition of probabilistic voting procedure and by candidate stability $p(a, C\backslash\{b\}, P) = p(a, C\backslash\{b\}, P') > \alpha_T$. Repeating the argument with the remaining candidates

$c \in \cup_{j \in S} top\left(C \backslash \{b\}, P_j\right)$, from the additivity of the weights $\{\alpha_T\}_{T \subseteq S}$, we obtain that for any $P \in \mathcal{P}^{\mathcal{V}}$, $b \in C$:

$$p\left(C \backslash \{b\}, P\right) = \sum_{i \in S} \alpha_i d_i \left(C \backslash \{b\}, P\right).$$

Finally, candidate stability implies that $p(C, P) \in \bigcap_{b \in C} L_p(b, C, P)$.

■

Proof of Theorem 3.3.

In order to prove sufficiency we will check that the conditions of Theorem 3.2 imply p is well defined. Assume $\mathcal{C} = \{a, b, c\}$ and take any $P \in \mathcal{P}^{\mathcal{V}}$. Let S_a be the set of vetoers who prefer a to b and c, and define analogously S_b and S_c. Notice first that:

$$\begin{aligned} \alpha_{S_a} &\leq \min_{x \in \{b,c\}} \{p(a, C \backslash \{x\}, P)\}, \\ \alpha_{S_b} &\leq \min_{y \in \{a,c\}} \{p(b, C \backslash \{y\}, P)\}, \\ \alpha_{S_a} &\leq \min_{z \in \{a,b\}} \{p(c, C \backslash \{z\}, P)\}, \end{aligned} \tag{3.1}$$

On the other hand, as S_a, Sb and S_c form a partition of S, items i)and iii) imply that

$$\alpha_{S_a} + \alpha_{S_b} + \alpha_{S_c} \geq 1. \tag{3.2}$$

From (3.2) we know there is $\lambda \in \mathcal{L}$ such that $\lambda(a) \leq \alpha_{S_a}$, $\lambda(b) \leq \alpha_{S_b}$ and $\lambda(c) \leq \alpha_{S_c}$. Moreover, by (3.1), $\lambda \in \bigcap_{b \in \mathcal{C}} L_p(b, C, P)$, and hence $\bigcap_{b \in \mathcal{C}} L_p(b, C, P)$ is always non-empty and p is well defined.

To see that a probabilistic voting procedure satisfying the conditions of the theorem is candidate stable is immediate since $p(\mathcal{C}, P)$ always belongs to $\bigcap_{b \in \mathcal{C}} L_p(b, P)$. Unanimity is also easy to check from i).

The proof of necessity is parallel to the proof of Theorem 3.2. If p is a candidate stable at the agenda $C = \{a, b, c\}$ and unanimous probabilistic voting procedure we can construct an auxiliary voting correspondence that is unanimous and candidate stable according to the leximin extension. Then there are a group of voters S holding veto power over p. Thus, only Pareto efficient candidates according to the preferences of the vetoers may receive positive probability and a candidate receives probability 1 if and only if she is the top candidate for all the vetoers. (This implies item i).)

We can apply directly the results in Claim 3.1 of the proof of Theorem 3.2 since we have only used three candidates in its proof. Therefore, when there are only two candidates at stake, say a and b, and a subset of the vetoers T prefer a to b, while the remaining vetoers prefer b to a then $p(a, \{a,b\}, P) = \alpha_T$ while $p(b, \{a,b\}, P) = \alpha_{(S\backslash T)} = (1-\alpha_T)$, and these weights are independent of the names of the candidates and of the preferences of the remaining voters.

Let us check monotonicity. It is trivially fulfilled when $\#S = 2$. So take $\#S \geq 3$, and assume there are $T \subset S$ and $i \in (S\backslash T)$ such that $\alpha_T > \alpha_{T\cup\{i\}}$. Without lost of generality, take the profile $P \in \mathcal{P}^{\mathcal{V}}$, such that aP_jbP_jc for all $j \in J$, bP_iaP_ic and bP_kcP_ka for all $k \in S\backslash(T\cup\{i\})$. Then as candidate c is Pareto dominated according to the preferences of the vetoers, $p(\mathcal{C}, P) = p(\mathcal{C}\backslash\{c\}, P)$ and $p(a, \mathcal{C}, P) = \alpha_T$ and $p(b, \mathcal{C}, P) = (1-\alpha_T)$. However $p(a, \mathcal{C}\backslash\{b\}, P) = \alpha_{(T\cup\{I\})} < \alpha_T$ which violates candidate stability. Repeating the argument as many times as necessary, we obtain that $\alpha_T \leq \alpha_{T'}$ whenever $T \subset T' \subset S$.

Now, we have to check sub-additivity. Take two arbitrary sets of vetoers $T, T' \subset S$, such that $T \cap T'$ and consider a profile $P \in \mathcal{P}^V$ such that $a = top(C, P_i)$, $b = top(C\backslash\{a\}, P_i)$ for all $i \in T$, $b = top(C, P_j)$ $c = top(C\backslash\{b\}, P_j)$ for all $j \in T'$, $c = top(C, P_k)$ and $a = top(C\backslash\{c\}, P_k)$ for all $k \in S\backslash(T\cup T')$. By candidate stability $p(a, C, P) \leq p(a, C\backslash\{b\}, P) = \alpha_T$, $p(b, C, P) \leq p(b, C\backslash\{c\}, P) = \alpha_{T'}$ and $p(c, C, P) \leq p(c, C\backslash\{a\}, P) = \alpha_{S\backslash(T\cup T')}$. As $p(C, P) \in \mathcal{L}$, adding up the three inequalities we obtain:

$$\alpha_T + \alpha_{T'} + \alpha_{S\backslash(T\cup T')} \geq 1. \qquad (**)$$

Finally, as $\alpha_{S\backslash(T\cup T')} = \left(1 - \alpha_{(T\cup T')}\right)$, (**) implies $\alpha_T + \alpha_{T'} \geq \alpha_{(T\cup T')}$. As the choice of T and T' was arbitrary this suffices to prove the result.

Just to conclude with necessity, notice that candidate stability implies that for any $P \in \mathcal{P}^{\mathcal{V}}$, $p(C, P) \in \bigcap_{b\in\mathcal{C}} L_p(b, C, P)$.

■

3.5 Conclusions

A few comments on possible extensions to this work are in order.

3.5.1 Overlap between Voters and Candidates

This work only covers the case in which the sets of voters and candidates are disjoint. It is evident that in many real life public decision processes the candidates are also voters. Nevertheless, many problems arise when one tries to model the participation of the candidates also as voters, both in probabilistic as in deterministic environments. Firstly, unanimity loses its bite when candidates can vote, since they are always supposed to support their own election. Hence, it becomes necessary to introduce stronger versions of unanimity. But the main problem that we face is to attain clear implications of candidate stability. For instance it is possible to construct degenerate candidate stable voting procedures by selecting the worst candidate of the candidate who decides to withdraw. This rule would be candidate stable but it would not satisfy strong unanimity conditions. A possibility is to impose stability conditions (namely Lemma 3.1) in the same fashion of [18] and [19], but then the strategic interpretation of the framework would not be clear. We want to remark that by imposing Lemma 3.1, and allowing the candidates to vote, we would obtain the following result, a probabilistic voting procedure satisfies Lemma 3.1 and strong unanimity then it is a *random dictatorship* in which no candidate have veto power.

We want to remark that in the case in which candidates can vote, we cannot use the results in [34] to get the characterization. When candidates are allowed to vote, voters' preferences are restricted by the self-preference of the candidates, while in [34] only unrestricted linear preferences are considered. Nevertheless, we can follow the reasonings in the proof of Theorem 3.2 and to extend them to variable agendas applying Theorem 4 in [35], which does cover the case of voting candidates. (Theorem 2.4 of the previous chapter.)

3.5.2 Relaxing Unanimity

We conclude the discussion with a remark on the role of unanimity. Unanimity plays a crucial role in the proofs in [35], and thus we cannot dispense with it in order to obtain the results in this chapter. However it would be interesting to know what kind of candidate stable probabilistic voting procedures are ruled out by its requirements. Notice that candidate stability has no bite when only two candidates can be elected. (If a candidate leaves the election, the remaining one wins, and it cannot be better than the winner when both candidates are at stake by the self preference of the candidates.) Probabilistic combinations of single-valued candidate stable voting procedures are candidate stable probabilistic voting procedures. Thus we can construct non-unanimous candidate stable probabilistic voting procedures by mixturing voting procedures with only two alternatives in the range. Many of these probabilistic voting procedures are by no means interesting. However, other attractive candidate stable but non-unanimous probabilistic voting procedures can be constructed in this fashion, as the next concluding example shows.[11]

Example 3.4. *Let $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ and define for all $a, b \in \mathcal{C}$, for all $i \in \mathcal{V}$, and for all $P \in \mathcal{P}^{\mathcal{V}}$ the function $s_i^a : (\mathcal{C} \backslash a) \times 2^{\mathcal{C}} \backslash \varnothing \times \mathcal{P} \to \{0,1\}$, in the following way,*

$$s_i^a(b, A, P) = \begin{cases} 1 & \text{if } aP_ib \text{ and } a \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Now let the probabilistic voting procedure p be such that for all $A \in \{C, \{C\backslash\{b\}\}_{b \in C}\}$, for $a \in \mathcal{C}$ and for $P \in \mathcal{P}^{\mathcal{V}}$, $p(a, A, P) = \frac{1}{\#\mathcal{V}(\#C-1)} \sum_{i \in \mathcal{V}} \sum_{b \in C\backslash\{a\}} s_i^a(b, A, P)$.

