Tesi doctoral
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Essays on Bargaining with Outside Options

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Chapter 1. Bargaining with Outside Options: an introduction.

1.1 Introduction

A bargaining problem is a situation in which agents have the possibility of concluding a mutually beneficial agreement while there is a conflict of interests about which agreement to conclude and no agreement may be imposed on any agent without her approval. Agreement may be delayed or never be reached. In real life, an impasse in negotiations is not the only route that may lead to a failure of agreement; they may terminate in disagreement when one of the bargainers unilaterally abandons the negotiation table to take up an opportunity elsewhere. This could be the case of an alternative job in a wage negotiation, a judicial intervention in a divorce proceeding, an alternative buyer/seller in a trade, etc. In all these situations the agent's decision to take up her outside option is not an accident, it is a strategic decision.

The game-theoretic literature on bargaining with outside options has addressed the important issue of the impact of such options on the outcome of the bilateral negotiations, by studying how outcomes change depending on who and when may opt out. Most of the initial models are extensions of Ariel Rubinstein's alternating offer bargaining game, and they specify at which points of the bargaining process a player is allowed to abandon her partner. It is a well established result that bargaining procedures may have a significant impact on the efficiency and uniqueness of the bargaining outcome. If a player is allowed to opt out after she has rejected an offer made by the other player, then an outside option affects the bargaining outcome only if the value of this option is larger than the equilibrium share in the game without the possibility to opt out. However, if a player can opt out each time her offer is rejected, then we a multiplicity of equilibria may exist, including some with significant delay.

Indeed, many questions about the impact of the outside options on negotiations remain open. Models of bargaining with outside options usually assume that the payoffs resulting from the outside options are independent of the actions taken by bargainers during the negotiation process. However, in many negotiation contexts, the outside option does depend on what the parties have done during the negotiation phase. One such context is that of negotiations in presence of a third party, an arbitrator. The possibility that bargainers may call in an arbitrator to solve the dispute may be considered from a game-theoretic point of view an external option that bargainers have. The value of this outside option is not exogenous, since arbitrators generally consider the views and actions of the bargainers in their arbitrated outcomes.

This thesis makes several contributions to the theory of bargaining with outside options, emphasizing situations in which outside options arise by the intervention of arbitrators. In the remainder of this chapter, I survey the literature and I summarize the results.

1.2 Opting out. The role of procedures.

Rubinstein's model of bargaining has been consecrated as the fundamental extensive form to study bargaining situations. It specifies a procedure of bargaining where the players take turns to make offers to each other until an agreement is secured. Although making offers and counteroffers lies at the heart of many real-life situations, it is also true that in most situations agents are not constrained, as Rubinstein assumed, to bargain until they reach an agreement, but they can freely quit whenever they wish so and take up an opportunity elsewhere.

An outside option is defined to be the best alternative that a player can command if she withdraws unilaterally from the bargaining process. Clearly, the bargainer decision to take up her outside option is a strategic decision. She can use her outside option to gain leverage by threatening to leave the negotiation. The first strategic models of bargaining with outside options show that the credibility of the opting out threat depends on the rules of the bargaining process that include matters such as who can opt out and when.

Outside options can be incorporated in Rubinstein model by modifying the extensive form. Shaked and Sutton (1984) and Binmore, Shaked and Sutton (1988) proposed a first modification that consisted in allowing, at each node of the game where a player has to respond to an offer, the additional alternative of withdrawing from the negotiation and enforcing the outside option. Consider the following simplified version of this game where only player 2 has an outside opportunity. The structure of the negotiation is the following: First player 1 proposes a division of the pie $x = (x_1, x_2)$ such that $x_1 + x_2 = 1$. Player 2 may accept this proposal, reject it and opt out, or reject and continue bargaining. If player 2 decides to opt out player 1 receives 0 and player 2 receives $b > 0$. If player 2 rejects and continues bargaining, play moves into the next period, when it is player 2's turn to make an offer that player 1 may accept or reject. In the event of rejection, another period elapses, and once again it is player 1´s turn to make an offer. If $0 < \delta < 1$ is the common discount factor, then the following results hold:

1. *If* $b \leqslant \frac{\delta}{1+\delta}$ *then the game has a unique subgame perfect equilibrium. That is, player* 1 *always proposes the agreement* $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$ *and accepts any proposal in which* $x_1 \geq \frac{\delta}{1+\delta}$, *and player* 2 *always proposes* $(\frac{\delta}{1+\delta}, \frac{1}{1+\delta})$ *and never opts out, and accepts any offer in which* $x_2 \geq \frac{\delta}{1+\delta}$.

2. *If* $b > \frac{\delta}{1+\delta}$, *then the game has a unique subgame perfect equilibrium*

in which player 1 *always proposes* $(1-b, b)$ *and accepts any proposal in which* $x_1 \geq \delta(1-b)$ *and player* 2 *always proposes* $(\delta(1-b), 1-\delta(1-b))$, *accepts any proposal such that* $x_2 \geq b$ *and rejects and opts out if* $x_2 < b$.

In the unique subgame perfect equilibrium (SPE), players reach an agreement at time 0. Although player 2 does not take up her outside option, its presence does influence the equilibrium partition of the pie; if the outside option of player 2 is less than or equal to the share she receives in the SPE of Rubinstein's model then the outside option has no influence on the SPE partition. On the other hand, if the outside option strictly exceeds her Rubinsteinian SPE's share, then her SPE share is equal to her outside option. This result is known as the *Outside Option Principle*.

As modeled above, a player cannot leave the bargaining table without first listening to an offer from her opponent, who therefore always has a last chance to save the situation. It was Shaked (1994) that recognized that the Outside Option Principle did not resist a minor change of the procedure. He showed that, if one of the players may opt out each time an offer is rejected, the strategic consequences are markedly different. Intuitively, a player then has the opportunity to make an offer with a threat that the offer is final. A simplified version of Shaked' model is presented by Osborne and Rubinstein (1990); player 2 may opt out only after player 1 rejects her offer, in which case player 2 gets b and player 1 gets nothing. The following results are obtained:

1. *If* $b < \frac{\delta^2}{1+\delta}$ *then the game has a unique subgame perfect equilibrium. That is, Player* 1 *always proposes the agreement* $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$ *and accepts any proposal in which* $x_1 \geq \frac{\delta}{1+\delta}$, *and player* 2 *always proposes* $(\frac{\delta}{1+\delta}, \frac{1}{1+\delta})$ *and never opts out, and accepts any offer in which* $x_2 \geq \frac{\delta}{1+\delta}$.

2. *If* $\frac{\delta^2}{1+\delta} \leqslant b < \delta^2$ *then there are many subgame perfect equilibria. In particular, for every* $\xi \in \left[1-\delta, 1-\frac{b}{\delta}\right]$ *there is a subgame perfect equilibrium that ends with immediate agreement on* $(\xi, 1-\xi)$. *In every*

subgame perfect equilibrium player 2's payoff is at least $\frac{\delta}{1+\delta}$.

3. *If* $\delta^2 \leqslant b < 1$ *there is a unique subgame perfect equilibrium, in which player* 1 *always proposes* $(1-\delta, \delta)$ *and accepts any offer and player* 2 *always proposes* $(0,1)$ *accepts any offer in which* $x_2 \geqslant \delta$*, and always opts out.*

If the outside option payoff's of player 2 is sufficiently large or sufficiently small the game has a unique equilibrium. However, there exists an intermediate range of outside options for which there exists multiple equilibria. In particular, there are equilibria in which the player $2'$ payoff exceeds the value of her outside option. This is so, because in these equilibria, the threat of opting out is credible, that is, the outside option of player 2 exceeds her continuation value in case she does not leave the game.

Ponsati and Sakovics (1998) showed that Shaked'assumption that only one of the players has the opportunity to take her outside option, incurs in a significant loss of generality. They consider the Rubinstein's alternating offer bargaining game and they add the possibility that both players leave the negotiation after a rejection, in which case they obtain a payoff of b_i $i = 1, 2$. Assuming that $b_1 + b_2 \leq 1$ and δ_i $i = 1, 2$ as the discount factor they get the following result:

1. If either $b_1 > \delta_1(1-b_2)$ *or* $b_2 > \delta_2(1-b_1)$ *there is a unique subgame perfect equilibrium outcome that consists in an immediate agreement at* $(1 - b_2, b_2)$.

2. If $b_i \leqslant \delta_i(1 - b_j)$ $i = 1, 2$ *the outcomes that can be supported by a subgame perfect equilibrium are either immediate efficient agreements that give Player* 1 *a payoff in* $[1 - \delta_2(1 - b_1), 1 - b_2]$ *or, for any period* $t > 0$*, any agreement that gives Player* 1 *a share in*

$$\left[(1 - \delta_2(1 - b_1))\delta_1^{-t}, 1 - (1 - \delta_1(1 - b_2))\delta_2^{-t}\right].$$

Contrary to Shaked' result, they find that, even if both players get zero from opting out ($b_i = 0$ for $i = 1, 2$), there is a continuum of subgame perfect equilibrium outcomes. And, as usual in this type of models, the existence of multiple equilibria allows for some of significant delay. The delayed agreements are supported by strategies where, up to the equilibrium date of agreement, only non-serious offers are made.

Finally, Mariotti and Manzini (2001) present a model where the decision to opt out must be reached by consensus; in other words, either player can veto the decision of her opponent to opt out. Their outside option is an arbitrator who is called by both players to settle the dispute. The structure of the proposed game is the following; first player i, $i = 1, 2$, proposes a partition of the pie which player j can either accept or reject. If player j rejects, she can either follow with a counteroffer in the subsequent round, or propose to opt out. The opting out payoffs are $(b_1 - \varepsilon, b_2 - \varepsilon)$ with $\varepsilon \leqslant \min[b_1, b_2]$ and $b_1 + b_2 = 1$. If opting out is proposed, player i has to decide whether to accept, in which case the game ends with the players receiving the opting out payoff, or to reject and let player i again propose a partition of the surplus in the following round.

Their results show that, even if a consensus is needed, the possibility of opting out drives the outcome of the negotiations:

1. $\forall(b_1, b_2) \in [0,1] \times [0,1]$, $\forall\delta \in (0,1)$, *and* $\forall\varepsilon \leq \frac{(1-\delta)\min[b_1,b_2]}{1+\delta}$ *there is a subgame perfect equilibrium in which agreement is reached immediately on the partition* $(b_1 + \varepsilon, b_2 - \varepsilon)$.

2. If $b_i - \varepsilon > \frac{\delta}{1+\delta}$ $\forall i$, *then* $\forall\delta \in (0,1)$,*and* $\forall\varepsilon \leq \frac{(1-\delta)\min[b_1,b_2]}{1+\delta}$ *the unique subgame perfect equilibrium payoff is* $(b_1 + \varepsilon, b_2 - \varepsilon)$.

3. If $b_i - \varepsilon \leqslant \frac{\delta}{1+\delta}$ *for some* i, *there is a subgame perfect equilibrium in which agreement is reached immediately on the partition* $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$.

4. If $b_i - \varepsilon \leqslant \frac{\delta}{1+\delta}$ *for some* i *and* $\varepsilon > \frac{(1-\delta)\min[b_1,b_2]}{1+\delta}$ *then the unique subgame perfect equilibrium payoff is* $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$.

If ε is sufficiently small, there exists a subgame perfect equilibrium where the negotiated agreement is reached immediately on the partition

$(b_1 + \varepsilon, b_2 - \varepsilon)$ and this equilibrium is unique when the opting out partition is not exceedingly favorable to one of the bargainers. When the opting out partition is particularly favorable to one of the two players, the standard Rubinstein outcome can also be supported in equilibrium. And, as in the previous models, the presence of two extreme equilibria supports a a continuum of equilibrium partitions, and allows for the possibility of delays.

1.3 The nature of the outside option.

All models presented in the previous section assume that the payoffs resulting from the outside options are exogenous to the bargaining process and well known. In many negotiation contexts though, the value of outside option does depend on the outcomes of other negotiations or on what the players have done during the negotiation phase or are uncertain.

When the outside option is another negotiation.

The central common idea underlying in these models is that, agents of opposite types meet in pairs, via a matching process, and then the matched pair engages in bilateral bargaining, according to some particular procedure, over the terms of the trade. A main objective is to explore the conditions under which the equilibrium outcome of the game approximates the Walrasian outcome of the market. The great variety of models present in the literature differ in their treatment of several key issues. First, there is the information structure, that is, the specification of what does a player know about the events in other bargaining sessions. Second, there is the modeling of the search technology through which the bargainers get matched. And finally, the detailed structure of the pairwise bargaining games. I will not do an exhaustive examination of the existing models (see Osborne and Rubinstein (1990) for an excellent survey of this literature). For illustrative purposes I will limit my discussion to Bester (1988) and Muttoo (1993).

Bester (1988) presents a bargaining approach to equilibrium price dispersions. It considers a one-commodity market in which there is a cost for the consumer to switching from one store to another. There are two types of agents in the model, producers and consumers. The producers' type are represented by a characteristic $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ and the quality of the producer θ' supply is represented by a function $q(\theta)$. Each consumer only knows the distribution of the producers' characteristics $F(\theta)$. She desires to purchase a single unit of the good and chooses at random one of the sellers who are in the market. The central assumption is that the price at each store, $p(\theta)$, is determined in an equilibrium of a bargaining game between the producer and the consumer. The bargaining procedure is described by a modified version of the Rubinstein' model. First one of the two parties is selected randomly to propose a price at which it is willing to exchange the good. With probability $1 - \lambda$ the first proposal is made by the producer and the consumer may accept or reject. If she rejects she can make a counteroffer or she can quit the actual producer and start negotiations with another producer, but at some cost. Entering the market or switching from one producer to another requires Δ time units. The consumer's expected gain from entering the market at t is denoted by $\delta^t v$. The consumer's ability to quit bargaining so as to search for another producer is incorporated into this game as an outside option. The value of this option depends upon the expected bargaining outcome in all other stores. The market equilibrium simultaneously determines the set of producers who operate in the market, the price at each store, and the consumers'expected utility level v. The set of producers who operate in equilibrium depends upon the value of the consumers' outside option v. In equilibrium, the set of stores which are active in the market consists of all producers θ for whom $q(\theta) \geq v$. It is described by $h(\theta, v) = 1$ if $q(\theta) \geq v$ and $h(\theta, v) = 0$ otherwise. Denote as Γ_p the bargaining game at which the first proposal is made by the producer and Γ_c the one at which the first proposal is made by the consumer. The following results hold:

1. If $q(\theta) \geq v$, then, the equilibrium prices of the bargaining games Γ_p and Γ_c are unique and are given by $p_p^*(\theta) = \min\left[\frac{q(\theta)}{1+\delta}, q(\theta) - v\right]$ and $p_c^*(\theta) = \min\left[\frac{\delta q(\theta)}{1+\delta}, \delta(q(\theta) - v)\right]$ respectively.

2. The consumer's expected utility of entering the market is given by $v = \frac{\int_H \delta^{\triangle}[q(\theta)-p(\theta)]h(\theta,v)dF(\theta)}{\int_H h(\theta,v)dF(\theta)}$ where $p(\theta)$ is the expected price for commodity θ, so that $p(\theta) = (1 - \lambda + \lambda\delta)\min\left[\frac{q(\theta)}{1+\delta}, q(\theta) - v\right]$.

An important feature of the equilibrium is that the consumer never actually quits bargaining. She makes use of her outside option at most to enforce a lower price. As long as $q(\theta) \geq v$ the consumer cannot improve upon the equilibrium outcome by searching for another store. Bester shows that, competition among producers is reduced if the consumer incurs high delay costs in breaking off negotiations with a producer. Due to this lack of competition, prices at different store do not reflect only differences in qualities but also a certain degree of monopoly power on the part of the producers. Differences between the sellers'types create price dispersions and the number of active producers increases with higher search costs. And the market equilibrium converges to the competitive equilibrium under perfect information when search costs become small.

Muttoo (1993) proposes a market game in which the bargaining procedure is a Rubinstein-type infinite horizon process, with the added feature that a matched agent can choose to opt out after rejecting her opponent's offer and rejoin the matching process. At any point in time a player will either have left the market after having executed a transaction, or be unmatched and thus, will be taking part in the random matching process, or be matched and thus, will be in a bargaining process. The market considered in this model opens with a finite number S of identical sellers and a finite number of identical buyers B, where $B > S \geq 1$. The rate at which a match occurs in $nm\lambda$ given that n sellers and m buyers are taking part in the random matching process and λ is exogenously given. When a match occurs a seller and a buyer are paired randomly, in such a way that each of the n sellers and each of the m buyers are picked with equal probability. After they are matched at t they begin a bargaining according to the following procedure; one of the players is selected randomly to be the proposer. The responder either accepts or rejects this first proposal. If she rejects she may counteroffer at $t + \Delta$ or abandon the proposer and rejoin the matching process at t. If a seller (resp.buyer) agrees at t on a price p he gets a payoff of pe^{-rt}(resp. $(1-p)e^{-rt}$) where r is the rate of time preference. The following result is obtained:

The market game has a unique subgame perfect equilibrium. In equilibrium, the ith matched seller and buyer (where $i = 1, 2....S$) reach agreement immediately on the price x_i or y_i according to whether it is a the buyer or the seller who is selected to be the proposer. The equilibrium prices $(x_i, y_i)_{i=1}^{S}$ are defined inductively by the following equations:

$$x_i = \max\{V_i^s, R_i^s\} \qquad 1\text{-}y_i = \max\{V_i^b, R_i^b\}$$

$$V_i^s = \lambda(B-i+1)(\tfrac{x_i+y_i}{2} + (S-i)V_{i+1}^s)r + (B-i+1)(S-i+1)\lambda$$

$$V_i^b = \lambda(S-i+1)(\tfrac{1-x_i+1-y_i}{2} + (B-i)V_{i+1}^b)r + (B-i+1)(S-i+1)\lambda$$

with $V_{S+1}^s = V_{S+1}^b = 0$ *and*

$$R_i^s = e^{-r\Delta}\left[q = i\sum^{S}\tfrac{x_q+y_q}{2}P_i^q\right] \qquad R_i^b = e^{-r\Delta}\left[q = i\sum^{S}\tfrac{1-x_q+1-y_q}{2}P_i^q\right]$$

where P_i^q is the probability that $q-i$ matches occur in Δ units of time given that $S-i$ sellers and $B-i$ buyers are taking part in the matching process.

For each $i = 1, ...S$, $V_i^s(resp.V_i^b)$ is the equilibrium value of the outside option to a seller (resp., buyer) who has just become part of the *ith* match. Muthoo analyzes this unique subgame perfect equilibrium and finds that, under frictionless conditions, the market is Walrasian if it is the case that a seller can credibly threaten a buyer that she can go though an infinite number of bargaining-relationships in a negligible amount of time. Moreover, if it takes the seller a non-negligible amount of time to go through an infinite number of bargaining-relationships, then, since time is valuable, a matched buyer can extract some positive surplus, thus generating non-Walrasian outcomes.

History-dependent outside options.

Compte and Jehiel (1997) analyze the effect of an outside option which value does depend on what players have done during the negotiation phase. In their model, two players negotiate on the partition of a pie of size one. Each player may either make in turn a concession on what has not been conceded yet or opt out. The game ends when either there is nothing left to be conceded or one of the players opts out. After opting out, players receive a payoff that depends on the total concessions received by each player. They denote as X_i^t the total concession that player i has received until t, the value of the outside option as $v_i^{out}(X_i, X_j)$ and the efficiency loss resulting from the option phase as $\gamma(X_1, X_2) = 1 - v_1^{out}(X_1, X_2) - v_2^{out}(X_1, X_2)$. Their main results are the following:

1. When it is party i's turn to move, in any subgame perfect equilibrium, either player i opts out or player i makes a concession no greater than $\gamma(X_1, X_2)/\lambda_j$ (where λ_j is a lower bound on $\frac{\partial v_j^{out}}{\partial X_j}$).

2. If $\overline{\gamma} < 1 - \delta^{\underline{n}-1}$ then in equilibrium, the first mover opts out right away, where $\overline{\gamma}$ is an upper bound on $\gamma(X_1, X_2)$ and $\underline{n}$ is the smallest integer greater than $i\min\frac{\lambda_i}{\overline{\gamma}}$.

Their main finding is that the presence of history-dependent outside options may force equilibrium concessions to be gradual, which in turn may be responsible for delays in reaching agreements. The size of these concessions depends on the extent to which conceding increases the other player's outside option and/or the efficiency of the outside option. However, delaying the agreement is costly. When the inefficiency associated with the outside option $\gamma(\bullet)$ is low, many rounds of negotiation are necessary to reach an agreement if no player is to opt out. If the efficiency loss which results from the delay is larger than the efficiency loss induced by the outside option, the players have incentives to opt out.

Uncertain outside options

Sutton (1986) analyzes a bargaining model where the decision of opting out is not always available. In a modified version of the Rubinstein model (1982), a player can choose to opt out only when a random event occurs with probability p. He shows that, in equilibrium, agreement is reached at t=0 and, if both options are small, both players strictly prefer to continue bargaining rather than take up their options, when they are available. However, if the options are sufficiently big, both options are worth taking, when available.

Wolinsky (1987) presents a model where players may search for outside opportunities during the bargaining process and this search is costly. The bargaining positions of the players are determined endogenously by the players' decisions concerning the intensities with which they search and the acceptance criteria they apply to alternatives. He finds that the outcome of the bargaining does not depend only on the parties' relative efficiency in interrupted search, but also on how aggressively each player credibly threatens to search in the event that the agreement is delayed.

Vislie (1988) adds uncertainty about the presence of a second potential bargainer to the model of Shaked and Sutton (1984) and derives the corresponding unique equilibrium. In his model, one seller of some indivisible object has the possibility to bargain with two types of buyers, which differ in their valuation of the object, and their likelihood for entering the market. The seller has the possibility of of abandoning the negotiation with one type of buyer and start negotiation with the other type with some probability. If the random event fails to occur, the seller is constrained to continue bargaining with the same type of buyer. The equilibrium determines not only the price that will be paid but also who the buyer will be. Vislie shows that if the ratio between the two reservation prices is sufficiently high, in equilibrium the seller prefers to wait for the second type of buyer to turn up.

Ponsati and Sakovics (1999) deal with the uncertainty about the size of the outside options. In their model the value of the outside options are random variables distributed according to some conditional distribution function that is common knowledge between the players. There are three important dates: $\underline{T}$ is the date at which the outside option is available, T^* is the date at which the uncertainty is revealed and the realization of the random outside options become common knowledge, and T^{**} is the date at which the options cease to be available if they have not been taken before it. They find that if the distribution of the outside option has a large variance then the players may find in their interest to delay agreement until their information about their outside opportunities improves. If the revelation date is too far in the future then the players will prefer to end the game immediately.

1.4 Plan of the the Tesis: Exploring New Models of Bargaining with Outside Options.

In the present thesis I study three models of bargaining with history-dependent and uncertain outside options. Each chapter is presented as a self-contained paper. The reader should have no problem alterning their order.

Chapter 2 and chapter 3 study the dynamics of bargaining when players have the possibility of calling an arbitrator in order to solve the dispute. Arbitration is modelled as a history-dependent outside option since the arbitrator generally considers the views of the players in order to make his arbitrated outcome. And chapter 4 explores the role of uncertain outside options in a War of Attrition.

Chapter 2, analyzes the effects of arbitration in negotiations when the use of this institution is voluntary. We consider a bargaining by concessions model where the parties have the possibility of calling an arbitrator with the consent of the other party. I show that introducing arbitration distorts the negotiated outcome. This distortion depends on the relative costs of implementation of the partition obtained by negotiating and the one obtained by arbitrating. If the arbitration cost is small relative to the cost of negotiation then the negotiated partition approximates the one proposed by the arbitrator, and in extreme cases arbitration is used in equilibrium. However players do not always choose the most efficient method to solve their dispute: sometimes they negotiate when it would be more efficient to use arbitration.

Chapter 3 studies the effects of different arbitration procedures on the bargaining outcome and its efficiency, in a bargaining model where players make non-increasing demands and an arbitrator is called if and only if negotiations are declared broken. Two arbitration procedures are analyzed: *the conventional arbitration* (CA) where the arbitrator is free to choose a settlement and *the final-offer arbitration* (FOA) where the arbitrator is constrained to pick one of the players' last offers. I show that, if players are sufficiently patient and the arbitrator follows a Final-Offer Arbitration procedure, the equilibrium negotiated outcome may involve some delay. But if he follows a Conventional Arbitration procedure, in equilibrium, players always use the arbitrator to solve the dispute.

Finally, chapter 4 discusses the role played by the outside options in negotiations when there is incomplete information about their existence. I examine a War of Attrition where players enjoy private information about their possibility of leaving the negotiation to take an outside option. There are two types of players: a weak type who has a valueless outside option-she always prefers conceding rather than opting out- and a strong type who has a valuable outside option that she prefers to take rather than conceding. The main message that emerges from the analysis of this game is that uncertainty about the possibility that the opponent opts out improves efficiency, since it increases the equilibrium probability of concession. More precisely, if the probability that the opponent is strong is relatively high, in equilibrium, the negotiation eventually ends with a sure concession. On the other extreme, if the likelihood of a weak opponent is high, strong types will eventually leave the negotiation and opt out with probability 1 leaving weak types to play from that time on the inefficient symmetric equilibrium of the classical War of Attrition. Even in this case, the probability of concession along the uncertainty phase of the equilibrium play increases.

Chapter 2. Concession Bargaining and Costly Arbitration

2.1 Introduction

Arbitration is an extended procedure of dispute resolution in which bargainers accept the decision of a neutral third party when direct negotiation has failed. It is used in divorce proceedings, the settlement of grievances in union-management contracts, the dissolution of partnerships, and international trade. It is sometimes included as a clause in contracts and sometimes it is imposed by law. The impact of this institution in the form, frequency, and outcome of negotiations has been treated in the industrial relations literature and its performance has been studied empirically.

From a game-theoretic point of view, it is natural to view arbitration as an external option that bargainers have during the negotiation process. The game-theoretic literature on bargaining with outside options[1] discusses how outcomes change depending on the value of the options, as well as on the identity and strategic role of players that may opt out. In this literature, external options are independent of the actions taken during the negotiation process, an unsatisfactory feature if one wants to address the role of arbitration as an outside option. In fact, the empirical studies display strong evidence that the history of negotiations prior to the intervention of arbitrators crucially affects the arbitrated outcomes.[2]

[1] See Shaked and Sutton (1984), Binmore et al. (1989), Shaked(1994) and Ponsati and Sakovics (1998).

[2] See Ashenfelter and Bloom (1984), Farber and Bazerman, (1986).

Compte and Jehiel (1997) and Mariotti and Manzini (2001) precede us in exploring bargaining models with the possibility of arbitration. Compte and Jehiel (1997) analyze the effects of an arbitrator modelled as an outside option which value does depend on the history of the negotiation process, assuming that either party can unilaterally impose the use of arbitration. The later assumption might be justified in situations where prior to beginning negotiations parties commit to allow each other this possibility. In many cases, however, parties negotiate without this prior commitment. The decision of using an arbitrator is then voluntary and the agreement of both parties is necessary before the arbitrator is called.[3] Mariotti and Manzini (2001) point out that the voluntary nature of arbitration has been overlooked in the literature on arbitration, a literature largely influenced by the U.S. institutional setting where arbitration is often compulsory. In Mariotti and Manzini (2001) the decision to call an arbitrator must be reached by consensus; the decisions of their arbitrators, however, are fixed and history independent, and their cost are time invariant.

In this chapter, we consider a model with the following distinct features:

1. Proposals take the form of concessions, that cannot be claimed back, and each round of concession takes an interval of time. Players enjoy what has been conceded to them only when the negotiation is over.

2. At any point of the process, a player can propose to call the arbitrator. If the opponent agrees then the arbitrated outcome is implemented. Otherwise the bargaining process continues. Thus, players have veto power over the opponent's decision to opt out.

[3] In U.K, the labour conflicts are resolved by an arbitrator only with the consent of both parties.

3. The arbitrator observes the sequence of concessions during the negotiation and assigns to each player the total concession granted by the opponent plus an equal share of the unlocated surplus.[4]

4. Arbitration is costly because it takes time. The span of time needed by the arbitrator to implement or decide the arbitrated outcome is measured as a portion of the time interval necessary between concessions.

