Capacitat analítica i nuclis de Riesz

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Introduction

In this dissertation we will study several questions concerning the natural capacity γ_α related to the signed vector valued Riesz kernels $x/|x|^{1+\alpha}$ in $\mathbb{R}^n$, where $0 < \alpha < n$. It is defined as follows. For a compact set $E \subset \mathbb{R}^n$ and $0 < \alpha < n$, set

$$\gamma_\alpha(E) = \sup |< T, 1 >| \,, \tag{1}$$

where the supremum is taken over all real distributions T supported on E such that for $1 \leq i \leq n$, the i-th Riesz potential of T, $T * \frac{x_i}{|x|^{1+\alpha}}$ is a function in $L^\infty(\mathbb{R}^n)$ and $\sup\limits_{1\leq i\leq n} \left\| T * \frac{x_i}{|x|^{1+\alpha}} \right\|_\infty \leq 1$.

These capacities can be understood as being certain real variable versions of analytic capacity. The notion of analytic capacity was introduced in 1947 by L. Ahlfors [A] to study removable singularities of bounded analytic functions. Recall that the analytic capacity of a compact subset E of the plane is defined by

$$\gamma(E) = \sup |f'(\infty)|,$$

where the supremum is taken over those analytic functions on $\mathbb{C}\setminus E$ such that $|f(z)| \leq 1$, for $z \notin E$. Ahlfors proved that a set is removable for bounded analytic functions if and only if it has zero analytic capacity.

When working with analytic capacity one uses often properties of the Cauchy kernel $1/z$ such as oddness and homogeneity, but it is unclear how important analiticity actually is. In fact, one can define analytic capacity without making any reference to analyticity in the form

$$\gamma(E) = \sup | < T, 1 > |,$$

where the supremum is taken over all complex distributions T supported on E such that the Cauchy potential of T, $f = \frac{1}{z} * T$, is a function in $L^\infty(\mathbb{C})$ satisfying $\|f\|_\infty \leq 1$.

Thus clearly γ_α can be considered as a real variable version of analytic capacity associated to the vector valued Riesz kernel $\frac{x}{|x|^{1+\alpha}}$ in $\mathbb{R}^n$. For $n \geq 2$ and $\alpha = n-1$, γ_{n-1} is Lipschitz harmonic capacity (see [V1], [Par], [MP] and [Vo]).

We introduce now the analytic capacity γ_+ of a compact set E as

$$\gamma_+(E) = \sup \mu(E),$$

where the supremum is taken over all positive measures supported on E such that the Cauchy transform $f = \frac{1}{z} * \mu$ is a function in $L^\infty(\mathbb{C})$ with $\|f\|_\infty \leq 1$. Then, by definition, for $n = 2$, writing $\frac{1}{z} = \frac{x}{|z|^2} - i\frac{y}{|z|^2}$, where $z = x + iy$, we have

$$\gamma_+(E) \leq \gamma_1(E) \leq \gamma(E).$$

Hence due to the celebrated Theorem of X. Tolsa [T2], saying that

$$\gamma(E) \leq C\gamma_+(E), \tag{2}$$

we get that on compact subsets of the plane, these three capacities are comparable. Inequality (2), was first proved for generalized four-corners Cantor sets by J. Mateu, X. Tolsa and J. Verdera ([MTV]). To prove (2) for any compact set one needs to overcome formidable technical complications and introduce new ideas.

According to the classical case, if we want to study how the set function γ_α behaves, we first have to take into account the role played by the "size" of the set. More precisely in terms of Hausdorff dimension (denoted by dim) we have:

1. If $\dim(E) > \alpha$ then $\gamma_\alpha(E) > 0$.
2. If $\dim(E) < \alpha$ then $\gamma_\alpha(E) = 0$.

This says that the critical situation occurs in dimension α, in accordance with the classical case.

An interesting fact of the capacities γ_α, $0 < \alpha < n$, is that they behave differently when dealing with integer or non-integer indexes α. For example let α be an integer and E a compact subset of an α-dimensional smooth surface with positive α-dimensional Hausdorff measure. Then one can show that $\gamma_\alpha(E) > 0$, which means that, in particular, there exist sets with finite α- dimensional Hausdorff measure and positive γ_α. In [MP] the authors study $\gamma_{n-1}(E)$ on sufficiently regular hypersurfaces, for example on Lipschitz graphs or bilipschitz images of $\mathbb{R}^{n-1}$. More precisely, it is shown that on such surfaces, γ_{n-1} is comparable to $\mathcal{H}^{n-1}$. (see [Par] or [MP]).

In constrast to this, in the first part of this dissertation we show that for $0 < \alpha < 1$, the capacity γ_α vanishes on compact sets $E \subset \mathbb{R}^n$ with finite α-Hausdorff measure, namely we show that

Theorem A. *Let $0 < \alpha < 1$ and let $E \subset \mathbb{R}^n$ be a compact set with $\mathcal{H}^\alpha(E) < \infty$. Then $\gamma_\alpha(E) = 0$.*

The techniques that we use are based on the so called symmetrization method and they do not extend to indexes $\alpha > 1$. We will explain these techniques below. However, we can extend the previous result to indexes $1 < \alpha < n$, $\alpha \notin \mathbb{Z}$, assuming Ahlfors-David regularity of the sets we are dealing with. Recall that a set E in $\mathbb{R}^n$ is Ahlfors-David regular of dimension α if there exists a constant $C > 0$ such that

$$C^{-1}r^\alpha \leq \mathcal{H}^\alpha(E \cap B(x,r)) \leq Cr^\alpha,$$

whenever $x \in E$ and $r \in (0, d(E))$. By $d(E)$ we denote the diameter of the set E. Notice that a compact Ahlfors-David regular set of dimension α satisfies $\mathcal{H}^\alpha(E) < \infty$.

The statement of the precise result is

Theorem B. *Let $E \subset \mathbb{R}^n$ be a compact Ahlfors-David regular set of non-integer dimension α, $0 < \alpha < n$. Then $\gamma_\alpha(E) = 0$.*

The proofs of the above two Theorems are based on the relation between the capacity γ_α of a set E and the L^2-boundedness of the $\alpha-$Riesz operators on E. Before explaining this relationship, we will first deal with the corresponding one between analytic capacity and L^2-boundedness of the Cauchy integral operator.

We have already said that having zero analytic capacity is equivalent to the fact of being removable for bounded analytic functions. But this is far from giving any geometric characterization of such sets. This has been called traditionally the Painlevé problem. In 1967, Vitushkin conjectured that for sets E with finite $\mathcal{H}^1$ measure, $\gamma(E) = 0$ is equivalent to the fact that $\mathcal{H}^1(E \cap \Gamma) = 0$, for every rectifiable curve Γ. Notice that if the above statement is true, then Painlevé's problem is solved for sets of finite length. The proof of the first half of Vitushkin's conjecture was done in 1977 by A. Calderón [C], who proved the L^2-boundedness of the Cauchy integral on Lipschitz graphs with small enough Lipschitz constant . From this result it follows that if $E \subset \mathbb{C}$ is a compact set with $0 < \mathcal{H}^1(E) < \infty$ and $\gamma(E) = 0$, then $\mathcal{H}^1(E \cap \Gamma) = 0$ for any rectifiable curve $\Gamma \subset \mathbb{C}$. Thus it was clear that the removability of a set E was closely related to the L^2-boundedness of the Cauchy operator on E. The proof of the other implication came years later after the discovery of the very important identity, see (3) below, relating the L^2-norm of the Cauchy integral operator and Menger curvature. We shall explain now this relationship, that is, the symmetrization method, which has been a really useful tool for the study of analytic capacity and L^2- boundedness of the Cauchy integral operator (see [V3], [MV] and [MMV] for example ; the survey papers [D3], [V5] and [T5] contain many other references as well as the book [Pa2]).