The voting procedure p is clearly candidate stable at C since it is a combination of voting procedures with only two alternatives in their range. It is easy to see that p is no more than a probabilistic version of the Borda count in which the drop of a candidate is equivalent to being the last candidate for all the voters.

[11] See Barberà [4] for more on the construction of "nice" probabilistic decision schemes.

Chapter 4

Strategy-Proofness and Residual Resoluteness: A Note on Duggan-Schwartz (2000)

4.1 Introduction

Electoral processes can be modeled as social choice correspondences. A social choice correspondence is a voting rule that selects a set of alternatives for each profile of voters' preferences. Although it seems natural to think of a unique alternative as the outcome of an election, in many situations it is rather restrictive to rule out the possibility of ties among several alternatives. We can interpret a social choice correspondence as a first screening device which narrows the social agenda to a smaller set of alternatives. The voters know this set of alternatives, but they are not aware of how the final resolution of the social choice is to be solved.

The study of manipulability of social choice correspondences has experienced a recent revival of interest. In the last years, several papers have addressed this topic, proposing different definitions of strategy-proofness. These definitions arise from different assumptions to how the agents' preferences on alternatives are extended to sets of alternatives. In spite of the differences all of these papers obtain negative results. This is the case for Ching and Zhou [12], Duggan and Schwartz [16],[1] Barberà, Dutta and Sen [6] or Benoit [8].

Among these works, DS introduce a very weak notion of strategy-proofness. Yet, they attain the most negative result. Only dictatorial social choice correspondences are non manipulable and respect two mild conditions on their range, citizens sovereignty and residual resoluteness. *Citizens Sovereignty* requires that all alternatives should be included among those chosen at some profile of preferences. *Residual Resoluteness* compels the election to be single-valued when all voters but one report identical preferences, and the remaining individual's preferences only differ from those of the others in the order of the first pair of alternatives. Although both conditions seem certainly weak, residual resoluteness may be quite stringent for small societies. In fact, it is responsible for this apparent paradox.

This note tries to emphasize the role played by residual resoluteness in DS's theorem by presenting an alternative proof. We state, firstly, DS's problem for a society with only two voters, in which Residual Resoluteness becomes rather stringent. In this context, we can use the intuitive arguments proposed by Schmeidler and Sonnenschein [38] for the proof of the Gibbard-Satterthwaite theorem. An induction argument applies to extend the result to arbitrary finite the societies.

Additionally, we use the same techniques to provide necessary conditions for non manipulability when residual resoluteness condition is substituted by ontoness (Theorem 4.3.)[2] We see that strategy-proof and onto social choice election must endow some individuals with veto power. There is a group of voters who always include their best preferred alternatives in the chosen set. Moreover, when they agree on their best alternative no other alternative is selected.[3]

Although a complete characterization is not attained, the result has interesting insights. Firstly, we can interpret Theorem 4.3 as a test of robustness of DS's theorem. Indeed, only rules that centralize the power of decision in an arbitrary group of individuals are strategy-proof and onto. This fact provides further evidence on the strength of the Gibbard-Satterthwaite Theorem. Moreover, the notion of strategy-proofness employed by DS is quite weak since voters only care about the best and the worst alternative in a set. This implies that we can obtain other results

[1]Henceforth, we refer to Duggan and Schwartz [16] as DS.

[2]A social choice correspondence is onto if all single alternatives are elected at some configuration of preferences.

[3]The first condition is stated as an impossibility result in DS, and proved in Duggan and Schwartz [15].

on non-manipulability of social choice correspondences as corollaries to our Theorem 4.3.

In order to illustrate this later point, we introduce a new family of preferences over sets of alternatives. These preferences are naturally related to the domain proposed by DS and they include the logic of the "discerning individuals" criterion proposed by Barberà [3]. This domain of preferences generates a definition of non-manipulability stronger than DS's one but weaker than others based on Expected Utility rationality. In this new environment we can provide a complete characterization of the family of strategy-proof and onto social choice correspondences. We will see that only dictatorial or bidictatorial rules are onto strategy-proof in this scenario. (Theorem 4.4.)

Finally, we propose a further extension to the analysis in DS. DS study voting rules that consider voters' preferences over alternatives as the input of the social choice. However, Barberà, Dutta and Sen [6] have proposed a more general approach. Given that the voters are endowed with preferences over sets of alternatives, we could construct social choice rules that use the information contained in these preferences. Unfortunately, we will see that in this more general framework, the results are parallel to those in Theorem 4.3. Even when voters can report their preferences over sets of alternatives, an arbitrary group of voters hold veto power over the social choice.

The remainder of the paper is organized as follows. In the next section we introduce notation and definitions. In Section 4.3 we present our alternative proof of DS's Theorem and provide necessary conditions for strategy-proof correspondences when Residual Resoluteness is substituted by ontoness. In Section 4.4, we analyze the consequences of introducing a stronger extension of preferences, while in Section 4.5 we present the alternative framework in which voters are allowed to express their preferences over sets of alternatives. Finally, in the concluding section, we relate our work with the existing literature.

4.2 Definitions

Voters, Alternatives and Preferences over Alternatives.

Consider a finite society, $\mathcal{N}$, of N individuals (voters), with $N \geq 2$. Let $A = \{x, y, z, ...\}$, be a finite set of mutually incompatible alternatives. Voters are endowed with linear preferences over the set A. Let $\mathcal{P}$ denote the set of linear preferences over A. $P_i \in \mathcal{P}$ refers to a preference relation over A for individual i.[4] We write $P_i = \langle x, y, z \rangle$ to mean xP_iyP_iz. We call $P \in \mathcal{P}^N$ a preference profile.

Let $\mathcal{A}$ denote the set of all non empty subsets of A. For any $X \in \mathcal{A}$, $max\,(X, P_i)$ and $min\,(X, P_i)$ are, respectively, the best and worst alternatives of X according to the preferences P_i. For any $P_i \in \mathcal{P}$, $X \in \mathcal{A}$, $P_i\mid_X$ refers to the restriction of P_i to X. For any $X \in \mathcal{A}$, we say $P_i \in \mathcal{P}_X$ if all alternatives in X are preferred to the remaining alternatives according P_i. For any $I \subset \mathcal{N}$, $P \in \mathcal{P}^N$, P_I refers to the restriction of P to the members of I.[5] Finally, for any $I \subset \mathcal{N}$, $-I$ denotes the set $\mathcal{N} \backslash I$.

Preferences over Sets of Alternatives.

Let $\mathcal{D}$ be the set of all orderings on $\mathcal{A}$. We denote by $\succsim_i \in \mathcal{D}$ an arbitrary preference relation over sets of candidates. Although we assume strict preference over alternatives, we do not rule out the possibility of indifferences among sets. Hence for each $\succsim_i \in \mathcal{D}$, $\succ_i$ refers to the strict component of $\succsim_i$, while $\sim_i$ refers to the indifference term that is immediately defined once $\succ_i$ is specified.

Voters' preferences over sets are naturally restricted. They must be consistent with the original preferences over alternatives. Moreover, they should be coherent with the interpretation of sets of alternatives as a first stage social selection.

For any set $X \in \mathcal{A}$, a probability assessment over X, λ_X is a mapping $\lambda_X : X \longrightarrow (0, 1)$, such that $\sum_{x \in X} \lambda_X(x) = 1$. We exclude degenerate lotteries assigning null probability to some alternative $x \in X$ in order to be coherent with our interpretation of sets as possible final choices of the society. A utility function u_i is a mapping from A to $\mathbb{R}$. It fits with preferences P_i if for all $x, y \in A$, xP_iy iff $u_i(x) > u_i(y)$.

For any $P_i \in \mathcal{P}$, we say the ordering $\succsim_i \in \mathcal{D}$ is (DS) consistent with P_i, $\succsim_i \in \mathcal{D}^{DS}(P_i)$, if and only if for all $X, Y \in \mathcal{A}$, $X \succ_i Y$ if and only if for any pair of probability assessment over X and

[4] A linear order on A is a complete, transitive, antisymmetric binary relation on A. For any $P_i \in \mathcal{P}$, once the strict component is defined, the weak component, R_i, is defined in the usual way.

[5] Abusing notation, we say $P_I \in \mathcal{P}_X^I$, if for all $i \in I$, $P_i \in \mathcal{P}_X$.

Y, λ_X, λ_Y there is a utility function fitting P_i such that:

$$\sum_{x \in X} \lambda_X(x) u_i(x) > \sum_{y \in Y} \lambda_Y(y) u_i(y).$$

Social Choice Correspondences and Duggan-Schwartz's Theorem.

Following DS, we study aggregation rules that for each profile of voters' preferences over alternatives select a non empty set of alternatives.

Social Choice Correspondences. A social choice correspondence f is a mapping from the set of preference profiles to sets of alternatives, $f : \mathcal{P}^N \longrightarrow \mathcal{A}$.

For any social choice correspondence f, R_f denotes the range of the social choice correspondence, that is $R_f = \{X \in \mathcal{A}$ such that there is $P \in \mathcal{P}^N, f(P) = X\}$. Finally, r_f denotes the set of alternatives belonging to some element of the range. That is, $r_f = \{x \in A$, such that there is $P \in \mathcal{P}^N$ with $x \in f(P)\}$.

We focus on the analysis of strategic incentives of voters. We are interested in social choice correspondences that provide incentives to the voters to reveal their true preferences. This idea is captured by the notion of strategy-proofness.