One may think that, if a player has the right to veto arbitration, she will use this right when the arbitrated outcome is unfavorable, neutralizing the presence of the arbitrator. Contrary to this intuition, we will show that the presence of an arbitrator may have a strong influence on the bargaining outcome, even when the consent of both parties is required. The impact of arbitration depends on the relative costs of implementing outcomes under the two procedures. When negotiation is cheaper than arbitration, that is, when it takes longer for the arbitrator to do her job than for one party to concede, the classical Rubinstein partition prevails in equilibrium. When arbitration resolves matters substantially faster than a negotiation process, then the arbitrator is called at the beginning of the game. When arbitration is quicker than negotiation but not by a great deal, the arbitrator is not used. Still, the potential presence of the arbitrator affects the outcome and the negotiated shares approach the (1/2,1/2) allocation than the arbitrator imposes if she intervenes at the beginning of the game.

In Compte and Jehiel (1997) arbitration has a negative effect negotiations because they may force equilibrium concessions to be gradual, which in turn may delay agreements. Abandoning the assumption of a pure outside option in favor of one where the consent of both parties is needed changes this result dramatically. In our framework (apart from

[4]This is as in Compte and Jehiel (1997).

the one period loss inherent to the structure of the concession game) there are no delays. A different type of inefficiency appears; players may insist on negotiating when arbitration would be more efficient.

The remainder of the chapter is organized as follows. In the next section the model is presented. In section 3 we study the equilibria of this game. Conclusions are gathered in the last section.

2.2 The model

Two players, $i = 1, 2$, bargain to share one unit of surplus. The game takes place over time and players are risk neutral and impatient. Each period $t = 0, 1, 2....$ players may offer each other, in alternating order with player 1 moving first, mutual concessions or they may agree to resort to arbitration. Thus, at each t, and given the cumulative concessions in periods 0 to $t-1$, player i must either offer to concede a non-negative additional portion of the surplus or she can propose to call an arbitrator, a proposal that j must accept or reject. Unless i concedes all the contested surplus at t or an agreement to resort to arbitration arises, the game continues with j moving first at $t+1$.

Perpetual disagreement yields a zero payoff to both players. All other outcomes of the game are one of the following: $N(x, 1-x, t)$ a negotiated agreement in period t allocating x to 1 and $(1-x)$ to 2, $0 \leqslant x \leqslant 1$; or $A(x, y, t)$ an agreement in period t to use arbitration after 1 has conceded y and 2 has conceded x, $0 \leqslant x, y \leqslant 1$, and $x + y < 1$.

Under a negotiated agreement $N(x, 1-x, t)$ each player enjoys the accumulated concessions at the date of agreement, i.e. payoffs are

$$\left(x\delta^t, (1-x)\delta^t\right),$$

where $0 < \delta < 1$.

On the other hand, in arbitration players obtain the concessions received prior to arbitration plus an equal share of the contested surplus. Arbitration is costly, usually in the form of direct fees paid to the arbitrator. Since these fees generally increase in the time spent by the arbitrator, we assume that implementing the arbitrated termination takes some interval of real time, and we treat the costs of arbitration as delay costs. That is, while $1-\delta$, $\delta = \exp(-r)$, measures the cost imposed by a one period of delay in the negotiations, a share of the surplus obtained under an arbitrated outcome has a cost $1-\alpha$, $\alpha = \exp(-rh)$, where h is the real time interval of delay imposed by the arbitrator. Note that $\alpha \leqslant \delta$ if and only if $h \geqslant 1$, i.e. if and only if arbitration takes longer than one round of bargaining. Hence, payoffs upon an arbitrated termination $A(x, y, t)$ are

$$\left(\alpha\left(x + \frac{1-x-y}{2}\right)\delta^t, \alpha\left(y + \frac{1-x-y}{2}\right)\delta^t\right).$$

Following Compte and Jehiel (1995) we denote by C_i^t the amount conceded by i to j at its turn t, by x_i^t the cumulative concession from player j to player i prior to time t and X^t denote the amount of pie that has not yet been conceded $X^t = 1 - x_1^t - x_2^t$. A *bargaining state* is a triple (x_1, x_2, X) indicating the shares conceded to each player and the contested surplus. At period t and at the bargaining state (x_1, x_2, X), if it is player 1's turn, she may:

1. Concede the rest of the pie X terminating the game at $(x_1, x_2 + X, t)$.

2. Concede $C_1 < X$, leading to a continuation game at $t+1$ with a new bargaining state, $(x_1, x_2 + C_1, X - C_1)$.

3. Propose arbitration. If 2 accepts then the game terminates at $(x_1 + \frac{1-x_1-x_2}{2}, x_2 + \frac{1-x_1-x_2}{2}, t)$. If 2 rejects, then the game continues at $t+1$ with the same bargaining state (x_1, x_2, X).

Ours is a bargaining game of complete information and infinite horizon. In spite of the close relationship of the present game to the standard bargaining games of alternating proposals, there are important differences that it is worthwhile to clarify. A first and fundamental difference is that players cannot claim back what they concede. Consequently, after each positive concession, the set of continuation strategies available to the players changes because the set of possible proposed partitions is smaller. Moreover, since equilibria where the first mover concedes all can easily be ruled out, negotiated agreements take at least one period. Second, proposing arbitration is not equivalent to conceding one half of the contested surplus. Rejecting arbitration prompts a continuation game at $t + 1$ with the bargaining state unchanged, while a concession $C_i = \frac{X}{2}$ leads to continuation game at $t+1$ with a bargaining state with contested surplus $\frac{X}{2}$. Moreover, if arbitration is accepted, then the delay cost is different.

A *pure strategy* of player i specifies the action at each subgame: a concession or an arbitration proposal if i moves first at t, or a reply to the opponent's proposal of arbitration if she moves second. In general, strategies are extremely complex since actions (concessions and proposals of arbitration) at any subgame may depend arbitrarily on the entire history of actions up to that point, and the set of histories is large. However the bargaining state summarizes all information of a history that is relevant to a player's choice.[5] Consequently, we will constraint attention to *stationary strategies*, that is, those in which the actions at each subgame depend exclusively on the bargaining state, being constant with respect to the particular history prior to attaining that subgame. An *equilibrium* will be a profile of stationary strategies that constitute a subgame perfect equilibrium.

Before we proceed to characterize the equilibrium of our game it is useful to discuss the concession game when arbitration is not a possibility. In the absence of arbitration only one player moves at each period

[5]See Osborne and Rubinstein (90) pag 41

and she can either concede the rest of the pie, or make a partial concession $C_i^t \in (0, X^t)$. The equilibrium outcome is the standard Rubinstein partition, attained with one period of delay.

Proposition 1 *(Compte and Jehiel). In equilibrium, player* 1 *concedes* $\delta 1 + \delta$ *and player* 2 *concedes the rest,* $11 + \delta$, *in the following period.*

Proof. See Compte and Jehiel (1997).

Without arbitration players concede up to the point where the opponent, given that she is impatient, is willing to terminate the game by conceding what is left. Since payoffs are only realized upon agreement, players do not benefit from the concessions they receive until the game ends. Therefore a player that has been granted a concession becomes effectively more impatient, delay is more costly for her that than for an opponent that has still nothing assured. If the first concession is large enough the optimal response is to terminate by conceding the rest of the pie.

We now turn our attention to the effect that voluntary arbitration has in the preceding concession game.

2.3 Equilibria with Arbitration

Arbitration is a voluntary outside option. Consequently, analyzing the equilibrium behavior of players under voluntary arbitration parallels the analysis of a bargaining game with outside options. The crucial insight is that outside options are not always relevant, and this is likewise with arbitration. However, since arbitration is an option of endogenous value, varying at the different states of bargaining, its relevance is more delicate than that of fixed outside options.

At any bargaining state, the first mover controls the rate at which payoffs are discounted. If she concedes, payoffs are discounted by δ; if she proposes arbitration and the opponent accepts, payoffs are discounted by α. In the later case, her bargaining power is limited by the veto power of opponent. In equilibrium, arbitration is rejected if the responder expects that the payoff of continuing the negotiation is greater than the payoff from the arbitrated termination. At a given bargaining state *the arbitration option is active* for player i when the best response of the opponent is to accept it.

Strategies specify actions at each bargaining state (a concession or the proposal of arbitration and thresholds of acceptance of the opponent's proposal of arbitration). In order to characterize equilibria we identify sets of bargaining states for which players have the same optimal actions. For each set of bargaining states (x_1, x_2, X), optimal actions will be identified by sequential elimination of weakly dominated strategies.

Since proposing arbitration is dominated in states where the opponent optimally rejects it, the acceptance rules can be omitted and the characterization of equilibria requires only that we specify the optimal action of the first mover at each bargaining state. A nice feature of the present approach is that it is easily represented graphically. In Figure 1 bargaining states (x_1, x_2, X) are represented on a plane, where we place $X = 1 - x_1 - x_2$ and $\rho = x_2 - x_1$ respectively in the horizontal and vertical axis. A concession $0 < C_1 < X$ of player 1 to player 2, changing the bargaining state from (x_1, x_2, X) to $(x_1, x_2 + C_1, X - C_1)$, is a move upwards on the plane from point $a = (X, \rho)$ to point $b = (X', \rho') = (X - C_1, \rho + C_1)$. A concession $0 < C_2 < X$ of player 2 to player 1 is a move downwards from $a = (X, \rho)$ to $c = (X'', \rho'') = (X - C_2, \rho - C_2)$.dtbpFU212.625pt279.125pt0ptFigure 1Figure

The following terminology simplifies the exposition. We say that arbitration is *slow* when $(\alpha,\delta)\in S$, where

$$S=S_1\cup S_2,$$

$$S_1=\left\{(\alpha,\delta) such that \alpha\leqslant Max\left\{\delta,\frac{2\delta}{1+2\delta}\right\}\right\},$$

and

$$S_2=\left\{(\alpha,\delta) such that \delta<\alpha<Min\left\{2\delta^2,\frac{2\delta}{1+\delta}\right\}\right\}.$$

Arbitration is *quick* when $(\alpha,\delta)\in Q$, where

$$Q=Q_1\cup Q_2\cup Q_3,$$

$$Q_1=\left\{(\alpha,\delta) such that Max\left\{2\delta^2,1+2\delta-\sqrt{1+4\delta}\right\}\leqslant\alpha<\frac{2\delta}{1+\delta}\right\},$$

$$Q_2=\left\{(\alpha,\delta) such that \alpha\geqslant Max\left\{2\delta^2,\frac{2\delta}{1+\delta}\right\}\right\},$$

and

$$Q_3=\left\{(\alpha,\delta) such that \frac{2\delta}{1+\delta}<\alpha<2\delta^2\right\}.$$

In the intermediate case that $(\alpha,\delta)\in I$,

$$I=\left\{(\alpha,\delta) such that Max\left\{2\delta^2,\frac{2\delta}{1+2\delta}\right\}\leqslant\alpha<1+2\delta-\sqrt{1+4\delta}\right\}.$$

we refer to *medium speed.* The partition of the space of parameters into slow, speedy and medium speed arbitration is represented in Figure 2.

dtbpFU405pt275pt0ptFigure 2Figure

In Proposition 2 that follows we characterize equilibrium actions when arbitration is slow.

Proposition 2. *Under slow arbitration the optimal actions are as follows.*

(i) When $(\alpha, \delta) \in S_1$

state	*i*	*j*
$x_i \geqslant \delta(x_i + X) i = 1, 2$	X	X
$x_i \geqslant \delta(x_i + X) and x_j < \delta(x_j + X)$	X	0
$x_i < \delta(x_i + X) i = 1, 2, x_i \geqslant \frac{\delta}{1+\delta}$	X	0
$x_i, x_j < \frac{\delta}{1+\delta}$	$C_i^* = \delta 1 + \delta - x_j$	$C_j^* = \delta 1 + \delta - x_i$

(ii) When $(\alpha, \delta) \in S_2$ *and the state* (x_1, x_2, X) *satisfies* $\alpha(x_j + \frac{X}{2}) < \delta^2(x_j + X)$*:*

state	i	j
$x_i \geqslant \alpha(x_i + \frac{X}{2})$	X	0
$x_i \geqslant \frac{\delta}{1+\delta}$, $x_i < \alpha(x_i + \frac{X}{2})$	X	0
$x_i < \frac{\delta}{1+\delta} i = 1, 2, x_i \leqslant \frac{2\delta^2 - \alpha}{\alpha(+\delta)} and x_j > \frac{2\delta^2 - \alpha}{\alpha(1+\delta)}$	C_i^*	$(*)$
$x_i < \frac{\delta}{1+\delta} x_i > \frac{2\delta^2 - \alpha}{\alpha(1+\delta)} i = 1, 2$	$(*)$	$(*)$
$x_i < \frac{\delta}{1+\delta} and x_i \leqslant \frac{2\delta^2 - \alpha}{\alpha(1+\delta)} i = 1, 2$	C_i^*	C_j^*

$(*)$ concedes $C_i = Max\left[C_i^*, C_i^{\diamondsuit}\right]$, where $C_i^* = \frac{\delta}{1+\delta} - x_j$ and $x_j + C_i^{\diamondsuit} = \alpha(x_j + \frac{X + C_i^{\diamondsuit}}{2})$.

Otherwise, if the state (x_1, x_2, X) *is such that* $\alpha(x_i + \frac{X}{2}) \geqslant \delta^2(x_i + X)$ $i = 1, 2$ *the optimal actions are coincide with those stated in Proposition 3 for parameters* $(\alpha, \delta) \in I$*.*

Proof. We prove (i). The proof of (ii) is in the appendix.

We examine the optimality of the proposed actions at each subset of states.

STEP 1: *If* $x_i \geqslant \delta(x_i + X)$ *for* $i = 1, 2$, *both players concede* X*.*

Let 1 move first (and note that the symmetric argument holds when 2 moves first). She must take one of the following actions: a) concede X and receive a payoff of x_1; b) concede nothing obtaining at most $\delta(x_1 + X)$ in the continuation; c) concede $0 < C_1 < X$, obtaining at most $\delta(x_1+X-C_1)$; d) Propose arbitration, which pays at best (when 2 accepts) $\alpha(x_1+\frac{X}{2})$. Since $x_1 \geqslant \delta(x_1+X) > \delta(x_1+X-C_1)$ and $x_1 > \alpha(x_1+\frac{X}{2})$ implying that the optimal action is to concede X.[6] This last inequality is straight forward when $\alpha \leqslant \delta = Max\left\{\delta, \frac{2\delta}{1+2\delta}\right\}$. When $\alpha \leqslant \frac{2\delta}{1+2\delta} = Max\left\{\delta, \frac{2\delta}{1+2\delta}\right\}$, $x_1 \geqslant \delta(x_1+X)$ is equivalent to $x_1 - x_2 \leqslant 1 - \frac{1+\delta}{1-\delta}X$. But $1-\frac{1+\delta}{1-\delta}X < 1 - \frac{1}{1-\alpha}X$ since $\alpha 2 < \delta 1+\delta$. And $x_1 - x_2 < 1 - \frac{1}{1-\alpha}X$ is equivalent to $x_1 > \alpha(x_1+\frac{X}{2})$.

STEP 2: *If $x_2 \geqslant \delta(x_2+X)$ and $x_1 < \delta(x_1+X)$ player 2 concedes X and player 1 concedes nothing.*

Consider player 2. Since she faces the same situation as in Step 1 she will optimally concede X. Consider now the game from the perspective of player 1 : if she concedes nothing, player 2 will concede X in the following period, and 1 can obtain $\delta(x_1+X)$. This payoff is greater than what 1 would get if she concedes X since $\delta(x_1+X) > x_1$. Likewise $\delta(x_1+X) > \delta(x_1+X-C_1)$ implying that C_1, $0 < C_1 < X$ is dominated. To call the arbitrator, with the consent of player 2, is dominated as well since $\delta(x_1+X) > \alpha(x_1+\frac{X}{2})$. If $Max\left[\delta, \frac{2\delta}{1+2\delta}\right] = \delta \geqslant \alpha$ this inequality is straight forward. Assume now that $Max\left[\delta, \frac{2\delta}{1+2\delta}\right] = \frac{2\delta}{1+2\delta} \geqslant \alpha$. If $x_1 < \delta(x_1+X)$ or $x_2 - x_1 > 1 - \frac{1+\delta}{1-\delta}X \geqslant 1 - \frac{\delta}{\alpha-\delta}X$ since $\frac{\alpha}{2} < \frac{\delta}{1+\delta}$. But $x_2 - x_1 > 1 - \frac{\delta}{\alpha-\delta}X$ is equivalent to $\delta(x_1+X) > \alpha(x_1+\frac{X}{2})$.

Step 1 and 2 are displayed in Figure 3.dtbpFU304.25pt298.6875pt0ptFigure 3Figure

STEP 3: *If $x_i < \delta(x_i+X)$, $i=1,2$ and $x_2 \geqslant \frac{\delta}{1+\delta}$ player 2 concedes X and player 1 concedes nothing.*

A) We first consider (α,δ) such that $11+\delta \leqslant 2(1-\alpha)2-\alpha$.

Player 2 may a) concede X and obtain a payoff of x_2; b) concede $\widetilde{C}_2$ and obtain $\delta(x_2+X-\widetilde{C}_2)$, where $\widetilde{C}_2$ is the minimal concession such that

[6] If $x_1 = \delta(x_1+X)$ player 1 will be indifferent between conceding X and conceding nothing if the optimal strategy of his opponent will be to concede X at his turn. In this case we will assume that player 1 will concede X.

1 concedes all the contested surplus at $t+1$, i.e. $x_1+ \widetilde{C_2}= \delta(x_1+X)$ by Step 1; c) concede $C_2 >\widetilde{C_2}$, that yields a payoff of $\delta(x_2+X-C_2)$ since at the new bargaining state $x_1+C_2 > \delta(x_1+X)$ holds and, by Step 1, player 1 will optimally concede $X-C_2$; d) concede $0 \leqslant C_2 <\widetilde{C_2}$; and e) propose arbitration, that yields payoff of $\alpha(x_2+\frac{X}{2})$, provided that 1 accepts it. Note that a) dominates b) and c): concessions $\widetilde{C_2}$ or $C_2 >\widetilde{C_2}$ are dominated by concession X since $x_2 \geqslant \frac{\delta}{1+\delta} > \delta(x_2+X- \widetilde{C_2}$ $) > \delta(x_2+X-C_2)$. At the same time a) dominates arbitration as well provided that $11+\delta \leqslant 2(1-\alpha)2-\alpha$. The later inequality implies that if $x_2 < \delta(x_2+X)$, $x_1 < \delta(x_1+X)$ and $x_2 \geqslant \frac{\delta}{1+\delta}$ then $x_2 \geqslant \alpha(x_2+\frac{X}{2})$ since $(1-\alpha)x_2 \geqslant (1-\alpha)\frac{\delta}{1+\delta} \geqslant \frac{\alpha}{2(1+\delta)} \geqslant \frac{\alpha}{2}X$.

Thus it only remains to show that a) dominates d) as well. A concession $C_2 <\widetilde{C_2}$ leads to a new bargaining state that may lie in the subset of states described in Step 2 (where player 1 concedes nothing at her turn) that yield payoff $\delta^2 x_2$ that is dominated by a). Concession C_2 may be small enough that the subsequent bargaining state still lies in the set of bargaining states that we are presently examining. On the other hand, following $C_2 <\widetilde{C_2}$, 2 cannot expect from 1 a concession greater than $\widetilde{C_1}$, the concession that will makes player 2 ready to finish the game at her next turn. Therefore, conceding $C_2 <\widetilde{C_2}$ pays 2 at most $\delta^2(x_2+ \widetilde{C_1})$. If $x_2 \geqslant \delta^2(x_2+ \widetilde{C_1})$, or substituting $\widetilde{C_1}$, $x_2 \geqslant \delta^3(x_2+X)$, concession X dominates C_2, $0 \leqslant C_2 <\widetilde{C_2}$. If $x_2 \geqslant \delta^3(x_2+X)$ holds then a) dominates d).

If the optimal action for player 2 is to concede X, then, it is easy to check that $x_1 < \delta(x_1+X)$ implies that the optimal action of player 1 is to concede nothing.

Suppose that $x_2 < \delta^3(x_2+X)$ is not satisfied, a situation displayed in Figure 4. Consider a bargaining state such as point **d** in Figure 4 and examine the payoffs attained by concessions C_2, $0 \leqslant C_2 \leqslant X$ (arbitration is a dominated strategy). If $C_2 < X$ is such player 1 will concede nothing at her turn, player 2 obtains a final payoff of $\delta^2 x_2$. If C_2 is such that at the new bargaining state $x_2 < \delta(x_2 + X)$ $x_1 < \delta(x_1 + X)$, $x_2 \geqslant \frac{\delta}{1+\delta}$ and $x_2 < \delta^3(x_2 + X)$, the largest concession 2 can expect in response from player 1 is C_1', a concession that makes player 2 ready to concede the rest of the pie, that is, $x_2 + C_1' = \delta^3(x_2 + X)$. Then, if $x_2 \geqslant \delta^2(x_2 + C_1')$, or substituting C_1', $x_2 \geqslant \delta^4(x_2 + X)$, X dominates any $C_2 < X$.

dtbpFU354.625pt295.9375pt0ptFigure 4Figure

Again; $x_2 \geqslant \delta^4(x_2 + X)$ may or may not be satisfied. But it is strictly satisfied for some subset. For any δ we there is a natural number $n \geqslant 3$ such that $\delta^{n+1}(x_2 + X) < x_2 < \delta^n(x_2 + X)$. In bargaining states satisfying $x_2 \geqslant \delta(x_2 + X)$ $x_1 < \delta(x_1 + X)$ $x_2 \geqslant \frac{\delta}{1+\delta}$ and $\delta^n(x_2 + X) < x_2$, conceding X dominates any other action for player 2 . In bargaining states such that $x_2 < \delta(x_2 + X)$ $x_1 < \delta(x_1 + X)$ $x_2 \geqslant \frac{\delta}{1+\delta}$ and $x_2 < \delta^n(x_2 + X)$ the greatest concession player 2 can expect from player 1 is C_1 such that $x_2 + C_1 = \delta^n(x_1 + X)$. Therefore the maximal expected payoff of from a concession smaller than X, is $\delta^2(x_2 + C_1) = \delta^{2+n}(x_2 + X) < \delta^{n+1}(x_2 + X)$, and this completes Step 3 A).

B) Let $11 + \delta > 2(1 - \alpha)2 - \alpha$.

Arbitration is no longer dominated by conceding X. Nevertheless there is a subset of states for which $x_2 > \alpha(x_2 + \frac{X}{2})$; the equilibrium actions in this subset are X for player 2 and 0 for player 1.

dtbpFU354.625pt295.9375pt0ptFigure 5Figure

Let us now consider states such that $x_2 < \alpha(x_2 + \frac{X}{2})$. Player 2 prefers arbitration rather than conceding X. However the acceptance of player 1 is needed. It is easy to check $\delta^2(x_1 + X) > \alpha(x_1 + X2)$[7]. Rejecting arbitration, player 1 can ensure a payoff of $\delta^2(x_1 + X - C_1^\diamond)$ since she may follow rejection with a concession $C_1^\diamond$ such that $x_2 + C_1^\diamond =$

[7]In this set of bargaining states it is satisfied $x_2 - x_1 \geq \frac{2\delta - \alpha(1+\delta)}{\alpha(1+\delta)}$. But $\frac{2\delta - \alpha(1+\delta)}{\alpha(1+\delta)} > 1 - \frac{\delta^2}{\alpha - \delta^2} X$ since $\alpha \leq \delta$. Then $x_2 - x_1 > 1 - \frac{\delta^2}{\alpha - \delta^2} X$, that is equivalent to $\alpha(x_1 + \frac{X}{2}) \leq \delta^2(x_1 + X)$.

$\alpha\,(x_2 + X + C_1^{\diamond}2)$ and at the new bargaining state 2 will concede the rest of the pie in the next period. If $\delta^2(x_1 + X - C_1^{\diamond}) > \alpha(x_1 + X2)$, then, player 1 will optimally reject the arbitration proposal. For the bargaining states sufficiently near to the set of bargaining states such that player 2 concedes X this condition is satisfied since $C_1^{\diamond}$ is very small (as bargaining state s in Figure 5). To propose arbitration is then a dominated option for player 2.

To complete Step 3 B), we repeat the arguments in Step 3 A).

STEP 4: *If* $x_i < \frac{\delta}{1+\delta}$ $i = 1, 2$, *player* i *concedes* C_i^* *such that* $x_1 + C_i^* = \frac{\delta}{1+\delta}$.

At these bargaining states player i can make a concession such ensuring her a payoff $\delta 1 + \delta$. This concession is C_i^* such that $x_i + C_i^* = \frac{\delta}{1+\delta}$. This concession is such that j responds conceding the rest of the pie (by Step 3). To concede more than C_i^* is clearly dominated. To concede less is also dominated; in that case, at the new bargaining state, player i cannot expect to receive from player j more than C_j^*, such that $x_i + C_j^* = \delta 1 + \delta$. By conceding less than C_i^*, player 1 can get, at most, $\delta^3 1 + \delta < \frac{\delta}{1+\delta}$. Arbitration is dominated as well; since it pays (if the opponent accepts) $\alpha\,(x_1 + X2) < \frac{\delta}{1+\delta}$.

Steps 1 to 4 cover all possible states.

Propositions 3 and 4 specify the optimal actions when the speed of arbitration is respectively intermediate and quick.

Proposition 3. *When arbitration proceeds at intermediate speed the optimal actions are:*

state	i	j
$x_i \geqslant \delta(x_i + X) i = 1,2$	X	X
$x_i \geqslant \delta(x_i + X) and x_j \geqslant \delta(x_j + X)$	X	0
$x_i < \delta(x_i + X) i = 1,2, x_i \geqslant \frac{\delta}{1+\delta} and x_i \geqslant \alpha(x_i + \frac{X}{2})$	X	0
$x_i < \delta(x_i + X) i = 1,2, x_i \geqslant \frac{\delta}{1+\delta} and x_i < \alpha(x_i + \frac{X}{2})$	A	$C_j^{\Diamond}$
$x_i < \frac{\delta}{1+\delta} i = 1,2 and x_i - x_j \geqslant \frac{2\delta}{\alpha(1+\delta)} - 1$	A	$C_j^{\Diamond}$
$x_i < \frac{\delta}{1+\delta} i = 1,2 and 1 - \frac{2\delta}{\alpha(1+\delta)} < x_i - x_j < \frac{2\delta}{\alpha(1+\delta)} - 1$	$(*)$	$(*)$

$(*)$ concedes $C_i = Max\left[C_i^*, C_i^{\Diamond}\right]$ where $x_j + C_i^* = \frac{\delta}{1+\delta}$ and $x_j + C_i^{\Diamond} = \alpha(x_j + \frac{X+C_i^{\Diamond}}{2})$.

Proof. See appendix.

Proposition 4. *The optimal actions when arbitration is quick are as follows:*

(i) Let $(\alpha, \delta) \in Q_1$. For states satisfying $x_i \geqslant \frac{\delta}{1+\delta}$ the optimal actions coincide whit those that prevail when $(\alpha, \delta) \in I$. Otherwise

state	i	j
$x_i \geqslant 1 - \frac{\delta(2-\alpha)}{\alpha(1+\delta)}$ i=1,2	C_i^*	C_j^*
$x_i - x_j \geqslant \frac{2\delta}{\alpha(1+\delta)} - 1$	A	$C_j^{\Diamond}$
$1 - \frac{2\delta}{\alpha(1+\delta)} < x_i - x_j < \frac{2\delta}{\alpha(1+\delta)} - 1$ $and x_i \geqslant 1 - \frac{\delta(2-\alpha)}{\alpha(1+\delta)}$	C_i^*	$(*)$
$1 - \frac{2\delta}{\alpha(1+\delta)} < x_i - x_j < \frac{2\delta}{\alpha(1+\delta)} - 1$ $and x_i < 1 - \frac{\delta(2-\alpha)}{\alpha(1+\delta)} i - 1, 2$	$(*)$	$(*)$

$(*)$ Arbitration if $\alpha(x_i + \frac{X}{2}) > \delta(x_i + X - C_i)$ and C_i otherwise with $C_i = Max\left\{C_i^*, C_i^{\Diamond}\right\}$. $C_i^{\Diamond}$ is defined as $x_j + C_i^{\Diamond} = \alpha(x_j + \frac{X+C_i^{\Diamond}}{2})$ and C_i^* as $x_j + C_i^* = \frac{\delta}{1+\delta}$.