In 1995, M. Melnikov [Me] rediscovered the Menger curvature when he was studying a discrete version of analytic capacity. Let z_1, z_2 and z_3 be three non collinear points in $\mathbb{C}$ (in particular z_1, z_2 and z_3 are distinct). Then the Menger curvature $c(z_1, z_2, z_3)$ of z_1, z_2 and z_3 is the inverse of the radius of the circle passing through z_1, z_2 and z_3. When the three points are collinear, we set $c(z_1, z_2, z_3) = 0$. Then one finds out, by an elementary computation that

$$c(z_1, z_2, z_3)^2 = \sum_\sigma \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}} \tag{3}$$

where the sum is taken over the six permutations of the set $\{1, 2, 3\}$. In particular this formula shows that the sum over σ on the right hand side is real and non-negative.

On the other hand, when one tries to extend the previous identity to higher dimensions, nothing similar occurs for the Riesz kernel $k_\alpha(x) = x/|x|^{1+\alpha}$ with $\alpha > 1$ (see [F] where he shows it for integers α with $\alpha > 1$). In this dissertation we show that for $0<\alpha<1$, when symmetrizing the vector valued Riesz kernel k_α, we still obtain a positive quantity.

For $0<\alpha<n$ consider the analogue of the right hand side in (3) for the Riesz kernel k_α,

$$\sum_\sigma \frac{x_{\sigma(2)}-x_{\sigma(1)}}{|x_{\sigma(2)}-x_{\sigma(1)}|^{1+\alpha}}\frac{x_{\sigma(3)}-x_{\sigma(1)}}{|x_{\sigma(3)}-x_{\sigma(1)}|^{1+\alpha}}, \tag{4}$$

where the sum is taken over the six permutations of the set $\{1,2,3\}$. Observe, however, that if σ is a transposition of two numbers in $\{1,2,3\}$ then the term one obtains is one of the three terms associated to the permutations $(1,2,3),\ (2,3,1),\ (3,1,2)$. Thus (4) is exactly

$$2\ p_\alpha(x_1,x_2,x_3),$$

where $p_\alpha(x_1,x_2,x_3)$ is defined as the sum in (4) taken only on the three permutations $(1,2,3),\ (2,3,1),\ (3,1,2)$.

In the first chapter it is shown that when $x_1,\ x_2$, and x_3 are three distinct points in $\mathbb{R}^n$, then

$$\frac{2-2^\alpha}{L(x_1,x_2,x_3)^{2\alpha}} \le p_\alpha(x_1,x_2,x_3) \le \frac{2^{1+\alpha}}{L(x_1,x_2,x_3)^{2\alpha}}, \tag{5}$$

where $L(x_1,x_2,x_3)$ is the largest side of the triangle determined by the three points x_1, x_2 and x_3. Notice that in particular this means that, for $0<\alpha<1$, $p_\alpha(x_1,x_2,x_3)$ is a positive quantity. Also, when $\alpha=1$ the left hand side in (5) is identically zero and the inequality becomes trivial.

We will make here a small computation to illustrate the phenomenon of change of signs when $1<\alpha<n$.

Notice first that if $x_1,\ x_2,\ x_3\in\mathbb{R}^n$, (4) can be written as

$$p_\alpha(x_1,x_2,x_3)=\frac{\cos(\theta_{23})|x_2-x_3|^\alpha+\cos(\theta_{13})|x_1-x_3|^\alpha+\cos(\theta_{12})|x_1-x_2|^\alpha}{|x_1-x_2|^\alpha|x_1-x_3|^\alpha|x_2-x_3|^\alpha},$$

where θ_{ij} is the angle opposite to the side x_ix_j in the triangle determined by $x_1,\ x_2,\ x_3$.

Denote by $l_{ij}=|x_i-x_j|$, for $i\neq j,\ i,j\in\{1,2,3\}$.

Let $n=2$ and take x_1, x_2 and x_3 on the x-axis such that $l_{12}>l_{13}>l_{23}$, then

$$p_\alpha(x_1,x_2,x_3)=\frac{l_{23}^\alpha+l_{13}^\alpha-l_{12}^\alpha}{l_{12}^\alpha l_{13}^\alpha l_{23}^\alpha}<0,$$

because $\alpha>1$ and $l_{12}=l_{23}+l_{13}$. On the other hand, if the three angles are in $[0,\pi/2]$ and $l_{12}\ge l_{13}\ge l_{23}$. Then we have

$$p_\alpha(x_1,x_2,x_3)=\frac{1}{l_{12}^\alpha l_{13}^\alpha}\left(\cos(\theta_{23})+\cos(\theta_{13})\frac{l_{13}^\alpha}{l_{23}^\alpha}+\cos(\theta_{12})\frac{l_{12}^\alpha}{l_{23}^\alpha}\right)$$

$$\geq \frac{1}{l_{12}^\alpha l_{13}^\alpha}\left(\cos(\theta_{23})+\cos(\theta_{13})+\cos(\theta_{12})\right)\geq \frac{1}{l_{12}^\alpha l_{13}^\alpha}>0.$$

We return now to the classical setting. For a positive Borel measure μ in $\mathbb{C}$, the curvature of μ is defined as

$$c^2(\mu)=\iiint c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z).$$

This notion was introduced by M. Melnikov in [Me], where he showed that

$$\gamma(E)\geq C\sup\frac{\|\mu\|^{3/2}}{\left(\|\mu\|+c^2(\mu)\right)^{1/2}}, \tag{6}$$

for some absolute constant C and the supremum taken over all positive finite Radon measures μ supported on E with linear growth, that is, $\mu(B(x,r))\leq r$ for all discs $B(x,r)\subset\mathbb{C}$. In particular the equivalence (6) shows that $\gamma(E)>0$ provided that E supports some positive finite Radon measure with $\mu(B(x,r))\leq r$ for all $x\in\mathbb{C}$, $r>0$, and $c^2(\mu)<\infty$.

On the other hand, due to a result of M. Melnikov and J. Verdera [MV], it turns out that the curvature of a measure is closely related to the Cauchy integral of this measure. They showed that for a measure μ with linear growth, the $L^2(\mu)$-norm of the Cauchy integral of the positive finite measure μ, is comparable to $c^2(\mu)+\|\mu\|$, that is

$$\|\mathcal{C}(\mu)\|^2_{L^2(\mu)}\approx c^2(\mu)+\|\mu\|, \tag{7}$$

where the notation $A\approx B$ means, as it is usual, that for some constant C one has $C^{-1}B\leq A\leq CB$.

Using Menger curvature and previous work of M. Christ [Ch2], P. Mattila, M. Melnikov and J. Verdera [MMV] proved the Vitushkin conjecture for Ahlfors-David regular sets, namely that compact sets $E\subset\mathbb{C}$ with $0<\mathcal{H}^1(E)<\infty$ and $\mathcal{H}^1(E\cap\Gamma)=0$ for any rectifiable curve $\Gamma\subset\mathbb{C}$, have $\gamma(E)=0$. The solution of the conjecture for any compact set was obtained by G. David in [D2]. To prove Vitushkin's conjecture without any regularity condition, it has been necessary to study the L^2-boundedness of the Cauchy integral operator with respect to measures which are non-doubling. Recall that a measure μ is doubling if $\mu(2B)\leq C\mu(B)$ for all balls, where $2B$ is the ball concentric with B but with double radius. A non-homogeneous Calderón-Zygmund theory has been developed (see [T1], [NTV3]). In fact, the solution of the Vitushkin's conjecture follows from a $T(b)$-type Theorem for non-doubling measures.