Strategy-Proofness. The social choice correspondence f is (DS) manipulable iff there are a individual $i \in \mathcal{N}$, profiles P, $P' = (P_{-i}, P_i')$, and $\succsim_i \in \mathcal{D}^{DS}(P_i)$ such that $f(P') \succ_i f(P)$. We say that f is (DS) strategy-proof if and only if it is not (DS) manipulable.

Finally, we also study several mild conditions regarding the range of social choice correspondences.

Citizens Sovereignty. The social choice correspondence f satisfies citizens sovereignty if for all $x \in A$ there is $P \in \mathcal{P}^N$ such that $x \in f(P)$. In others terms $A = r_f$.

Ontoness. The social choice correspondence f fulfills ontoness if for all $x \in A$ there is some $P \in \mathcal{P}^N$, such that $x = f(P)$. ($A \subseteq R_f$.)

Residual Resoluteness. The social choice correspondence f satisfies residual resoluteness if for all $x, y \in A$, $\bar{P} \in \mathcal{P}_{\{x,y\}}$, $j \in \mathcal{N}$ and $P \in \mathcal{P}^N$, such that for all $i \in \mathcal{N} \backslash j$, $P_i = \bar{P}$, and $P_j \in \mathcal{P}_{\{x,y\}}$, $P_j |_{A \backslash \{x,y\}} = \bar{P} |_{A \backslash \{x,y\}}$, $f(P)$ is a singleton.

Dictatorship. The social choice correspondence f is dictatorial if there is an individual i such that for all $P \in \mathcal{P}^N$, $f(P) = max(r_f, P_i)$.

Ontoness condition is stronger than Citizen Sovereignty, since it implies that all singleton sets belong to the range. Residual Resoluteness does not implies nor is implied by ontoness, but it also compels a single-valued choice at some preferences profile.

In order to close this section we introduce the main theorem in DS.

Theorem 4.1 (Duggan-Schwartz [16]). *Consider a society with at least three candidates. A social choice correspondence satisfies strategy-proofness, citizens' sovereignty and residual resoluteness if and only if it is dictatorial.*

4.3 The Role of Residual Resoluteness

In this section we propose an alternative proof of the main result of DS. It follows the intuitive arguments introduced by Schmeidler and Sonnenschein [38] in their proof of the Gibbard-Satterthwaite Theorem. We will see that Citizen Sovereignty is not necessary to get the result. It is enough to assume that the range of the social choice correspondence includes three alternatives to get the result. Moreover, this proof highlights the decisive role of residual resoluteness condition. Using the same techniques we provide necessary conditions for strategy-proof and onto social choice correspondences and then we can check the robustness of DS results. (Theorem 4.3.)

Before presenting our alternative proof of DS Theorem, we introduce three lemmata on the implications of strategy-proofness and residual resoluteness. The first implication of Lemma 4.1 and Lemma 4.2 are also stated in DS. Lemma 4.3 could be proved using Theorem 1 in [8]. We include the proofs for the sake of completeness.

Lemma 4.1. *A social choice correspondence f is strategy-proof if and only if for all $i \in \mathcal{N}$ and for all $P, P' = (P_i', P_{-i}) \in \mathcal{P}^N$:*

$$max\left(f\left(P\right), P_i\right) R_i max\left(f\left(P'\right), P_i\right) \text{ and } min\left(f\left(P\right), P_i\right) R_i min\left(f\left(P'\right), P_i\right).$$

Proof. We first proof that any strategy-proof social choice correspondence f satisfies that for any $i \in \mathcal{N}$ and all preference profiles $P, P' = (P_i', P_{-i}) \in \mathcal{P}^N$, $max(f(P), P_i) R_i max(f(P'), P_i)$ and also $min(f(P), P_i) R_i min(f(P'), P_i)$. Assume to the contrary, f is strategy-proof but for some voter i and profiles $P, P' = (P_{-i}, P_i') \in \mathcal{P}^N$, $max\left(f\left(P'\right), P_i\right) P_i max\left(f\left(P\right), P_i\right)$. Notice that, for any arbitrary pair of probability assessments $\lambda_{f(P)}, \lambda_{f(P')}$ we can find a utility function u_i fitting P_i, such that $u_i\left(max\left(f\left(P'\right), P_i\right)\right)$ is high enough while $u_i(y)$ is low enough for any alternative $y \in (f(P) \cup f(P')) \backslash max(f(P'), P_i)$ to get:

$$\sum_{x' \in f(P')} \lambda_{f(P')}\left(x'\right) u_i\left(x'\right) > \sum_{x \in f(P)} \lambda_{f(P)}\left(x\right) u_i\left(x\right).$$

A similar argument applies to prove that it is not possible that $min\left(f\left(P'\right), P_i\right) P_i min\left(f\left(P\right), P_i\right)$.

Now, we prove the converse statement. Assume that for all $i \in \mathcal{N}$; $P, P' = (P_{-i}, P_i') \in \mathcal{P}^N$, $max\left(f\left(P\right), P_i\right) R_i max\left(f\left(P'\right), P_i\right)$, and $min\left(f\left(P\right), P_i\right) R_i min\left(f\left(P'\right), P_i\right)$. Now, we check that this suffices for f being strategy-proof. Consider an arbitrary $i \in \mathcal{N}$ and profiles of preferences $P, P' = (P_{-i}, P_i') \in \mathcal{P}^N$. We have to consider two cases: either $f\left(P\right)$ or/and $f\left(P'\right)$ are singletons, or both $f(P)$ and $f(P')$ contain more than one alternative.

In the first case, it must hold that $f(P) R_i f(P')$, and i cannot manipulate. If both $f\left(P\right)$ and $f\left(P'\right)$ are not singletons, then let $max(f\left(P\right), P_i) = x$ and $min(f\left(P'\right), P_i) = y$. Notice that for all $a \in f\left(P\right) \cup f\left(P'\right)$, $x R_i a R_i y$. Assume to the contrary that voter i can manipulate f at profile P reporting P_i'. Consider now $\lambda_{f(P)}$ and $\lambda_{f(P')}$ such that $\lambda_{f(P)}\left(x\right) = 1 - \epsilon$, and $\lambda_{f(P')}\left(y\right) = 1 - \epsilon$, for some $\epsilon < \frac{1}{2}$. For some u_i' fitting P_i it must be the case that:

$$\sum_{x \in f(P)} \lambda_{f(P)}(x) u_i'(x) < \sum_{x' \in f(P')} \lambda_{f(P')}(x') u_i'(x'). \tag{*}$$

Notice that if (*) holds, it must also hold for another utility function fitting P_i, $\hat{u}_i$, such that for all $a \in A$ $\hat{u}_i(a) = u_i'(a) - u_i'(y)$. Notice that, $\sum_{x \in f(P)} \lambda_{f(P)}\left(x\right) \hat{u}_i\left(x\right) > \left(1 - \epsilon\right) \hat{u}_i\left(x\right)$. On the other hand, $\sum_{x' \in f(P')} \lambda_{f(P')}\left(y'\right) \hat{u}_i\left(y'\right) < \epsilon u_i\left(x\right)$. But this leads to a contradictions since we have assumed that $\epsilon < \frac{1}{2}$. Therefore, our assumption was not true, and i cannot manipulate f at profile P. As the choice of i, P and P' was arbitrary, this suffices to show that f is strategy-proof.

■

Lemma 4.1 presents the main implications of DS strategy-proofness. A social choice correspondence is DS strategy-proof if it is non-manipulable when voters have maximin or maximax preferences over sets. By using Lemma 4.1, we can also describe the preferences orderings in the

DS domain. Voters only care about the worst the best alternatives in a set. Moreover, when two sets share the same best alternative and the same worst alternative, a voter with DS preferences must be indifferent between them.

For any $P \in \mathcal{P}^N$ and $X \subseteq A$, $top\,(X, P)$ denotes the minimal set of alternatives in X such that all voters prefer any alternative in $top\,(X, P)$ to the remaining ones. In the following lemma we will see that strategy-proofness and residual resoluteness imply both together unanimity in the range. When the voters agree in the best alternative in the range, the social choice consist in this single alternative.

Lemma 4.2. *If f satisfies strategy-proofness and residual resoluteness, then, for any $P \in \mathcal{P}^N$, $f\,(P) \subseteq top\,(r_f, P)$.*

Proof. Take $P \in \mathcal{P}^N$ such that for all $i, j \in \mathcal{N}$, $P_i = P_j$ and let $max\,(r_f, P_i) = x$. As $x \in r_f$, there is $P' \in \mathcal{P}^N$, such that $x \in f\,(P')$. Moreover $x \in f\,\left(P_1, P'_{-1}\right)$, since by strategy-proofness $max\left(f\left(P_1, P'_{-1}\right), P_1\right) R_1 max\,(f\,(P'), P_1) = x$. Repeating the argument, we know that $x \in f\,(P)$. Residual Resoluteness implies that $f\,(P)$ contains a single alternative. Hence, we have $f\,(P) = \{x\}$. A similar argument implies that for all $P^* \in \mathcal{P}^N$ such that for all $i, j \in \mathcal{N}$, $max\,(r_f, P_i^*) = max\left(r_f, P_j^*\right) = x$, then $f\,(P^*) = \{x\}$. Just notice that by strategy-proofness, $min(f(P_{-i}, P_i^*), P_i^*) R_i^* min\,(f(P), P_i^*) = x$, and we can repeat the argument as many times as necessary to get the result. Therefore,as the choice of x was arbitrary, we have seen that if f is strategy-proof and residual resolute, f is unanimous in its range.