(ii) When $(\alpha, \delta) \in Q_2$

state	i	j
$x_i \geqslant \alpha(x_i + \frac{X}{2}) i = 1,2$	X	X
$\alpha(x_i + \frac{X}{2}) \geqslant \delta(x_i + X) i = 1,2$ $x_i \geqslant \alpha(x_i + \frac{X}{2}) and x_j < \alpha(x_j + \frac{X}{2})$	X	A
$\alpha(x_i + \frac{X}{2}) \geqslant \delta(x_i + X)$,$x_i < \alpha(x_i + \frac{X}{2}) i = 1,2$	A	A
$\alpha(x_i + \frac{X}{2}) \geqslant \delta(x_i + X)$,$\alpha(x_j + \frac{X}{2}) < \delta(x_j + X)$, $x_i \geqslant \alpha(x_i + \frac{X}{2}) and x_j < \alpha(x_j + \frac{X}{2})$	X	0
$\alpha(x_i + \frac{X}{2}) \geqslant \delta(x_i + X)$,$\alpha(x_j + \frac{X}{2}) < \delta(x_j + X)$ $x_i < \alpha(x_i + \frac{X}{2}) i = 1,2$	A	$(*)$
$\alpha(x_i + \frac{X}{2}) < \delta(x_i + X) i = 1,2$	$(*)$	$(*)$

$(*)$ Arbitration if $\alpha(x_i + \frac{X}{2}) > \delta(x_i + X - C_i^{\Diamond})$ and $C_i^{\Diamond}$ otherwise, where $x_j + C_i^{\Diamond} = \alpha(x_j + \frac{X + C_i^{\Diamond}}{2})$.

(iii) Let $(\alpha, \delta) \in Q_3$. For states satisfying $\alpha(x_i + \frac{X}{2}) \geqslant \delta^2(x_i + X)$ $i = 1,2$ the optimal actions are the ones specified for Q_2. Otherwise

state	*i*	*j*
$x_i \geqslant \alpha(x_i + \frac{X}{2})$ and $\alpha(x_i + \frac{X}{2}) \geqslant \delta^2(x_i + X)$	X	0
$x_i < \alpha(x_i + \frac{X}{2})$ and $\alpha(x_i + \frac{X}{2}) \geqslant \delta^2(x_i + X)$	A	$(*)$
$\alpha(x_j + \frac{X}{2}) < \delta^2(x_j + X)$	$(*)$	$(*)$

$(*)$ Arbitration if $\alpha(x_i + \frac{X}{2}) > \delta(x_i + X - C_i^{\Diamond})$ and $C_i^{\Diamond}$ otherwise, where $x_j + C_i^{\Diamond} = \alpha(x_j + \frac{X + C_i^{\Diamond}}{2})$.

Proof. See appendix.

With a full characterization of the optimal actions at each possible bargaining state the full characterization of the equilibrium outcomes is straight forward. It suffices to observe that the optimal actions at the initial state must necessarily yield either an arbitrated termination or a negotiated agreement that occurs in two steps of concession, and the measure of these concessions are given in Propositions 2 or 3 depending on the values of (α, δ).

We are now ready to state our main result, characterizing the equilibrium outcome for all possible parameters α and δ.

Proposition 5. *There is a unique equilibrium. Arbitration prevails if and only if it is quick, and this outcome occurs at $t = 0$. Otherwise a negotiated agreement is reached at $t = 1$. The negotiated partition that prevails when arbitration is slow is $(\frac{1}{1+\delta}, \delta 1 + \delta)$, otherwise the split is $(\frac{2(1-\alpha)}{2-\alpha}, \frac{\alpha}{2-\alpha})$.*

Proof. Let $(\alpha, \delta) \in S_1$, at $t = 0$ the bargaining state $(0, 0, 1)$ is such that $x_i < \frac{\delta}{1+\delta}$ for $i = 1, 2$. Then by Proposition 2 the first mover concedes $\frac{\delta}{1+\delta}$. The game reaches the bargaining state $(0, \frac{\delta}{1+\delta}, \frac{1}{1+\delta})$ at which, Proposition 2 prescribes that the opponent concedes the rest of the pie.

When $(\alpha, \delta) \in S_2, I, Q$ the result follows immediate from Propositions 2, 3 and 4 in the same fashion.

Observe that $\alpha > \delta$ may hold even for parameters (α, δ) in the area of slow arbitration. In this case, arbitration is the superior procedure to solve the dispute since an arbitrated termination would take less than a bargaining round. Still, players ignore it in equilibrium.

Under slow arbitration, the equilibrium consists in two consecutive concessions; the first mover concedes $\delta 1 + \delta$ and the other player concedes the rest, $\frac{1}{1+\delta}$, in the following period. The negotiated equilibrium outcome is independent of the possibility of arbitration and depends only on the cost of negotiation.

To gain some intuition consider, for example, the case $\alpha \leqslant \delta$. After 1 concedes $\frac{1}{1+\delta}$, i.e. at the bargaining state $(0, \frac{\delta}{1+\delta}, \frac{1}{1+\delta})$, player 2 has the opportunity to propose arbitration. If the arbitrator is slow (if $\alpha \leqslant \delta$ and $\frac{2(1-\alpha)}{2-\alpha} \geqslant \frac{1}{1+\delta}$) player 2 will never propose arbitration because conceding the rest of the pie yields a payoff $\frac{\delta}{1+\delta}$ that strictly dominates the payoff of arbitration $\alpha(\frac{\delta}{1+\delta}+\frac{1}{2(1+\delta)})$. If arbitration proceeds at a medium speed, it still takes longer that a round of negotiation but by not so much, that is, $(\frac{2(1-\alpha)}{2-\alpha} < \frac{1}{1+\delta})$. In this situation the payoff to 2 in the arbitrated termination are greater than $\frac{\delta}{1+\delta}$ dominating any other continuation; but player 1 will never agree to arbitration, because she can attain a higher payoff rejecting arbitration and forcing player 2 to concede the rest of the pie next period. In either case the equilibrium strategies will be as if it were impossible to use an arbitrator and (as in Rubinstein (1982)) the relative impatience of the players is the only force driving the equilibrium concessions.

The equilibrium under intermediate speed arbitration consists also in two consecutive concessions; the first mover concedes $\frac{\alpha}{2-\alpha}$ and the other player the rest of the pie, $\frac{2(1-\alpha)}{2-\alpha}$, in the next period.

When we consider an intermediate speed arbitrator, the possibility of opting for arbitration may have a dramatic effect on the equilibrium behavior of the players. Suppose the bargaining state is $(0, \frac{\delta}{1+\delta}, \frac{1}{1+\delta})$. If there is no arbitrator, the impatience of the opponent dictates that she concedes the rest of the pie. But if $2\delta^2 < \alpha$ player 2 can call the arbitrator at her turn and get a higher payoff. Player 1 cannot reject this arbitration proposal; if she refuses she can get at most $\frac{\delta^2}{1+\delta}$ (she does not concede at her turn waiting for a concession of $\frac{1}{1+\delta}$ at the next period) while she gets $\frac{\alpha}{2(1+\delta)} > \frac{\delta^2}{1+\delta}$ under arbitration. Thus, in state $(0, \frac{\delta}{1+\delta}, \frac{1}{1+\delta})$ proposing arbitration dominates conceding the rest of the pie. Anticipating that in some bargaining states player 2 credibly threatens with the use of arbitration player 1 is forced to concede, in equilibrium, a share larger than $\frac{\delta}{1+\delta}$.

The opponent's threat to call an arbitrator drives the dynamics of the concessions in equilibrium; the first mover makes a concession that leaves the opponent indifferent between conceding the rest of the pie and calling an arbitrator. Notice that it is only the cost of arbitration which determines the equilibrium partition. As the cost of arbitration decreases the equilibrium share of the opponent increases. Therefore, a voluntary outside option may reduce the first mover advantage. This player could call the arbitrator at $t = 0$, but the payoff she would get in an arbitrated termination is smaller than what she gets by making the equilibrium concession.

Following the argument, if the arbitrator is very fast, so that the first mover must concede almost all, she will choose to use arbitration. In the equilibrium under a quick arbitration the arbitrator is called at $t = 0$.

The threat that the opponent calls the arbitrator without resistance is credible in almost all the bargaining states. The first concession necessary to avoid bargaining states where the opponent can credibly call the arbitrator is so large that the first mover prefers to call the arbitrator right away rather than conceding. Here there is no inefficiency- the most efficient method is used to share the pie.

Proposition 5 shows that an endogenous outside option affects the equilibrium outcome differently than when the outside option is exogenous. When is exogenous, players may agree at the shares that yield the same payoff as the exogenous option. In contrast, players do not negotiate the arbitrated outcome; either they opt out to arbitration or they use the threat of arbitration to attain a larger share.

We can also infer some interesting comparative statics regarding this effect of arbitration costs on the efficiency of the negotiation process. On one hand, when arbitration cost is high relative to the negotiation cost, we see that a negotiated outcome is reached in two rounds which are quite efficient. On the other hand, we also see that, when arbitration cost is very small the arbitrator is called in an efficient way. Inefficiency arises for intermediate ranges of arbitration costs. In these cases, players negotiate without appealing to the arbitrator institution, when it would be more efficient to do so.

Figures 6 and 7 display the equilibrium share of the first mover for each arbitration cost α respectively for $\delta = \frac{1}{3}$ and $\delta = \frac{3}{4}$.

dtbpFU253.875pt209.0625pt0ptFigure 6Figure

In Figure 6, we see that arbitration does not affect the equilibrium share for $\alpha < \frac{2}{5}$. It does for $\frac{2}{5} < \alpha < \frac{5-\sqrt{13}}{3}$. For $\alpha > \frac{5-\sqrt{13}}{3}$ arbitration is used in equilibrium.dtbpFU253.875pt194.25pt0ptFigure 7Figure

In Figure 7, the final partition is unaffected by the cost of arbitration; is either $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$, or $(\frac{1}{2}, \frac{1}{2})$. The inefficiency is located in the range of $\alpha \in \left[\frac{3}{4}, \frac{6}{7}\right]$. Here players negotiate even though it would be more efficient to arbitrate.

Thus, the effect of the arbitration procedure on the equilibrium partition depends of the relative patience of the players. When players are patient this effect is nil. As they become more impatient, however these costs may completely determine the negotiated shares in equilibrium.

2.4 Conclusions

We have explored the impact of arbitration on negotiations, addressing the equilibrium features of bargaining games with endogenous and voluntary outside options requiring mutual consent. We conclude that voluntary arbitration has two distinctive effects:

a) It alters the negotiated partition of the surplus relative to the situation in which arbitration is unavailable. This only occurs if the relative cost of arbitration is not too high, since arbitration turns irrelevant when it is excessively costly. When arbitration is relevant, the negotiation positions of the players approach those sustained by the arbitrator and this implies a reduction of the first mover advantage. As the cost of arbitration vanishes players immediately resort to arbitration, and an equal split of the surplus prevails.

b) The two procedures may not be used efficiently. For some range of parameters α and δ, players negotiate even though arbitration is the more efficient scheme.

Chapter 3. Bargaining over Finite Alternatives and Arbitration

3.1 Introduction

In negotiations, agreement may be delayed or never reached. The strategic approach to bargaining has analyzed what causes these inefficiencies using game theoretic models of non-cooperative bargaining. The first successful attempt to justify equilibrium outcomes with delayed agreements comes from the literature on bilateral bargaining with incomplete information. The main idea is that delays occur because they are used as a signalling device. By delaying the negotiated agreement a player can persuade his opponent about the type of bargainer he is (his valuation or his discount factor).

But delay can arise in equilibrium of complete information games. Delays can appear when we enlarge the set of alternatives available to the players during the negotiations; when each player can choose when to make an offer (Sakovics, 1993), when one player can destroy surplus after her own offer is rejected (Fernandez and Glazer 1991), or when bargainers can freely quit whenever they wish so, even if their outside option is zero (Ponsati and Sakovics, 1998). The present chapter adds to results on delays in bargaining under complete information. Under certain conditions, the possibility of an arbitrator intervention to solve the dispute induces delay.

This chapter is also related with the industrial relations literature which has been concerned with the effect of arbitration procedures on the efficiency and outcome of negotiations. Two arbitration rules are in wide use: *the conventional arbitration* (CA) where the arbitrator is free to choose a settlement and *the final-offer arbitration* (FOA) where the arbitrator is constrained to pick one of the negotiators' last offers. Some empirical studies that study the rules followed by arbitrators [8] found that the negotiated settlement rates are much higher under FOA than under CA. The theoretical reasons for this are not clear and controversy is unsolved about the extent to which arbitration inhibits genuine bargaining (this phenomena is referred as the chilling effect of arbitration) and more generally about the effect on the positions taken by the two sides during the bargaining phase.[9]

The theoretical literature has used the idea of the "contract zone" to explore the positions taken by bargainers facing alternative arbitration rules. In these static models players simultaneously make a final proposal to the arbitrator who then makes his decision. Since they do not allow for an explicit temporal setting they do not permit the study of the dynamics of the negotiation process that may take place before final proposals. By contrast in this chapter we propose a sequential bargaining with which we can analyze this dynamics. Specifically a bargaining model of alternating offers where the set of possible partitions is finite and have the possibility of asking for an arbitrator that solves the dispute using a procedure known by the players.

Our main finding is that if players are sufficiently patient and the arbitrator uses a Final-Offer Arbitration type-procedure the negotiated equilibrium outcome may involve some delay. Under FOA, if players perceive that if they don't make concessions they may be punished by the arbitrator choosing the opponent'offer, they will never use arbitration in equilibrium. Under this threat they will decrease their demands progressively until an agreement on an equal partition of the surplus

[8]See Ashenfelter and Bloom (1984), Bazerman and Farber (1985) and Bloom (1986)

[9]For a discussion and survey see Farber and Bazerman (87).

is reached. However if the arbitrator uses a Conventional Arbitration procedure and splits the difference between the players last offers, there will be a tendency to use arbitration in order to settle disputes without wasting time.

The chapter is organized in three sections. In the next section, the model is presented. In the third section we analyze the bargaining game under the two arbitration rules. Conclusions are presented in the last section.

3.2 The model

The following bargaining situation is studied. Two players bargain about how to share N units of surplus that will be available only when they reach an agreement. An agreement is denoted by a pair of positive integers y_i $i = 1, 2$, that indicates the number of units assigned to player i. That is, the set of possible agreements are the elements of the set:

$$Y = \{(y_1, y_2) \mid y_1 + y_2 \leq N, and y_1, y_2\ are positive integers\}$$

and the set of efficient agreements Y^e are $(y_1, y_2) \in Y$ such that $y_1 + y_2 = N$.

We assume that offers are made sequentially. The game proceeds as follows: in each period t, one of the players, say i, selects an offer and player j either accepts it or rejects it. An offer (or a proposal) is a partition of the N units. If the offer is accepted, then the bargaining ends, and the agreement is implemented. If the offer is rejected, then play moves to period $t + 1$ where player j proposes and player i accepts or rejects. The game ends when a proposal is accepted or when the arbitrator is called. In order to call the arbitrator a player must break the negotiation. We consider that a player breaks the negotiation when she demands at least as many units as in her previous proposal. Moreover, at $t = 0$ the first mover breaks the negotiation if she asks for N units.

If the arbitrator is called she imposes a partition and the game ends.[10]

Players are assumed to be risk neutral and impatient. Their impatience is modeled by a common discount factor, normalized to be δ per unit of time. If at time t players reach an agreement on a partition (y_1, y_2) payoffs are:

$$(\delta^t y_1, \delta^t y_2)$$

If players break the negotiation at t and the arbitrator is called, a partition (y_{1a}, y_{2a}) with $y_{1a} + y_{2a} = N$ prevails at $t+1$ and payoffs are:

$$(\delta^{t+1} y_{1a}, \delta^{t+1} y_{2a})$$

The partition (y_{1a}, y_{2a}) depends on the rule that the arbitrator uses. We will study two arbitration rules or procedures. Under Final-Offer Arbitration the arbitrated partition is equal to the last proposal of the player that does not break the negotiation. Under Conventional Arbitration the the arbitrated partition splits the difference[11] between the players last two offers.[12]

The present a bargaining games are of games of complete information and infinite horizon. A *pure strategy* of player i specifies the action

[10]In the context of international negotiations the arbitration phase is exogeneously triggered after a deadline or after an impasse where an impasse may understood as a phase without any significant concession from either party.

[11]If $(N - x_2, x_2)$ is the last offer of player 1 and $(x_1, N - x_1)$ the last offer of player 2, then the arbitrated partition will be $(x_1 + \frac{N-x_1-x_2}{2}, x_2 + \frac{N-x_2-x_1}{2})$ whenever $N - x_1 - x_2 \geqslant 2$. If $N - x_1 - x_2 = 1$, we assume that the arbitrator favors the player that did not break the negotiation, says player 1, by implementing the partition $(N - x_2, x_2)$.

[12]The empirical literature has found that a more accurate description of arbitrators' actual behavior is one in which the final choice of the arbitrator also depends on his own expertise of the case. Yet a dependence on the parties positions exists. Since arbitrators are chosen (at least to some extent) by the players, and arbitrators care about their chances to be hired again, the arbitrator must use the information provided by players' offers.

at each subgame: a proposal if player i moves first at t or a reply to the opponent's proposal if she moves second. In general, strategies are extremely complex since actions at any subgame depend arbitrarily on the entire history of actions up to that point. We will, however, restrict our attention to stationary strategies, that is, strategies in which actions depend only on the bargaining state. *A bargaining state* is defined as the triple (x_1, x_2, X) where $(N - x_2, x_2)$ and $(x_1, N - x_1)$ are the last offer of player 1 and 2 respectively, and $X = N - x_1 - x_2$ is the contested surplus. A *pure stationary strategy* specifies the offer a player makes as proposer and a reply to the opponent's proposal as responder for each bargaining state. *An equilibrium* will be a profile of stationary strategies that constitute a subgame perfect equilibrium.

3.3 Equilibria under Final Offer Arbitration

Consider the bargaining game under FOA. If player 1 breaks the negotiations at some t when the bargaining state is (x_1, x_2, X), the arbitrated the partition $(x_1, x_2 + X) = (x_1, N - x_1)$ is implemented one period later.

We denote as Z_1 the smallest number of units such that $Z_1 \geq \delta(Z_1 + 1)$. And as $I(X)$ an indicator function that takes values of $I(X) = 0$ if X is even and $I(X) = 1$ if X is odd.

In the proposition that follows we characterize the equilibrium moves at each bargaining state.

Proposition 6: *Assume that the arbitrator uses a FOA procedure and let N and δ such that $Z_1 > \frac{N}{2}$. The optimal actions at each possible bargaining state are as follows:*

At (x_1, x_2, X) player i accepts any offer that gives him a number of units $x_i + l \geq Z_1$, or $x_i + X - 1$ if $x_i + 1 < Z_1$. Otherwise, she rejects and she demands a number of units:

state	i demands
$x_j \geq Z_1 - 1$	$x_i + X - 1$
$x_i \leq x_j < Z_1 - 1$ and $v_i(x_1, x_2, X) \leq N - Z_1$	$N - Z_1$
$x_i \leq x_j < Z_1 - 1$ and $v_i(x_1, x_2, X) > N - Z_1$	$x_i + X - 1 - I(X)$
$x_j < x_i < Z_1 - 1$ and $v_i(x_1, x_2, X) \leq N - Z_1$	$x_i + X - 1$
$x_j < x_i < Z_1 - 1$ and $v_i(x_1, x_2, X) > N - Z_1$	$x_i + X - 1 - I(X)$
$x_i \geq Z_1 - 1$	$x_i + 1$

$v_i(x_1, x_2, X) = \delta^{\frac{X-2-I(X)}{2}}(x_i + \frac{X-I(X)}{2})$

Proof. See appendix.■

Under FOA players never break the negotiation process in order to force arbitration because they can always replicate the arbitrated partition without the one period delay needed for arbitration. Thus, under a FOA, an impasse in the negotiation process never occurs because, at each state, players are forced to agree or decrease their demands.

With a full characterization of the optimal actions at each possible bargaining state, we are now ready to characterize the equilibrium outcomes. At the initial state $(0, 0, N)$, the first mover either proposes $(N - Z_1, Z_1)$ and the opponent accepts or she demands $N - 1 - I(N)$ units and the opponent rejects. After a path of consecutive offers and rejections players finally agree on the partition $(\frac{N-I(N)}{2}, \frac{N+I(N)}{2})$.

Proposition 7*: Assume that the arbitrator uses a FOA procedure and let δ and N such that Z_1¿$\frac{N}{2}$. If $\delta^{\frac{N-2-I(N)}{2}}(\frac{N-I(N)}{2}) > N - Z_1$, in equilibrium the partition $\left(\frac{N-I(N)}{2}, \frac{N+I(N)}{2}\right)$ is accepted after $\frac{N-2-I(N)}{2}$ periods. Otherwise, the partition $(N - Z_1, Z_1)$ is accepted at t=0.*

Proof. Assume first that $N = 3$ and without loss of generality, let player 1 be the first mover. At the initial bargaining state $(0, 0, 3)$ player

1 proposes $(1,2)$ since $v_1(0,0,N) \geq N - Z_1$ and player 2 accepts since $x_2 + 1 < Z_1$.

Now consider $N > 3$ and N even. At the initial bargaining state $(0,0,N)$ $x_1 = x_2 < Z_1 - 1$. If $v_1(0,0,N) = \delta^{\frac{N}{2}-1}\frac{N}{2} \leq N - Z_1$, the optimal proposal of player 1 is $(N - Z_1, Z_1)$ which player 2 optimally accepts. If $v_1(0,0,N) = \delta^{\frac{N}{2}-1}\frac{N}{2} > N - Z_1$, player 1 proposes $(N - 1, 1)$ and player 2 rejects since $x_2 + 1 < \frac{N}{2} < Z_1$. At $t = 1$, the new bargaining state is $(0, 1, N-1)$ with $x_1 < x_2 < Z_1 - 1$, $N - 1$ odd and $\delta^{\frac{N-4}{2}}(1 + \frac{N-2}{2}) > \delta^{\frac{N}{2}-1}\frac{N}{2} > N - Z_1$. Player 2 proposes $(2, N-2)$ and player 1 accepts if $N = 4$ and rejects if $N > 4$ since $2 \leq \frac{N}{2} < Z_1$. At $t = 2$ the bargaining state is $(2,1,N-3)$ with $x_2 < x_1 < Z_1 - 1$ and $\delta^{\frac{N-6}{2}}(2 + \frac{N-4}{2}) > \delta^{\frac{N}{2}}(\frac{N}{2}) > N - Z_1$. Player 1 proposes $(N-3, 3)$ and player 2 accepts if $N = 6$, and rejects otherwise. And so on, and so forth. At $t = \frac{N}{2} - 1$ the bargaining state is $(\frac{N}{2} - 1, \frac{N}{2} - 2, 3)$ and player 1 proposes $(\frac{N}{2}, \frac{N}{2})$ and player 2 accepts it.$\blacksquare$

Under a FOA, the efficiency of the equilibrium outcome may be affected. If players are sufficiently patient, in equilibrium players delay an agreement on the partition $(\frac{N}{2}, \frac{N}{2})$ (or the 'almost equal' partition if N is odd). The threat of arbitration forces players to decrease their demands until an agreement is reached. However if they are sufficiently impatient, the first mover will make a proposal that the opponent will not reject.

The intuition is the following: suppose that the bargaining state is such that $x_1 \geq Z_1 - 1$ and $x_2 < Z_1 - 1$. Recall that Z_1 the smallest number of units such that $Z_1 \geq \delta(Z_1 + 1)$. Since $x_2 < Z_1$, and a player's demand must be reduced in at least one unit to avoid arbitration, player 2 finds worthy to delay the agreement one period if she can gain at least one more unit. Player 1, however, prefers to finish the negotiation by accepting player 2's demand of $N - x_1 - 1$ units.

If we are at a bargaining state with $x_i < Z_1 - 1$ for $i = 1, 2$, both players have incentives to delay the agreement and decrease their demands by the minimal amount. That behavior is responsible for delay in reaching the agreement. However, if the resulting payoff for player 1 is smaller than $N - Z_1$, she will choose to propose the partition offering a number of units that 2 will not reject, i.e. Z_1. There will be no rejection since if player 2 does reject, at the next turn, any proposal different than $(N - Z_1 - 1, Z_1 + 1)$ will be rejected by player 1 leaving player 2 clearly worse off.

This arbitration procedure has effect on the negotiation equilibrium behavior even under the present assumption that arbitration consumes one period to be implemented. In the present model arbitration is an outside option that has a "negative value". Opting out is not a credible threat since it is a dominated strategy for any subgame. It is, nevertheless, an impasse solving device that prevents that players stick to a proposal paralyzing negotiations.

In the absence of arbitration, Van Damme, Selten and Winter (1990) have shown that, for given N if $Z_1 \geq N$, any immediate efficient agreement can be supported as a SPE. The argument is quite simple; any offer $(x, N - x)$ can be sustained as an equilibrium: If player i wants more than x she must ask for at least $x + 1$. But if δ is sufficiently large, then player j may optimally reject this offer. Since there is a multiplicity of equilibria, alternative equilibria that involve delays and even perpetual disagreement can be supported as a SPE. Constructing this equilibria, however, requires the use of non-stationary strategies.

Adding arbitration has an immediate consequence: strategies that sustain an immediate agreement on $(x, N - x)$ cannot be supported as SPE strategies anymore. If player 1 asks for x, rejecting this proposal is profitable for player 2. After rejection she can propose, at her turn, $(x - 1, N - x + 1)$. Player 1 cannot reject this proposal without punishment by the arbitrator. If she rejects, the game reaches the bargaining state

$(x-1, N-x, 1)$ and if she asks for more than $x-1$ the arbitrator is called. Thus, the offer of player 2 is such that player 1 cannot reject; and player 2 gets a payoff $\delta(N-x+1) > N-x$ since $Z_1 \geq N$.

Under FOA, if agreement is not immediate, delay is related to the number of units N. When agreement is immediate, the first mover gets a share smaller than her opponent's since $N - Z_1 < Z_1$. Hence, under certain conditions, if the set of alternatives is finite, the introduction of a FOA eliminates first-mover advantage.

3.4 Equilibria under Conventional Arbitration

Let us now consider Conventional Arbitration. At the bargaining state (x_1, x_2, X), if the negotiation is broken by player 1, the arbitrated partition

$$
\begin{array}{l}
(x_1 + \frac{X-I(X)}{2}, x_2 + \frac{X+I(X)}{2}) if X \geqslant 2 \\
(x_1, x_2 + 1) if X = 1
\end{array}
$$

is implemented one period later.

We will denote as Z_l the smallest number of units such that $Z_l \geq \delta(Z_l + 1)$. Notice that if $Z_1 > \frac{N}{2}$ then $Z_2 \geq N$.

With this arbitration procedure, at any bargaining state (x_1, x_2, X) a player can guarantee herself a payoff of $\delta(x_i + \frac{X-I(X)}{2})$. Then, if she chooses to continue the negotiation by offering her opponent $x_j + y$ with $y \geq 1$ she must expect a payoff greater than $\delta(x_i + \frac{X-I(X)}{2})$.

Proposition 8 that follows characterizes the equilibrium actions at each bargaining state and shows that, contrary to what happens with under FOA, there are bargaining states from which players may find optimal to force arbitration.

Proposition 8: *Assume that the arbitrator uses a CA procedure and let N and δ such that $Z_1 > \frac{N}{2}$. The optimal actions at each possible bargaining state are as follow:*

For (x_1, x_2, X) with $X < 6$ players follow the optimal actions specified in proposition 1.

For (x_1, x_2, X) with $X \geq 6$, player i accepts any offer that gives him a number of units $x_i + l \geq Max\{Z_1, x_i + X - 5\}$. Otherwise, she rejects. And she demands a number of units:

state	*i demands*
$x_j \geq Z_1 - 1$ and $x_i + 5 \geq v_i^a(x_i, x_j, X)$	$x_i + 5$
$x_j \geq Z_1 - 1$ and $x_i + 5 < v_i^a(x_i, x_j, X)$	$x_i + X$
$x_i \leq x_j < Z_1 - 1,\ x_i + 5 \geq N - Z_1 \geq v_i^a(x_i, x_j, X)$	$N - Z_1$
$x_i \leq x_j < Z_1 - 1, x_i + 5 \geq N - Z_1$ $and N - Z_1 < v_i^a(x_i, x_j, X)$	$x_i + X$
$x_i \leq x_j < Z_1 - 1,\ v_i^a(x_i, x_j, X) \leq x_i + 5 < N - Z_1$	$x_i + 5$
$x_i \leq x_j < Z_1 - 1, x_i + 5 < N - Z_1$ $and x_i + 5 < v_i^a(x_i, x_j, X)$	$x_i + X$
$x_j < x_i < Z_1 - 1, \delta Z_1 > v_i^a(x_i, x_j, X)$ $and x_j + 5 \geq N - Z_1 \geq v_j^a(x_i, x_j + 1, X - 1)$	$x_i + X - 1$
$x_j < x_i < Z_1 - 1, \delta Z_1 \leq v_i^a(x_i, x_j, X)$ $and x_j + 5 \geq N - Z_1 \geq v_j^a(x_i, x_j + 1, X - 1)$	$x_i + X$
$x_j < x_i < Z_1 - 1,\ x_j + 5 < N - Z_1$	$x_i + X$
$x_i \geq Z_1 - 1$	$x_i + X$

$v_i^a(x_1, x_2, X) = \delta(x_i + \frac{X - I(X)}{2})$
$v_j^a(x_1, x_2 + 1, X - 1) = \delta(x_j + 1 + \frac{X - 1 - I(X-1)}{2})$

Proof. See Appendix.■

The equilibrium outcomes are easily characterized from Proposition 8. Our main result for Conventional Arbitration is that when players are sufficiently patient, in equilibrium they will always resort to arbitration when the number of units N to share is sufficiently large.