X. Tolsa [T1] showed a $T(1)$-Theorem for the Cauchy integral operator with respect to an underlying measure μ which is not assumed to satisfy the standard doubling condition. Namely he proved that if μ is a continuous positive Radon measure with linear growth and such that $c^2(\mu_{|B}) \leq C\mu(B)$ for all discs $B \subset \mathbb{C}$, then the Cauchy integral operator is bounded on $L^2(\mu)$. Notice that, by the result of [MV], see (7), the conditions required to the measure μ in Tolsa's $T(1)$-Theorem are equivalent to

$$\int_B |\mathcal{C}_\epsilon(\chi_B\mu))|^2 \, d\mu \leq C\mu(B), \text{ for all discs } B \subset \mathbb{C}, \tag{8}$$

where

$$\mathcal{C}_\epsilon(\mu)(y) = \int_{|y-z|>\epsilon} \frac{d\mu(z)}{z-y}, \, y \in \mathbb{C}$$

is the truncated Cauchy integral operator.

We remark that if the measure μ is doubling, then condition (8) is easily seen to be equivalent to requiring that $\mathcal{C}_\epsilon(\mu)$ belongs to $\mathrm{BMO}(\mu)$, uniformly in ϵ. Hence the result can be understood as a $T(1)$-Theorem for a continuous positive measure non necessarily doubling (see [D1] for the standard formulation of the $T(1)$-theorem). The same result has been proved, independently, by Nazarov Treil and Volberg, where fairly more general Calderón-Zygmund operators are considered. The arguments in [T1] are of complex analytic nature and exploit Menger curvature, which is a tool very specific for the Cauchy kernel. In [V4] there is an alternative proof of the same result, which also uses Menger curvature but does not use any complex analysis.

After David's solution of Vitushkin's conjecture, F. Nazarov, S. Treil and A. Volberg [NTV3] proved a $T(b)$-Theorem useful for dealing with analytic capacity. Their theorem gives also an alternative solution to the Vitushkin's conjecture.

These $T(b)$-Theorems on non-homogeneous spaces have been an important ingredient, together with the curvature of a measure for the impressive progress made recently to solve old important problems concerning analytic capacity (see [D2], [MTV], [NTV3], [T2] and [T4]).

As we said before, the proofs of Theorems A and B are based on the relation between the capacity γ_α and the L^2-boundedness of the $\alpha-$Riesz operators. David and Semmes have studied singular integrals on integral dimensions and rectifiable sets. If $\alpha \in \mathbb{Z}$ and μ is a surface measure on a sufficiently nice α-dimensional surface, for example a Lipschitz graph, then the α-Riesz operator is bounded on $L^2(\mu)$. It turns out that if E is an Ahlfors-David regular set and the operators corresponding to the Riesz kernels $x_i/|x|^{1+\alpha}$, $0 < \alpha < n$, $i = 1, ..., n$ are bounded on $L^2(\mathcal{H}^\alpha_{|E})$, then α must be an integer (see [Vi]). The result in [Vi] is proved by using an approach based on tangent measures, in which the Ahlfors-David regularity condition is strongly used. If we knew how to generalize Vihtilä's result, namely, if we could show that for $\alpha \notin \mathbb{Z}$, compact sets $E \subset \mathbb{R}^n$ with finite $\mathcal{H}^\alpha$-measure have unbounded $\alpha-$Riesz transform in $L^2(\mathcal{H}^\alpha_E)$, then we would be able to extend Theorem A to all indexes $0 < \alpha < n$ such that $\alpha \notin \mathbb{Z}$. However, we do not know how to prove this result for general sets with finite non-integer α-Hausdorff measure. On the other hand, we do have a result when $0 < \alpha < 1$. The proof of Theorem A is based on the following. Given a positive finite Radon measure μ, set

$$p_\alpha(\mu) = \iiint p_\alpha(x_1, x_2, x_3) d\mu(x_1) d\mu(x_2) d\mu(x_3).$$

If we assume that $\mu(B(x,r) \leq Cr^\alpha$, then, arguing as in [MV], but with $p_\alpha(\mu)$ instead of with $c^2(\mu)$, we obtain that the $L^2(\mu)$-norm of the $\alpha-$Riesz operator is comparable to $p_\alpha(\mu) + \|\mu\|$, that is,

$$\|R_\alpha(\mu)\|^2_{L^2(\mu)} \approx p_\alpha(\mu) + \|\mu\| \tag{9}$$

In the first Chapter of this dissertation, we can show that for $0 < \alpha < 1$,

$$p_\alpha(\mu) = +\infty, \tag{10}$$

for every positive Borel measure μ with

$$0 < \limsup_{r \to 0} \frac{\mu(B(x,r))}{r^\alpha} < \infty$$

for μ almost all $x \in \mathbb{R}^n$. (10) holds because for $0 < \alpha < 1$, in contrast with the case $\alpha = 1$, the positivity of the permutations comes with lower bounds (see (5) above).

Thus, due to (9), for such a measure μ, the $\alpha-$Riesz operator is not bounded in $L^2(\mu)$. Namely, there are no sets of finite α-dimensional measure where the α-Riesz kernel is bounded. Hence one gets zero capacity for all such sets. All this happens without the Ahlfors-David regularity assumption. The fact that $p_\alpha(x_1, x_2, x_3) \geq 0$ only for $0 < \alpha \leq 1$, explains our restriction on α in Theorem A. In proving Theorem A we also have to adapt a deep recent result of Nazarov, Treil and Volberg [NTV3] on the L^2 boundedness of singular integrals with respect to very general measures. One cannot apply directly the Nazarov Treil and Volberg $T(b)$-Theorem in [NTV3] to our $\alpha-$Riesz operators. It is necessary to find an apropriate substitute for what they call suppressed operators. In Section 2 of the first Chapter, it is shown that at least there are two suitable versions of such suppressed operators that work for the $\alpha-$Riesz transforms, $0 < \alpha < n$.

The proof of Theorem B follows the ideas of a well known result of Christ [Ch2] stating that if an Ahlfors-David regular set E of dimension 1 in the plane has positive analytic capacity then the Cauchy integral operator is bounded in $L^2(F, \mathcal{H}^1)$, where F is another Ahlfors-David regular set such that $\mathcal{H}^1(E \cap F) > 0$. Then the main difficulty for us to adapt Christ's result lies in the fact that if α is non-integer then, according to Vihtilä's result there are no Ahlfors-David regular sets E on which the α-dimensional Riesz operator is bounded in the space $L^2(E, \mathcal{H}^\alpha)$. This prevents us from adapting directly Christ's arguments and then we are forced to find a way around, which turns out to be rather lengthy and involved. To prove Theorem B, we suppose that we have positive γ_α capacity and then by means of a stopping time argument, in which we use the set itself to construct pieces of the new one, we manage to construct an Ahlfors-David regular set F whose intersection with the initial set E has positive $\alpha-$Hausdorff measure and where the $\alpha-$Riesz operator is bounded in $L^2(F, \mathcal{H}^\alpha)$. Thus Vihtilä's result gives the desired contradiction.