In order to close the proof, select an arbitrary profile $P \in \mathcal{P}^N$. Take any $x \in top\,(r_f, P)$ and construct the profile P' in such a way that $x = max\,(A, P_i)$ for all $i \in \mathcal{N}$, while $P\,|_{A \setminus \{x\}} = P'\,|_{A \setminus \{x\}}$. By unanimity in the range we have that $f\,(P') = \{x\}$, and by strategy-proofness, we know that $min\left(f\left(P_1, P'_{-1}\right)\right) R_1 x$. Repeating the argument iteratively,

$$min\left(f\left(P_1, ..., P_i, P'_{i+1}, ...\right), P_i\right) R_i min\left(f\left(P_1, ..., P'_i, P'_{i+1}, ...\right), P_i\right).$$

This implies that for all $i \in \mathcal{N}$, $min\,(f\,(P), P_i) \subseteq X$, and hence $f\,(P) \subseteq top\,(r_f, P)$.

■

As an immediate aftermath of the previous lemma we can state that any alternative $x \in r_f$, also $x \in R_f$. ($r_f \subseteq R_f$.)

At this point, we have to introduce a new condition on the range of f, $N-m$ unanimity in the range. It is clearly stronger than unanimity in the range. Lemma 4.3 proves that it is incompatible with strategy-proofness.

$N-m$ Unanimity in the Range. Let $m \in \mathbb{N}$, $m < \frac{N}{2}$. A social choice correspondence is $N-m$ unanimous in the range iff for all $x \in r_f$, $P \in \mathcal{P}^N$, such that there is $J \subseteq \mathcal{N}$, with at least $N-m$ members and $max\left(r_f, P_j\right) = x$ for all $j \in J$, $f\left(P\right) = \{x\}$.

Notice that in order to $N-m$ unanimity in the range become meaningful, the restriction on $m < \frac{N}{2}$ has to be included. If not, there would be preference profiles for which two different groups of $N-m$ members would have power to impose their best alternatives, which is not possible.

Lemma 4.3. *If f is strategy-proof, f is not $N-1$ unanimous in the range.*

Proof. Assume to the contrary that there is a social choice correspondence f strategy-proof and $N-1$ unanimous. Now, we prove that if f is strategy-proof and $N-1$ unanimous, f is also $N-2$ unanimous. Assume it is not. Then, without any lost of generality, there exist $x, z \in A$ and $P \in \mathcal{P}^N$, s.t. for all $j = 1, ..., N-2$; $max(A, P_j) = x$; but $z \in f(P)$. Pick $\hat{P} \in \mathcal{P}^N_{\{x,z\}}$ and $y \in (A \backslash \{x, z\})$ such that $P \mid_{A \backslash \{x,y,z\}} = \hat{P} \mid_{A \backslash \{x,y,z\}}$, $max(A \backslash \{x\}, \hat{P}_i) = z$ for all $i \in (\mathcal{N} \backslash \{N-1, N-2\})$, $max(A, \hat{P}_j) = z$ for $j \in \{N-1, N\}$ while $max(A \backslash \{x, z\}, \hat{P}_k) = y$, for all $k \in \mathcal{N}$. By Lemma 2, $f(\hat{P}) \subseteq \{x, z\}$, and from the iterative application of strategy-proofness we have that $x \neq f(\hat{P})$. Thus $z \in f(\hat{P})$. Consider now, $P^* \in \mathcal{P}^N_{\{x,y,z\}}$ s.t. $P^* \mid_{A \backslash \{x,y,z\}} = \hat{P} \mid_{A \backslash \{x,y,z\}}$, $P^* \mid_{\{x,z\}} = \hat{P} \mid_{\{x,z\}}$, and for all $i \neq N$, $xP^*_i y$; while for all $j \neq N-1$, $yP^*_j z$. (That is $P^*_i = \langle x, y.z \rangle$ for all $i = 1, ..., N-2$, $P^*_{N-1} = \langle z, x, y \rangle$ and $P^*_N = \langle y, z, x \rangle$.) Construct now $P' \in \mathcal{P}^N$ by dropping z to the third position in every voter preference while maintaining the remaining position unaltered.(That is $P' \in \mathcal{P}^N_{\{x,z\}}$, $P' \mid_{\{x,z\}} = \hat{P} \mid_{\{x,z\}}$.) By $N-1$ unanimity $f(P') = \{x\}$. Construct analogously $P'' \in \mathcal{P}^N$ from $\hat{P}$ by dropping x to the third position and again by $N-1$ unanimity $f(P'') = \{y\}$. Finally, construct $P''' \in \mathcal{P}^N$ from $\hat{P}$ by dropping y to the third position. As $z \in f(\hat{P})$, also $z \in f(P''')$. Notice now that, from the iterated application of strategy-proofness, $f(P') = \{x\}$ implies that $y \notin f(P^*)$. Analogously, $f(P'') = \{y\}$ implies $z \notin f(P^*)$. Hence, as Lemma 4.2 implies $f(P^* \subseteq \{x, y, z\}$, necessarily $f(P^*) = \{x\}$. Nevertheless, if $f(P^*) = \{x\}$, strategy-proofness implies that $f(P''') = \{x\}$, which is a contradiction. Thus, if f is $N-1$ unanimous in the range and strategy-proof, f is $N-2$ unanimous in the range.

Repeating the arguments of the previous paragraph, we can prove that if f is $N-2$ unanimous in the range, f is also $N-3$ unanimous in the range, and so on. We can continue this process till we reach that f is $N-m'$ unanimous in the range for some $m' \geq \frac{N}{2}$, which is not possible. ■

Theorem 4.2. *Let $\#r_f \geq 3$. A social choice correspondence f satisfies strategy-proofness and residual resoluteness if and only if f is dictatorial.*

Proof. It is immediate to see that dictatorial rules satisfy residual resoluteness and strategy-proofness. So, we focus on necessity.

Let $N = 2$ and $A = r_f = \{x, y, z\}$. Following the arguments in [38], we present in Figure 4.1 the 6×6 grid containing the results of the social choice for the 36 admissible preference profiles. By Lemma 4.2, we know f is unanimous. Moreover, residual resoluteness implies that f is single-valued at several preference profiles. (We remark it by staring the boxes 1, 3, 8, 11, 13, 15, 22, 24, 26, 29, 34 and 36.)

Notice first that $z \notin f(P_i^1, P_j^3)$ since xP_j^3z and by strategy-proofness:

$$min\left(f\left(P_i^1, P_j^3\right), P_j^3\right) R_j^3 min\left(f\left(P_i^1, P_j^1\right), P_j^3\right) = \{x\}.$$

(We remark it by writing $-z$ at the right inferior corner of the box 3.) Analogously, x cannot be selected in boxes 18, 23, 24, 28, 33 and 34; y in 5, 11, 12, 25, 26, and 32 and z in 4, 9, 13, 14, and 19.

Notice now that residual resoluteness implies that $f(P_i^1, P_j^3)$ must be a singleton. We have two symmetrical possibilities, either $f(P_i^1, P_j^3) = \{x\}$ or $f(P_i^1, P_j^3) = \{y\}$. Assume $f(P_i^1, P_j^3) = \{y\}$. By strategy-proofness $min(f(P_i^1, P_j^4), P_j^4) R_j^4 min(f(P_i^1, P_j^3), P_j^4) = y$. This implies that $f(P_i^1, P_j^4) = \{y\}$. Notice now that $x \notin f(P_i^2, P_j^3)$, since xP_i^1y and by strategy-proofness $\{y\} = max(f(P_i^1, P_j^3), P_i^1) R_i^1 max(f(P_i^2, P_j^3), P_i^1)$. Moreover, as $z \notin f(P_i^2, P_j^3)$, $f(P_i^2, P_j^3) = \{y\}$ and also $f(P_i^2, P_j^4) = \{y\}$. In the same fashion, we can see that, $z \notin f(P_i^5, P_j^4)$. Therefore, it follows that for all $P' \in \mathcal{P}^2$ such that $max(A, P_j') = y$, $f(P') = \{y\}$.

$i\backslash j$	$\langle x, y, z\rangle$	$\langle x, z, y\rangle$	$\langle y, x, z\rangle$	$\langle y, z, x\rangle$	$\langle z, x, y\rangle$	$\langle z, y, x\rangle$
$P^1 = \langle x, y, z\rangle$	1 $\star$ x	2 x	3 $\star$ y $-z$	4 y $-z$	5 z $-y$	6 z
$P^2 = \langle x, z, y\rangle$	7 x	8 $\star$ x	9 y $-z$	10 y	11 $\star$ z $-y$	12 z $-y$
$P^3 = \langle y, x, z\rangle$	13 $\star$ x $-z$	14 x $-z$	15 $\star$ y	16 y	17 z	18 z $-x$
$P^4 = \langle y, z, x\rangle$	19 x $-z$	20 x	21 y	22 $\star$ y	23 z $-x$	24 $\star$ z $-x$
$P^5 = \langle z, x, y\rangle$	25 $\star$ x $-y$	26 x $-y$	27 y	28 $\star$ y $-x$	29 z	30 z
$P^6 = \langle z, y, x\rangle$	31 x $-y$	32 x	33 $\star$ y $-x$	35 y $-x$	35 z	36 $\star$ z

Figure 4.1: Theorem 4.2: The 2 voters and 3 alternatives case.

The same arguments apply to prove that, $x \notin f(P_i^2, P_j^6)$, since $y = min(f(P_i^2, P_j^4), P_j^6)P_j^6 x$ and $f(P_i^2, P_j^6) = \{z\}$. Following step by step the arguments in the previous paragraph, we see that for all $P'' \in P^2$ such that $max(A, P_j'') = z$, $f(P'') = \{z\}$. Similarly, we can prove that for all $P''' \in P^2$ such that $max(A, P_j''') = x$, $f(P''') = \{x\}$. This shows that if $f(P_i^1, P_j^3) = \{y\}$, j is a dictator. If we had assumed $f(P_i^1, P_j^3) = \{x\}$, we would have obtained that voter i is the dictator.