Proposition 9: *Assume that the arbitrator uses a CA procedure and let N and δ such that $Z_1 > \frac{N}{2}$. Then if $N \geq 6$, in equilibrium the first mover asks for all the N units, the opponent rejects, and the arbitrator is called at t=1.*

Proof. Notice that if $Z_1 > \frac{N}{2}$, then $N - Z_1 < \frac{N}{2} < Z_1$ and $N - Z_1 < \delta(N - Z_1 + 1) \leq \delta(\frac{N}{2})$. If $N \geq 6$, at the initial bargaining state $(0, 0, N)$ with $x_1 = x_2 = 0 < Z_1 - 1$ and $N - Z_1 < \delta(\frac{N}{2}) = v_i^a(0, 0, N)$, the optimal action of the first mover will be to force arbitration by asking for the N units.■

Remark 1. *For $N < 6$ the optimal strategies of the players are the same as the specified strategies in proposition 1 and the equilibrium outcome is described in proposition 2.*

In order to be true we must prove that forcing arbitration is a dominated strategy for any player at any bargaining state (x_1, x_2, X) with $X < 6$. At bargaining states (x_1, x_2, X) with $X = 1, 2, 3$ arbitration pays either δx_i or $\delta(x_i + 1)$. To force arbitration is clearly a dominated strategy since a player can always ask for one unit and the opponent will accept this partition without wasting one period. At (x_1, x_2, X) with $X = 4, 5$ arbitration pays $\delta(x_i + 2)$. It is easy to check that arbitration is a dominated alternative for player i at the bargaining states such that $x_i \geq Z_1 - 1$ and at the bargaining states such that $x_j \geqslant Z_1 - 1$. And at the bargaining states satisfying $x_i \leq x_j < Z_1 - 1$ or $x_j < x_i < Z_1 - 1$, arbitration gives the player, at most, the same payoff as the one she gets if she follows the strategies specified in table 1.

If players are sufficiently patient, that is if $Z_1 > \frac{N}{2}$, for any bargaining state, players are ready to delay the agreement at least one period in order to get two or more units since $N < Z_2$. The conventional arbitrator implements a partition that gives the player half of contested units X one period later. Then, for X large enough opting out becomes an attractive alternative.

Unlike under the Final Offer Arbitration, the outside option has a "positive value" and may be a credible threat for a player in some bargaining states. Players never make proposals such that the opponent will optimally reject to opt out for arbitration. Since avoiding these bargaining states requires very generous proposals, players prefer arbitration right away.[13] Let us illustrate this point. Suppose we are at the bargaining state (x_1, x_2, X) and that player 1 proposes $(x_1 + X - l, x_2 + l)$. If l is not sufficiently large, 2 opponent will optimally reject and break the negotiation at the next turn since $x_2 + l < \delta^2(x_2 + l + \frac{X-l-I(X-l)}{2})$ for $\frac{X-l-I(X-l)}{2} \geq 4$ given that $N < Z_2$.

3.5 Conclusions

In this chapter we have analyzed the effect of two different arbitration rules on negotiations with a finite set of agreements. Under a Final Offer Arbitration, where the arbitrator imposes the last offer of the agent not breaking negotiations, we found that, in equilibrium, players will negotiate the partition of the pie and will never resort to arbitration. Under a Conventional Arbitration, where the arbitrator splits the difference between the last offers of the players, arbitration is used in equilibrium. These results are compatible with the observed higher rate of negotiated settlements under FOA than under CA and support the intuition that the later scheme leads to arbitration as the dominant settlement mode

[13]This result is consistent with Compte and Jehiel (1997) that explore the present model where the set of alternatives is continuum. One important result is that due to the form of outside option, equilibrium concessions are gradual, and delay may be an equilibrium outcome. When the efficiency loss in case of delayed agreement is too high then, players use the arbitrator right away.

(a phenomenon known as the narcotic effect).

Arbitration systems and arbitration provisions take a wide variety of forms. Although we do not claim full generality for our approach, our model does capture the basic features of actual arbitration and thus provides sound theoretical basis for a number of claims commonly made about the effects of arbitration.

Chapter 4. Opting Out in the War of Attrition

4.1 Introduction

This aim of this chapter is to study the role played by the outside options in negotiations when there is incomplete information about their existence. For this purpose we focus our analysis on the War of Attrition since this is the simplest model of conflict that yields inefficient equilibria under complete information. It is well known that, in a symmetric War of Attrition without outside options, the unique symmetric equilibrium consists in players randomizing at a constant probability between conceding and not conceding, a very inefficient outcome indeed. We show that the presence of uncertain outside options improves efficiency.

The relevance of outside opportunities available to the players on the outcome of a negotiation has been well established in models of bargaining with complete information (Shaked and Sutton (1984), Binmore et al.(1986), Shaked (1987), and Ponsati and Sakovics (1998)). In these models the decision of a bargainer to take up her outside option is a strategic decision and outcomes depend crucially on who has this possibility and when. If it is the responder who has the outside opportunity, then, in the unique subgame perfect equilibrium, this player obtains a payoff equal to the value of her option if this is larger than her equilibrium share in the game without the possibility to opt out. Otherwise, the option has no effect on the outcome (this is known as Outside Option Principle, see Shaked and Sutton (1984)). But if it is the proposer who can threaten to take her outside option, she can appropriate the entire surplus making a take-it-or-leave-it offer and, in this case, there is multiplicity of equilibria for a range of outside options.

Considering uncertainty about outside options is a natural extension of the literature that deserves attention. Nevertheless, bargaining models devoted to that subject are scarce.[14] Wolinsky (1987) presents a model where players may search for outside opportunities during the bargaining process. He shows that the outcome of the bargaining does not depend only on the players' relative efficiency in searching, but also on how aggressively each party can credibly threaten to search in the event that the agreement is delayed. Vislie (1988) extends Shaked and Sutton's model (1984) by allowing the presence of a second random outside option for the seller, and finds the conditions under which the equilibrium price is affected by this random appearance. And finally, Ponsati and Sakovics (1999) analyze a bargaining game where both players have outside options but they are uncertain about their size. In all these models players do not know with certainty either the existence or the size of their own outside options. By contrast, in this chapter we present a model where players enjoy private information about their possibilities of opting out, but they do not know their opponent' opportunities.

We carry out our analysis within the simple framework of a War of Attrition, a situation where there are only two available agreements and each player favors one of them. The decision problem of each player consists in deciding when to give in by accepting her opponent´s favorite agreement. The distinctive feature of our model is that, since outside options are present yielding takes two forms: a player can give in by accepting her opponent's favorite agreement, or by contrast, she can

[14]The literature of pretrial negotiation has an apparent similarity to models of sequential bargaining with incomplete information and outside options. In these models of litigation and pretrial negotiation (see Spier (1992), Wang et al, (1994)) only the plaintiff can opt out forcing the trial. But an important difference is that in bargaining models both players would like to come to an agreement immediately while in pretrial negotiation the plaintiff would like to settle as soon as possible and the defendant to pay as late as possible.

give up, taking her option, and leaving her current partner to take her outside payoff as well. Both players have private information about their own outside options and are impatient in that delaying is costly. There are two types of players: a weak type who has no outside option (or whose outside option is without value) and a strong type who has a valuable outside option that she prefers to take rather than conceding.

We show that introducing the possibility of opting out in a War of Attrition has a dramatic effect on the outcomes.[15] We find that, if the probability of facing a weak opponent is sufficiently low, in equilibrium, the negotiation will surely end at some future date, since weak types eventually become sufficiently pessimistic about the prospect of reaching their preferred agreement so that, in fear that the opponent might opt out, they concede with probability 1. On the other extreme, if the likelihood of a weak opponent is high, strong types eventually opt out with probability 1, leaving weak types to play, from that time on, the symmetric inefficient equilibrium of the complete information War of Attrition. Even in this case, the probability of concession along the uncertainty phase of the equilibrium play increases.

The following section presents our bargaining model and characterizes equilibria of this game. In section 3 we turn to an asymptotic analysis of this game considering the limit as $\delta \rightarrow 1$, and carry out comparative statics. Conclusions are presented in the last section.

[15] In a very different model, Compte and Jehiel (2000) find also that outside options have a positive effect on bargaining. They show that the existence of outside options may cancel out the effect of obstinacy in bargaining.

4.2 The model

The following bargaining situation is studied. Two players bargain about how to share one unit of surplus that will be available only when they reach an agreement. An agreement is denoted by x, where x indicates the portion of the surplus assigned to player 1. There are only two possible agreements; either $x = 1 - a$ or $x = a$ with $0 < a < \frac{1}{2}$. Players may also decide to break the negotiation by opting out, in which case, they receive a payoff b_i $i = 1, 2$.

In this game there are three possible bargaining outcomes; either an agreement is reached, or negotiations break, or perpetual disagreement prevails.

Players are assumed to be risk neutral and impatient. Their impatience is modeled by a common discount factor, normalized to be δ per unit of time. And the payoffs are as follows: if players perpetually disagree, they both receive zero payoff. If only player i concedes at time t, then player i gets $a\delta^t$ and player j gets $(1-a)\delta^t$. If both players concede at the same time each players gets $a\delta^t$[16]. And if either or both players opt out, payoffs are $b_i\delta^t$ for $i = 1, 2$.

Each player i has private information about the value of her outside opportunity, which can be either $b_i = 0$ or $b_i = b$, $a < b < 1 - a$. A player with no outside option (or whose outside option is 0) is a weak type, denoted as W, and a player with an outside option $b > 0$ is a strong type, denoted as S. Strong types always prefer opting out rather than conceding and weak types prefer conceding rather than opting out. The players entertain beliefs about each other's type and they are represented by an initial probability $0 < \pi_0^i < 1$, that is, the probability that player i is weak. We assume that these probabilities are common knowledge

[16]This assumption is computationally convenient. Results do not change substantially if we assume that in the case that both players concede at the same time a lottery is used to decide the outcome.

and we set $\pi_0^i = \pi_0$ for simplicity.

The game is played in discrete time, starting at $t = 0$. At each time (a stage), both players decide simultaneously either: (i) to propose her preferred agreement, (ii) to concede by proposing her opponent's favorite agreement or (iii) to leave the negotiation and opt out. The game ends whenever a player or both, at the same time, concedes or opts out. Otherwise, disagreement occurs, discounting applies and the game proceeds to a new stage.

The history h_t observed by the players is just the fact that no player has yielded before t (no player has conceded or has opted out).

A strategy $\sigma_i(\tau)$ of player i with type $\tau = W, S$ is defined as a pair of sequences $\sigma_i(\tau) = \{\alpha_t^i(\tau), \beta_t^i(\tau)\}_{t=0}^{\infty}$ where $\alpha_t^i(\tau)$ is the probability of conceding at t and $\beta_t^i(\tau)$ is the probability of opting out at t, given that no player yields before that time. Let $\sigma = (\sigma_i(W), \sigma_i(S), \sigma_j(W), \sigma_j(S))$.

A system of beliefs π^i for player i maps each observed history into some probability measure on the types W and S of player j. Let $\Pi = (\pi^i, \pi^j)$.

Given a strategy-belief profile (σ, Π), the expected payoff of player i of not conceding at t, conditional on the history h_t, is

$$V_t^{iW} = \pi_t^j \alpha_t^j (1-a) + \delta \left[1 - \pi_t^j \alpha_t^j - (1-\pi_t^j)\beta_t^j\right] V_{t+1}^{iW},$$

and the expected payoff of not opting out at t is

$$V_t^{iS} = \pi_t^j \alpha_t^j (1-a) + (1-\pi_t^j)\beta_t^j b_i + \delta \left[1 - \pi_t^j \alpha_t^j - (1-\pi_t^j)\beta_t^j\right] V_{t+1}^{iS}.$$

Since we are interested on the role played by outside options on the efficiency and outcome of the War of Attrition, we find appropiate to examine the Symmetric Perfect Bayesian Equilibria of this game given that inefficiency arises in a War of Attrition when players are constrained to use symmetric strategies.

The *Symmetric Perfect Bayesian Equilibrium* (SPBE) is defined in the usual way. A strategy-belief profile (σ, Π) is a SPBE if, at any stage of the game, strategies are optimal given the beliefs, and the beliefs are obtained from equilibrium strategies and observed actions using Bayes' rule:

$$\pi_t^i = \pi_{t-1}^i(1-\alpha_{t-1}^i)\pi_{t-1}(1-\alpha_{t-1}^i) + (1-\pi_{t-1}^i)(1-\beta_{t-1}^i).$$

Notice that π_t^i is not defined if $\alpha_{t-1}^i = \beta_{t-1}^i = 1$. If the optimal strategy tells a player to concede and opt out at some t with probability 1, then to stay at $t+1$ is a probability 0 event and Bayes' rule does not pin down posterior beliefs. Any posterior beliefs are then admissible. Symmetry in strategies implies that $\alpha_t^i = \alpha_t^j = \alpha_t$ and $\beta_t^i = \beta_t^j = \beta_t$.

Since in a SPBE a weak type will never opt out and a tough type will never concede, in an abuse of terminology, we will identify the probabilities of conceding α_t with the strategy of the weak type, and the probabilities of opting out β_t with the strategy of the tough type.

The first result is quite straight forward.

Proposition 10. *There is no SPBE in pure strategies.*

Proof. See Appendix.■

We next turn attention to profiles where players randomize. In a SPBE in mixed strategies, it must be true that the payoff of conceding at t, conditional on the opponent not having conceded or opted out previously, must be equal to the payoff of conceding at t+1. At the same time, the payoff of opting out at t, conditional on the opponent not having yielded in before, must be equal to the payoff of opting out at t+1:

$$\begin{aligned} a &= (1-a)\pi_t\alpha_t + a\delta(1-\pi_t\alpha_t - (1-\pi_t)\beta_t), \\ b &= (1-a)\pi_t\alpha_t + b(1-\pi_t)\beta_t + b\delta(1-\pi_t\alpha_t - (1-\pi_t)\beta_t). \end{aligned} \tag{1}$$

Next lemma points out that, in a SPBE it is not possible to have both types yielding at the same time with probability 1. And if the equilibrium is such that weak types concede with probability 1 at some t, then strong types certainly opt out at t+1.

Lemma 1. *If* $\{\alpha_t\}_0^\infty$ *and* $\{\beta_t\}_0^\infty$ *are SPBE, then:*
(i) there is no t *such that* $\alpha_t = \beta_t = 1$
(ii) If $\alpha_t = 1$ *and* $0 < \beta_t < 1$ *then* $\beta_{t+1} = 1$.

Proof. Statement (i) indicates that, in equilibrium, it is not possible that both types yield at the same time with probability 1. If the strategy of the opponent is to concede and to opt out at some t with probability 1, then a strong player will have always incentives to wait one period since $b < (1-a)\pi_t + b(1-\pi_t)$, breaking the symmetry of the strategies. Statement (ii) establishes that, if the weak type strategy yields a period t probability of conceding of unity, then to opt out at t+1 dominates doing so in t+2, since waiting until period t+2 discounts their payoff and provides no additional probability that a weak type will make a concession.■

In a SPBE, both types distribute concessions across time. The equilibrium strategies are characterized by the pair of difference equations (1). To simplify notation let,

$$H = ab(1-\delta)a\delta(1-a-\delta b) + b(1-\delta)(1-a-\delta a),$$
$$G = (1-\delta)(1-a)(b-a)a\delta(1-a-\delta b) + b(1-\delta)(1-a-\delta a).$$

Our system of equations (1) can be rewritten as

$$\alpha_t \pi_t = H.$$
$$\beta_t(1-\pi_t) = G.$$

Substituting these expressions on the posterior probability π_t, we have the difference equation that rules the posterior:

$$\pi_t - \pi_{t-1} 1 - H - G + H 1 - H - G = 0.$$

Solving this difference equation with the initial condition π_0,

$$\pi_t = \frac{H}{H+G} + (\pi_0 - \frac{H}{H+G})\,(11 - H - G)^t\,.$$

For what follows we analyze the different profiles that can be sustained as equilibria.

Concession Equilibria

A *Concession Strategy Profile* is a strategy profile where weak types eventually concede with probability 1.

Define $\underline{T}$ as the natural number that solves:

$$\tfrac{H}{H+G} + (\pi_0 - \tfrac{H}{H+G})(11 - H - G)^t \leq H \leq \tfrac{H}{H+G} + (\pi_0 - \tfrac{H}{H+G})(11 - H - G)^{t-1}.$$

Our result, stated below as Proposition 11, shows that if the initial probability of facing a weak type is $\pi_0 \in \left(0, \frac{H}{H+G}\right)$ in equilibrium players will not continue in the game indefinitely. Instead, we can identify a period $\underline{T}$, which depends upon the parameters of the game (a, b, δ, π_0), with the property that weak types will never delay play beyond period $\underline{T}$ and strong types never stay beyond $\underline{T}+1$. Moreover, if $\pi_0 \in (0, H]$, the game ends at $\underline{T} = 0$.

A Concession Equilibrium is described by finite pairs of sequences $\{\alpha_t\}_{t=0}^{\underline{T}}$ and $\{\beta_t\}_{t=0}^{\underline{T}+1}$ identifying the indifference valuations in each period and a sequence of beliefs $\{\pi_t\}_{t=0}^{\underline{T}}$. The posterior π_t deteriorates over time; as time passes, players become more pessimistic about their opponents being a weak type. That fact will naturally affect the probability of conceding α_t which increases over time and the probability of opting out β_t which decreases. At some time $t = \underline{T}$ the probability that her opponent is strong is so high that a weak type optimally concedes with probability 1 since the chance to receive her preferred agreement is too small. And, as stablished on Lemma 1, a strong type will opt out at $\underline{T}+1$ with probability 1 if this agent infers that her opponent is strong.

If the period $\underline{T}$ is reached by which a weak type would have conceded, the strong type infers that the opponent is as strong as she is. If both players prefer opting out rather than conceding, they will leave the negotiation immediately since there is no possibility to receive $1-a$ from their opponents and delaying their way out only decreases their payoffs.

The formal statement of this result follows:

Proposition 11. *If* $\pi_0 \in (0, H]$, *there is a unique SPBE such that* $\alpha_t = \beta_{t+1} = 1\, \forall t \geq 0$ *and* $\beta_0 = b(1-\delta) - \pi_0(1-a-\delta b)b(1-\delta)(1-\pi_0)$. *And if* $\pi_0 \in \left(H, \frac{H}{H+G}\right)$, *the unique SPBE is such that*:

$$\alpha_t = \frac{H}{\frac{H}{H+G}+(\pi_0-\frac{H}{H+G})(11-H-G)^t}, \forall t < \underline{T},$$
$$\beta_t = \frac{G}{1-\left[\frac{H}{H+G}+(\pi_0-\frac{H}{H+G})(11-H-G)^t\right]}, \forall t \leq \underline{T},$$

and $\alpha_t = \beta_{t+1} = 1\ \forall t \geq \underline{T}$.

Proof. We prove Proposition 11 for $\pi_0 \leq H$. The rest is detailed in the appendix.

Let us check first the optimal response of both types to the opponent's strategy $(\{\alpha_t\}_0^\infty \{\beta_t\}_0^\infty)$ such that $\alpha_t = \beta_{t+1} = 1\ \forall t \geq 0$ and $\beta_0 = \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$. Notice that, given this strategy of the opponent, $\pi_t = 0\ \forall t \geq 1$.

A weak type concedes optimally at t=0 if:

$$a > \pi_0\alpha_0(1-a) + a\delta(1-\pi_0\alpha_0-(1-\pi_0)\beta_0), fort = 0.$$

$$a > \pi_t\alpha_t(1-a) + a\delta(1-\pi_t\alpha_t-(1-\pi_t)\beta_t), \forall t \geq 1.$$

The second inequality is automatically satisfied since $\pi_t = 0$. And the first inequality is satisfied since $\pi_0 \leqslant H$.

Consider now a strong type. If the strategy of her opponent is to concede at t=0 with probability 1, then, by Lemma 1, she will opt out with probability 1 at t=1. And at t=0 she opts out with probability $\beta_0 = \frac{1}{1-\pi_0} - (\frac{\pi_0}{1-\pi_0})\frac{(1-a-\delta b)}{b(1-\delta)}$ since $b = \pi_0(1-a) + b(1-\pi_0)\beta_0 + b\delta(1-\pi_0)(1-\beta_0)$.

Now we prove that if $\pi_0 \leq H$, then the unique SPBE must be $(\{\alpha_t\}_0^\infty, \{\beta_t\}_0^\infty)$ such that $\alpha_t = 1\ \forall t \geq 0$ and $\beta_0 = \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$ $\beta_t = 1\ \forall t \geq 1$. To see that, indeed this is the unique SPBE, we explore all the other possible candidates.

First, assume that there is a SPBE $(\left\{\tilde{\alpha}_t\right\}_0^\infty, \left\{\tilde{\beta}_t\right\}_0^\infty)$ with $0 < \tilde{\alpha}_0 < 1$, $0 < \tilde{\beta}_0 < 1$ and $\tilde{\beta}_0 \neq \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$. If these were equilibrium strategies, then it must be true that:

$$\pi_0\, \tilde{\alpha}_0 = H.$$
$$(1-\pi_0)\, \tilde{\beta}_0 = 1 - G.$$

But since $\pi_0 \leq H$ then $\tilde{\alpha}_0 \geq 1$, a contradiction.

Second, assume that $(\left\{\tilde{\alpha}_t\right\}_0^\infty, \left\{\tilde{\beta}_t\right\}_0^\infty)$ is an equilibrium with $\tilde{\alpha}_0 = 1$ and $0 < \tilde{\beta}_0 < 1$ with $\tilde{\beta}_0 \neq \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$. Then if these strategies constitute a SPBE, it must be true that:

$$a > (1-a)\pi_0 + a\delta(1-\pi_0)(1-\tilde{\beta}_0),$$

and

$$b = (1-a)\pi_0 + b(1-\pi_0)\, \tilde{\beta}_0 + b\delta(1-\pi_0)(1-\tilde{\beta}_0).$$

But if $\tilde{\beta}_0 \neq \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$, the second condition is violated; either a strong type will deviate by opting out at t=0 if $\tilde{\beta}_0 < \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$ or by never opting out if $\tilde{\beta}_0 > \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$.

Finally, assume that $(\left\{\tilde{\alpha}_t\right\}_0^\infty, \left\{\tilde{\beta}_t\right\}_0^\infty)$ is an equilibrium with $0 < \tilde{\alpha}_0 < 1$ and $\tilde{\beta}_0 = \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$. Then,

$$a = (1-a)\pi_0\, \tilde{\alpha}_0 + a\delta(1-\pi_0\, \tilde{\alpha}_0 - (1-\pi_0)\, \tilde{\beta}_0).$$

$$b = (1-a)\pi_0\, \tilde{\alpha}_0 + b(1-\pi_0)\, \tilde{\beta}_0 + b\delta(1-\pi_0\, \tilde{\alpha}_0 - (1-\pi_0)\, \tilde{\beta}_0).$$

But if $\overset{\sim}{\beta}_0 = \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$ the first condition is not satisfied since $\overset{\sim}{\alpha}_0 >$ 1.■

Opting Out Equilibria

An *Opting Out Profile* is characterized by strong types taking their outside opportunities at some time with probability 1, leaving weak types to play as in the complete information War of Attrition from that time on.

Define as $\overline{T}$ the natural number that solves:

$$HH+G+(\pi_0-HH+G)(11-H-G)^{t-1} \leq 1-G \leq HH+G+(\pi_0-HH+G)(11-H-G)^{t}.$$

The next proposition shows that, if the probability of facing a weak type is relatively high, the optimal strategy of a strong type is such that she opts out at period $\overline{T} \geqslant 0$ with probability 1, and the optimal strategy of a weak type, from time $\overline{T}$ on, is to concede with a constant probability.

Proposition 12. *If* $\pi_0 \in \left(\frac{H}{H+G}, 1-G\right)$, *the unique SPBE is such that:*

$$\alpha_t = \frac{H}{\frac{H}{H+G}+(\pi_0-\frac{H}{H+G})(11-H-G)^t}, \quad \forall t \leq \overline{T},$$
$$\beta_t = \frac{G}{1-\left[\frac{H}{H+G}+(\pi_0-\frac{H}{H+G})(11-H-G)^t\right]}, \quad \forall t < \overline{T},$$

$\beta_t = 1$ *and* $\alpha_{t+1} = \alpha = \frac{a(1-\delta)}{1-a-\delta a}$ $\forall t \geq \overline{T}$.

And if $\pi_0 \in [1-G, 1)$ $\beta_t = 1$ $\forall t \geqslant 0$ *and* $\alpha_0 = a(1-\delta\pi_0)(1-a-\delta a)\pi_0$, $\alpha_t = \alpha = a(1-\delta)(1-a-\delta a)$ $\forall t \geq 1$.

Proof. See Appendix.■

In an Opting Out Equilibrium, weak types place a small probability of concession at each period. The posterior of facing a weak type opponent π_t increases over time, but the probability α_t that weak types concede decreases. In equilibrium, there will be some time $t = \overline{T}$ such that the optimal concession probability of the weak types cannot induce strong types to stay in the game beyond $\overline{T}$ since the payoff they get by opting out at that time, b, is greater than the expected payoff of waiting an aditional period for $(1-a)$. After $\overline{T}$ the posterior probability of facing a weak opponent is 1. Players that are still at the negotiation table recognize themselves as weak types and thus, from that period $\overline{T}$ on, they play the Symmetric Perfect Equilibrium of the complete information War of Attrition without outside options. In this continuation the equilibrium concession probability remains constant over time at $\alpha_{t+1} = \frac{a(1-\delta)}{1-a-\delta a}$ $\forall t \geq \overline{T}$.

On the other hand, Proposition 12 also tells us that if the initial probability of facing a weak type is close to 1, that is, if $\pi_0 \in [1-G, 1)$, then, in equilibrium, strong types opt out with probability 1 at $\overline{T} = 0$. In this case, even if the probability of facing a weak opponent is very high, the probability of receiving the preferred agreement is sufficiently low to make it worthwhile for a strong type to leave the negotiation immediately.

In an Opting Out Equilibrium players try to screen each other´s type by prolonging the game and thus imposing a delay cost on the opponent, as well as on themselves. After some time, strong types are convinced that they will never receive their preferred agreement and decide to opt out. From that moment on, nothing can convince players that the other will ever concede for sure, and thus they adopt the symmetric equilibrium strategies of the classical War of Attrition.

Pooling Equilibrium

The next proposition establishes the unique combination of parameters (a, b, δ, π_0) for which the SPBE is pooling. Players follow strategies such that both types randomize at the same constant rate between yielding and not yielding. Therefore, there is no learning and $\underline{T} = \infty$.

Proposition 13. *If* $\pi_0 = \frac{H}{H+G}$*, the unique symmetric PBE is* $\alpha_t = \beta_t = H + G$, $\forall t \geq 0$.

Proof. See Appendix.■

If the probability of facing a weak opponent is exactly $\pi_0 = \frac{H}{H+G}$, in equilibrium, both types remain indifferent about conceding and opting out at every time. That is, in terms of randomized strategies, each player believes, at each time, that the probabilities that the opponent concedes or opts out at subsequent times are exactly so as to make continuation marginally worthwhile. No information is revealed along this equilibrium. No player updates his beliefs about the weakness of her opponent since if players concede and opt out at each time with the same probability, the posterior π_t is constant over time.

The next table summarizes our results so far:

dtbpF356.5625pt183.4375pt0ptFigure

Our characterization of the unique SPBE allows meaningful comparative statics results. We carry out this exercise for the limit, as $\delta \rightarrow 1$. This is the object of the next section.