The capacity $\gamma_{\alpha,+}$ is defined as γ_α, but the supremum in (1) taken only over positive measures. In Chapter 3, we will prove that for $0 < \alpha < n$, $\gamma_{\alpha,+}$ is countably semiadditive. This can be used, coupled with an idea of Pajot, to extend Theorem B to a more general setting. Pajot proved in [Pa1] that for $0 < \alpha < n$ and $\alpha \in \mathbb{Z}$, a compact set with finite α-Hausdorff measure and satisfying some density condition can be covered by a countable union of Ahlfors-David regular sets of dimension α. This result extends directly to any $0 < \alpha < n$ and so we can prove the following

Theorem C. *Let $0 < \alpha < n$, $\alpha \notin \mathbb{Z}$ and let $E \subset \mathbb{R}^n$ be a compact set with $\mathcal{H}^\alpha(E) < \infty$, such that for all $x \in E$,*

$$0 < \theta_*^\alpha(x, E) \leq \theta^{*\alpha}(x, E) < \infty.$$

Then $\gamma_\alpha(E) = 0$.

Recall that the quantities $\theta_*^\alpha(x, E)$ and $\theta^{*\alpha}(x, E)$ are the lower and upper densities of E in x. They are defined by

$$\theta_*^\alpha(x, E) = \liminf_{r \to 0} \frac{\mathcal{H}^\alpha(E \cap B(x, r))}{r^\alpha}$$

and

$$\theta^{*\alpha}(x, E) = \limsup_{r \to 0} \frac{\mathcal{H}^\alpha(E \cap B(x, r))}{r^\alpha}.$$

It seems that one can extend the results concerning analytic capacity and Menger curvature to results involving the capacity γ_α and the quantity p_α, for $0 < \alpha < 1$. In this fashion, we obtain in Chapter 2 a characterization of the capacity $\gamma_{\alpha,+}$, $0 < \alpha < 1$, in terms of the quantity p_α. We need to introduce first some notation. For a positive Radon measure μ in $\mathbb{R}^n$, $0 < \alpha < 1$ and $x \in \mathbb{R}^n$, set

$$p_\alpha^2(\mu)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p_\alpha(x, y, z) d\mu(y) d\mu(z),$$

$$M_\alpha \mu(x) = \sup_{r > 0} \frac{\mu(B(x, r))}{r^\alpha}$$

and

$$U_\alpha^\mu(x) = M_\alpha \mu(x) + p_\alpha(\mu)(x).$$

Notice that $p_\alpha(\mu) = \int_{\mathbb{R}^n} p_\alpha^2(\mu)(x) d\mu(x)$. The potential U_α^μ is analogue to the one introduced in [V4]. The energy associated to this potential is

$$E_\alpha(\mu) = \int_{\mathbb{R}^n} U_\alpha^\mu(x) d\mu(x).$$

Then we obtain

Lemma D. *For each compact set $K \subset \mathbb{R}^n$ and $0 < \alpha < 1$ we have*

$$\gamma_{\alpha,+}(K) \approx \sup_\nu \frac{1}{E_\alpha(\nu)},$$

where the supremum is taken over the probability measures ν supported on K.

The proof of $\gamma \approx \gamma_+$ in [T2] (or the same result for the continuous analytic capacity proved in [T3]), involves Calderón-Zygmund theory on a non-homogeneous setting, localization of the Cauchy kernel and the fact that the symmetrization of the Cauchy kernel gives a non-negative quantity. These three ingredients remain valid when instead of γ, we consider the capacity γ_α for $0 < \alpha < 1$. The localization of the $\alpha-$Riesz potential for $0 < \alpha < n$ is proved in Chapter 2. We show that for a given infinitely differentiable function φ_Q supported on $2Q$, $Q \subset \mathbb{R}^n$ being a cube, the fact that the potential $T * \frac{x}{|x|^{1+\alpha}}$, $0 < \alpha < n$, is a function in $L^\infty(\mathbb{R}^n)$ implies that $\varphi_Q T * \frac{x}{|x|^{1+\alpha}}$ belongs also to $L^\infty(\mathbb{R}^n)$. Hence one can adapt the proof in [T2] (taking into account some adjustments introduced in [T3]) to get that

Theorem E. *For compact sets $K \subset \mathbb{R}^n$ and $0 < \alpha < 1$, we have*

$$\gamma_\alpha(K) \approx \gamma_{\alpha,+}(K).$$

Thus using Lemma D, we have a description of γ_α in terms of the energy E_α. Moreover, for a positive measure μ in $\mathbb{R}^n$ and $0 < \alpha < 1$, due to the lower inequality in (5) we get the following equivalence,

$$E_\alpha(\mu) \approx \int \int_0^\infty \left(\frac{\mu(B(x,r))}{r^\alpha} \right)^2 \frac{dr}{r} d\mu(x). \tag{11}$$

Notice that the expression on the right hand side of (11) is nothing but the energy associated to the Wolff potential $W^\mu_{\frac{2}{3}(n-\alpha),\frac{3}{2}}$ of the measure μ. Recall that given $1 < p < \infty$ and $0 < sp \leq n$, the Wolff potential $W^\mu_{s,p}$ is defined by (see [AH], p. 45)

$$W^\mu_{s,p}(x) = \int_0^\infty \left(\frac{\mu(B(x,r))}{r^{n-sp}} \right)^{p'-1} \frac{dr}{r}, \quad x \in \mathbb{R}^n,$$

where $p' = p/(p-1)$ is the exponent conjugate to p. Let

$$E_{s,p}(\mu) = \int \int_0^\infty \left(\frac{\mu(B(x,r))}{r^{n-sp}} \right)^{p'-1} \frac{dr}{r} d\mu(x)$$

be the energy associated to the Wolff potential $W^{\mu}_{s,p}$. Recall now that for any compact set $K \subset \mathbb{R}^n$ and $p,\ s$ as above, the Riesz capacity $C_{s,p}$ is defined by

$$C_{s,p}(K) = \inf\{\|\varphi\|_p^p :\ \varphi * \frac{1}{|x|^{n-s}} \geq 1 \ \text{ on } K\},$$

where the infimum is taken over all compactly supported infinitely differentiable functions on $\mathbb{R}^n$.

By Wolff's inequality ([AH], Theorem 4.5.4, p.110) we have that

$$C^{-1} \sup_{\mu} \frac{1}{E_{s,p}(\mu)^{p-1}} \leq C_{s,p}(K) \leq C \sup_{\mu} \frac{1}{E_{s,p}(\mu)^{p-1}}, \tag{12}$$

where C is a positive constant depending only on s and p and the supremum is taken over the probability measures μ supported on K. Hence using Theorem E, Lemma D, (11) and (12) we obtain

Theorem F. *Given a compact set $K \subset \mathbb{R}^n$ and $0 < \alpha < 1$,*

$$C^{-1} C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}(K) \leq \gamma_\alpha(K) \leq C C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}(K).$$

It is well-known that the capacity $C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}$ vanishes on sets with finite $\alpha-$dimensional Hausdorff measure. Thus, we recover here Theorem A. Moreover $C_{s,p}$ is a subadditive set function (almost by definition), and consequently, γ_α is semiadditive for $0 < \alpha < 1$, that is, given compact sets K_1 and K_2,

$$\gamma_\alpha(K_1 \cup K_2) \leq C\left\{\gamma_\alpha(K_1) + \gamma_\alpha(K_2)\right\}, \tag{13}$$

for some constant C depending only on α and n. In fact γ_α is countably semiadditive. For $\alpha = 1$ and $n = 2$ inequality (13) is still true and is a remarkable result obtained in [T2]. For $\alpha = n-1$ and any $n \geq 2$, (13) has been shown very recently in [Vo] by adapting the techniques used in [T2] (for $\alpha = n-1$ the symmetrization method does not give a positive quantity, thus in [Vo] one has to circumvent this difficulty in order to adapt Tolsa's result).