Hence, we have proved that if there are only two voters and three alternatives, any social choice correspondence satisfying strategy-proofness and residual resoluteness is a dictatorship. We have to extend the result to arbitrary societies. The induction argument on the number of alternatives in the range is standard and then it is omitted.[6] We present now, the induction to arbitrary finite sets of voters.

Induction Basis. There is $m \geq 2$, such that for all $N' \leq m$, if f satisfies strategy-proofness and residual resoluteness then f is dictatorial. Now, we prove our theorem for $N = m + 1$.

Let f be a $m + 1$ voters social choice correspondence satisfying strategy-proofness and ontoness. Construct an auxiliary social choice correspondence $h_1 : \mathcal{P}^2 \longrightarrow \mathcal{A}$, in such a way that $h_1(P_1, P_2) = f(P_1, P_2, ..., P_2)$. It is clear that h_1 satisfies residual resoluteness, since f also does, and its range contains more than 2 alternatives by unanimity in the range. Moreover, 1 cannot manipulate h_1 since f is strategy-proof. Moreover, 2 cannot manipulate, given that, for all $P_1, P_2, P_2' \in \mathcal{P}$, it holds:

$$max\left(f\left(P_1, P_2, ..., P_2\right), P_2\right) R_2 max\left(f\left(P_1, P_2', P_2, ...\right), P_2\right) R_2 ... max\left(f\left(P_1, P_2', ..., P_2'\right), P_2\right),$$
$$min\left(f\left(P_1, P_2, ..., P_2\right), P_2\right) R_2 min\left(f\left(P_1, P_2', P_2, ...\right), P_2\right) R_2 ... min\left(f\left(P_1, P_2', ..., P_2'\right), P_2\right).$$

Thus, we get that for all $P, P' \in \mathcal{P}^2$, on the one hand $max\left(h_1\left(P\right), P_2\right) R_2 max\left(h_1\left(P'\right), P_2\right)$, and also $min\left(h_1\left(P\right), P_2\right) R_2 min\left(h_1\left(P'\right), P_2\right)$, and h_1 is strategy-proof. This is enough to see that h_1 satisfies the induction hypotheses, and therefore either 1 or 2 are dictators for h_1. Let us check that if 1 is a dictator for h_1, she is also for f. Just pick P_1 and P_2' such that $max\left(r_f, P_1\right) = min\left(r_f, P_2'\right)$. For all $\left(P_1, P_{-1}\right) \in \mathcal{P}^N$, we get $f\left(P_1, P_{-1}\right) = max\left(r_f, P_1\right)$, since

[6] See, for instance, [38].

$h_1(P_1,P_2) = max\,(r_f,P_1)$ and by strategy-proofness:

$$max\left(f\left(P_1,P_2',...,P_2'\right),P_2'\right)R_2'max\left(f\left(P_1,P_2,P_2',...\right),P_2'\right)R_2'...max\left(f\left(P_1,P_2,...,P_n\right),P_2'\right),$$
$$min\left(f\left(P_1,P_2',...,P_2'\right),P_2'\right)R_2'min\left(f\left(P_1,P_2,P_2',...\right),P_2'\right)R_2'...min\left(f\left(P_1,P_2,...,P_n\right),P_2'\right).$$

Analogously, we can construct N ancillary social choice correspondences $h_i\left(P_i,P_{i+1}\right) = f\left(P_{i+1},...,P_i,P_{i+1},...,P_{i+1}\right)$, for all $i = 1,...,N$. Notice that it must be the case that either there is $i^* \in \mathcal{N}$ such that i^* is a dictator for h_{i^*}; or whenever $N-1$ individuals agree in their preferences, f chooses their most preferred alternative. This later possibility implies that f is $N-1$ unanimous in the range which is not possible, since it would contradict Lemma 4.3. Therefore, the first case must hold and the arguments in the previous paragraph suffice to show that f is a dictatorship.

■

Our proof of DS's main Theorem highlights the role played by residual resoluteness in the result. In fact, we can see that in small societies, it is enough to assume that a singleton is selected at a non unanimous profile to obtain the dictatorial result. Residual resoluteness condition goes even further since it compels single-valued choice at many preference profiles. (In fact, to the half of the possible preference profiles if there are only two voters and three candidates.) From the result for small societies, it is not difficult to find the induction argument leading to the general impossibility theorem. So, we can say that residual resoluteness is "too much" resoluteness. If we want to test the robustness of the Gibbard-Satterthwaite theorem in multivalued environments, we should not include residual resoluteness assumption in the analysis.

In the following theorem, we provide two necessary conditions for strategy-proof and onto correspondences. The first one is already stated in DS while the second condition is new.[7] Basically, item i) says there is a group of voters (oligarchs or vetoers) that always obtain their most preferred alternatives among the selected ones. Moreover, item ii) asserts that these oligarchs have indeed veto power. We will see that whenever they unanimously agree in a best alternative, this is uniquely elected. Thus, even when there exist non dictatorial strategy-proof

[7] Item i) is proved in [15].

and onto social choice correspondences, it is only at the cost of concentrating the power of decision in an arbitrary set of individuals. The weak notion of strategy-proofness presented by DS is not mild enough to avoid the negative consequences of the Gibbard-Satterthwaite Theorem.

Theorem 4.3. *Let f be a social choice correspondence satisfying strategy-proofness and ontoness, then there is a set of voters $S \subseteq \mathcal{N}$, such that for all $P \in \mathcal{P}^N$:*

i) *For all $x \in A$, if $max(A, P_i) = x$ for some $i \in S$, $x \in f(P)$.*

ii) $f(P) \subseteq top(A, P_S)$.

Proof. Notice first that the lemmata previous to the proof of Theorem 4.2 remain valid. Moreover, with the arguments included in the proof of Lemma 4.2, we can see that a strategy-proof and onto social choice correspondence is unanimous.

We follow the same strategy we have followed in the proof of Theorem 4.2. We can focus on small societies with only two voters and three alternatives. In this environment, it is not difficult to see that any strategy-proof and onto social choice correspondence is either dictatorial or both individuals can include their best alternatives in the chosen set. Notice that any single valued choice at a non unanimous profile implies directly that f is dictatorial. If singleton choice is only attained at unanimous profiles, it follows immediately that both voters must have veto power, and include always their best preferred alternatives in the chosen set. Furthermore, unanimity implies that when ever both voters agree in their top alternative, no other alternative is selected.

The standard induction argument on the number of alternatives applies. Hence, we only have to extend the result to groups of voters.

Induction Basis. There is $m \geq 2$, such that for all $N' \leq m$, if f satisfies strategy-proofness and ontoness, there is a set $S \subseteq \mathcal{N}$ such that:

i) For all $P \in \mathcal{P}^N$ and for all $x \in A$, if there is $i \in S$, with $max(A, P_i) = x$, then $x \in f(P)$.

ii) For all $P \in \mathcal{P}^N$, if there is $x \in A$ such that $max(A, P_i) = x$ for all $i \in S$, $f(P) = x$.

We now prove that this is also true for $N = m + 1$.

Take any pair of voters $i, j \in \mathcal{N}$. Define the $N-1$ social choice correspondence $g_{ij} : \mathcal{P}^{N-1} \to \mathcal{A}$ in such a way that for all $P \in \mathcal{P}^{N-1}$, $g_{ij}(P) = f(P_1, ..., P_i, P_i, ..., P_n)$. That is, g_{ij} is the restriction of f to the profiles in which individuals i and j report the same preferences. It is evident that g_{ij} satisfies ontoness from f's unanimity. Moreover, following the arguments in the induction step of Theorem 4.2, it is easy to check that g_{ij} is also strategy-proof. Hence, g_{ij} satisfies the conditions of the induction hypothesis and we know there is a group of individuals $S \subseteq \mathcal{N} \backslash \{j\}$ holding veto power over g_{ij}. They always include their best preferred alternatives in the social outcome. Moreover, whenever they agree on the best alternative, no other alternative selected. In order to obtain the conclusion for f we have to consider three cases:

Case a): i is not a vetoer for g_{ij} $(i \notin S)$.

Case b): i is a dictator for g_{ij} $(i = S)$.

Case c): i is a vetoer for g_{ij} $(i \subset S)$.

Case a). If $i \notin S$ for g_{ij} we can prove, that S is also an oligarchy for f. It is easy to see that all members of S always include their best alternatives in the chosen set. Take $P_{-j} \in P^{N-1}$, such that there is $x \in A$, and $k \in S$, with $x = max(A, P_k) = min(A, P_i)$. Then, $x \in f(P_1, ..., P_i, P_i, ..., P_N) = g_{ij}(P_{-j})$, and as f is strategy-proof, for any $P_i', P_j' \in \mathcal{P}$:

$$x = min\left(f\left(P_1, ..., P_i, P_i, ..., P_N\right), P_i\right) R_i min\left(f\left(P_1, ..., P_i', P_i, ..., P_N\right), P_i\right).$$

Repeating the reasoning as many times as necessary, we reach the desired conclusion. Finally, item ii) follows directly form the same arguments.