4.3 Comparative Statics.

In this section we conduct comparative statics by analyzing the effects of change in the parameters in the limit of the game as the interval between periods becomes arbitrarily small. Let the length of each period in real time be denoted by Δ, $0 << 1$(there are $\frac{1}{\Delta}$ periods per unit of time), so that we can replace the term δ by $e^{-\Delta}$. We are interested in the limit of SPBE as $\Delta \rightarrow 0$.

It is easily checked that $HH + G$ is independent of Δ and that $\Delta \rightarrow 0\lim H = 0$, $\Delta \rightarrow 0\lim G = 0$. The limit period $\underline{T}$, beyond which weak types will never continue in the negotiation in a Concession Equilibrium (see Proposition 11) and the limit period $\overline{T}$, beyond which strong types will surely opt out in an Opting Out Equilibrium (see Proposition

12), are given as functions[17] of the parameters of the game (a, b, π_0) as:

$$\underline{T}(\pi_0, a, b) = -a(1-a-b)(b-a(1-a))\ln\left[b(a-\pi_0)+a\pi_0(1-a)ab\right], for \pi_0 \in (0, HH+G),$$

$$\overline{T}(\pi_0, a, b) = -a(1-a-b)(b-a(1-a))\ln\left[b(\pi_0-a)-a\pi_0(1-a)(1-a)(b-a)\right] for \pi_0 \in (HH$$

Next figure displays $\underline{T}$ and $\overline{T}$ as functions of π_0 for a representative case ($a=\frac{1}{4}$, $b=\frac{3}{8}$).

dtbpF379.875pt148.5625pt0ptFigure

We want to evaluate how $\underline{T}$ and $\overline{T}$ change as the result of changes in the parameters (a, b, π_0). We carry out this exercise in order to measure the effect on those parameters changes in the efficiency. We conjecture that efficiency improves as $\underline{T}$ decreases and $\overline{T}$ increases. A general proof for this conjecture is work in progress.

Next proposition establishes that an increase in the likelihood that the opponent has a valuable outside option reduces $\underline{T}$ and increases $\overline{T}$.

Proposition 14. *$\underline{T}$ decreases and $\overline{T}$ increases as π_0 decreases.*

Proof. See Appendix.■

Next we will analyze the effect of an increase of a on the limit periods $\underline{T}$ and $\overline{T}$. Define the following sets of parameters:

$$S_1 = \left\{(a, b) such that b \leq (1-a)^2\right\},$$

$$S_2 = \left\{(a, b) such that b > (1-a)^2 and \frac{a(1-a-b)(b-a^2)}{(1-a)(b-a)(b-(1-a)^2)} > 1\right\},$$

$$S_3 = \left\{(a, b) such that b > (1-a)^2 and 0 < \frac{a(1-a-b)(b-a^2)}{(1-a)(b-a)(b-(1-a)^2)} < 1\right\}.$$

[17]See appendix for the derivation of these functions.

Let $x = b(\pi_0 - a) - a\pi_0(1-a)(1-a)(b-a)$ and $\widetilde{x}$ be the solution to:

$$(b-(1-a)^2)\ln[x]+a(1-a-b)(b-a^2)(1-a)(b-a)(\frac{1}{x}-1) = 0. \qquad (2)$$

Proposition 15. *(i) $\underline{T}$ decreases as a increases. (ii) $\overline{T}$ increases as a increases $\forall(a,b) \in S_1 \cup S_2$. If $(a,b) \in S_3$, then $\frac{\partial \overline{T}}{\partial a} \geq 0$ if $x \in \left(0, \widetilde{x}\right]$ and $\frac{\partial \overline{T}}{\partial a} < 0$ if $x \in (\widetilde{x}, 1)$.*

Proof. See Appendix.■

We see that efficiency improves as a increases if a and b are close since an increase on a reduces the time at which weak types concede with probability 1 in a Concession Equilibrium, and increases the time at which strong types opt out with probability 1 in an Opting Out Equilibrium. However, the effect of the concession payoff a on $\overline{T}$ when a and b are far, depends on the relationship between a, b and π_0. In the next example we find the initial probability π_0^* such that $\frac{\partial \overline{T}}{\partial a} \geqslant 0$ if $\pi_0 \in \left(\frac{H}{H+G}, \pi_0^*\right]$ and $\frac{\partial \overline{T}}{\partial a} < 0$ if $\pi_0 \in (\pi_0^*, 1)$.

a	b	π_0^*
$\frac{1}{3}$	$\frac{58}{100}$	0.722556
$\frac{1}{3}$	$\frac{6}{10}$	0.631353
$\frac{1}{3}$	$\frac{62}{100}$	0.562209

If the value of the outside option is only slightly greater than the concession payoff, then the range of probabilities π_0 for which an increase of the size of the concession payoff improves efficiency is bigger.

Finally we analyze the effect of an increase in the value of the outside option, b, on $\underline{T}$ and $\overline{T}$. Define $y = b(a - \pi_0) + a\pi_0(1-a)ab$ and let $\widetilde{y}$ the solution to:

$$\ln[y] + a(1-a-b)b(1-a)(\frac{1}{y} - 1) = 0. \qquad (2)$$

Proposition 16. *(i)* $\overline{T}$ *increases as* b *increases. (ii)* $\frac{\partial \underline{T}}{\partial b} \leq 0$ *if* $y \in \left(0, \widetilde{y}\right]$ *and* $\frac{\partial \underline{T}}{\partial b} > 0$ *if* $y \in (\widetilde{y}, 1)$.

Proof. See Appendix.■

If the value of the outside option increases, strong types take longer to opt out with probability 1 in an Opting Out Equilibrium. However, the effect of b on $\underline{T}$, is not clear cut. In this case the sign of this derivative will depend on the relationship between a, b and π_0. Since is not possible to find an analytical solution to the equation (2), we make some numerical computations. Notice that finding $\widetilde{y}$ is equivalent to find the initial probability π_0^* such that $\frac{\partial \underline{T}}{\partial b} < 0$ if $\pi_0 \in (0, \pi_0^*]$ and $\frac{\partial \underline{T}}{\partial b} > 0$ if $\pi_0 \in (\pi_0^*, \frac{H}{H+G})$. The following table shows some numerical examples:

a	b	π_0^*
$\frac{1}{3}$	$\frac{2}{5}$	0.638251
$\frac{1}{4}$	$\frac{2}{5}$	0.413369
$\frac{1}{5}$	$\frac{2}{5}$	0.301117

We see that the difference between a and b matters. If the value of the outside option is only slightly greater than the concession payoff, then the range of probabilities π_0 for which an increase of the size of the outside option improves efficiency is bigger.

4.4 Conclusions.

In this chapter we have explored the effect of the private information about outside options on the outcomes of negotiations. In order to address this issue we analyzed a War of Attrition allowing players to leave the negotiation in order to opt out and we characterized the Symmetric Perfect Bayesian Equilibrium of this game. There are two types of players: a weak type who has a valueless outside option-she always prefers conceding rather than opting out- and a strong type who has a valuable outside option that she prefers to take rather than conceding. We show that uncertainty about the possibility that the opponent opts out improves efficiency, since it increases the equilibrium probability of concession. More precisely, if the probability that the opponent is strong is relatively high, in equilibrium, the negotiation eventually ends with a sure concession. In these cases, we are able to identify a time $\underline{T}$ at which a player with a valueless outside option, will concede with probability 1, and a player with an outside option will wait to obtain a concession until $\underline{T}+1$; then, she will opt out with probability 1. On the other extreme, if the likelihood of a weak opponent is high, strong types stay in the game for a while and eventually leave the negotiation and opt out with probability 1. From that date $\overline{T}$ on, weak types play the (inefficient) symmetric equilibrium of the classical War of Attrition with complete information. Even in this case, the probability of concession by weak types along the uncertainty phase of the equilibrium play increases.

Appendix

Proof of Proposition 2 (ii).

STEP 1: *For states that satisfy* $\alpha(x_i + \frac{X}{2}) \geq \delta^2(x_i + X)$, $i = 1, 2$, *the statement is proved as in the proof of Proposition 3 that follows.*

These states displayed in Figure 8.

dtbpFU354.625pt245pt0ptFigure 8Figure

STEP 2: *If* $x_2 \geq \alpha(x_2 + \frac{X}{2})$ *and* $\alpha(x_1 + \frac{X}{2}) < \delta^2(x_1 + X)$ *player* 2 *concedes* X *and player* 1 *concedes nothing.*

For player 2 arbitration is dominated by conceding X since $x_2 \geq \alpha(x_2 + \frac{X}{2})$. If player 2 concedes $\widetilde{C}_2$ such that $x_1 + \widetilde{C}_2 = \delta(x_1 + X)$ she gets a payoff of $\delta(x_2 + X - \widetilde{C}_2) < x_2$ since $x_2 \geq \frac{\delta}{1+\delta}$. If she concedes nothing (or make a concession C_2 such that the new bargaining state belongs to the same set), she will get, at most, $\delta^2(x_2 + \widetilde{C}_1)$ and $x_2 > \delta^2(x_2 + \widetilde{C}_1)$. Thus, player 2 concedes X.

For player 1 conceding 0 dominates any other concession $0 < C_1 \leq X$ since player 2 will concede at her turn the rest of the pie, and player 1 will get $\delta(x_1 + X) > x_1$. To propose arbitration is also a dominated alternative since $\delta(x_1 + X) > \delta^2(x_1 + X) > \alpha(x_1 + \frac{X}{2})$.

STEP 3: *If* $x_2 \geq \frac{\delta}{1+\delta}$, $x_2 < \alpha(x_2 + \frac{X}{2})$ *and* $\alpha(x_1 + \frac{X}{2}) < \delta^2(x_1 + X)$ *player* 2 *concedes* X *and player* 1 *concedes nothing.*

If player 2 concedes $C_2, 0 \leq C_2 < X$, the game continues. If C_2 is such that, at the new bargaining state, player 1 concedes nothing player 2 gets a final payoff of $\delta^2 x_2 < x_2$. If C_2 is such that, at the new bargaining state, the maximal concession she can expect from her opponent is $C_1^\diamond$, player 2 gets at most $\delta^2(x_2 + C_1^\diamond) < x_2$, or equivalently $x_2 > \frac{\delta^2\alpha}{2-\alpha-\alpha\delta^2}X$ ($x_2 \geq \frac{\delta}{1+\delta} > \frac{\delta^2\alpha}{2-\alpha-\alpha\delta^2}\frac{1}{(1+\delta)}$ since $\frac{\alpha}{2} < \frac{\delta}{1+\delta}$ and $\frac{\delta^2\alpha}{2-\alpha-\alpha\delta^2}\frac{1}{(1+\delta)} \geq \frac{\delta^2\alpha}{2-\alpha-\alpha\delta^2}X$

since $X \leq \frac{1}{1+\delta}$). Thus, for player 2 conceding X dominates any other concession. On the other hand, 2 does not propose arbitration because player 1 will reject it since rejection gives player 1 at least $\delta^2(x_1+X) > \alpha(x_1+\frac{X}{2})$.

For player 1, to concede nothing dominates arbitration since $\delta(x_1+X) > \delta^2(x_1+X) > \alpha(x_1+\frac{X}{2})$; and any concession $0 < C_1 \leq X$ is dominated as well since $\delta(x_1+X) > \delta(x_1+X-C_1)$.

Steps 2 and 3 are summarized in Figure 9.

dtbpFU354.625pt245pt0ptFigure 9Figure

STEP 4: *If* $x_i < \frac{\delta}{1+\delta}$ $i=1,2$, $x_1 \leq \frac{2\delta^2-\alpha}{\alpha(1+\delta)}$ *and* $x_2 > \frac{2\delta^2-\alpha}{\alpha(1+\delta)}$ *player* 1 *concedes* C_1^* *such that* $x_2+C_1^* = \frac{\delta}{1+\delta}$ *and player 2 makes a concession* $C_2 = Max\,\{C_2^*, C_2^\diamond\}$ *where* $x_1+C_2^* = \frac{\delta}{1+\delta}$ *and* $x_1+C_2^\diamond = \alpha(x_1+\frac{X+C_2^\diamond}{2})$.

This set corresponds to set 1 in Figure 9. If player 1 concedes C_1^* the opponent concedes the rest of the pie in the next turn, and she gets a final payoff of $\frac{\delta}{1+\delta}$. To propose arbitration is dominated by conceding C_1^* since $\alpha(x_1+\frac{X}{2}) < \frac{\alpha}{2} < \frac{\delta}{1+\delta}$. Clearly to make a concession $C_1 > C_1^*$ too. Suppose now that player 1 concedes $C_1 < C_1^*$. At the new bargaining state the opponent may optimally:

(i) call the arbitrator and player 1 would get a final payoff of $\alpha\delta(x_1+\frac{X-C_1}{2}) < \frac{\delta}{1+\delta}$.

(ii) concede at most C_2^* and player's 1 final payoff is $\frac{\delta^3}{1+\delta} < \frac{\delta}{1+\delta}$.

(iii) concede at most $C_2^\diamond$ and player 1 gets at most $\delta^2(x_1+C_2^\diamond) < \frac{\delta}{1+\delta}$.

Then, the best alternative for player 1 is to concede C_1^*.

In order to prove that player 2 concedes $C_2 = Max\{C_2^*, C_2^\diamond\}$ we use the same argument as in step 5 of proposition 3.

STEP 5: *If* $x_i < \frac{\delta}{1+\delta}$ $x_i > \frac{2\delta^2-\alpha}{\alpha(1+\delta)}$ $i=1,2$ *and* $\alpha(x_1+\frac{X}{2}) < \delta^2(x_1+X)$ *both players concede* $C_i = Max\,\{C_i^*, C_i^\diamond\}$ *with* C_i^* *defined as* $x_j+C_i^* = \frac{\delta}{1+\delta}$ *and* $C_i^\diamond$ *defined as* $x_j+C_i^\diamond = \alpha(x_j+\frac{X+C_i^\diamond}{2})$.

This set corresponds to set 2 in Figure 9. We use here the same argument as the one used in step 5 proposition 3 to prove that player 2 concedes $C_2 = Max\,[C_2^*, C_2^\diamond]$.

STEP 6: *If* $x_i < \frac{\delta}{1+\delta}$ *and* $x_i \leq \frac{2\delta^2-\alpha}{\alpha(1+\delta)}$ $i = 1, 2$ *then both players concede* C_i^* *defined as* $x_j + C_i^* = \frac{\delta}{1+\delta}$.

This set corresponds to set 3 in Figure 9. We use here the same argument as in step 4 to prove that player i concedes C_i^*.

Proof of Proposition 3.

STEP 1: *For states such that* $x_i \geq \delta(x_i + X)$ *for at least one player the proof follows as in Steps 1 and 2 of the proof of Proposition 2 (i).*

STEP 2: *For states such that* $x_i < \delta(x_i + X)$ *for* $i = 1, 2$, $x_2 \geq \frac{\delta}{1+\delta}$ *and* $x_2 \geq \alpha(x_2 + X2)$ *the proof follows as in step 3 of the proof of Proposition 2 (i).*

STEP 3: *For states such that* $x_i < \delta(x_i + X)$ *for* $i = 1, 2$, $x_2 \geq \frac{\delta}{1+\delta}$ *and* $x_2 < \alpha(x_2 + X2)$ *player* 2 *proposes arbitration and player* 1 *concedes* $C_1^\diamond$ *such that* $x_2 + C_1^\diamond = \alpha(x_2 + \frac{X+C_1^\diamond}{2})$.

If $x_2 < \alpha(x_2 + \frac{X}{2})$ conceding X is a dominated strategy for player 2 if the proposal of arbitration cannot be vetoed by her opponent. Player 1 cannot reject this proposal since if she does she will get, at most, a payoff of $\delta^2(x_1+X) < \alpha(x_1+\frac{X}{2})$ since $\alpha \geq Max\left\{2\delta^2, \frac{2\delta}{1+2\delta}\right\}$. To propose arbitration also dominates to make any other concession $0 \leq C_2 < X$.

Consider now player 1. If she makes a concession $C_1^\diamond$ such that $x_2 + C_1^\diamond = \alpha(x_2 + \frac{X+C_1^\diamond}{2})$, at the new bargaining state player 2 optimally concedes the rest of the pie (by step 2). To make a concession $C_1 > C_1^\diamond$ is clearly dominated. If she concedes C_1, $0 \leq C_1 < C_1^\diamond$, the new state is such that $x_i < \delta(x_i + X)$ for $i = 1, 2$, $x_2 \geq \frac{\delta}{1+\delta}$ and $x_2 < \alpha(x_2 + X2)$, at which player 2 (optimally) proposes arbitration. Thus, if player 1 concedes $0 \leq C_1 < C_1^\diamond$ she gets at most $\alpha\delta(x_1 + \frac{X-C_1}{2}) < \alpha(x_1 + \frac{X}{2})$. And finally, to propose arbitration is dominated by conceding $C_1^\diamond$ since $\alpha(x_1+\frac{X}{2}) \leq \alpha(\frac{\alpha(1+\delta)-\delta}{\alpha(1+\delta)}) < \frac{2\delta(1-\alpha)}{2-\alpha} \leq \delta(x_1+X-C_1^\diamond)$ given that $11+\delta \geq 2(1-\alpha)2-\alpha$. Thus, proposing $C_1^\diamond$ dominates the other alternatives.

To characterize optimal actions for states such that $x_i < \frac{\delta}{1+\delta}$ for $i = 1, 2$ it is convenient to consider the following partition into subsets L_1,L_2^+, its symmetric L_2^- and L_3 defined as follows:

$$L_1 = \left\{ (x_1, x_2, X) such that x_i < \frac{\delta}{1+\delta}, x_i \geq 1 - \frac{\delta(2-\alpha)}{\alpha(1+\delta)} i = 1, 2 \right\}$$

$$L_2^+ = \left\{ (x_1, x_2, X) such that x_i < \frac{\delta}{1+\delta} i = 1, 2 and x_2 - x_1 \geq \frac{2\delta}{\alpha(1+\delta)} - 1 \right\}$$

$$L_2^- = \left\{ (x_1, x_2, X) such that x_i < \frac{\delta}{1+\delta} i = 1, 2 and x_2 - x_1 \leq 1 - \frac{2\delta}{\alpha(1+\delta)} \right\}$$

$$L_3 = \left\{ (x_1, x_2, X) \notin L_1, L_2^+, L_2^- and x_i < \frac{\delta}{1+\delta} i = 1, 2 \right\}.$$

Figure 10 displays L_1,L_2^+, L_2^- and L_3:

dtbpFU354.625pt295.9375pt0ptFigure 10Figure

STEP 4: *If* $(x_1, x_2, X) \in L_1$ *player* 2 *concedes* C_2^* *such that* $x_1 + C_2^* = \frac{\delta}{1+\delta}$ *and player* 1 *concedes* C_1^* *such that* $x_2 + C_1^* = \frac{\delta}{1+\delta}$.

If 2 concedes C_2^* she gets a payoff of $\frac{\delta}{1+\delta}$ because player 1, at her turn, will concede the rest, $X - C_2^*$. Any another $C_2 \neq C_2^*$ is dominated. The case $C_2 > C_2^*$ is clear, while if $C_2 < C_2^*$ player 2 cannot expect more than $\frac{\delta^3}{1+\delta} < \frac{\delta}{1+\delta}$. Finally, C_2^* dominates arbitration since $\alpha(x_2 + X2) \leq \frac{\delta}{1+\delta}$. By the same argument we prove that 1 concedes C_1^*.

STEP 5: *If* $(x_1, x_2, X) \in L_2^+$ *player* 2 *proposes arbitration and player* 1 *concedes* $C_1^\diamond$ *such that* $x_2 + C_1^\diamond = \alpha(x_2 + \frac{X + C_1^\diamond}{2})$.

Take player 2 and consider in turn states such as **e** and **f** in Figure 11.

dtbpFU354.625pt295.9375pt0ptFigure 11Figure

Let the state be **e**. If 2 concedes C_2^* she gets $\frac{\delta}{1+\delta}$. To concede more than C_2^* is clearly dominated. To concede less than C_2^* is also dominated: A concession $C_2 < C_2^*$ leads to a new state in L_1, L_3 or still in L_2^+. If the new state lies in L_1 player 2 can get at most $\frac{\delta^3}{1+\delta}$, if it lies in L_3 or L_2^+ the concession of player 1 is at most $C_1^\diamond$ such that $x_2 + C_1^\diamond = \alpha(x_2 + \frac{X+C_1^\diamond}{2})$, and 2 gets at most $\delta^2(x_2 + C_1^\diamond)$. Hence, arbitration dominates any other alternative since $\delta^2(x_2 + C_1^\diamond) \leq \frac{\delta^3}{1+\delta} < \frac{\delta}{1+\delta} \leq \alpha(x_2 + X2)$. Player 1 must accept arbitration since $\alpha(x_1 + \frac{X}{2}) > \delta^2(x_1 + X) > \delta^2(x_1 + X - C_1^\diamond)$ and $\alpha(x_1 + \frac{X}{2}) > \delta x_1$. Consider now states such as **f**. If 2 concedes $C_2^\diamond$ she gets a payoff $\delta(x_2 + X - C_2^\diamond)$. To concede more than $C_2^\diamond$ is dominated. If she concedes $C_2 < C_2^\diamond$, the new bargaining state lies in L_3 or L_2^-. If it lies in L_3, player 2 can expect at most $\frac{\delta^3}{1+\delta}$ since $\delta^2(x_2 + C_1^\diamond) < \frac{\delta^3}{1+\delta}$ and $\frac{\delta^3}{1+\delta} < \delta(x_2 + X - C_2^\diamond)$. At the same time, to concede $C_2^\diamond$ is dominated by arbitration since $\delta(x_2 + X - C_2^\diamond) < \delta(x_2 + X - C_2^*) = \frac{\delta}{1+\delta} \leq \alpha(x_2 + X2)$. If the new state lies in L_2^- player 2 cannot expect more than $\alpha\delta(x_2 + \frac{X-C_2}{2}) < \alpha(x_2 + \frac{X}{2})$.

Let us now see that player 1 must concede $C_1^\diamond$. Note first that arbitration is dominated since $\alpha(x_1 + X2) \leq \frac{\delta}{1+\delta} \leq \delta(x_1 + X - C_1^\diamond)$. Moreover, while conceding $C_1 > C_1^\diamond$ is obviously dominated, a concession $C_1 < C_1^\diamond$ leads the game to a state where player 2 proposes arbitration and player 1 accepts. Since this alternative pays $\delta\alpha(x_1 + X - C_1 2) < \alpha(x_1 + X2) \leq \frac{\delta}{1+\delta} \leq \delta(x_1 + X - C_1^\diamond)$, player 1 must concede $C_1^\diamond$.

STEP 6: *If* $(x_1, x_2, X) \in L_3$ *player* 1 *concedes* $C_1 = Max\,\{C_1^*, C_1^\diamond\}$ *and player* 2 *concedes* $C_2 = Max\,\{C_2^*, C_2^\diamond\}$ *where* $x_j + C_i^* = \frac{\delta}{1+\delta}$ *and* $x_j + C_i^\diamond = \alpha(x_j + \frac{X+C_i^\diamond}{2})$.

dtbpFU354.625pt295.9375pt0ptFigure 12Figure

Consider player 1. If $C_1^* \geq C_1^\diamond$ (as in state **g** in Figure 12) player 1 concedes C_1^*, getting a final payoff of $\frac{\delta}{1+\delta}$. This alternative dominates arbitration since $\alpha(x_1 + X2) \leq \frac{\delta}{1+\delta}$. To concede more than C_1^* is easily ruled out. To concede less leads to a state in L_1 or L_3. If the new state is in L_1, then 2 responds conceding C_2^* and 1 obtains $\frac{\delta^3}{1+\delta} < \frac{\delta}{1+\delta}$. If the new state remains in L_3, 1 can expect at most $\delta(x_1 + X - C_2^\diamond) \leq \frac{\delta^2\alpha}{2-\alpha}($

$C_2^\diamond$ is at most $\frac{\alpha}{2-\alpha}$). But $\frac{\delta^2\alpha}{2-\alpha} < 2\delta(1-\alpha)2-\alpha < \delta 1+\delta$.[18]

If $C_1^* < C_1^\diamond$ (as in state **h** in Figure 12) player 1 concedes $C_1^\diamond$, and obtains $\delta(x_1 + X - C_1^\diamond)$. Arbitration is dominated since $\alpha(x_1 + X2) < \delta(x_1 + X - C_1^\diamond)$. To concede more than $C_1^\diamond$ is dominated. And to concede less that $C_1^\diamond$ is dominated as well since the new state is either in L_2^+ or in L_3. If the new state is in L_2^+ player 2 calls the arbitrator at her turn. If the new state remains in L_3, the greatest concession that 1 can expect from 2 is $C_2 = Max\{C_2^*, C_2^\diamond\}$, and this pays at most $Max\{\delta^3 1+\delta, \delta^2\alpha 2-\alpha\} \leq 2\delta(1-\alpha)2-\alpha \leq \delta(x_1 + X - C_1^\diamond)$.

Observe now that 2 faces the same situation as player 1; then, for player 2 to concede $C_2 = Max\{C_2^*, C_2^\diamond\}$ dominates all the other alternatives.

We have exhausted all possible bargaining states for $(\alpha, \delta) \in I$.

Proof of Proposition 4

Consider $(\alpha, \delta) \in Q_1$. Unless the state is in L_3, the optimal action for parameters Q_1 are the same as the optimal actions for I.

STEP 1: *If $(x_1, x_2, X) \in L_3$ player i concedes $C_i = Max\left\{C_i^\diamondsuit, C_i^*\right\}$ if $\delta(x_i + X - C_i) \geq \alpha(x_i + \frac{X}{2})$. Otherwise she proposes arbitration.*

Consider player 1. As in step 6 of proposition 3 if player 1 considers to make a concession that must be $C_1 = Max\left\{C_1^\diamondsuit, C_1^*\right\}$. This alternative dominates to propose arbitration if $\delta(x_1+X-C_1) \geq \alpha(x_1+\frac{X}{2})$. If $\delta(x_1 + X - C_1) < \alpha(x_1 + \frac{X}{2})$ player 1 proposes arbitration and player 2 accepts this proposal. Accepting pays $\alpha(x_2+\frac{X}{2}) > Max\{\delta^2(x_2 + X), \delta x_2\}$ since $\alpha \geq 2\delta^2$ and $\alpha > \delta$.

Let us now consider when $(\alpha, \delta) \in Q_2$.

STEP 1: *If $x_i \geq \alpha(x_i + X2)$ for $i = 1, 2$ both players concede X.*

Arbitration is clearly dominated for both players. To concede $0 \leq$

[18] Since $\frac{\alpha}{2} < \frac{2\delta(1-\alpha)}{2-\alpha}$ and $\delta < \alpha$, then $\alpha \leq \frac{2}{3}$. That means that $\frac{\alpha}{2-\alpha} < \frac{2(1-\alpha)}{2-\alpha}$ and $\frac{\delta^2\alpha}{2-\alpha} < 2\delta(1-\alpha)2-\alpha$.

$C_i < X$ is dominated as well, since $x_i \geq \alpha(x_i + X2) > \delta(x_i + X) > \delta(x_i + X - C_i)$.

STEP 2: *If* $\alpha(x_i + \frac{X}{2}) \geq \delta(x_i + X)$ *for* $i = 1, 2$, $x_2 \geq \alpha(x_2 + X2)$ *and* $x_1 < \alpha(x_1 + X2)$ *player* 2 *concedes* X *and player* 1 *proposes arbitration.*

The optimal strategy of player 2 is to concede X as in Step 1. Player 1 prefers to propose arbitration rather than conceding C_1, $0 \leq C_1 \leq X$ since $\alpha(x_1 + X2) > x_1 \geq \delta(x_1 + X) > \delta(x_1 + X - C_1)$. Player 2 accepts the arbitration proposal since if she rejects, at her turn, she concedes X getting a payoff of $\delta x_2 < \alpha(x_2 + X2)$ since $\alpha > \delta$.

STEP 3: *If* $\alpha(x_i + \frac{X}{2}) \geq \delta(x_i + X)$, $x_i < \alpha(x_i + X2)$ *for* $i = 1, 2$,*both players propose arbitration.*

Consider player 2. To propose arbitration dominates to concede C_2, $0 \leq C_2 \leq X$ since $\alpha(x_2 + X2) > x_2$ and $\alpha(x_2 + X2) \geq \delta(x_2 + X) > \delta(x_2 + X - C_2)$. Player 1 accepts this proposal, since if she rejects she will get at most $\max[\delta x_1, \delta^2(x_1 + X)] < \alpha(x_1 + X2)$. The same argument applies for player 1.