Another interesting consequence is the bilipschitz invariance of the capacity γ_α for $0 < \alpha < 1$. This means that if $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a bilipschitz homeomorphism of $\mathbb{R}^n$, namely,

$$L^{-1}|x-y| \leq |\phi(x) - \phi(y)| \leq L|x-y|,$$

for $x,\ y \in \mathbb{R}^n$ and for some constant $L > 0$, then for compact sets K one has

$$C^{-1}\gamma_\alpha(K) \leq \gamma_\alpha(\phi(K)) \leq C\gamma_\alpha(K),$$

where C depends only on L, α and n.

Very recently, X. Tolsa [T4] has proved that the analytic capacity is also bilipschitz invariant, which solves the Painlevé problem of characterizing geometrically sets that are removable for bounded analytic functions. The problem of the bilipschitz invariance of analytic capacity, first appeared in [V2]. J. Garnett and J. Verdera [GV] proved it before for generalized Cantor type sets.

This dissertation consists in three articles which form the three next chapters. Thus each one is completely self-contained. It is organized as follows. The first paper is entitled *Potential theory of signed Riesz kernels: capacity and Hausdorff measure*, and we show there that the symmetrization of the Riesz kernels gives a positive quantity. This result together with a $T(b)$ type Theorem of Nazarov, Treil and Volberg [NTV3], is used to prove Theorem A. Theorem B is also proved in this Chapter by means of a lengthy delicate stopping time argument.

In the second preprint, *Signed Riesz capacities and Wolff potentials*, we prove that for $0 < \alpha < 1$, the capacity γ_α is equivalent to the well known capacity $C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}$ of non-linear potential theory.

In the third paper, *Sets with vanishing signed Riesz capacity*, we deal with the proof of Theorem C.

Finally in the last chapter, we state some open problems connected to the questions considered in this dissertation.

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Chapter 1

Potential theory of signed Riesz kernels: capacity and Hausdorff measure.

1.1 Introduction.

The has been recently substantial progress on the problem of understanding the nature of analytic capacity (see [D2], [MTV] and [T2]). Recall that the analytic capacity of a compact subset E of the plane is defined by

$$\gamma(E) = \sup |f'(\infty)|,$$

where the supremum is taken over those analytic functions on $\mathbb{C} \setminus E$ such that $|f(z)| \leq 1$, for $z \notin E$. It is easily shown that sets of zero analytic capacity are the removable sets for bounded analytic functions.

In [D2] one proves Vitushkin's conjecture, namely the statement that among compact sets of finite length (one dimensional Hausdorff measure) the sets of zero analytic capacity are precisely those that project into sets of zero length in almost all directions. Equivalently, by Besicovitch theory, these are the purely unrectifiable sets, that is, the sets that intersect each rectifiable curve in zero length. In [MTV] one characterizes the Cantor sets of vanishing analytic capacity and in [T2] the semiadditivity of analytic capacity is proven.

When dealing with analytic capacity, very often one finds oneself working with the Cauchy kernel $1/z$ and not using at all analyticity. Indeed, analytic capacity itself can be easily expressed without making any reference to analyticity in the form

$$\gamma(E) = \sup |< T, 1 >| \tag{1.1}$$

where the supremum is taken over all complex distributions T supported on E such that the Cauchy potential of T, $f = \frac{1}{z} * T$, is a function in $L^\infty(\mathbb{C})$ satisfying $\|f\|_\infty \leq 1$. It seems, then, interesting to try to isolate properties of analytic capacity that depend only on the basic characteristics of the Cauchy kernel, such as oddness or homogeneity.

With this purpose in mind we start in this paper the study of certain real variable versions of analytic capacity related to the Riesz kernels in $\mathbb{R}^n$. Their definition is as follows. Given $0 < \alpha < n$ and a compact subset E of $\mathbb{R}^n$, set

$$\gamma_\alpha(E) = \sup | < T, 1 > |$$

where the supremum is taken over all real distributions T supported on E such that for $1 \leq i \leq n$, the i-th α-Riesz potential $T * \frac{x_i}{|x|^{1+\alpha}}$ of T is a function in $L^\infty(\mathbb{R}^n)$ and $\|T * \frac{x_i}{|x|^{1+\alpha}}\|_\infty \leq 1$. When $n = 2$ and $\alpha = 1$, writing $\frac{1}{z} = \frac{x}{|z|^2} - i\frac{y}{|z|^2}$ with $z = x + iy$, we obtain $\gamma_1(E) \leq \gamma(E)$ for all compact sets E. According to Tolsa's Theorem [T2] one has

$$\gamma(E) \leq C\gamma_+(E),$$

for all compact sets E, where $\gamma_+(E)$ is defined by the supremum in (1.1) where now one requires T to be a positive measure supported on E (with Cauchy potential bounded almost everywhere by 1 on $\mathbb{C}$). Thus, on compact subsets of the plane γ and γ_1 are comparable, in the sense that for some positive constant C one has

$$C^{-1}\gamma_1(E) \leq \gamma(E) \leq C\gamma_1(E).$$

Therefore our set function γ_α can be viewed as a real variable version of analytic capacity associated to the vector valued kernel $\frac{x}{|x|^{1+\alpha}}$. Of course one can think of other possibilities : for example, one can associate in a similar fashion a capacity γ_Ω to a scalar kernel of the form $K(x) = \frac{\Omega(x)}{|x|^\alpha}$ where Ω is a real valued smooth function on $\mathbb{R}^n$, homogeneous of degree zero. We will not pursue this issue here.

In section 3 we compare the capacity γ_α to Hausdorff content. We obtain quantitative statements that in particular imply that if E has zero α -dimensional Hausdorff measure, then it has also zero γ_α capacity. In the other direction one gets that if E has Hausdorff dimension larger than α then γ_α is positive. Then the critical situation occurs in dimension α, in accordance with the classical case.

The main contribution of this paper is the discovery of an interesting special behaviour of γ_α for non integer indexes α. When α is an integer and E is a compact subset of an α-dimensional smooth surface, then one can see that $\gamma_\alpha(E) > 0$ provided $\mathcal{H}^\alpha(E) > 0$, $\mathcal{H}^\alpha$ being α-dimensional Hausdorff measure (see [MP], where it is shown that if E lies on a Lipschitz graph, then $\gamma_{n-1}(E)$ is comparable to the $(n-1)-$Hausdorff measure $\mathcal{H}^{n-1}(E)$). In particular, there are sets of finite α-dimensional Hausdorff measure $\mathcal{H}^\alpha(E)$ and positive $\gamma_\alpha(E)$. It turns out that this cannot happen when $0 < \alpha < 1$.

Theorem 1.1. *Let $0 < \alpha < 1$ and let $E \subset \mathbb{R}^n$ be a compact set with $\mathcal{H}^\alpha(E) < \infty$. Then $\gamma_\alpha(E) = 0$.*

Notice that the analogue of the above result in the limiting case $\alpha = 1$ is the difficult part of Vitushkin's conjecture : if E is a purely unrectifiable planar compact set of finite length, then $\gamma(E) = 0$. We do not know how to prove Theorem 1.1 for a non integer $\alpha > 1$. Even for an integer $\alpha > 1$ we do not know if the natural analogue of Vitushkin's conjecture is true. However we do have a result in the Ahlfors-David regular case. Recall that a closed subset E of $\mathbb{R}^n$ is said to be Ahlfors-David regular of dimension d if it has, locally, finite and positive d-dimensional Hausdorff measure in a uniform way:

$$C^{-1}r^d \leq \mathcal{H}^d(E \cap B(x,r)) \leq Cr^d, \text{ for } x \in E, \ \ r \leq d(E),$$

where $B(x,r)$ is the open ball centered at x of radius r and $d(E)$ is the diameter of E. Notice that if E is a compact Ahlfors-David regular set of dimension d, then $\mathcal{H}^d(E) < \infty$.