Case b). Assume now i is a dictator for g_{ij}. We will show that either i (or j) is a dictator for f, or i and j are the only vetoers for f. Fix an arbitrary restricted profile $P_{-\{i,j\}} \in \mathcal{P}^{N-2}$, and define the social choice correspondence $h : P^2 \longrightarrow \mathcal{A}$ in such a way that for all $P_{\{i,j\}} \in \mathcal{P}^2$, $h(P_i, P_j) = f\left(P_i, P_j, P_{-\{i,j\}}\right)$. As we know that i is a dictator for g_{ij}, h is unanimous and also onto. As f is strategy-proof, h also is. Hence, we have that h satisfies the induction conditions and then either i (or j) is a dictator, or i and j form an oligarchy for h. Therefore, when all individuals but i and j, report the preference profile $P_{-\{i,j\}}$, i and (or) j are an oligarchy for f. As the choice of $P_{-\{i,j\}}$ was arbitrary, we know that it also holds for any $P'_{-\{i,j\}} \in \mathcal{P}^{N-2}$.

It only remains to prove that if i is a dictator for $P_{-\{i,j\}}$, she is also for any$P^*_{-\{i,j\}}$. Assume *"ad contrarium"*, then there are an individual $k \notin \{i,j\}$ and profiles $P_{-\{i,j\}}$, $P^*_{-\{i,j\}} = \left(P^*_k, P_{-\{i,j,k\}}\right)$ such that i is a vetoer for $f\left(P_i, P_j, P_{-\{i,j\}}\right)$, but i is not for $P^*_{-\{i,j\}}$. Take P_i and P_j such that $max\left(A, P_i\right) = min\left(A, P_k\right) \neq max\left(A, P_j\right)$. As i is a vetoer for h at $P_{-\{i,j\}}$, $max\left(A, P_i\right) \in f\left(P\right)$, but it does not belong to $f\left(P_{-k}, P^*_k\right)$, this implies that $min\left(f\left(P^*\right), P_k\right) P_k min\left(f\left(P\right), P_k\right)$, contradicting f's strategy-proofness. We can repeat the argument iteratively to obtain the desired result.

Case c). Let S be the oligarchy for g_{ij} and $S' = S\backslash\{i\}$. We will prove that either S (or $S' \cup \{j\}$) f are vetoers for f, or $S \cup \{j\}$ is an oligarchy for f.

An already well known argument applies to prove that for all $P \in \mathcal{P}^N$, for all $x \in A$, if $max\left(A, P_k\right) = x$ for some $k \in S'$, $x \in f\left(P\right)$.

We provide the proof for societies with three voters. The induction argument for larger societies follows the same steps and then it is omitted. So assume $N = \{1, 2, 3\}$, and define g_{12}, g_{13}, and g_{23}; by the induction hypotheses we know that either one of them is dictatorial or 1 and 3 are vetoers for g_{12}, 1 and 2 are vetoers for g_{13}, and 1 and 3 are vetoers for g_{23}. In the former case, the arguments in cases a) and b) apply to show that f is oligarchical. In the last case, since all voters are vetoers for some auxiliary correspondence g_{ij}, we know that all voters can include their most preferred alternatives in the chosen set, which proves item i). Item ii) follows directly from unanimity.

Finally, item ii) of the Theorem follows from minimal modifications from the proof of Lemma 4.2.

■

Many voting rules satisfy the requirements of Theorem 4.3. Among other relevant instances we can mention the Pareto Correspondence, which includes the set of all Pareto undominated alternatives. It can be defined in the following way, for any $P \in \mathcal{P}^N$ and any $X \in \mathcal{A}$, $Pareto\left(X, P\right) = \{x \in X, s.t.$ there is no $y \in X$, yP_ix for all $i \in \mathcal{N}\}$. In the case of the Pareto correspondence, all voters would have veto power over the social choice.

Theorem 4.3 proves that any strategy-proof and onto social choice correspondence must endow some individuals with veto power. However, as it is shown by the following example, Theorem 4.3 does not imply that the voters out of the oligarchy cannot affect the social choice outcome.

Example 4.1. *Let* $A = \{x, y, z\}$, $\mathcal{N} = \{1, 2, 3\}$, *and*

$$f(P) = \begin{cases} \cup_{i \in N} max(A, P_i) & \text{if } max(A, P_3) \in Pareto\left(A, P_{\{1,2\}}\right), \\ max(A, P_1) \cup max(A, P_2) & \text{otherwise.} \end{cases}$$

It is easy to check that f *is onto and strategy-proof. Notice that voter* 3 *is not a vetoer but her preferences are relevant for the social choice.*

4.4 A Characterization Result

Theorem 4.3 does not provide a characterization of the family of (DS) strategy-proof and onto social choice correspondences. Nevertheless, this result is quite useful, since definition of strategy-proofness proposed by DS is not very stringent. This fact implies that we can obtain other negative results derived from stronger extensions of preferences as corollaries to Theorem 4.3.

We introduce a new domain of preferences over sets, naturally related to DS's one, that we call Extended Duggan-Schwartz preferences (DS+). It adds the concept of "discerning individuals" proposed in [3] to the notion of maximin and maximax preferences. Basically, a discerning individual i with preferences over alternatives x, y and z, $P_i = \langle x, y, z \rangle$, either prefers the set $\{x, y, z\}$ to the set $\{x, z\}$, or is indifferent between them , or prefers $\{x, z\}$ to $\{x, y, z\}$. We state this formally.

Extended Duggan-Schwartz Preferences. For any $P_i \in \mathcal{P}$, $\mathcal{D}^+(P_i)$ denotes the set of preferences over sets consistent with the preference order over alternatives P_i in the Extended DS domain. For any $P_i \in \mathcal{P}$, we say that the preference order over sets $\succsim_i \in \mathcal{D}$ is DS+ consistent with P_i, $\succsim_i \in \mathcal{D}^+(P_i)$, if and only if for all $X, Y \in \mathcal{A}$ such that $X \succ_i Y$ one of the following cases holds:

i) $max(X, P_i) P_i max(Y, P_i)$.

ii) $min(X, P_i) P_i min(Y, P_i)$.

iii) $X = max(Y, P_i) \cup min(Y, P_i)$ or $Y = max(X, P_i) \cup min(Y, P_i)$.

The social choice correspondence f is DS+ manipulable if there are $i \in \mathcal{N}$, $P, P' = (P_{-i}, P_i') \in \mathcal{P}^N$, $\succsim_i \in \mathcal{D}^+(P_i)$, such that $f(P') \succ_i f(P)$. Conversely, f is DS+ strategy-proof if f is not DS+ manipulable.

It is clear that for any $X, Y \in \mathcal{A}$, $P_i \in \mathcal{P}$, if there exist $\succsim_i \in \mathcal{D}^{DS}(P_i)$ such that $X \succ_i Y$, there also is $\succsim_i' \in \mathcal{D}^+(P_i)$, with $X \succ_i' Y$, while the contrary is not true. Hence, DS+ strategy-proofness is more compelling than (DS) strategy-proofness.

In the following lemma we characterize the family of DS+ strategy-proof and onto social choice correspondences.

Bidictatorship. The social choice correspondence f is bidictatorial if there are $i, j \in \mathcal{N}$ such that for all $P \in \mathcal{P}^N$, $f(P) = max(r_f, P_i) \cup max(r_f, P_j)$.

Theorem 4.4. *A social choice function f satisfies DS+ strategy-proofness and ontoness if and only if f is either dictatorial or bidictatorial.*

Proof. It is immediate to see that dictatorial and bidictatorial rules are onto and DS+ strategy-proof. Now, we focus in the proof of necessity side.

Firstly, we have to remark that the results of Theorem 4.3 are valid for DS+ strategy-proofness since it is a stronger condition than (DS) strategy-proofness.

We start proving that $\#S \leq 2$. Assume the contrary, then there exist at least three vetoers $j, k, l \in S$. Take $P \in \mathcal{P}^N$, such that $P_j = \langle x, y, z \rangle$, and $P_i = \langle y, x, z \rangle$ for all $i \in N \backslash j$. By the arguments in Theorem 4.3, we know that $f(P) = \{x, y\}$. Pick, $P_k' = \langle y, z, x \rangle$, again by Theorem 4.3, we have that $\{x, y\} \subseteq f(P_{-k}, P_k') \subseteq \{x, y, z\}$. If $f(P_{-k}, P_k') = \{x, y, z\}$, then there is $\succsim_k \in \mathcal{D}^+(P_k')$, such that $f(P) = \{x, y\} \succ_k \{x, y, z\}$, contradicting DS+ strategy-proofness. Hence, $f(P_{-k}, P_k') = \{x, y\}$. Take now, $P_k'' = \langle z, y, x \rangle$, then $f(P_{-k}, P_k'') = \{x, y, z\}$, since by item i) of Theorem 3, all vetoers include their best preferred alternatives in the chosen set. This leads to a contradiction with DS+ strategy-proofness, since we can find $\succsim_k' \in \mathcal{D}^+(P_k')$, such that $f(P_{-k}, P_k'') \succ_k' f(P_{-k}, P_k')$. Hence, our assumption was incorrect and $\#S \leq 2$.