Next figure shows steps 1,2 and 3 of proposition 4 (ii).

dtbpFU354.625pt245pt0ptFigure 13Figure

STEP 4: *If* $x_2 \geq \alpha(x_2 + X2)$, $x_1 < \alpha(x_1 + X2)$, $\alpha(x_2 + X2) \geq \delta(x_2 + X)$ *and* $\alpha(x_1 + X2) < \delta(x_1 + X)$ *player* 2 *concedes* X *and player* 1 *concedes nothing.*

Player 2 concedes X since dominates to propose arbitration ($x_2 \geq \alpha(x_2 + X2)$) and conceding C_2,$0 \leq C_2 < X$ ($x_2 \geq \alpha(x_2 + X2) \geq \delta(x_2 + X) > \delta(x_2 + X - C_2)$). Player 1 concedes nothing since her opponent concedes the rest of the pie next period and she gets a payoff of $\delta(x_1 + X) > \alpha(x_1 + X2) > x_1$ and $\delta(x_1 + X) > \delta(x_1 + X - C_1)$.

STEP 5: *If* $x_i < \alpha(x_2 + X2)$ *for i=1,2,* $\alpha(x_2 + X2) \geq \delta(x_2 + X)$ *and* $\alpha(x_1 + X2) < \delta(x_1 + X)$ *player* 2 *proposes arbitration and player* 1 *concedes* $C_1^\diamond$ *such that* $x_2 + C_1^\diamond = \alpha(x_2 + \frac{X + C_1^\diamond}{2})$ *if* $\delta(x_1 + X - C_1^\diamond) \geq \alpha(x_1 + X2)$ *and proposes arbitration if* $\delta(x_1 + X - C_1^\diamond) < \alpha(x_1 + X2)$.

Consider player 2. she proposes arbitration since this alternative dominates conceding C_2, $0 \leq C_2 \leq X$ since $\alpha(x_2 + X2) > x_2$ and $\alpha(x_2 + X2) \geq \delta(x_2 + X) > \delta(x_2 + X - C_2)$. Player 1 accepts this proposal since $\alpha(x_1 + X2) > Max\,\{\delta^2(x_1 + X), \delta x_1\}$.

Now consider player 1. Player 1 will not concede X since it is dominated by proposing arbitration. If player 1 proposes arbitration player 2 accepts since $\alpha(x_2 + X2) > \max[\delta x_2, \delta^2(x_2 + X)]$. She can also make a concession $C_1^\diamond$ defined as $x_2 + C_1^\diamond = \alpha(x_2 + \frac{X+C_1^\diamond}{2})$ such that player 2 will concede the rest of the pie next period. She will not concede more or less than this amount. Obviously, to concede more than $C_1^\diamond$ is a dominated strategy. And to make a concession $C_1 < C_1^\diamond$ too since, at the new bargaining state, player 2 will optimally propose arbitration. Thus player 1 optimally concedes $C_1^\diamond$ if $\alpha(x_1 + X2) < \delta(x_1 + X - C_1^\diamond)$ and optimally proposes arbitration if $\alpha(x_1 + X2) \geq \delta(x_1 + X - C_1^\diamond)$.

STEP 6: *If* $x_i < \alpha(x_2+X2)$ *and* $\alpha(x_i+X2) < \delta(x_i+X)$ *for* $i = 1, 2$. *player* i *makes a concession* $C_i^\diamond$ *defined as* $x_j + C_i^\diamond = \alpha(x_j + \frac{X+C_i^\diamond}{2})$ *if* $\delta(x_i+X-C_i^\diamond) \geq \alpha(x_i+X2)$ *and proposes arbitration if* $\delta(x_i+X-C_i^\diamond) \geq \alpha(x_i + X2)$.

To prove this step we apply the same argument as the one applied to player 1 in step 5.

And finally we will prove the optimal actions when $(\alpha, \delta) \in Q_3$.

The graphical representation for Q_3 is displayed in Figure 14.

dtbpFU354.625pt245pt0ptFigure 14Figure

We will not prove the optimal actions of both players for bargaining states satisfying $\alpha(x_i + \frac{X}{2}) \geq \delta^2(x_i + X)$ for $i = 1, 2$ since the proof is identical than for Q_2. Without loss of generality we will prove the optimal actions for the bargaining sets that satisfy $\alpha(x_1 + \frac{X}{2}) < \delta^2(x_1 + X)$.

STEP 1: *If* $x_2 \geq \alpha(x_2 + X2)$ *and* $\alpha(x_2 + \frac{X}{2}) \geq \delta^2(x_2 + X)$ *player* 2 *concedes* X *and player* 1 *concedes nothing.*

For player 2 conceding X dominates arbitration and conceding C_2, $0 \leq C_2 < X$ since $x_2 \geq \alpha(x_2+X2)$ and $x_2 \geq \delta(x_2+X) > \delta(x_2+X-C_2)$. Player 1 concedes nothing since she gets $\delta(x_1+X) > x_1$ and $\delta(x_1+X) > \alpha(x_1+\frac{X}{2})$.

STEP 2: *If* $x_2 < \alpha(x_2+X2)$ *and* $\alpha(x_2+\frac{X}{2}) \geq \delta^2(x_2+X)$ *player* 2 *proposes arbitration and player* 1 *concedes* $C_1^{\diamond}$ *such that* $x_2 + C_1^{\diamond} = \alpha(x_2+\frac{X+C_1^{\diamond}}{2})$ *if* $\delta(x_1+X-C_1^{\diamond}) \geq \alpha(x_1+X2)$ *and proposes arbitration if* $\delta(x_1+X-C_1^{\diamond}) \geq \alpha(x_1+X2)$.

In order to prove this assertion suppose player 2 concedes $C_2 < X$. she knows that some concessions lead the game to bargaining states where player 1 optimally proposes arbitration and others concessions lead the game to bargaining states where player 1 concede $C_1^{\diamond}$. Thus, if she considers conceding, she gets either $\delta\alpha(x_2+\frac{X-C_2}{2})$ or $\delta^2(x_2+C_1^{\diamond})$. But proposing arbitration dominates to conceding since $\alpha(x_2+\frac{X}{2}) \geq \delta\alpha(x_2+\frac{X-C_2}{2})$, then $\alpha(x_2+\frac{X}{2}) \geq \delta^2(x_2+C_1^{\diamond})$ and $\alpha(x_2+\frac{X}{2}) > x_2$. Player 1 accepts this proposal since $\alpha(x_1+X2) > Max\left[\delta^2(x_1+X-C_1^{\diamond}), \delta x_1\right]$.

Now consider player 1. Player 1 will not concede X since it is dominated by proposing arbitration. If player 1 proposes arbitration player 2 accepts since $\alpha(x_2+X2) > \max\left[\delta x_2, \delta^2(x_2+X-C_2^{\diamondsuit})\right]$. She can also make a concession such that player 2 will concede the rest of the pie next period, that is, $C_1^{\diamond}$ defined as $x_2 + C_1^{\diamond} = \alpha(x_2+\frac{X+C_1^{\diamond}}{2})$. She will not concede more or less than this amount. Obviously, to concede more than $C_1^{\diamond}$ is a dominated strategy. And to make a concession $C_1 < C_1^{\diamond}$ too since, at the new bargaining state, player 2 will optimally propose arbitration. Thus player 1 optimally concedes $C_1^{\diamond}$ if $\alpha(x_1+X2) < \delta(x_1+X-C_1^{\diamond})$ and optimally proposes arbitration if $\alpha(x_1+X2) \geq \delta(x_1+X-C_1^{\diamond})$.

STEP 3: *If* $\alpha(x_2+\frac{X}{2}) < \delta^2(x_2+X)$ *player* i *concedes* $C_i^{\diamond}$ *if* $\delta(x_1+X-C_i^{\diamond}) \geq \alpha(x_i+X2)$ *and proposes arbitration if* $\delta(x_i+X-C_i^{\diamond}) \geq \alpha(x_i+X2)$.

The proof of this step is straight forward using the same argument as the one applied to prove the optimal action of player 1 in step 2

Proof of Proposition 6

Define as $P_l = \{(x_1, x_2, X) such that X = l\}$. We use the method of induction to prove proposition 1. First, we prove that these strategies are optimal for P_l with $l \leq 4$. We assume that are optimal for P_l such that $l < n$ and prove that are optimal for P_n. Without loss of generality we consider $i = 1$ and $j = 2$.

(i) For $(x_1, x_2, X) \in P_1, P_2$ player 1 proposes $(x_1, x_2 + 1)$ and $(x_1 + 1, x_2 + 1)$ respectively and player 2 accepts. The proof is straight forward.

(ii) Take now any bargaining state that belongs to P_3. The equilibrium strategies of players are:

state	i demand	j response
$x_j \geq Z_1 - 1$	$x_i + 2$	A
$x_i \leq x_j < Z_1 - 1$	$x_i + 1$	A
$x_j < x_i < Z_1 - 1$	$x_i + 1$	A
$x_i \geq Z_1 - 1$	$x_i + 1$	A

STEP 1: *If* $x_2 \geq Z_1 - 1$ *player* 1 *proposes* $(x_1 + 2, x_2 + 1)$ *and player* 2 *accepts.*

Suppose that player 2 rejects player's 1 offer. The new bargaining state belongs to P_2 and player 2 offers $(x_1 + 2$,$x_2 + 1)$ and player 1 accepts. Player 2 gets a payoff of $\delta(x_2 + 2) < x_2 + 1$ since $x_2 \geq Z_1 - 1$. Clearly, player 1 will not make a different offer since is the best offer she can make.

STEP 2. *If* $x_1 \leq x_2 < Z_1 - 1$ *player* 1 *proposes* $(x_1 + 1, x_2 + 2)$ *and player* 2 *accepts.*

Player's 2 optimal response is clear since if he rejects he will have to make the same proposal, $(x_1 + 1, x_2 + 2)$ to avoid arbitration.

Suppose player 1 proposes $(x_1 + 2, x_2 + 1)$. Player 2 optimally rejects this offer and at the new bargaining state that belongs to P_2 he proposes

(x_1+1, x_2+2) which will be accepted by player 1. Player 2 gets $\delta(x_2+2) > x_2+1$ since $x_2+1 < Z_1$. Then, if player 1 proposes (x_1+2, x_2+1) she gets $\delta(x_1+1) < x_1+1$.

STEP 3. *If* $x_2 < x_1 < Z_1 - 1$ *player* 1 *proposes* (x_1+1, x_2+2) *and player* 2 *accepts.*

Here the situation is identical to 2).

STEP 4. *If* $x_1 \geq Z_1 - 1$ *player* 1 *proposes* (x_1+1, x_2+2) *and player* 2 *accepts*.

Player 2 accepts (x_1+1, x_2+2) for the same reason as in 2). If player 1 offers (x_1+2, x_2+1), player 2 will optimally reject it and propose, at next turn, (x_1+1, x_2+2) which will be accepted by player 1. Player 1 gets $\delta(x_1+X) < x_1+1$.

We have exhausted all the bargaining states (x_1, x_2, X) that belong to P_3.

(iii) Consider now all the bargaining states belonging to P_4. The equilibrium strategies are:

bargaining state	i demands	j response
$x_j \geq Z_1 - 1$	$x_i + 3$	A
$x_i \leq x_j < Z_1 - 1$ and $\delta(x_i+2) \leq N - Z_1$	$N - Z_1$	A
$x_i \leq x_j < Z_1 - 1$ and $\delta(x_i+2) > N - Z_1$	$x_i + 3$	R
$x_j < x_i < Z_1 - 1$	$x_i + 3$	R
$x_i \geq Z_1 - 1$	$x_i + 1$	A

STEP 1: *If* $x_2 \geq Z_1 - 1$ *player* 1 *proposes* (x_1+3, x_2+1) *and player* 2 *accepts.*

Suppose player 2 rejects this offer. A rejection leads the game to a new bargaining state that belongs to P_3 with $x_2+1 > Z_1 - 1$. Thus, if player 2 rejects he gets $\delta(x_2+2) < x_2+1$ since $x_2+1 \geq Z_1$.

Player 1 will not make a different proposal since this is the best she can get.

STEP 2: *If* $(x_1, x_2, X) \in P_4$, $x_1 \leq x_2 < Z_1 - 1$ *and* $N - Z_1 > \delta(x_1 + 2)$ *player* 1 *proposes* $(N - Z_1, Z_1)$ *and player* 2 *accepts.*

If player 2 rejects the offer, the new bargaining state may belong to P_1, P_2 or P_3 with $x_2 > Z_1 - 1$. If it belongs to P_2 or P_3 player 2 gets a payoff of $\delta(Z_1 + 1) \leq Z_1$. If the new bargaining state belongs to P_1 player 2 gets $\delta Z_1 < Z_1$.

Now consider the alternatives to player 1. If $N - Z_1 > \delta(x_1 + 2)$ and $x_1 < Z_1 - 1$, then either $N - Z_1 = x_1 + 3$ or $N - Z_1 = x_1 + 2$. If $N - Z_1 = x_1 + 3$ player 1 will not consider to make a different proposal since this is the best share she can get. And if $N - Z_1 = x_1 + 2$ and player 1 proposes $(x_1 + 3, x_2 + 1)$, player 2 will optimally reject. The new bargaining state belongs to P_3 with $x_1 < x_2 + 1 < Z_1$ and player 1 will get a final payoff of $\delta(x_1 + 2) < x_1 + 2 = N - Z_1$.

STEP 3: *If* $x_1 \leq x_2 < Z_1 - 1$ *and* $N - Z_1 < \delta(x_1 + 2)$ *player* 1 *proposes* $(x_1 + 3, x_2 + 1)$ *and player* 2 *rejects.*

First, we will prove that player 2 will optimally reject this proposal. By rejecting player's 1 offer, the game reaches the bargaining state belonging to P_3 with $x_2 + 1 < Z_1$. Player 2 will optimally propose $(x_1 + 2, x_2 + 2)$ and player 1 will accept it. Then, by rejecting, player 2 gets $\delta(x_2 + 2) > x_2 + 1$ since $x_2 + 1 < Z_1$. Player 1 gets a payoff of $\delta(x_1 + 2)$.

If $N - Z_1 < \delta(x_1 + 2)$ and $x_1 \leq x_2 < Z_1 - 1$ then $N - Z_1 = x_1 + 1$. If player 1 proposes $(x_1 + 1, x_2 + 3)$ player 2 accepts and player 1 gets $x_1 + 1 < \delta(x_1 + 2)$. If player 1 proposes $(x_1 + 2, x_2 + 2)$, player 2 optimally rejects and propose and his turn $(x_1 + 1, x_2 + 3)$ which will be accepted by player 1. Player's 1 final payoff will be $\delta(x_1 + 1) < \delta(x_2 + 2)$.

STEP 4: *If* $x_2 < x_1 < Z_1 - 1$ *player* 1 *proposes* $(x_1 + 3, x_2 + 1)$ *and player* 2 *rejects.*

Player 2 rejects player's 1 offer $(x_1 + 3, x_2 + 1)$ and, at the new bargaining state that belongs to P_3 with $x_2 + 1 \leq Z_1 - 1$, he proposes

$(x_1{+}2, x_2{+}2)$ which player 1 accepts. He gets a payoff of $\delta(x_2{+}2) > x_2{+}1$ since $x_2 + 1 < Z_1$.

By proposing $(x_1 + 3, x_2 + 1)$ player 1 gets $\delta(x_1 + 2)$. Instead if she proposes $(x_1 + 1, x_2 + 3)$ player 2 will accept and player 1 final payoff is $x_1 + 1 < \delta(x_1 + 2)$. And if she proposes $(x_1 + 2, x_2 + 2)$, player 2 rejects, the new bargaining state belongs to P_2 and player 1 gets a final payoff of $\delta(x_1 + 1) < \delta(x_1 + 2)$.

STEP 5: *If* $x_1 \geq Z_1 - 1$ *player* 1 *proposes* $(x_1 + 1, x_2 + 3)$ *and player* 2 *accepts.*

Player 2 accepts $(x_1 + 1, x_2 + 3)$ since $x_2 + 3$ is the best share he can get. If player 1 either proposes $(x_1 + 2, x_2 + 2)$ or $(x_1 + 3, x_2 + 1)$ he will receive a final payoff of $\delta(x_1 + 1) < x_1 + 1$.

We have exhausted all the bargaining states $(x_1, x_2, X) \in P_4$

(iv) And finally, assume players follow the strategies specified above for any bargaining state (x_1, x_2, X) such that $X < n$. We will show that for any bargaining state $(x_1, x_2, X) \in P_n$, the optimal strategies must be the proposed ones. Without loss of generality we will prove them for n even:

Bargaining state	i demand	j response
$x_j \geq Z_1 - 1$	$x_i + n - 1$	A
$x_i \leq x_j < Z_1 - 1$ and $\delta^{n2-1}(x_i + \frac{n}{2}) \leq N - Z_1$	$N - Z_1$	A
$x_i \leq x_j < Z_1 - 1$ and $\delta^{n2-1}(x_i + \frac{n}{2}) > N - Z_1$	$x_i + n - 1$	R
$x_j < x_i < Z_1 - 1$	$x_i + n - 1$	R
$x_i \geq Z_1 - 1$	$x_i + 1$	A

STEP 1: *If* $x_2 \geq Z_1 - 1$ *player 1 proposes* $(x_1 + n - 1, x_2 + 1)$ *and player 2 accepts.*

If player 1 proposes $(x_1 + n - 1, x_2 + 1)$ player 2 accepts since if he rejects this offer, at the new bargaining state that belongs to P_{n-1} with $x_2 + 1 > Z_1 - 1$, he optimally proposes $(x_1 + n - 2, x_2 + 2)$ and player 1 accepts. He gets a payoff of $\delta(x_2+2) \leq x_2+1$ since $x_2+1 \geq Z_1$[19].

If player 1 makes a different proposal she will get a smaller payoff since x_1+n-1 is the best share she can get.

STEP 2: *If* $x_1 \leq x_2 < Z_1-1$, *and* $\delta^{\frac{n}{2}-1}(x_1+\frac{n}{2}) > N - Z_1$ *player* 1 *proposes* (x_1+n-1, x_2+1) *and player* 2 *rejects.*

If player 2 rejects, then the game reaches the bargaining state, $(x_1, x_2+1, n-1) \in P_{n-1}$ with $x_2+1 \leq Z_1-1$ and $n-1$ odd. From this bargaining state on players follow the specified strategies. The next table resumes the path of the game after player´s 2 rejection.

for $i = 0, 1, 2, 3...$

At, $t+2i+1$, $(x_1, x_2, X) \in P_{n-1-4i}$ and $\delta^{\frac{n-4i-2}{2}-1}(x_2+2i+1+\frac{n-4i-2}{2}) > N - Z_1$ since $x_2 > x_1$.

state	$x_1+2i < x_2+2i+1 < Z_1-1$	$x_2+2i+1 = Z_1-1$
2 proposes	$(x_1+2i+2, x_2+n-2i-2)$	$(x_1+n-2i-2, x_2+2i+2)$
1 responds	R	A

At $t+2i+2$ $(x_1, x_2, X) \in P_{n-3-4i}$ and $\delta^{\frac{n-4i-4}{2}-1}.(x_1+2i+2+\frac{n-4i-4}{2}) > \delta^{\frac{n}{2}-1}.(x_1+\frac{n}{2}) > N - Z_1$.

state	$x_1+2i+2 < x_2+2i+1 < Z_1-1$	$x_1+2i+2 = Z_1-1$
1 proposes	$(x_1+n-2i-3, x_2+2i+3)$	$(x_1+2i+3, x_2+n-2i-3)$
2 responds	$A if x_2+2i+3 = Z_1$ $R if x_2+2i+3 < Z_1$	A

At $(x_1+\frac{n}{2}-1, x_2+\frac{n}{2}-2, 3)$ player 1 proposes $((x_1+\frac{n}{2}, x_2+\frac{n}{2})$ and player 2 accepts.

If the game ends at some time where player 2 is the proposer, then his final payoff is $\delta^{2i+1}(x_2+2i+2)$ and $\delta^{2i+1}(x_2+2i+2) > \delta^{2i}(x_2+2i+1)$

[19] It could be that $\delta(x_2+2) = x_2+1$ if $x_2+1 = Z_1$ and Z_1 is such that $Z_1 = \delta(Z_1+1)$. In this case, player 2 will be indifferent between accepting or rejecting the proposal of player 1. We assume that he accepts.

since $x_2 + 2i + 1 < Z_1$. Thus, player 2 optimally rejects the initial offer $(x_1 + n - 1, x_2 + 1)$.

If the game ends at some t where player 1 is the proposer, then player's 2 final payoff is either $\delta^{2i+2}(x_2+2i+3)$ or $\delta^{2i+2}(x_2+n-2i-3)$. In both cases, player 2 gets a higher payoff rejecting the initial offer rather than accepting it since $x_2 + 1 < Z_1$ and either $x_2 + 2i + 3 = Z_1$ or $x_2 + n - 2i - 3 = N - Z_1$.

Suppose now that player 1 makes a different proposal (x_1+n-k, x_2+k), with n and $k \geq 1$ even. If $x_2 + k \geq Z_1$ player 2 accepts this proposal and player 1 would get a payoff of $x_1 + n - k < N - Z_1 < \delta^{\frac{n}{2}-1}(x_1 + \frac{n}{2})$. If $x_2 + k < Z_1$, then player 2 rejects this offer and the continuation game is as follows:

At t+1 $(x_1, x_2 + k, n - k)$

state	$x_1 < x_2 + k < Z_1 - 1$	$x_2 + k = Z_1 - 1$
2 proposes	$(x_1 + 1, x_2 + n - 1)$	$(x_1 + n - k - 1, x_2 + k + 1)$
1 responds	R	A

At t+2 $(x_1 + 1, x_2 + k, n - k - 1)$ and $x_1 + 1 < x_2 + k < Z_1 - 1$

state	$\delta^{\frac{n-k-2}{2}-1}(x_1 + 1 + \frac{n-k-2}{2}) > N - Z_1$	$\delta^{\frac{n-k-2}{2}-1}(x_1 + 1 + \frac{n-k-2}{2}) \leq N - Z_1$
1 proposes	$(x_1 + n - k - 2, x_2 + k + 2)$	$(N - Z_1, Z_1)$
2 responds	$A if x_2 + k + 2 = Z_1$ $R if x_2 + k + 2 < Z_1$	A

At t+3 $(x_1 + 1, x_2 + k + 2, n - k - 3)$

state	$x_1 + 1 < x_2 + k + 2 < Z_1 - 1$	$x_2 + k + 2 = Z_1 - 1$
2 proposes	$(x_1 + 3, x_2 + n - 3)$	$(x_1 + n - k - 3, x_2 + k + 3)$
1 responds	R	A

At t+4 $(x_1 + 3, x_2 + k + 2, n - k - 5)$, $x_1 + 3 < x_2 + k + 2 < Z_1 - 1$ and

$$\delta^{\frac{n-k-6}{2}}(x_1 + 3 + \tfrac{n-k-6}{2}) > \delta^{\frac{n-k-2}{2}-1}(x_1 + 1 + \tfrac{n-k-2}{2}) > N - Z_1.$$

1 proposes	$(x_1+n-k-4, x_2+k+4)$
2 responds	$Aifx_2+k+4=Z_1$ $Rifx_2+k+4<Z_1$

At t+5 $(x_1+3, x+k+4, n-k-7)$

state	$x_1+3<x_2+k+4<Z_1-1$	$x_2+k+4=Z_1-1$
2 proposes	$(x_1+5, x_2+n-k-5)$	$(x_1+n-k+5, x_2+k+5)$
1 responds	R	A

....

At $(x_1+\frac{n-k}{2}-2, x_2+k-1+\frac{n-k}{2}, 3)$ player 2 proposes $(x_1+\frac{n-k}{2}, x_2+k+\frac{n-k}{2})$ and player 1 accepts.

In all the cases player 1 gets a payoff smaller than $\delta^{\frac{n}{2}-1}(x_1+\frac{n}{2})$ since $\delta^{\frac{n}{2}-1}(x_1+\frac{n}{2})>N-Z_1$ and $\delta^{\frac{n}{2}-1}(x_1+\frac{n}{2})>\delta^{\frac{n-k}{2}-1}(x_1+\frac{n-k}{2})$. Then, the best proposal player 1 can make is (x_1+n-1, x_2+1).

STEP 3: *If* $x_1\leq x_2<Z_1-1$ *and* $\delta^{\frac{n}{2}-1}(x_1+\frac{n}{2})<N-Z_1$ *player* 1 *proposes* $(N-Z_1, Z_1)$ *and player* 2 *accepts.*

Suppose that player 2 rejects the offer. The game reaches the bargaining state $(x_1, Z_1, n-Z_1)$ with $x_2=Z_1$ and $X=n-Z_1<n$. Player 2 proposes (x_1+n-Z_1+1, Z_1+1) and player 1 accepts. By rejecting player's 1 offer player 2 gets $\delta(Z_1+1)\leq Z_1$.

Consider now player 1. Suppose she offers the opponent Z_1-k. If player 1 wants to avoid arbitration, $Z_1-k=x_2+g$ with $g>0$. Assume that g is even. Player 2 optimally rejects this offer since $x_2+g<Z_1$ and the game reaches the bargaining state $(x_1, x_2+g, n-g)$ with $x_1<x_2+g<Z_1-1$, and $\delta^{\frac{n-g-2}{2}-1}(x_1+1+\frac{n-g-2}{2})\lesseqgtr N-Z_1$. Following the path after player 2 rejection, we can compute the final payoff of player 1 as in step 2. This will be either $\delta^r(N-Z_1)$ with $r\geq 2$ or $\delta^{\frac{n-g}{2}-1}(x_1+\frac{n-g}{2})$ both smaller than $N-Z_1$.

STEP 4: *If* $x_2 < x_1 < Z_1 - 1$ *player* 1 *proposes* $(x_1 + n - 1, x_2 + 1)$ *and player* 2 *rejects this proposal.*

First we will prove that for player 2 it is optimal to reject the proposal of player 1. After rejection, the bargaining state is $(x_1, x_2 + 1, n - 1)$ with $x_2 + 1 \leq x_1 < Z_1 - 1$ and either $\delta^{\frac{n}{2}-1}(x_2 + 1 + \frac{n-2}{2}) \leq N - Z_1$ or $\delta^{\frac{n}{2}-1}(x_2+1+\frac{n-2}{2}) > N-Z_1$. If $\delta^{\frac{n-2}{2}}(x_2+1+\frac{n-2}{2}) \leq N-Z_1$ he proposes $(Z_1, N-Z_1)$ and player 1 accepts. His final payoff is $\delta(N-Z_1) > x_2+1$ since $x_2+1 < Z_1-1$ and $N-Z_1 \geq x_2+2$. However, if $\delta^{\frac{n-2}{2}}(x_2+1+\frac{n-2}{2}) > N-Z_1$, he proposes (x_1+2, x_2+n-2) and the game follow a path such that player 2 will receive a payoff of $\delta^{\frac{n}{2}}(x_2+\frac{n}{2})$. To reject is better than to accept player's 1 proposal since $x_2 + 1 < \delta^{\frac{n}{2}}(x_2 + \frac{n}{2})$.

Now consider player's 1 strategy and assume that $\delta^{\frac{n}{2}-1}(x_2+1+\frac{n-2}{2}) \leq N - Z_1$. If she proposes (x_1+n-1, x_2+1) she gets δZ_1. If she makes a different proposal, say (x_1+n-k, x_2+k), player 2 accepts if $x_2+k \geq Z_1$. In that case, player 1 gets $x_1 + n - k \leq N - Z_1 < \delta Z_1$. Player 2 rejects if $x_2+k < Z_1$. At the new bargaining state $(x_1, x_2+k, n-k)$ with $n-k$ even players follow the specified strategies.

1) If $x_2+k \leq x_1 < Z_1-1$ and $\delta^{\frac{n-k}{2}-1}(x_2+k+\frac{n-k}{2}) \leq N-Z_1$ player 2 proposes, at his turn, $(Z_1, N-Z_1)$ and player 1 accepts it. Player 1 gets δZ_1.

2) If $x_2 + k \leq x_1 < Z_1 - 1$ and $\delta^{\frac{n-k}{2}-1}.(x_2 + k + \frac{n-k}{2}) > N - Z_1$, player 2 proposes (x_1+1, x_2+n-1) and player 1 rejects. Player 1 will receive a payoff of $\delta^r(Z_1)$ with $r \geq 2$ or $\delta^{\frac{n-k}{2}-1}(x_1 + \frac{n-k}{2})$, both smaller than δZ_1.