Theorem 1.2. *Let $E \subset \mathbb{R}^n$ be a compact Ahlfors-David regular set of non-integer dimension α, $0 < \alpha < n$. Then $\gamma_\alpha(E) = 0$.*

In proving Theorem 1 we use a deep recent result of Nazarov, Treil and Volberg [NTV3] on the L^2 boundedness of singular integrals with respect to very general measures (see section 2 below for a statement). As a technical tool we also need a variant of the well known symmetrization method relating Menger curvature (see section 2 for a definition) and the Cauchy kernel (see [Me], [MV] and [MMV]). Symmetrization of the kernels $\frac{x}{|x|^{1+\alpha}}$, leads to a non-negative quantity only for $0 < \alpha \leq 1$. For $\alpha = 1$ this is Menger curvature and for $0 < \alpha < 1$ a description can be found in Lemma 1.15 below. However, non-negativity and homogeneity seem to be more relevant facts than having exact expressions for the symmetrized quantity. The lack of non-negativity for $\alpha > 1$ is the reason that explains the restriction on α in Theorem 1.1.

The proof of Theorem 1.2 follows the line of reasoning of a well known result of Christ [Ch2] stating that if an Ahlfors-David regular set E of dimension 1 in the plane has positive analytic capacity then the Cauchy integral operator is bounded in $L^2(F, \mathcal{H}^1)$, where F is another Ahlfors-David regular set such that $\mathcal{H}^1(E \cap F) > 0$. The main difficulty for us lies in the fact that if α is non-integer then, according to a result of Vithila [Vi] there are no Ahlfors-David regular sets E on which the α-dimensional Riesz operator is bounded in the space $L^2(E, \mathcal{H}^\alpha)$. This prevents us from adapting directly Christ's arguments and then we are forced to find a way around, which turns out to be rather laborious.

Throughout all the paper, the letter C will stand for an absolute constant that may change at different occurrences.

If $A(X)$ and $B(X)$ are two quantities depending on the same variable (or variables) X, we will say that $A(X) \approx B(X)$ if there exists $C \geq 1$ independent of X such that $C^{-1}A(X) \leq B(X) \leq CB(X)$ for every X.

In section 2 one can find statements of some auxiliary results and the basic notation and terminology that will be used throughout the paper. As we already mentioned above, in section 3 we compare γ_α to Hausdorff content. Theorem 1 is proven in section 4 and Theorem 2 in section 5.

1.2 L^2 boundedness of singular integral operators.

A function $K(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y) : x = y\}$ is called a Calderón-Zygmund kernel if the following holds:

1. $|K(x,y)| \leq C|x-y|^{-\alpha}$ for some $0 < \alpha < n$ (α not necessarily integer) and some positive constant $C < \infty$.

2. There exists $0 < \varepsilon \leq 1$ such that for some constant $0 < C < \infty$,

$$|K(x,y) - K(x_0,y)| + |K(y,x) - K(y,x_0)| \leq C\frac{|x-x_0|^{\varepsilon}}{|x-y|^{\alpha+\varepsilon}},$$

   if $|x-x_0| \leq |x-y|/2$.

Let μ be a Radon measure on $\mathbb{R}^n$. Then the Calderón-Zygmund operator T associated to the kernel K and the measure μ is formally defined as

$$Tf(x) = T(f\mu)(x) = \int K(x,y)f(y)d\mu(y).$$

This integral may not converge for many functions f, because for $x = y$ the kernel K may have a singularity. For this reason, we introduce the truncated operators T_ε, $\varepsilon > 0$:

$$T_\varepsilon f(x) = T_\varepsilon(f\mu)(x) = \int_{|x-y|>\varepsilon} K(x,y)f(y)d\mu(y).$$

We say that the singular integral operator T is bounded in $L^2(\mu)$ if the operators T_ε are bounded in $L^2(\mu)$ uniformly in ε.

The maximal operator T^* is defined as

$$T^*f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|.$$

Let $0 < \alpha < n$ and consider the Calderón-Zygmund operator R_α associated to the antisymmetric vector valued Riesz kernel $x/|x|^{1+\alpha}$.

For the proof of Theorem 1.1 a deep result of Nazarov, Treil and Volberg will be needed. First we introduce some more notation. We say that $B(x,r)$ is a non-Ahlfors disk with respect to some constant $M > 0$ if $\mu(B(x,r)) > Mr$. A disk $B(x,r)$ is non-accretive with respect to some bounded function b if for some fixed positive constant ε we have $\left|\int_{B(x,r)} b d\mu\right| < \varepsilon\mu(B(x,r))$.

Let ϕ be some non negative Lipschitz function with Lipschitz constant 1 and consider the antisymmetric Calderón-Zygmund operator K_ϕ associated to the suppressed kernel k_ϕ:

$$k_\phi(x,y) = \frac{\overline{x-y}}{|x-y|^2 + \phi(x)\phi(y)}.$$

The kernel k_ϕ has the very important property of being well suppressed (we are borrowing the terminology from [NTV3]) at the points where $\phi > 0$, that is

$$|k_\phi(x,y)| \le \frac{1}{\max\{\phi(x),\phi(y)\}}. \tag{1.2}$$

We will state now a $T(b)$ Theorem of [NTV3] for the Cauchy kernel.

Theorem 1.3. *Let μ be a positive Radon measure on $\mathbb{C}$ with $\limsup_{r\to 0}\frac{\mu(B(x,r))}{r} < \infty$ for μ almost all x and b an $L^\infty(\mu)$ function with $|\int_{\mathbb{C}} b d\mu| = \gamma$. Let $M > 0$, $B > 0$, an open set $H \subset \mathbb{C}$ with $\mu(H^c) > 0$ and $\phi : \mathbb{C} \to [0,\infty)$ a Lipschitz function with constant 1 such that:*

1. *Every non-Ahlfors disk and every non-accretive disk is contained in H.*
2. *$\phi(x) \ge dist(x, H^c)$.*
3. *$K^*_\theta b(x) \le B$ for μ almost all x and for every Lipschitz function θ with constant 1 such that $\theta \ge \phi$.*

Then K_ϕ is bounded in $L^2(\mu)$. In particular, if $F = \{x : \phi(x) = 0\}$, the Cauchy transform is bounded in $L^2(\mu_{|F})$.

One can use this result to give an alternative proof of Vitushkin's conjecture (see [NTV3]).

To use their result for the $\alpha-$Riesz transform R_α, $0 < \alpha < n$, we need an appropriate version of the suppressed kernels associated to the Riesz α-operator R_α. We have found that the following kernel does the job:

$$k_{\phi,\alpha}(x,y) = \frac{x-y}{|x-y|^{1+\alpha}}\Big(\frac{|x-y|^2}{|x-y|^2+\phi(x)\phi(y)}\Big)^N \tag{1.3}$$

where $N = \min\{m \in \mathbb{N} \,:\, \alpha \le m\}$. That is, $N = \alpha$ if $\alpha \in \mathbb{N}$ and $N = [\alpha]+1$ if $\alpha \notin \mathbb{N}$, where $[\alpha]$ denotes the integer part of α. Notice that $k_{\phi,1} = k_\phi$.

For the sake of completeness we state the properties of the kernel $k_{\phi,\alpha}$ on a separate lemma.