By item ii) of Theorem 4.3, it is immediate that whenever $\#S = 1$, f is dictatorial. Item ii) of Theorem 3 implies that for all $P \in \mathcal{P}^N$ such that $max(A, P_i) = max(A, P_j)$ for all $i, j \in S$, then $f(P) = \cup_{i \in S} max(A, P_i)$. Now, we will prove that the result holds for any preference profile. Without lost of generality let $S = \{1, 2\}$ and consider a profile $P \in \mathcal{P}^N$, such that $max\,(A, P_1) = x$, $max\,(A, P_2) = y$. Take now $P_1', P_2' \in \mathcal{P}$ such that y is raised in $1's$ preferences to the second position, while x is raised to the second position in voter 2's preferences. That is, $max\,(A\backslash\{x\}, P_1') = y$ and $P_1 \mid_{A\backslash\{y\}} = P_1' \mid_{A\backslash\{y\}}$ while $max\,(A\backslash\{y\}, P_2') = x$ and $P_2 \mid_{A\backslash\{x\}} = P_2' \mid_{A\backslash\{x\}}$. Item ii) of Theorem 4.3 implies that $f\,(P_S', P_{-S}) = \{x, y\}$. Notice that as $y \in f\,(P_1, (P_2', P_{-S})$, we also have that $min\,(f\,(P_1, P_2', P_{-S})\,, P_1) = y$, since by (DS+) strategy-proofness,

$$min\left(f\,\left(P_1, P_2', P_{-S}\right), P_1\right) R_1 min\left(f\,\left(P_S', P_{-S}\right), P_1\right) = y.$$

Moreover, $f\,(P_1, P_2', P_{-S}) = \{x, y\}$, since if there exists a non-empty set $B = f\,(P_1, P_2', P_{-S})\backslash\{x, y\}$, there is $\succsim_1 \in \mathcal{D}^+\,(P_1)$, such that $f\,(P_S', P_{-S}) \succ_1 f\,(P_1, P_2', P_{-S})$, contradicting again DS+ strategy-proofness.

Repeating the argument for individual 2, we obtain that $f\,(P) = \{x, y\} = max\,(A, P_1) \cup max\,(A, P_2)$. This concludes the proof, as the choice of x, y and P was arbitrary.

■

Some final remarks are in order. We have to note that the application of the preferences of the "discerning individuals" to triples of alternatives is enough to show that there cannot be more than two vetoers. Nevertheless, we need to use this rationale for larger sets to obtain the bidictatorship characterization. On the other hand, Theorem 4.4 shows that a slight reinforcement of DS's definition of strategy-proofness leads immediately to very negative results.

4.5 An Alternative Approach: Reporting Preferences over Sets

All through the previous analysis, although we have endowed the voters with preferences over sets of alternatives, it has been assumed that the social choice only depends on their preferences over singleton sets. In this section, we will follow a more general approach introduced in [6]. We consider now the possibility of devising voting rules that select sets of alternatives according to the voters' preferences over sets. With this new scope, the social choice rule would be more flexible in order to adapt voters' preferences, since the society could use the full information contained in the preferences over sets. This setting coincides with the original Gibbard-Satterthwaite framework because the voters express their preferences over the very outcome of the social choice. Nevertheless, the negative result does not apply since the preferences over sets of the voters are restricted to be DS consistent.

First, we define the domain of all admissible preferences over sets. We denote the set of DS consistent orderings over sets by $\mathcal{D}^{DS}$. An ordering over $\mathcal{A}$, $\succsim_i \in \mathcal{D}$, is DS consistent, $\succsim_i \in \mathcal{D}^{DS}$ if and only if there is $P_i \in \mathcal{P}$ such that $\succsim_i \in \mathcal{D}^{DS}(P_i)$. The discussion after Lemma 4.1 in Section 4.2 presents the most relevant features of the preferences in the domain $\mathcal{D}^{DS}$. Namely, the voters only care about the best and the worst alternative in each set. It is easy to see that for any $P_i \in \mathcal{P}$ and any $\succsim_i \in \mathcal{D}^{DS}(P_i)$, $max(A, P_i) = max(\mathcal{A}, \succsim_i)$ and $min(A, P_i) = min(\mathcal{A}, \succsim_i)$. In fact, we only need the following features of the DS preference domain:

-i)(Weak Dominance Property:) For any $\succsim_i \in \mathcal{D}^{DS}$, $x, y \in A$, $\{x\} \succ_i \{y\}$ implies $\{x\} \succ_i \{x, y\} \succ_i \{y\}$.

-ii): For any $X \in \mathcal{A}$, there is $\succsim_i \in \mathcal{D}^{DS}$ such that any subset $X' \subseteq X$ is preferred to any set $Y \subseteq (A \backslash X)$.

From now on, we denote by $\succsim$ an arbitrary profile of voters' preferences over sets of alternatives, $\succsim \in [D^{DS}]^N$ refers to a profile of voters' preferences over sets of alternatives.

Social Choice Functions (over Sets). A social choice function (over sets) φ is a mapping from admissible preference profiles to non empty subsets of alternatives. Formally:

$$\varphi : [\mathcal{D}^{DS}]^N \to \mathcal{A}.$$

A social choice function is more general than a social choice correspondence as defined in the previous sections. A social choice correspondence is a social choice function subject to a strong invariance requirement. A social choice correspondence must yield the same result for any two profiles of preferences for which the preferences over singleton sets coincide.

The general approach we present here offers an additional advantage. As voters report preferences over sets, strategy-proofness is trivially defined, since we are dealing with social choice functions. The reported preferences contain the necessary information to check the profitability of any possible misrepresentation of the preferences.

A social choice function is manipulable in the domain $\mathcal{D}^{DS}$ if there are $i \in \mathcal{N}$, $\succsim, \succsim' = (\succsim_{-i}, \succsim_i') \in [\mathcal{D}^{DS}]^N$ such that $\varphi(\succsim') \succ_i \varphi(\succsim)$. We say φ is strategy-proof iff φ is not manipulable.

Finally, a social choice function is unanimous if for all $X \in \mathcal{A}$ and all $\succsim \in [\mathcal{D}^{DS}]^N$such that $X = max(\mathcal{A}, \succsim_i)$, for all $i \in \mathcal{N}$, $\varphi(\succsim) = X$.

Before introducing the results, we present a piece of notation that will become crucial in the proof of our Theorem 4.5. Given $\succsim_{-i} \in [\mathcal{D}^{DS}]^{N \setminus \{i\}}$, we define the option set for individual i, $o_i(\succsim_{-i})$, as the set of outcomes available to i when the remaining voters report the preferences $\succsim_{-i}$, that is:

$$o_i(\succsim_{-i}) = \{X \in \mathcal{A} \text{ such that there is } \succsim_i \in \mathcal{D}^{DS}, \varphi(\succsim_{-i}, \succsim_i) = X\}.$$

Abusing notation, for any $X \in \mathcal{A}$, we denote by $o_i(\succsim_{-i}, X)$ the set of outcomes in X available for voter i, that is $o_i(\succsim_{-i}, X) = o_i(\succsim_{-i}) \cap X$.

Remark 4.1. *If φ is strategy-proof $\varphi(\succsim) \in \bigcap_{i \in \mathcal{N}} max(o_i(\succsim_{-i}), \succsim_i)$*

The analysis of the structure of the option sets of the voters will be key of our analysis. We follow the line proposed in [6]. So we focus on two individual societies and investigate the structure of the option sets of the voters. Finally, we can apply standard induction arguments to extend the results to arbitrary societies.

We begin our analysis by taking an arbitrary unanimous and strategy-proof social choice correspondence φ defined on the domain $\mathcal{D}^{DS}$.

Theorem 4.5. *Let φ be a strategy-proof social choice function defined on the domain $\mathcal{D}^{DS}$, then there is a set of voters $S \subseteq N$, such that for any $\succsim \in [\mathcal{D}^{DS}]^N$:*

i) *If $x = max(\succsim_i, \mathcal{A})$ for some $i \in S$, $x \in \varphi(\succsim)$.*

ii) *If $x = max(\succsim_i, \mathcal{A})$ for all $i \in S$, $x = \varphi(\succsim)$.*

Proof. As we have already noted, we focus on the case with only two voters. The proof proceeds through a series of claims. We will start analyzing the singleton sets that are available to each voter, that is we study the structure of the sets $o_i(\succsim_j, A)$. Notice first that by unanimity, it is clear that for any $\succsim_1 \in \mathcal{D}^{DS}$, $max(\mathcal{A}, \succsim_1) \in o_2(\succsim_1)$.

Claim 4.1. *For any $\succsim_1, \succsim_1' \in \mathcal{D}^{DS}$ such that $max(\succsim_1) = max(\succsim_1') = \{a_1\}$ $o_2(\succsim_1, A) = o_2(\succsim_1', A)$.*

Proof. Assume to the contrary there is an alternative $\{a\} \in o_2(\succsim_1, A) \backslash o_2(\succsim_1', A)$. Consider $\succsim_2 \in \mathcal{D}^{DS}$ such that $\{a\} \succ_2 \{a, a_1\} \succ_2 \{a_1\} \succ_2 X$ for all $X \in \mathcal{A} \backslash (\{a\}, \{a, a_1\}, \{a_1\})$. Then $\varphi(R) = \{a\}$ while as $\{a\} \notin o_2(\succsim_1)$, but also by unanimity $\{a_1\} \in o_2(\succsim_1)$, we have that $\varphi(\succsim_1', \succsim_2)$ is either $\{a_1\}$ or $\{a, a_1\}$. Notice that by the Weak Dominance Property, for any $\succsim_1^* \in D^{DS}$ with $max(\succsim_1^*, A) = \{a_1\}$, we have that $\{a_1\} \succ_1^* \{a\}$ and $\{a, a_1\} \succ_1^* \{a\}$. This implies a contradiction with strategy-proofness, since voter 1 with preferences $\succsim_1$ can manipulate reporting $\succsim_1'$.

■

From now on, for any $\succsim_1 \in \mathcal{D}^{DS}$, $\{a_1\}$ denotes the best preferred set according preferences $\succsim_1$; $\{a_1\} = max(\mathcal{A}, \succsim_1)$.