3) If $x_1 < x_2 + k < Z_1 - 1$. Since $n - k$ is even player 2 proposes (x_1+1, x_2+n-1) and player 1 rejects. At $(x_1+1, x_2+k, n-k-1)$ player 1 proposes $(N-Z_1, Z_1)$ and player 2 accepts if $x_1+1 \leq x_2+k < Z_1-1$ and $\delta^{\frac{n-k-2}{2}-1}(x_1+1+\frac{n-k-2}{2}) < N-Z_1$. Player 1 will get a final payoff of $\delta^2(N-Z_1) < \delta Z_1$. Player 1 proposes $(x_1+n-k-2, x_2+k+2)$ if $x_1+1 \leq x_2 + k < Z_1 - 1$ and $\delta^{\frac{n-k-2}{2}-1}(x_1 + 1 + \frac{n-k-2}{2}) > N - Z_1$ and player 2

rejects. Following the path of the game after the rejection of his proposal by player 2, player´s 1 final payoff will be $\delta^{\frac{n-k-2}{2}+1}(x_1+1+\frac{n-k-2}{2}) < \delta Z_1$.

Now assume that $\delta^{\frac{n}{2}-1}(x_2+1+\frac{n-2}{2}) > N - Z_1$ and suppose player 1 proposes (x_1+n-k, x_2+k) with $k > 1$. If $x_2+k \geq Z_1$ player 2 accepts and player 1 will get payoff of $x_1+n-k < N-Z_1 < \delta^{\frac{n}{2}}(x_1+\frac{n}{2})$. If $x_2+k < Z_1$ player 2 rejects and the new bargaining state is $(x_1, x_2+k, n-k)$ with $x_1 \gtreqless x_2+k$. Then, if:

1) $x_2+k \leq x_1 < Z_1-1$, since $\delta^{\frac{n-k}{2}-1}.(x_2+k+\frac{n-k}{2}) > \delta^{\frac{n-1}{2}-1}.(x_2+1+\frac{n-2}{2}) > N-Z_1$, then the game follows a path such that the player's 1 final payoff will be $\delta^{\frac{n-k}{2}}(x_1+\frac{n-k}{2}) < \delta^{\frac{n}{2}}(x_1+\frac{n}{2})$.

2) $x_1 < x_2+k < Z_1-1$ player 2 proposes (x_1+1, x_2+n-1) and player 1 rejects. At $(x_1+1, x_2+k, n-k-1)$ player 1 proposes $(N-Z_1, Z_1)$ if $\delta^{\frac{n-k-2}{2}-1}(x_1+1+\frac{n-k-2}{2}) \leq N-Z_1$ and player 2 accepts. Then, the final payoff for player 1 will be $\delta^2(N-Z_1) < \delta^{\frac{n}{2}}(x_1+\frac{n}{2})$. And if $\delta^{\frac{n-k-2}{2}-1}(x_1+1+\frac{n-k-2}{2}) > N-Z_1$, player 1 will get a final payoff $\delta^{\frac{n-k}{2}}(x_1+\frac{n-k}{2}) < \delta^{\frac{n}{2}}(x_1+\frac{n}{2})$.

STEP 5: *If* $x_1 \geq Z_1 - 1$ *player 1 proposes* (x_1+1, x_2+n-1) *and player 2 accepts.*

If player 1 proposes (x_1+1, x_2+n-1), player 2 obviously accepts because it is the best share he can get. Suppose player 1 makes a different proposal (x_1+k, x_2+n-k), with $k > 1$, and k even. Player 2 will reject this proposal since $x_2+n-k < Z_1$ and the game reaches the bargaining state (x_1, x_2+n-k, k) with $X = k < n$ and $x_1 \geq Z_1$. At his turn, player 2 proposes (x_1+1, x_2+n-1) and his opponent accepts. Player 1 will get a payoff $\delta(x_1+1) < x_1+1$.$\blacksquare$

Proof of proposition 8

For any bargaining state (x_1, x_2, X) such that $X < 6$ the optimal strategies of the players are the same as the specified strategies in proposition 1. Notice that if a player forces arbitration at bargaining states

$(x_1, x_2, X) \in P_2, P_3$ she will get a payoff of $\delta(x_i + 1)$. But this is a dominated strategy since she can always make an offer asking for one unit and the opponent will accept this partition without wasting one period. If $(x_1, x_2, X) \in P_4, P_5$ arbitration pays $\delta(x_i + 2)$. It is easy to check that arbitration is a dominated alternative for player i at the bargaining states such that $x_i \geq Z_1 - 1$ and $x_j \geq Z_1 - 1$. And at the bargaining states satisfying $x_i \leq x_j < Z_1 - 1$ or $x_j < x_i < Z_1 - 1$, arbitration gives the player, at most, the same payoff as the one she gets if she follows the strategies specified in table 1.

Assume that $\delta^2(x_i + 4) > x_i + 1 \; \forall x_i$. We will prove the optimal actions for $X = 6, 7$. Thereafter we will assume that are optimal for $X < n$ and we will prove that are optimal for $X = n$.

(i) Consider any bargaining state $(x_1, x_2, X) \in P_6$. For these bargaining states $x_i + 5 > v_i^a = \delta(x_i + 3)$, and if $x_i \leq x_j < Z_1 - 1$ then $x_i + 5 > N - Z_1$ since $x_i + 6 = N - x_j > N - Z_1 + 1$.

state	i demand	j response
$x_j \geq Z_1 - 1$	$x_i + 5$	A
$x_i \leq x_j < Z_1 - 1 and N - Z_1 \geq v_i^a$	$N - Z_1$	A
$x_i \leq x_j < Z_1 - 1, x_i + 5 \geq N - Z_1 and N - Z_1 < v_i^a$	$x_i + 6$	R
$x_j < x_i < Z_1 - 1, x_j + 5 \geq N - Z_1 and \delta Z_1 > v_i^a$	$x_i + 5$	R
$x_j < x_i < Z_1 - 1, x_j + 5 \geq N - Z_1 and \delta Z_1 \leq v_i^a$	$x_i + 6$	R
$x_i \geq Z_1 - 1$	$x_i + 6$	R

STEP 1: *If* $x_2 \geq Z_1 - 1$ *player* 1 *proposes* $(x_1 + 5, x_2 + 1)$ *and player* 2 *accepts.*

If player 2 rejects player's 1 proposal, at the new bargaining state $(x_1, x_2 + 1, 5)$ player 2 will propose $(x_1 + 4, x_2 + 2)$ and player 1 will accept (by proposition 1). Then, a rejection pays $\delta(x_2 + 2) < x_2 + 1$ since $x_2 \geq Z_1 - 1$. For player 1 this accepted offer is the best deal he can get.

STEP 2: *If* $x_1 \leq x_2 < Z_1 - 1$ *and* $N - Z_1 \geq \delta(x_1 + 3)$ *player* 1 *proposes* $(N - Z_1, Z_1)$ *and player* 2 *accepts.*

If player 2 rejects, at the bargaining state $(x_1, Z_1, N - Z_1 - x_1)$ with $N - Z_1 - x_1 \leq 5$ she proposes $(N - Z_1 - 1, Z_1 + 1)$ and player 1 accepts (by proposition 1). But $Z_1 \geq \delta(Z_1 + 1)$.

For player 1 arbitration is a dominated strategy. Since $N - Z_1 \geq \delta(x_1 + 3)$ then either:

a) $N - Z_1 = x_1 + 5$. Player 1 will not make a different proposal since this is the best share he can get.

b) $N - Z_1 = x_1 + 4$. If player 1 proposes $(x_1 + 5, x_2 + 1)$ player 2 optimally rejects and at the new bargaining state $(x_1, x_2 + 1, 5)$ since $x_2 + 1 = Z_1 - 1$ player 2 proposes $(N - Z_1, Z_1)$ which is accepted by player 1. Player 1 gets $\delta(N - Z_1) < N - Z_1$.

c) $N - Z_1 = x_1 + 3$. If player 1 proposes either $(x_1 + 5, x_2 + 1)$ or $(x_1 + 4, x_2 + 2)$ player 2 will reject both and player 1 will end up with a payoff of $\delta^2(N - Z_1)$ and $\delta(N - Z_1)$ respectively.

STEP 3: *If* $x_1 \leq x_2 < Z_1 - 1$ *and* $N - Z_1 < \delta(x_1 + 3)$ *player* 1 *forces arbitration.*

If player 1 offers $(x_1 + 6, x_2)$ he breaks the negotiation and the arbitrator is called to implement the partition $(x_1 + 3, x_2 + 3)$ one period later. Instead, player 1 may propose either $(x_1 + 3, x_2 + 3)$ or $(x_1 + 4, x_2 + 2)$ or $(x_1 + 5, x_2 + 1)$. In all the cases, if the offered share to player 2 is $Z_1 - 1$, she will reject the offer and, at her turn, she will ask for Z_1. Player 1 will get $\delta(N - Z_1) < \delta(x_1 + 3)$. If the offered share to player 2 is smaller than $Z_1 - 1$, she will reject and player 1 will get a final payoff of $\delta^2(x_1 + 2)$ or $\delta^2(x_1 + 3)$ (by proposition 1) both smaller than $\delta(x_1 + 3)$.

STEP 4: *If* $x_2 < x_1 < Z_1 - 1$, $x_2 + 5 > N - Z_1$ *and* $\delta(x_1 + 3) < \delta Z_1$ *player* 1 *proposes* $(x_1 + 5, x_2 + 1)$ *and player* 2 *rejects.*

If player 2 rejects $(x_1 + 5, x_2 + 1)$, then at the bargaining state $(x_1, x_2 + 1, 5)$ with $x_2 + 1 \leq x_1 < Z_1 - 1$ and $N - Z_1 > \delta(x_2 + 3)$ she proposes

$(Z_1, N - Z_1)$ and player 1 accepts (proposition 1). Then, by rejecting the player's 1 offer she gets $\delta(N - Z_1) > x_2 + 1$, since $x_2 + 1 < N - Z_1 < Z_1$.

By proposing $(x_1 + 5, x_2 + 1)$, player 1 gets δZ_1. Suppose he makes a different proposal $(x_1 + n - k, x_2 + k)$. If the offered share is greater or equal to Z_1 player 2 accepts, but player 1 gets $x_1 + k < N - Z_1 < \delta Z_1$. If the offered share is equal to $Z_1 - 1$ player 2 rejects and proposes, at her turn, $(N - Z_1, Z_1)$ which will be accepted and player 1 will get a final payoff of $\delta(N - Z_1) < \delta Z_1$. If the offered share is smaller than $Z_1 - 1$, player 2 rejects. We use proposition 1 to find the final payoff of player 1 and check that in all cases these payoffs are smaller than δZ_1.

STEP 5: *If* $x_2 < x_1 < Z_1 - 1$ *and* $\delta(x_1 + 3) > \delta Z_1$ *player* 1 *forces arbitration.*

By forcing arbitration, player 1 gets $\delta(x_1 + 3)$. Suppose he makes an offer $(x_1 + n - k, x_2 + k)$. Player 2 accepts if $x_2 + k \geq Z_1$, but player 1 gets $x_1 + 6 - k < N - Z_1 < \delta Z_1 < \delta(x_1 + 3)$. If player 2 rejects because $x_2 + k \leq Z_1 - 1$, then the final payoff of player 1 will be smaller than $\delta(x_1 + 3)$. Since after rejection the new bargaining state $(x_1, x_2, X) \in P_l$ with $l \leq 5$, proposition 1 applies. It is easy to check that in all cases player 1 gets a payoff smaller than $\delta(x_1 + 3)$.

STEP 6: *If* $x_1 \geq Z_1 - 1$ *player* 1 *forces arbitration.*

Player 1 proposes $(x_1 + 6, x_2)$, the arbitrator is called and player 1 gets $\delta(x_1 + 3)$. If player 1 makes a different proposal $(x_1 + 6 - k, x_2 + k)$, player 2 rejects and, at $(x_1, x_2 + k, 6 - k)$ since $x_1 \geq Z_1 - 1$ and $6 - k \leq 5$, she proposes the partition $(x_1 + 1, x_2 + 5)$ which player 1 accepts. Player 1 gets a final payoff of $\delta(x_1 + 1) < \delta(x_1 + 3)$

(ii) Consider now any bargaining state $(x_1, x_2, X) \in P_7$. For these bargaining states $x_i + 5 > v_i^a = \delta(x_i + 3)$, and if $x_i \leq x_j < Z_1 - 1$ then $x_i + 5 \geq N - Z_1$ since $x_i + 7 = N - x_j > N - Z_1 + 1$. Assume first that $\delta^2(x_j + 4) > x_j + 1$. The optimal strategies are:

state	i demand	j response
$x_j \geq Z_1 - 1$	$x_i + 5$	A
$x_i \leq x_j < Z_1 - 1 and N - Z_1 \geq v_i^a$	$N - Z_1$	A
$x_i \leq x_j < Z_1 - 1, x_i + 5 \geq N - Z_1 and N - Z_1 < v_i^a$	$x_i + 7$	R
$x_j < x_i < Z_1 - 1, x_j + 5 \geq N - Z_1 and \delta Z_1 > v_i^a$	$x_i + 5$	R
$x_j < x_i < Z_1 - 1, x_j + 5 \geq N - Z_1 and \delta Z_1 \leq v_i^a$	$x_i + 7$	R
$x_i \geq Z_1 - 1$	$x_i + 7$	R

STEP 1: *If* $x_2 \geq Z_1 - 1$ *player* 1 *proposes* $(x_1 + 5, x_2 + 2)$ *and player* 2 *accepts.*

If player 2 rejects the proposal of player 1, at the new bargaining state $(x_1, x_2 + 2, 5)$ player 2 proposes $(x_1 + 4, x_2 + 3)$ being accepted by her opponent (see proposition 6). Then, by rejecting she gets $\delta(x_2+3) < x_2 + 2$ since $x_2 \geq Z_1 - 1$.

For player 1, forcing arbitration is a dominated strategy since $\delta(x_1 + 3) < x_1 + 5$. He may offer the partition $(x_1 + 6, x_2 + 1)$, but player 2 optimally rejects and forces arbitration at her turn since she will get $\delta^2(x_2 + 4) > x_2 + 1$.

STEP 2: *If* $x_1 \leq x_2 < Z_1 - 1$ *and* $N - Z_1 > \delta(x_1 + 3)$ *player* 1 *proposes* $(N - Z_1, Z_1)$ *and player* 2 *accepts.*

If she rejects, at the bargaining state $(x_1, Z_1, N - Z_1 - x_1)$ with $N - Z_1 - x_1 \leq 5$ player 2 proposes $(N - Z_1 - 1, Z_1 + 1)$ and player 1 accepts. But $Z_1 \geq \delta(Z_1 + 1)$.

For player 1 forcing arbitration is a dominated strategy since $N - Z_1 \geq \delta(x_1 + 3)$. Since $x_2 < Z_1 - 1$, then $N - Z_1$ may be equal to:

a) $N - Z_1 = x_1 + 5$. If player 1 proposes $(x_1 + 6, x_2 + 1)$, player 2 optimally rejects and forces arbitration at her turn since $x_2 + 1 = Z_1 - 1$. Player 1 gets $\delta^2(x_1 + 3) < x_1 + 5 = N - Z_1$.

b) $N - Z_1 = x_1 + 4$. If player 1 proposes $(x_1 + 6, x_2 + 1)$ player 2 rejects and at the new bargaining state $(x_1, x_2 + 1, 6)$ with $x_1 < x_2 + 1 < Z_1 - 1$,

$x_1 + 5 > N - Z_1$ and $\delta Z_1 = \delta(x_2 + 3) < v_2^a = \delta(x_2 + 4)$ player 2 forces arbitration and player 1 gets a final payoff of $\delta^2(x_1+3) < N - Z_1$. Instead, if player 1 proposes (x_1+5, x_2+2), player 2 rejects and proposes $(x_1 + 4, x_2 + 3)$ which is accepted by player 1 that gets a final payoff $\delta(N - Z_1) < N - Z_1$.

c) $N - Z_1 = x_1 + 3$. Player 1 may propose either $(x_1 + 6, x_2 + 1)$, or $(x_1 + 5, x_2 + 2)$ or $(x_1 + 4, x_2 + 3)$. It is easy to check that in the first two cases player 1 ends up with a payoff of $\delta^2(x_1 + 3)$ and in the last case $\delta(N - Z_1)$.

STEP 3: *If* $x_1 \leq x_2 < Z_1 - 1$ *and* $N - Z_1 < \delta(x_1+3)$ *player* 1 *forces arbitration.*

Player 1 gets $\delta(x_1+3)$ if he forces arbitration. Instead, player 1 may propose $(x_1 + 7 - k, x_2 + k)$ with $1 < k \leq 4$. If $x_2 + k = Z_1 - 1$, player 2 rejects the offer and asks for Z_1 at his turn. Player 1 gets a final payoff of $\delta(N - Z_1) < N - Z_1$. If $x_2 + k < Z_1 - 1$, player 2 rejects and since the number of units left to be negotiated is smaller than 6 proposition 1 applies and is easy to check that the final payoff of player 1 is smaller than $\delta(x_1 + 3)$. And finally, if player 1 proposes $(x_1 + 6, x_2 + 1)$ player 2 rejects and forces arbitration at her turn. Player 1 receives a payoff of $\delta^2(x_1 + 3) < \delta(x_1 + 3)$.

STEP 4: *If* $x_2 < x_1 < Z_1 - 1$, $N - Z_1 > \delta(x_2+3)$ *and* $\delta(x_1+3) < \delta Z_1$ *player* 1 *proposes* $(x_1 + 6, x_2 + 1)$ *and player* 2 *rejects.*

If player 2 rejects $(x_1 + 6, x_2 + 1)$, then at $(x_1, x_2 + 1, 6)$ he proposes $(Z_1, N - Z_1)$ since $x_2 + 1 \leq x_1 < Z_1 - 1$ and $N - Z_1 > \delta(x_2 + 3)$ and player 1 accepts. Then, rejection pays $\delta(N - Z_1) > x_2 + 1$.

By proposing $(x_1 + 6, x_2 + 1)$ player 1 gets δZ_1. Suppose she makes a different proposal, that is $(x_1 + 7 - k, x_2 + k)$ with $1 < k \leq 5$. If $x_2 + k = Z_1 - 1$ player 2 rejects and proposes, at his turn $(N - Z_1, Z_1)$ and player 1 gets $\delta(N - Z_1) < \delta Z_1$. If $x_2 + k < Z_1 - 1$, player 2 rejects. We use proposition 1 to find the final payoff of player 1. It is easy to check that player 1 will receive a payoff smaller than δZ_1.

STEP 5: *If* $x_2 < x_1 < Z_1 - 1$ *and* $\delta(x_1 + 3) > \delta Z_1$ *player* 1 *forces arbitration.*

By forcing arbitration, player 1 will get $\delta(x_1 + 3)$. If she makes a proposal $(x_1 + 7 - k, x_2 + k)$ with $1 \leq k \leq 5$ player 2 rejects if $x_2 + k < Z_1$, and accepts otherwise. If he accepts, then player 1 gets $x_1 + 7 - k < N - Z_1 < \delta Z_1 < \delta(x_1 + 3)$. If he rejects any proposal with $k > 1$, we know by proposition 1 the continuation game and it is easy to check that the payoff of player 1 is smaller than $\delta(x_1 + 3)$. If player 1 proposes $(x_1 + 6, x_2 + 1)$, player 2 surely rejects. Since $x_1 + 3 > Z_1$ then $x_2 + 4 < N - Z_1$ and by step 2 and 3 player 2 proposes $(Z_1, N - Z_1)$ if $x_2 + 4 < \delta(N - Z_1)$ and forces arbitration if $x_2 + 4 > \delta(N - Z_1)$. Player 1 still do not improve the arbitrated payoff.

STEP 6: *If* $x_1 \geq Z_1 - 1$ *player* 1 *forces arbitration.*

Player 1 forces arbitration and gets $\delta(x_1 + 3)$. Instead, if player 1 proposes $(x_1 + 1, x_2 + 6)$ player 2 accepts but player 1 gets $x_1 + 1 < \delta(x_1 + 3)$. If he makes a different proposal $(x_1 + 7 - k, x_2 + k)$, with $k \leq 5$, player 2 rejects. At the bargaining state $(x_1, x_2 + k, 7 - k)$ with $1 < k \leq 5$ player 2 proposes $(x_1 + 1, x_2 + 6)$ and player 1 accepts. Player 1 gets a final payoff of $\delta(x_1 + 1) < \delta(x_1 + 3)$. If $k = 1$ player 2 forces arbitration and player 1 gets $\delta^2(x_1 + 3) < x_1 + 3$.

(iii) Finally consider a bargaining state (x_1, x_2, X) with $6 \leq X < n$ and assume that players follow the proposed strategies. Then, take a bargaining state (x_1, x_2, n):

STEP 1: *If* $x_2 \geq Z_1 - 1$ *and* $x_1 + 5 \geq v_1^a = \delta(x_1 + \frac{n - I(n)}{2})$ *player* 1 *proposes* $(x_1 + 5, x_2 + n - 5)$ *and player* 2 *accepts.*

If player 2 rejects, at the new bargaining state $(x_1, x_2 + n - 5, 5)$ he proposes $(x_1 + 4, x_2 + n - 4)$ (by proposition 1) and player 1 accepts. But $\delta(x_2 + n - 4) < x_2 + n - 5$ since $x_2 \geq Z_1 - 1$.

If player 1 makes a different proposal, $(x_1 + n - k, x_2 + k)$, with

$n - k \geq 5$, then player 2 rejects. After a rejection, at the bargaining state $(x_1, x_2+k, n-k)$, since $x_2+k \geq Z_1 - 1$ player 2 forces arbitration and gets a payoff of $\delta^2(x_2 + k + \frac{n-k-I(n-k)}{2}) > x_2 + k$.[20] By proposing (x_1+n-k, x_2+k) player 1 gets a final payoff of $\delta^2(x_1+\frac{n-k+I(n-k)}{2}) < x_1+5$ since $\delta^2(x_1 + \frac{n-k+I(n-k)}{2}) < \delta(x_1 + \frac{n-I(n)}{2}) < x_1 + 5$. Finally, player 1 may consider to force arbitration but $\delta(x_1 + \frac{n-I(n)}{2}) \leq x_1 + 5$.

STEP 2: *If* $x_2 \geq Z_1 - 1$ *and* $x_1 + 5 < v_1^a = \delta(x_1 + \frac{n-I(n)}{2})$ *player* 1 *forces arbitration.*

Player 1 forces arbitration by offering (x_1+n, x_2) and he gets $\delta(x_1 + \frac{n-I(n)}{2})$. If he makes a different proposal, $(x_1 + n - k, x_2 + k)$, player 2 may accept or reject this offer. If $n - k < 5$ player 2 accepts and player 1 gets $x_1 + n - k < x_1 + 5 < \delta(x_1 + \frac{n-I(n)}{2})$. If $n - k \geq 5$ player 2 rejects and forces arbitration at her turn and player 1 gets $\delta^2(x_1 + \frac{n-k-I(n-k)}{2}) < \delta(x_1 + \frac{n-I(n)}{2})$.

STEP 3: *If* $x_1 \leq x_2 < Z_1 - 1$, $x_1 + 5 > N - Z_1$ *and* $\delta(x_1 + \frac{n-I(n)}{2}) < N - Z_1$ *player* 1 *proposes* $(N - Z_1, Z_1)$ *and player* 2 *accepts.*

If player 2 rejects $(N-Z_1, Z_1)$, at the new bargaining state, $(x_1, Z_1, N-Z_1 - x_1)$, player 2 only asks for 1 more unit, since the pie left to be negotiated $N - Z_1 - x_1 < 5$, and this proposal is accepted by player 1. Player 2 gets $\delta(Z_1 + 1) < Z_1$.

For player 1 arbitration is dominated by proposing $(N-Z_1, Z_1)$ since $\delta(x_1+\frac{n-I(n)}{2}) < N-Z_1$. If he makes a different proposal (x_1+n-k, x_2+k) with $x_1 + n - k > N - Z_1$ player 2 rejects it. The game reaches the bargaining state $(x_1, x_2 + k, n - k)$ with $x_1 + 5 > N - Z_1$ and either $Z_1 - 1 = x_2 + k$ or $x_1 < x_2 + k < Z_1 - 1$.

a) If $Z_1 - 1 = x_2 + k$ and $n - k \leq 5$ player 2 rejects and proposes, at her turn, $(N - Z_1, Z_1)$ which is accepted. Player 1 gets $\delta(N - Z_1) < Z_1$.

b) If $Z_1 - 1 = x_2 + k$ and $n - k > 5$, player 2 forces arbitration and

[20] If $n-k \geq 8$, $\delta^2(x_2+k+\frac{n-k-I(n-k)}{2}) > \delta(x_2+k+2) > x_2+k$. And if $n-k = 6, 7$, then $x_2 + k < \delta^2(x_2 + k + 3)$ since $x_2 + 1 < \delta^2(x_2 + 4)$.

player 1 gets a final payoff of $\delta^2(x_1 + \frac{n-k+I(n-k)}{2}) < \delta(x_1 + \frac{n-I(n)}{2}) < N - Z_1$.

c) If $x_1 < x_2 + k < Z_1 - 1$ then player 2 proposes $(x_1 + 1, x_2 + n - 1)$ if $\delta(x_1 + 1 + \frac{n-k-I(n-k)}{2}) < N - Z_1$ and $\delta Z_1 > \delta(x_2 + k + \frac{n-k-I(n-k)}{2})$. Player 1 rejects this offer and proposes at his turn $(N - Z_1, Z_1)$ that is finally accepted. 1 gets a final payoff of $\delta^2(N - Z_1) < N - Z_1$. Instead, if $\delta Z_1 < \delta(x_2 + k + \frac{n-k}{2})$ player 2 forces arbitration and player 1 gets getting a payoff of $\delta^2(x_1 + \frac{n-k+I(n-k)}{2}) < \delta(x_1 + \frac{n-I(n)}{2}) < N - Z_1$.

STEP 4: *If* $x_1 \leq x_2 < Z_1 - 1$, $x_1 + 5 > N - Z_1$ *and* $\delta(x_1 + \frac{n-I(n)}{2}) > N - Z_1$ *player* 1 *forces arbitration.*

By asking for all the rest of the pie, player 1 knows that an arbitrator is going to implement the partition $(x_1 + \frac{n-I(n)}{2}, x_2 + \frac{n+I(n)}{2})$ one period later, getting a final payoff of $\delta(x_1 + \frac{n-I(n)}{2})$. Other alternatives are dominated by this one. Player 1 may propose $(x_1 + n - k, x_2 + k)$ being k even. Player 2 accepts the proposal if $x_2 + k \geq Z_1$. But that means that $x_1 + n - k < N - Z_1 < \delta(x_1 + \frac{n}{2})$. Player 2 he rejects if $x_2 + k < Z_1$. Then, as we saw above, at the new bargaining position $(x_1, x_2 + k, n - k)$, player 2 finishes the game by proposing:

a) $(N - Z_1, Z_1)$ if $x_2 + k = Z_1 - 1$. Player 1 accepts it.

b) (x_1+1, x_2+n-1) if $x_1 < x_2+k < Z_1-1$, $\delta(x_1+1+\frac{n-k-1}{2}) < N-Z_1$ and $Z_1 > x_2 + k + \frac{n-k}{2}$. Player 1 rejects and proposes $(N - Z_1, Z_1)$ that is accepted.

c) forcing arbitration if $x_1 < x_2 + k < Z_1 - 1$ and $.Z_1 < x_2 + k + \frac{n-k}{2}$.

In all the cases it is easy to check that player 1 gets a higher payoff by forcing arbitration rather than making the proposal $(x_1 + n - k, x_2 + k)$.

STEP 5: *If* $x_1 \leq x_2 < Z_1 - 1$, $x_1 + 5 < N - Z_1$ *and* $x_1 + 5 < \delta(x_1 + \frac{n}{2})$ *player* 1 *forces arbitration*

If player 1 deviates making a proposal, $(x_1 + n - k, x_2 + k)$, player 2 may accept or reject it. If $n - k \leq 5$, player 2 will accept this proposal and player 1 will get a final payoff of $x_1 + n - k < x_1 + 5 < \delta(x_1 + \frac{n}{2})$. If

$n-k>5$, player 2 will reject and will force arbitration at his turn since $x_2+k<\delta^2(x_2+k+\frac{n-k}{2})$. Then, player 1 would better force arbitration first, rather than wait for the opponent to do it in the next turn.

STEP 6: *If* $x_1 \leq x_2 < Z_1-1$, $x_1+5<N-Z_1$ *and* $x_1+5 \geq \delta(x_1+\frac{n}{2})$ *player* 1 *proposes* (x_1+5, x_2+n-5) *and player* 2 *accepts it.*

If player 2 rejects player's 1 proposal, at the new bargaining state, he will propose (x_1+4, x_2+n-4) being accepted by player 1. Then, for player 2 it will be better to accept rather than reject it because $x_2+n-5>\delta(x_2+n-4)$ since $x_2+n-5>Z_1$.