Lemma 1.4. *The kernel $k_{\phi,\alpha}(x,y)$ is an antisymmetric Calderón-Zygmund kernel and is also well suppressed at the points where $\phi > 0$, that is,*

$$|k_{\phi,\alpha}(x,y)| \le \frac{1}{\max\{\phi(x)^\alpha,\phi(y)^\alpha\}}.$$

Proof. It is easy to see that this suppressed kernel satisfies $k_{\phi,\alpha}(x,y) = -k_{\phi,\alpha}(y,x)$ and $|k_{\phi,\alpha}(x,y)| \le |x-y|^{-\alpha}$. We show now that $|k_{\phi,\alpha}(x,y)| \le \dfrac{1}{\phi(x)^\alpha}$, for all x, y. Observe first that $\phi(y) \ge \phi(x) - |x-y|$, which implies that

$$|k_{\phi,\alpha}(x,y)| \leq \frac{1}{|x-y|^\alpha}\left(\frac{|x-y|^2}{|x-y|^2+\phi(x)(\phi(x)-|x-y|)}\right)^N$$

$$= \frac{1}{|x-y|^\alpha}\left(\frac{|x-y|^2}{|x-y|^2+\phi(x)(\phi(x)-|x-y|)}\right)^{N-\alpha}\left(\frac{|x-y|^2}{|x-y|^2+\phi(x)(\phi(x)-|x-y|)}\right)^{\alpha}$$

$$\leq \frac{1}{|x-y|^\alpha}\left(\frac{|x-y|^2}{|x-y|^2+\phi(x)(\phi(x)-|x-y|)}\right)^{\alpha}$$

$$= \frac{1}{|x-y|^\alpha}\left(\frac{|x-y|^2}{\phi(x)|x-y|+(\phi(x)-|x-y|)^2}\right)^{\alpha} \leq \frac{1}{|x-y|^\alpha}\left(\frac{|x-y|^2}{\phi(x)|x-y|}\right)^{\alpha}$$

$$= \frac{1}{\phi(x)^\alpha}.$$

Now we only need to show that

$$|\nabla_x k_{\phi,\alpha}(x,y)| \leq \frac{4N+\alpha+3}{|x-y|^{1+\alpha}}.$$

Set $P_\phi(x,y) = \dfrac{|x-y|^2}{|x-y|^2+\phi(x)\phi(y)}$ and write $\nabla_x k_{\phi,\alpha}(x,y) = A+B$, with

$$|A| = |P_\phi(x,y)|^N \left|\frac{|x-y|^{1+\alpha}-(1+\alpha)|x-y|^\alpha(x-y)}{|x-y|^{2(1+\alpha)}}\right| \leq \frac{\alpha+2}{|x-y|^{1+\alpha}}$$

and

$$|B| = N\,|P_\phi(x,y)|^{N-1}\,\frac{|2(x-y)\left(|x-y|^2+\phi(x)\phi(y)\right)-|x-y|^2\left(2(x-y)+\phi'(x)\phi(y)\right)|}{\left(|x-y|^2+\phi(x)\phi(y)\right)^2|x-y|^\alpha}$$

$$\leq N\left(\frac{|x-y|^2}{|x-y|^2+\phi(x)\phi(y)}\right)^N \frac{2\left(|x-y|^2+\phi(x)\phi(y)\right)+|x-y|\left(2|x-y|+\phi'(x)\phi(y)\right)}{|x-y|^{1+\alpha}(|x-y|^2+\phi(x)\phi(y))}$$

$$\leq N\left(\frac{|x-y|^2}{(|x-y|^2+\phi(x)\phi(y)}\right)^N \frac{4|x-y|^2+2\phi(x)\phi(y)+\phi(y)|x-y|}{|x-y|^{1+\alpha}(|x-y|^2+\phi(x)\phi(y))}$$

$$\leq \frac{4N}{|x-y|^{1+\alpha}}+\frac{\phi(y)}{|x-y|^{1+\alpha}}|k_\phi(x,y)| \leq \frac{4N+1}{|x-y|^{1+\alpha}},$$

where one uses (1.2) in the last inequality. □

Using this operators and adapting Theorem 1.3 one obtains the following result for the $\alpha-$Riesz transform R_α:

Theorem 1.5. *Let μ be a positive measure on $\mathbb{R}^n$ such that $\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^\alpha} < +\infty$ for μ almost all x and b an $L^\infty(\mu)$ function such that $|\int b d\mu| = \gamma_\alpha$. Assume that $R^*_\alpha\, b\,(x) < +\infty$ for μ almost all x. Then there is a set F with $\mu(F) \geq \frac{\gamma_\alpha}{4}$ such that the α-Riesz potential R_α is bounded in $L^2(\mu_{|F})$.*

Remark. *This set F corresponds to $\mathbb{C}\setminus H$ in Theorem 1.5. Namely F is the set where there are no problems, (every disk is Ahlfors and accretive and the maximal operator is uniformly bounded.)*

Remark. [A. Volberg, personal communication] *Instead of using the Calderón-Zygmund operator related to the suppressed kernel defined in (1.3), one can also use the operator related to the following suppressed kernel:*

$$k_{\phi,\alpha}(x,y) = \frac{k_\alpha(x,y)}{1 + k_\alpha^2(x,y)\phi^\alpha(x)\phi^\alpha(y)},$$

with $k_\alpha(x,y) = (x-y)/|x-y|^{1+\alpha}$.

For the proof Theorem 1.2, we need to define some sets Q^k_β that will be the analogues of the euclidean dyadic cubes. These "dyadic cubes" were introduced by M. Christ in [Ch2].

Let $E \subset \mathbb{R}^n$ be an Ahlfors-David regular compact set with $\mathcal{H}^\alpha(E) < \infty$. Let $\mu = \mathcal{H}^\alpha_{|E}$ and ρ the euclidean metric. Then (E,ρ,μ) is a space of homogeneous type, that is, (E,ρ) is a metric space and μ is a doubling measure, i.e. $\mu(B(x,2r) \leq C\mu(B(x,r))$, (see [Ch2]).

Theorem 1.6. *[Ch2] For a space of homogeneous type (E,ρ,μ) with μ as above, there exists a collection of Borel sets $\mathcal{Q}(E) = \{Q^k_\beta \subset E :\ k \in \mathbb{N},\ \beta \in \mathbb{N}\}$, and positive numbers $\delta \in (0,1)$, a_1, b_1 and η such that*

1. *$\mu(E \setminus \bigcup_\beta Q^k_\beta) = 0$ for each k,*
2. *If $l \geq k$ then either $Q^l_\gamma \subset Q^k_\beta$ or $Q^l_\gamma \cap Q^k_\beta = \emptyset$.*
3. *For each (k,β) and each $l < k$, there is a unique γ such that $Q^k_\beta \subset Q^l_\gamma$.*
4. *$d(Q^k_\beta) \leq \delta^k$, where $d(Q^k_\beta)$ denotes the diameter of the cube Q^k_β.*
5. *Each Q^k_β contains some ball $B(Q^k_\beta) = E \cap B(z^k_\beta, a_1\delta^k)$.*
6. *Each cube Q^k_β has a "small boundary", i.e. $\mu\{x \in Q^k_\beta :\ \rho(x, E\setminus Q^k_\beta) \leq t\delta^k\} \leq b_1 t^\eta \mu(Q^k_\beta)$ for every k, β and for every $t > 0$.*

We denote by $\mathcal{Q}^k(E) = \{Q^k_\beta \in \mathcal{Q}(E) :\ \beta \in \mathbb{N}\}$, $k \in \mathbb{N}$ the cubes of generation k in $\mathcal{Q}(E)$.