Claim 4.2. *For any $\succsim_1 \in \mathcal{D}^{DS}$ either $o_2(\succsim_1, \mathcal{A}_1) = \{a_1\}$ or $o_2(\succsim_1, \mathcal{A}_1) = \mathcal{A}_1$.*

Proof. Assume there are alternatives $a, a' \in A$ and $\succsim_1 \in \mathcal{D}^{DS}$ such that $a \in o_2(\succsim_1, A)$, $\{a'\} \notin o_2(\succsim_1, A)$. By the previous lemma, without loss of generality we can assume $\succsim_1$ is such that $\{a_1\} \succ_1 \{a_1, a'\} \succ_1 \{a'\} \succ_1 X$, for all $X \in \mathcal{A} \backslash (\{a_1\}, \{a_1, a'\}, \{a'\})$. Take now $\succsim_2 \in \mathcal{D}^{DS}$ such that $\{a'\} \succ_2 \{a, a'\} \succ_2 \{a\} \succ Y$ for all $Y \in \mathcal{A} \backslash (\{a\}, \{a, a'\}, \{a'\})$. By Remark 4.2, we know $\varphi(\succsim_1, \succsim_2)$ is either $\{a\}$ or $\{a, a'\}$. But this implies a contradiction. Since by unanimity $\{a'\} \in o_1(\succsim_2)$ and by the Weak Dominance Property $\{a'\} \succ_1 \{a, a'\}$ and also $\{a'\} \succ \{a\}$.

■

Now, we show that if the option set of an agent contains all single alternatives for some preference of the other agent, the same must hold for any other possible preference ordering. The whole set of alternatives is always included in her option set.

Claim 4.3. *If* $o_2(\succsim_1, A) = A$ *for some* $\succsim_1 \in \mathcal{D}^{DS}$*, then* $o_2(\succsim'_1, A) = A$ *for all* $\succsim'_1 \in \mathcal{D}^{DS}$.

Proof. Assume that there are $\succsim_1, \succsim'_1 \in \mathcal{D}^{DS}$ such that $o_2(\succsim_1, A) = \{a_1\}$ while $o_2(\succsim'_1, A) = A$. Take now $\succsim_2 \in \mathcal{D}^{DS}$ in such a way that $max(\succsim_2) = min(\succsim_1)$. This implies that $\varphi(\succsim'_1, \succsim_2) = min(\mathcal{A}, \succsim'_1)$, while $\varphi(\succsim_1, \succsim_2) \neq min(\mathcal{A}, \succsim'_1)$. Therefore, $\varphi(\succsim_1, \succsim_2) \succ'_1 \varphi(\succsim'_1, \succsim_2)$, contradicting again strategy-proofness.

■

Notice that if it holds that for any $\succsim_1 \in \mathcal{D}^{DS}$, $o_2(\succsim_1, A) = A$, then voter 2 is indeed a dictator since her maximal element in A is her maximal element in A. The same arguments apply to voter 1. Hence, either one of the voters is a dictator or the only singleton in the option set of one agent is the best alternative of the other agent. That is :

$$\text{For all } \succsim \in [\mathcal{D}^{DS}]^2, \; o_2(\succsim_1, A) = \{a_1\} \text{ and } o_1(\succsim_2, A) = \{a_2\}. \qquad (*)$$

In the following claims we investigate the possibility that φ is not dictatorial. So, we assume (*) holds. We will prove that any available set for voter 2 includes the best preferred alternative of voter 1.

Claim 4.4. *If * holds, for any* $\succsim_1 \in \mathcal{D}^{DS}$*if* $X \in o_2(\succsim_1)$*, then* $max(\succsim_1) \in X$.

Proof. Assume the contrary. Then, there are $\succsim_1 \in \mathcal{D}^{DS}$, $X \in \mathcal{A}$ such that $X \in o_2(\succsim_1)$ but $\{a_1\} \notin X$. Let $X^* \in \mathcal{A}$, be the set in $o_2(\succsim_1)$ with smaller cardinality. Notice that by (*) $\#X^* \geq 2$. Find now $\succsim_2 \in \mathcal{D}^{DS}$ such that any set containing alternatives in $(A \backslash X^*)$ is strictly preferred by any subset of X^* , and with $max(\succsim_2) = max(\succsim_1, X) = \{a\}$. Notice that, $X = max(\succsim_2, o_2(\succsim_1))$, and therefore $\varphi(\succsim_1, \succsim_2) = X$. Notice that, as $\succsim_1 \in \mathcal{D}^{DS}$ an ordering and from Lemma 1 we know that,

$$\{a\} = max(\succsim_1, X) \succ_1 \{max(\succsim_1, X), min(\succsim_1, X)\} \sim_1 X,$$

and by (*) $\{a\} \in o_1(\succsim_2)$ which contradicts $\varphi(\succsim_1, \succsim_2) = X$.

■

Again the same arguments can be applied to voter 1. Hence, if (*) holds, any element of $o_1(\succsim_2)$ contains the maximal element in $\mathcal{A}$ according to $\succsim_2$, and analogously, any element of

$o_2(\succsim_1)$ includes the best alternative according to $\succsim_1$. This implies that for any $\succsim \in [\mathcal{D}^{DS}]^2$, $\{max\,(\mathcal{A}, \succsim_1)\,, max(\mathcal{A}, \succsim_2)\} \subseteq \varphi(\succsim)$. Moreover, as we have focused in two voters societies, item ii) follows directly from unanimity. Hence, we have proved the Theorem for societies with only two voters.

The proof for larger societies uses an induction argument. It runs parallel to the induction step presented in Theorem 4.3 and it is omitted. ■

4.6 Related Literature

We want to conclude this work by relating our results with others existing in the literature apart of DS.

The interest on manipulable correspondences was initiated by the seminal contribution of Pattanaik [32] and followed by Gärdenfors [20], Kelly [26], Barberà [3] and [2] and Feldman [23] and [24]. All these papers present weak definitions of strategy-proofness but their results are obscured by a number of regularity conditions that lead to negative conclusions. (This conditions range from rationalizability in [32], [3], [24] and [26], to Condorcet consistency in [20] or strong version of positive responsiveness in [2].) We can highlight among them the work in [23] which shows that the Pareto correspondence is strategy-proof when voters are consistent with maximin or maximax preferences over sets.

Our article is also closely related to Feldman [25]. He considers that a social choice correspondences is strongly strategy-proof if and only if for all utility functions consistent with the true preferences of the voter, the expected utility of the voter is not higher reporting false preferences over alternatives when all alternatives in the chosen set are equally likely. He proves that only dictatorial or bidictatorial choice correspondences are unanimous and strongly strategy-proof. It is clear that (DS) strategy-proofness is weaker than strong strategy-proofness, since Feldman only requires the existence of a consistent utility function that makes profitable the misrepresentation of the preferences for even chance lotteries and not for any possible lottery. Moreover, the reader can check that strong strategy-proofness is equivalent to DS+ strategy-proofness when there are only three alternatives, but it is stronger when there are more. Therefore, his result is a direct corollary to our Theorem 4.3.

We have to mention the recent paper by Ching and Zhou [12]. They present a notion of strategy-proofness even stronger than Feldman's one, since they do not restrict the lotteries over the chosen sets to be even chance. In their context, that any strategy-proof rule is constant or dictatorial. As we rule out the possibility of constant social choice correspondences, we can say their Theorem 1 can "almost" be obtained from our Theorem 4.4.

In a different vein are the works of Campbell and Kelly [11], Benoit [8], and the already noted [6] . They propose a more general analysis as the one proposed in Section 5 and study rules that take the preferences over sets of the voters as inputs. The study in [6] generalizes the frameworks in [25] and [12] to social choice functions over sets. We have to note that the preference orderings we have employed in the proof of Theorem 4.5 are also admissible in their domains of preferences. Thus, their strategy-proofness condition is stronger than ours, and allows them to obtain a full characterization.

Finally we have to review the works in [11] and [8]. Both papers consider explicitly preferences over sets, but they focus their attention to domains of preferences over sets such that for each possible preference over singleton sets there is only one admissible preference on the power set of the alternatives. Therefore, in their domains, any social choice function is indeed a social choice correspondence. In [11] the domain of leximin preferences is studied. It is shown that only the union of the best alternatives of an arbitrary group of voters is strategy-proof and unanimous.[8] In [8] a more general environment is introduced since the preferences of the agents are not specified but for a reduced group of sets. Basically, the only restriction is that there exist admissible preferences such that the best set is a singleton, the second best set is a duple containing the best set, the third best set is the singleton contained in the second best set, and finally, the worst set is also a singleton. This vague description of the admissible preferences implies that Benoit's definition of strategy-proofness is weaker than DS's one, since any DS preference ordering satisfies Benoit's requirements. The main result in [8] is a reinforcement of

[8] An individual with leximin preferences compares sets of alternatives paying attention first to the worst alternative on each set and prefers the set with the best worst alternative. If they are the same, then she compares the second worst alternative and so on.

our Lemma 4.3, strategy-proofness and $N-1$ unanimity are incompatible even under severe restriction over the domain of admissible preferences over sets.

Although their results and ours are logically independent, we want to remark that the arguments following the Schmeidler and Sonnenschein's techniques that we have employed in the proofs of Theorem 4.2 and Theorem 4.3 apply without any modification to the settings in [11] and [8]. Moreover, when voters compare sets according to the leximin extension, it is easy to see that a unanimous and strategy-proof social choice correspondence is either dictatorial or bidictatorial, and standard extension arguments apply to extend the result to arbitrary societies and sets of alternatives. Nevertheless, in the framework of [8], we cannot extend the results to arbitrary numbers of alternatives since the restriction on the domain of preferences does not take into account the comparisons between many feasible sets.

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