If player 1 makes a different proposal, (x_1+n-k, x_2+k), with $n-k>5$, player 2 will reject and will force arbitration at his turn, because $x_2+k<\delta^2(x_2+k+\frac{n-k}{2})$. For player 1, this alternative is worse than to propose (x_1+5, x_2+n-5) since $x_1+5 \geq \delta(x_1+\frac{n}{2}) > \delta^2(x_1+\frac{n-k}{2})$. If $n-k<5$, then player 2 will accept the proposal and player 1 will get a smaller final payoff since $x_1+n-k<x_1+5$.

STEP 7: *If* $x_2<x_1<Z_1-1$, $x_2+5>N-Z_1$ *player* 1 *asks for* x_1+n-1 *if* $\delta(x_2+1+\frac{n-2}{2})<N-Z_1$ *and* $\delta Z_1 > \delta(x_1+\frac{n}{2})$ *and player* 2 *rejects.*

If player 1 proposes (x_1++n-1, x_2+1), player 2 will reject it, the game will reach $(x_1, x_2+1, n-1)$ with $x_2+5>N-Z_1$ and he will propose at his turn $(Z_1, N-Z_1)$, being accepted by his opponent. By rejecting, player 2 gets $\delta\,(N-Z_1)$ greater than the payoff he gets if he accepts, x_2+1. Player 1 gets a payoff of δZ_1.

To force arbitration is clearly dominated since $\delta Z_1 \geq \delta(x_1+\frac{n}{2})$. To make a different proposal, too. If he proposes (x_1+n-k, x_2+k), player 2 may accept or reject it. If he rejects, at the new bargaining state it happens that:

a) $Z_1-1=x_2+k$. Player 2 will propose, at his turn, $(x_1+n-k-1, x_2+k+1)$ and player 1 will accept it and $x_1+k<\delta(x_1+k+1)$. But player 1 will get $\delta(x_1+n-k-1)=\delta(N-Z_1)$ smaller than $\delta(x_1+\frac{n}{2})$.

b) $x_2 + k < x_1 < Z_1 - 1$. Player 2 will reject player's 1 offer and propose $(Z_1, N - Z_1)$ if $\delta(x_2 + k + \frac{n-k}{2}) < N - Z_1$, and he will force arbitration if the opposite happens. In the first case, player 1 accepts, and player 2 gets a payoff of $\delta(N - Z_1) > x_1 + k$. Player 1 get a final payoff of δZ_1, the same payoff as if he would propose $(x_1 + n - 1, x_2 + 1)$. In the second case player 2 forces arbitration if $\delta(x_2 + k + \frac{X-k}{2}) > N - Z_1$ getting a higher payoff than the one he gets by accepting the initial offer of player 1 since $\delta^2(x_2 + k + \frac{X-k}{2}) > x_2 + k$. For player 1 this is clearly worse since $\delta^2(x_1 + \frac{n-k}{2}) < \delta Z_1$.

c) $x_1 < x_2 + k < Z_1 - 1$.By rejecting the game reaches a bargaining state where, at player's 2 turn he will either propose $(x_1 + 1, x_2 + n - 1)$ knowing that the opponent will reject and propose, at his turn, $(N - Z_1, Z_1)$ or he will force arbitration. In the first case, by rejecting, player 2 gets $\delta^2 Z_1$ and in the second case $\delta^2(x_2 + k + \frac{X-k}{2})$. In both cases these payoffs are greater than $x_2 + k$. For player 1, to propose $(x_1 + n - k, x_2 + k)$ mean to get or $\delta^2(N - Z_1)$ or $\delta^2(x_1 + \frac{n-k}{2})$ both payoffs smaller than δZ_1 the one he gets by proposing $(x_1 + n - 1, x_2 + 1)$.

STEP 8: *If* $x_2 < x_1 < Z_1 - 1$, $x_2 + 5 > N - Z_1$ *player* 1 *forces arbitration if* $\delta Z_1 < \delta(x_1 + \frac{n}{2})$.

If he forces arbitration by proposing $(x_1 + n, x_2)$ he will get a payoff of $\delta(x_1 + \frac{n}{2})$. To make a different proposal $(x_1 + n - k, x_2 + k)$ leads the game to a bargaining state belonging to a), b), or c) and as we have just seen, player 2 optimally rejects this proposal. Player 1 gets a final payoff in case a) $x_1 + n - k$, in case b) δZ_1 or $\delta^2(x_1 + \frac{n-k}{2})$, and finally in case c) $\delta^2(N - Z_1)$ or $\delta^2(x_1 + \frac{n-k}{2})$. In all these cases these payoffs are smaller than the one player 1 gets by forcing arbitration right now.

STEP 9: *If* $x_2 < x_1 < Z_1 - 1$, $x_2 + 5 < N - Z_1$ *player* 1 *forces arbitration.*

If player 1 forces arbitration he gets $\delta(x_1 + \frac{n}{2})$. If he makes a proposal, $(x_1 + n - k, x_2 + k)$ and $x_2 + k < Z_1$ player 2 will reject, the game will reach a bargaining state where either $x_2+k+5 \lesseqgtr N-Z_1$. Suppose first that $x_2+k+5 > N-Z_1$and:

a) $Z_1-1 = x_2+k$. Player 2 rejects and asks for one more unit since $\delta Z_1 > Z_1-1$. Player 1 prefers to get $\delta(x_1+\frac{n}{2})$ by forcing arbitration rather than $\delta(N-Z_1)$.

b) $x_2+k \leq x_1 < Z_1-1$. Player 2 will reject theoffer since he can get a higher payoff by proposing $(N-Z_1, Z_1)$ if $\delta(x_2+k+\frac{n-k}{2}) < N-Z_1$. By rejecting he gets $\delta(N-Z_1)$ that is greater than x_2+k. If $\delta(x_2+k+\frac{n-k}{2}) > N-Z_1$ he will reject in order to force arbitration at his turn, getting a payoff of $\delta^2(x_2+k+\frac{n-k}{2}) > x_2+k$. In both cases player 1 get worse by trying with (x_1+n-k, x_2+k) rather than forcing arbitration since he gets or δZ_1 or $\delta^2(x_1+\frac{n-k}{2})$.

c) $x_1 < x_2+k < Z_1-1$. By rejecting, player 2 will ask for (x_1+1, x_2+n-1), knowing that player 1 will reject and propose, at his turn, $(N-Z_1, Z_1)$ if $\delta(x_1+1+\frac{n-k-2}{2}) < N-Z_1$ and $Z_1 > x_2+k+\frac{n-k}{2}$. Player 1 gets a final payoff of $\delta^2(N-Z_1)$ smaller than $\delta(x_1+\frac{X}{2})$. Player 2 will reject and force arbitration, at his turn, if $Z_1 < x_2+k+\frac{n-k}{2}$. Naturally, player 1 will prefer to force arbitration right now, rather than wait for the opponent to do it in the next period.

Now assume that $x_2+k+5 < N-Z_1$and:

a) $x_2+k \leq x_1 < Z_1-1$. If player 2 rejects the initial offer of player 1, he will propose, at his turn, $(x_1+n-k-5, x_2+k+5)$ or he will force arbitration. In the first case, player 1 gets $\delta(x_1+n-k-5)$ smaller than $\delta(x_1+\frac{n}{2})$ and in the second case, this alternative is clearly dominated by proposing arbitration right now.

b) $x_1 < x_2+k < Z_1-1$. If this happens, player 2 will force arbitration. For player 1 is better to force arbitration right now rather than wait for player 2 to do it in the next turn.

STEP 10: *If* $x_1 \geq Z_1-1$.*Player* 1 *forces arbitration.*

If player 1 forces arbitration he gets a payoff of $\delta(x_1+\frac{n}{2})$. If he makes a different proposal, for example, (x_1+n-k, x_2+k), player 2 may accept or reject it. If he rejects, the new bargaining state $(x_1, x_2+k, n-k)$, may have $n-k \lesseqgtr 5$. If $n-k \leq 5$, then player 2 proposes[21] (x_1+1, x_2+n-1) and player 1 accepts it. Then player 2 will reject player's 1 proposal since $\delta(x_2 + n - 1) > x_2 + k$ since $x_1 \geq Z_1 - 1$. For player 1, it will be better to force arbitration instead of proposing $(x_1 + n - k, x_2 + k)$ because $\delta(x_1 + \frac{n}{2}) > \delta(x_1 + 1)$. If $n - k > 5$, player 2 will reject and proposes $(x_1+n-k-5, x_2+k+5)$ and player 1 accepts it if $x_2+k+5 \geq \delta(x_2 + k + \frac{n-k}{2})$ and forces arbitration if the opposite happens. Then, arbitration dominates to propose $(x_1 + k, x_2 + n - k)$, since, in the first case $\delta(x_1 + \frac{n}{2}) > \delta(x_1 + n - k - 5)$[22]and in the second case $\delta(x_1 + \frac{n}{2}) > \delta^2(x_1 + \frac{n-k}{2})$.■

Proof of Proposition 10

A pure strategy for player i type $\tau = W, S$, is a time t_i^τ at which she plans to yield (to concede is she is weak and to opt out is she is strong) given than no player yields before that time. If a pure SPBE exists, then $t_i^W = t_j^W = t_W$ and $t_i^S = t_j^S = t_S$. Assume that $t_S \leq t_W$. Thus, strong types know that, in equilibrium, weak types do not concede before they opt out with certainty. Then, it is optimal for a strong type to opt out at period 0, so she avoids any discounting of the payoff. The same happens to a weak type, since she knows she is not going to get any concession from her opponent. Thus, if there is a SPBE with $t_S \leq t_W$, it must be $t_W = t_S = 0$. But this cannot be an equilibrium since strong types will deviate from this strategy by delaying at least one period the decision of opting out since $b < (1 - a)\pi_0 + b(1 - \pi_0)$.

[21]The game reaches the bargaining position with $X < 6$ .Players follow the strategies of proposition 1 that, in that case, specify to player 1 to ask for only one unit, and for player 2 to accept it.

[22]$x_2 + k + 5 \geq \delta(x_2 + k + \frac{X-k}{2})$ if $5 \geq \frac{X-k}{2}$ since $x_2 < Z_1 - 1$. Then, $\delta(x_1 + \frac{X}{2}) > \delta(x_1 + X - k - 5)$

The other potential equilibrium is $t_W < t_S$ in which case $t_W = 0$ and $t_S = x$ with $x \geq 1$. If weak types concede in equilibrium at t=0, then it must be true that $a \geq (1-a)\pi_0 + a\delta(1-\pi_0)$ or $\pi_0 \leq \frac{a(1-\delta)}{1-a-\delta a}$. Since $\pi_0 \leq \frac{a(1-\delta)}{1-a-\delta a} < \frac{b(1-\delta)}{1-a-\delta b}$, strong types deviate and opt out at t=0 since $b > (1-a)\pi_0 + b\delta(1-\pi_0)$.■

Proof of Proposition 11.b

Consider the equation that rules the posterior:

$$\pi_t = \frac{H}{H+G} + (\pi_0 - \frac{H}{H+G})\,(11 - H - G)^t\,.$$

If $\pi_0 < \frac{H}{H+G}$, π_t is decreasing over time and, thus $\alpha_t = H\pi_t$ increases. At some period t, α_t reaches the value of 1. We denote that time as $\underline{T}$. In order to identify the time $\underline{T}$ we must use:

$$\pi_{\underline{T}-1}\alpha_{\underline{T}-1} = H,$$

$$\pi_{\underline{T}}\alpha_{\underline{T}} \leq H.$$

Since $\alpha_{\underline{T}} = 1$ and $\alpha_{\underline{T}-1} < 1$ then $\pi_{\underline{T}-1} \geq H \geq \pi_{\underline{T}}$. Using the solution for π_t, $\underline{T}$ will be the natural number that solves:

$$HH+G+(\pi_0 - HH+G)(11-H-G)^t \leq H \leq HH+G+(\pi_0 - HH+G)(11-H-G)^{t-1}.$$

By lemma 1 we know that if $\alpha_{\underline{T}} = 1$ then $\beta_{\underline{T}+1} = 1$.■

Proof of Proposition 12

If $\frac{H}{H+G} < \pi_0 < 1-G$, then π_t is increasing over time and β_t increases until, at some point, it reaches the value of 1. We denote as $\overline{T}$ that time and

$$(1 - \pi_{T_H^*-1})\beta_{\overline{T}-1} = G,$$

$$(1-\pi_{\overline{T}})\beta_{\overline{T}} \leq G.$$

Since $\beta_{\overline{T}} = 1$ and $\beta_{\overline{T}-1} < 1$, then $1 - \pi_{\overline{T}-1} \geq G \geq 1 - \pi_{\overline{T}}$. Using the solution of π_t, $\overline{T}$ will be the natural number that solves:

$$HH + G + (\pi_0 - HH + G)(11 - H - G)^{t-1} \leq 1 - G \leq HH + G + (\pi_0 - HH + G)(11 - H - G)^t.$$

Since $\beta_{\overline{T}} = 1$, then $\pi_t = 1$ $\forall t \geq \overline{T} + 1$. Players that are still playing are weak types and thus $\alpha_t = \frac{a(1-\delta)}{1-a-\delta a}$ for $t \geqslant \overline{T} + 1$.

If $1 - G \leqslant \pi_0$, the SPBE is $\beta_t = 1$ $\forall t \geq 0$ and $\alpha_0 = a(1-\delta\pi_0)(1-a-\delta a)\pi_0$ $\alpha_t = a(1-\delta)1-a-\delta a$ $\forall t \geq 1$. Then, a weak type will optimally randomize between conceding and not conceding at each t if:

$$a = \pi_0\alpha_0(1-a) + a\delta(1 - \pi_0\alpha_0 - (1-\pi_0)\beta_0),$$

$$a = \pi_t\alpha_t(1-a) + a\delta(1 - \pi_t\alpha_t - (1-\pi_t)\beta_t)\forall t \geq 1.$$

Given these strategies, $\pi_t = 1$ $\forall t \geq 1$. We substitute $\beta_t = 1$ $\forall t \geq 0$ and $\alpha_0 = a(1-\delta\pi_0)(1-a-\delta a)\pi_0$ $\alpha_t = a(1-\delta)1-a-\delta a$ in those equations and check if they are satisfied $\forall t$.

Consider now a strong type. Given the opponent's strategy, $\beta_t = 1$ $\forall t \geq 0$ and $\alpha_0 = a(1-\delta\pi_0)(1-a-\delta a)\pi_0$ $\alpha_t = a(1-\delta)1-a-\delta a$ $\forall t \geq 1$ he will opt out with probability 1 from period 0 on if:

$$b > \pi_0\alpha_0(1-a) + b(1-\pi_0)\beta_0 + b\delta(1 - \pi_0\alpha_0 - (1-\pi_0)\beta_0) = 0,$$

$$b > \pi_t\alpha_t(1-a) + b(1-\pi_t)\beta_t + b\delta(1 - \pi_t\alpha_t - (1-\pi_t)\beta_t) fort \geq 1$$

Since $\pi_t = 1$ $\forall t \geq 1$, the second condition is satisfied if $b > \alpha_t(1-a) + b\delta(1-\alpha_t)$. Substituting α_t,

$$b > a(1-\delta)1-a-\delta a(1-a) + b\delta(1 - a(1-\delta)1-a-\delta a).$$

Or $\frac{b(1-\delta)}{1-a-\delta b} > \frac{a(1-\delta)}{1-a-\delta a}$ that is true since $b > a$.

At t=0 it must be satisfied that $b > \pi_0(1-a)\alpha_0 + b(1-\pi_0) + b\delta\pi_0(1-\pi_0\alpha_0)$. Substituting α_0 and β_0 it is easy to check that this equation is satisfied only if $\pi_0 \geq 1 - G$.

Now we will prove that if $\pi_0 \geq 1-G$, the unique symmetric SPBE is $\{\{\alpha_t\}_0^\infty \{\beta_t\}_0^\infty\}$ such that $\beta_t = 1\ \forall t \geq 0$ and $\alpha_0 = a(1-\delta\pi_0)(1-a-\delta a)\pi_0$, $\alpha_t = a(1-\delta)1-a-\delta a\ \forall t \geq 1$. We will explore all possible candidates and see that, indeed, this is the unique SPBE.

First, consider a SPBE $\left\{\left\{\overset{\wedge}{\alpha_t}\right\}_0^\infty, \left\{\overset{\wedge}{\beta_t}\right\}_0^\infty\right\}$ such that $0 < \overset{\wedge}{\alpha_t} < 1$ and $0 < \overset{\wedge}{\beta_t} < 1\ \forall t \geq 0$. Then,

$$a = (1-a)\pi_t \overset{\wedge}{\alpha_t} + a\delta(1-\pi_t \overset{\wedge}{\alpha_t} - (1-\pi_t) \overset{\wedge}{\beta_t}),$$

$$b = (1-a)\pi_t \overset{\wedge}{\alpha_t} + b(1-\pi_t) \overset{\wedge}{\beta_t} + b\delta(1-\pi_t \overset{\wedge}{\alpha_t} - (1-\pi_t) \overset{\wedge}{\beta_t}),$$

for $\forall t \geq 0$. At t=0 these conditions are rewritten as:

$$\overset{\wedge}{\alpha_0} \pi_0 = H,$$

$$\overset{\wedge}{\beta_0} (1-\pi_0) = G.$$

But since $\pi_0 \geq 1-G$, $\overset{\wedge}{\beta_0} \geq 1$ contradicting the assumption that $0 < \overset{\wedge}{\beta_t} < 1\ \forall t \geq 0$.

Second, assume that there is a SPBE such that $\overset{\wedge}{\beta_t} = 1\ \forall t \geq 0$ and $0 < \overset{\wedge}{\alpha_t} < 1$ such that $\overset{\wedge}{\alpha_0} \neq a(1-\delta\pi)(1-a-\delta a)\pi$, $\overset{\wedge}{\alpha_t} \neq a(1-\delta)1-a-\delta a$ $\forall t \geq 1$. Notice that if $\overset{\wedge}{\beta_0} = 1$ and $0 < \overset{\wedge}{\alpha_0} < 1$ then $\pi_1 = 1$. But this cannot be an equilibrium since a weak type will deviate and concede with probability 1 at $t = 1$ if $\overset{\wedge}{\alpha_1} < \frac{a(1-\delta)}{1-a-\delta a}$ since $a > (1-a) \overset{\wedge}{\alpha_1} + a\delta(1- \overset{\wedge}{\alpha_1})$ and will never concede if $\overset{\wedge}{\alpha_t} > \frac{a(1-\delta)}{1-a-\delta a}$. The same happens at t=0.

And finally, assume that there is a SPBE with $0 < \overset{\wedge}{\beta_t} < 1\ \forall t \geq 0$ and $\overset{\wedge}{\alpha_0} = a(1-\delta\pi_0)(1-a-\delta a)\pi_0$, $\overset{\wedge}{\alpha_t} = a(1-\delta)1-a-\delta a\ \forall t \geq 1$. In that case, at t=0, it must be true that:

$$a = (1-a)\pi_0 \overset{\wedge}{\alpha_0} + a\delta(1-\pi_0 \overset{\wedge}{\alpha_0} - (1-\pi_0) \overset{\wedge}{\beta_0})$$

$$b = (1-a)\pi_0 \overset{\wedge}{\alpha_0} + b(1-\pi_0) \overset{\wedge}{\beta_0} + b\delta(1-\pi_0 \overset{\wedge}{\alpha_0} - (1-\pi_0) \overset{\wedge}{\beta_0})$$

Substituting $\overset{\wedge}{\alpha_0} = a(1-\delta\pi_0)(1-a-\delta a)\pi_0$ in the first condition, it must be that $\overset{\wedge}{\beta_0} = 1$ contradicting that $0 < \overset{\wedge}{\beta_t} < 1 \ \forall t \geq 0.$■

Proof of Proposition 13

Now consider the case $\pi_0 = \frac{H}{H+G}$. Then $\pi_t = \pi_0$ and $\alpha_t = H\pi_0 = H+G$ and $\beta_t = \frac{G}{1-\pi_0} = H+G \ \forall t \geq 0.$■

The derivation of $\underline{T}$ and $\overline{T}$.

We reduce the length of each period to $0 < \Delta < 1$ (there are $\frac{1}{\Delta}$ periods per unit of time) and the term δ is replaced by $e^{-\Delta}$. Define:

$$H^{'} = ab(1-e^{-\Delta})ae^{-\Delta}(1-a-e^{-\Delta}b) + b(1-e^{-\Delta})(1-a-e^{\Delta}a)$$

$$G^{'} = (1-e^{-\Delta})(1-a)(b-a)ae^{-\Delta}(1-a-e^{-\Delta}b) + b(1-e^{-\Delta})(1-a-e^{-\Delta}a)$$

$$H^{'}H^{'} + G^{'} = abb - a(1-a)$$

$\frac{H^{'}}{H^{'}+G^{'}}$ is independent of Δ. It is easily checked that $\Delta \to 0 \lim H^{'} = 0$ and $\Delta \to 0 \lim 1 - G = 1.$.

Proposition 2 establishes that we can identify an ending period $\underline{T}$ at which the equilibrium probability of conceding is 1 if $\pi_0 \in \left(H^{'}, \frac{H^{'}}{H^{'}+G^{'}}\right)$. This $\underline{T}$ is the natural number that solves:

$$HH+G+(\pi_0 - HH+G)(11-H-G)^{\frac{t}{\Delta}} \leq H \leq HH+G+(\pi_0 - HH+G)(11-H-G)^{\frac{t}{\Delta}-\Delta}.$$

Or, for each possible expected delay $\underline{T}$ we have a compatible interval of π

$$\pi_0 \in \left(H^{'}H^{'} + G^{'}(1-(1-H^{'}-G^{'})^{\frac{t}{\Delta}}), H^{'}H^{'} + G^{'}(1-(1-H^{'}-G^{'})^{\frac{t}{\Delta}+\Delta}\right].$$

The size of this interval tends to 0 as $\Delta \to 0$. Hence, in the limit, we have a function

$$\pi_0 = \frac{H^{'}}{H^{'}+G^{'}}\left[1-e^{-tI}\right]$$

with $I = \frac{b-a(1-a)}{a(1-a-b)}$. Or, given the parameters of the game (a, b, π_0)

$$\underline{T} = \frac{-1}{I}\ln(1-H^{'}+G^{'}H^{'}\pi_0)$$

We consider now the interval of probabilities $\pi_0 \in \left(\frac{H^{'}}{H^{'}+G^{'}}, 1-G^{'}\right)$. Proposition 4 shows that, in equilibrium, strong types won't remain in the game beyond some period $\overline{T}$ that can be identified as the natural number that solves:

$$HH+G+(\pi_0-HH+G)(11-H-G)^{\frac{t}{\Delta}-\Delta} \leq 1-G \leq HH+G+(\pi_0-HH+G)(11-H-G)^{\frac{t}{\Delta}}$$

We compute the interval of probabilities for which $\overline{T} = \frac{t}{\Delta}$,

$$\pi_0 \in \left[H^{'}H^{'}+G^{'}+G^{'}H^{'}+G^{'}(1-H^{'}-G^{'})^{\frac{t}{\Delta}+\Delta}, H^{'}H^{'}+G^{'}+G^{'}H^{'}+G^{'}(1-H^{'}-G^{'})^{\frac{t}{\Delta}}\right]$$

As Δ goes to 0 the size of this interval tends also to 0 and

$$\pi_0 = \frac{H^{'}}{H^{'}+G^{'}} + \frac{G^{'}}{H^{'}+G^{'}}e^{-tI}$$

Therefore

$$\overline{T} = \frac{-1}{I}\ln\left[\pi_0(\frac{H^{'}+G^{'}}{G^{'}}) - \frac{H^{'}}{G^{'}}\right]$$

Proof of Proposition 14

We simply compute the partial derivatives of $\underline{T}$ and $\overline{T}$with respect to π_0. Denote as $y = b(a - \pi_0) + a\pi_0(1 - a)ab$ and $x = b(\pi_0 - a) - a\pi_0(1 - a)(1 - a)(b - a)$. Then,

$$\partial\underline{T}\partial\pi_0 = (1 - a - b)by > 0,$$

$$\partial\overline{T}\partial\pi_0 = -a(1 - a - b)(1 - a)(b - a)x < 0,$$

since $0 < y < 1$, $0 < x < 1$ and $a < b < 1 - a$.■

Proof of Proposition 15

The partial derivative $\partial\underline{T}\partial a$ is,

$$\partial\underline{T}\partial a = 1(b - a(1 - a))^2 \left[b(b - (1 - a)^2)\ln[y] - (1 - a - b)(b - a^2)(\frac{1}{y} - 1)\right].$$

In order to prove that $\partial\underline{T}\partial a < 0$ we will consider two cases:

(i) $b \geqslant (1 - a)^2$. The sign of the derivative is clearly negative since $0 < y < 1$ and $a < b < 1 - a$.

(ii) $b < (1 - a)^2$. We study the function

$$F(y) = b(b - (1 - a)^2)\ln[y] - (1 - a - b)(b - a^2)(\frac{1}{y} - 1)$$

It is easy to check that $F(1) = 0$, $F(0) = -\infty$ and $F(y)$ has a maximum on $y^* = \frac{-(1-a-b)(b-a^2)}{b^2-b(1-a)^2}$. Since $\frac{-(1-a-b)(b-a^2)}{b^2-b(1-a)^2} > 1$ then $F(y) < 0$ $\forall y \in (0, 1)$.

The derivative $\partial\overline{T}\partial a$ is

$$\partial\overline{T}\partial a = b(b - a(1 - a))^2 \left[(b - (1 - a)^2)\ln[x] + a(1 - a - b)(b - a^2)(1 - a)(b - a)(\frac{1}{x} - 1)\right]$$

We study the sign of this derivative and find two cases:

(i) $b \leq (1-a)^2$. Clearly $\partial\overline{T}\partial a > 0$ since $0 < x < 1$ and $a < b < 1-a$.

(ii) $b > (1-a)^2$. The sign of $\partial\overline{T}\partial a = signF(x)$ with $F(x) = (b-(1-a)^2)\ln[x]+a(1 - a - b)(b - a^2)(1 - a)(b - a)\left[\frac{1}{x} - 1\right]$. This function has a minimum at $x^* = \frac{a(1-a-b)(b-a^2)}{(1-a)(b-a)(b-(1-a)^2)}$ since $F^{''}(x^*) > 0$ and takes the values $F(0) = +\infty$ and $F(1) = 0$. Thus, if $x^* > 1$, then $\forall x \in (0,1)$ $F(x) > 0$. Otherwise, if $x^* < 1$, then $\partial\overline{T}\partial a > 0$ $\forall x \in (0, \widetilde{x})$ and $\partial\overline{T}\partial a < 0$ $\forall x \in (\widetilde{x}, 1)$ where $\widetilde{x}$ is the unique root of $F(x)$ on the range $x \in (0,1)$.■

Proof of Proposition 16

First, we compute the partial derivative of $\overline{T}$ with respect to b:

$$\partial\overline{T}\partial b = a(1-a)r(b-a(1-a))^2 \left[(1-a)\ln[x] - a^2(1-a-b)(b-a)(1-a)(\frac{1}{x}-1)\right] < 0,$$

Now we derive $\partial\underline{T}\partial b$ that is,

$$\partial\underline{T}\partial b = a(1-a)^2 r(b-a(1-a))^2 \left[\ln[y] + a(1-a-b)b(1-a)(\frac{1}{y}-1)\right].$$

It is clear that $sign\partial\underline{T}\partial b = signJ(y)$ with $J(y) = \ln[y] + a(1-a-b)b(1-a)(\frac{1}{y} - 1)$.

This function takes values $J(0) = +\infty$ and $J(1) = 0$ and its derivative $J^{'}(y^*) = 0$ with $y^* = a(1-a-b)b(1-a)$. It is easy to check that $J(y^*) < 0$ and that $J(y)$ is decreasing on $(0, y^*)$ and increasing on $(y^*, 1)$. Since $a(1-a-b)b(1-a) < 1$, then $J(y)$ has a unique root on the range $y \in (0,1)$. This root is the $\widetilde{y}$ that solves the equation

$$\ln[y] + a(1-a-b)b(1-a)(\frac{1}{y} - 1) = 0.$$

Then, $J(y) > 0$ if $y \in (0, \widetilde{y})$ and $J(y) < 0$ if $y \in (\widetilde{y}, 1)$.■

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