For the variant of the $T(b)$ Theorem that we need (see Theorem 20 in [Ch2]) we require the definitions of a dyadic para-accretive function and a dyadic BMO function.

Definition 1.7. *A function $b \in L^\infty(E)$ is said to be dyadic para-accretive if for every $Q^k_\beta \in \mathcal{Q}(E)$, there exists $Q^l_\gamma \in \mathcal{Q}(E)$, $Q^l_\gamma \subset Q^k_\beta$, with $l \leq k+N$ and*

$$|\int_{Q^l_\gamma} b d\mu| \geq c\mu(Q^l_\gamma)$$

for some fixed constants $c>0$ and $N \in \mathbb{N}$.

Definition 1.8. *A locally μ integrable function f belongs to dyadic $BMO(\mu)$ if*

$$\sup_Q \inf_{c\in\mathbb{C}} \frac{1}{\mu(Q)} \int_Q |f(z)-c| d\mu(z) < \infty,$$

where the supremum is taken over all dyadic cubes $Q \in \mathcal{Q}(E)$.

Theorem 1.9. *[Ch2] Let E be a space of homogeneous type with underlying doubling measure μ, b a dyadic para-accretive function and T a Calderón-Zygmund operator associated to an antisymmetric standard kernel. Suppose that $T(b)$ belongs to dyadic $BMO(\mu)$. Then T is a bounded operator in $L^2(\mu)$.*

A recent new approach to a variety of $T(b)$ Theorems can be found in [AHMTT].

For the proof of Theorem 1.2, the following result of Vihtilä will be also needed.

Theorem 1.10. *[Vi] Let μ be a nonzero Radon measure in $\mathbb{R}^n$ for which there exist constants $0<c_1\leq c_2<\infty$ such that*

$$c_1 r^\alpha \leq \mu(B(x,r)) \leq c_2 r^\alpha$$

for all $x \in spt(\mu)$ and $0<r<d(spt\mu)$. If R_α is an operator bounded in $L^2(\mu)$, then α is an integer.

This theorem was proved by using an approach based on tangent measures.

1.3 Relation between γ_α and Hausdorff content.

We need the following lemma.

Lemma 1.11. *If a function $f(x)$ has compact support and has continuous derivatives up to order n, then it is representable, for $0<\alpha<n$, in the form*

$$f(x) = \left(\sum_{i=1}^n \varphi_i * \frac{x_i}{|x|^{1+\alpha}} \right)(x), \quad x \in \mathbb{R}^n, \tag{1.4}$$

where φ_i, $i=1,...,n$, are defined by the formulas:

$$\varphi_i = c_{n,\alpha} \, \triangle^k \partial_i f * \frac{1}{|x|^{n-\alpha}} \quad for \;\; n = 2k+1,$$

$$\varphi_i = c_{n,\alpha} \, \triangle^k f * \frac{x_i}{|x|^{1+n-\alpha}} \quad for \;\; n = 2k,$$

in which $c_{n,\alpha}$ is a constant depending on n and α.

Proof. Assume first that $n = 2k+1$. Taking Fourier transform of the right hand side of (1.4) we get, for appropriate numbers $a_{n,\alpha}$ and $b_{n,\alpha}$,

$$\sum_{i=1}^{n} \widehat{\varphi}_i(\xi) \; a_{n,\alpha} \frac{\xi_i}{|\xi|^{1+n-\alpha}} = \sum_{i=1}^{n} c_{n,\alpha} |\xi|^{2k} \xi_i \; \widehat{f}(\xi) \; \frac{b_{n,\alpha}}{|\xi|^{\alpha}} \;\; a_{n,\alpha} \frac{\xi_i}{|\xi|^{1+n-\alpha}} = c_{n,\alpha} \, a_{n,\alpha} \, b_{n,\alpha} \, \widehat{f}(\xi).$$

Then (1.4) follows by choosing $c_{n,\alpha}$ so that $c_{n,\alpha} \, a_{n,\alpha} \, b_{n,\alpha} = 1$.
A similar argument proves (1.4) in the case $n = 2k$. □

We are now ready to describe the basic relationship between γ_α and Hausdorff content (the d-dimensional Hausdorff content will be denoted by M^d (see [G] for the definition and basic properties)).

Lemma 1.12. *If $\;0 < \alpha < n$ then there exist constants C and C_ε such that*

$$C_\varepsilon M^{\alpha+\varepsilon}(E)^{\frac{\alpha}{\alpha+\varepsilon}} \leq \gamma_\alpha(E) \leq C M^{\alpha}(E)$$

for any compact set $E \subset \mathbb{R}^n$ and $\varepsilon > 0$.

Proof. We proof first the second inequality. Let $\{Q_j\}_j$ be a covering of E by dyadic cubes $Q_j \subset \mathbb{R}^n$ with disjoint interiors. By a well known lemma (see [HP]) there exist functions $g_j \in C_0^\infty(2Q_j)$ satisfying $\sum_j g_j = 1$ in a neighborhood of $\cup_j Q_j$ and $|\partial^s g_j| \leq C_s l(Q_j)^{-|s|}$, $|s| \geq 0$. Here $s = (s_1, ... s_n)$, with $0 \leq s_i \in \mathbb{Z}$, $|s| = s_1 + s_2 + ... + s_n$ and $\partial^s = (\partial/\partial x_i)^{s_1} ... (\partial/\partial x_n)^{s_n}$.

Let T be a distribution with compact support contained in E and such that the i-th $\alpha-$Riesz potentials $T * \frac{x_i}{|x|^{1+\alpha}}$ of T are functions in $L^\infty(\mathbb{R}^n)$ with L^∞- norm not greater than 1, $1 \leq i \leq n$. Applying Lemma 1.11 to each g_j, we obtain functions φ_j^i satisfying (1.4) with f and φ_i replaced by g_j and φ_i^j respectively. Thus

$$| < T, 1 > | = | < T, \sum_j g_j > | \leq \sum_j | < T, g_j > | = \sum_j | < T, \sum_{i=1}^{n} \varphi_i^j * \frac{x_i}{|x|^{1+\alpha}} > |$$

$$\leq \sum_j \sum_{i=1}^{n} | < T * \frac{x_i}{|x|^{1+\alpha}}, \varphi_i^j > | \leq \sum_j \sum_{i=1}^{n} \int |\varphi_i^j(x)| dx.$$

Take $n=2k+1$ (for $n=2k$ the argument is similar) and write $k_\alpha(x)=|x|^{-n+\alpha}$. Let Q_0 be the unit cube centered at 0. Changing variables and using $|\partial^s g_j|\le C_s l(Q_j)^{-|s|}$ we get

$$|<T,1>|\le \sum_j\sum_{i=1}^n\int|\varphi_i^j(x)|dx=\sum_j\sum_{i=1}^n\int\left|\int_{2Q_j}\triangle^k\partial_i g_j(y)k_\alpha(x-y)dy\right|dx$$

$$=\sum_j\sum_{i=1}^n\left\{\int_{3Q_j}\left|\int_{2Q_j}\triangle^k\partial_i g_j(y)k_\alpha(x-y)dy\right|dx+\int_{\mathbb{R}^n\setminus 3Q_j}\left|\int_{2Q_j}g_j(y)\triangle^k\partial_i k_\alpha(x-y)\}dy\right|dx\right\}$$

$$\le n\sum_j l(Q_j)^\alpha\left\{C_n\iint_{3Q_0\times 2Q_0}k_\alpha(x-y)dydx+C_0\iint_{(\mathbb{R}^n\setminus 3Q_0)\times 2Q_0}\frac{1}{|x-y|^{2n-\alpha}}dydx\right\}$$

$$\le C\sum_j l(Q_j)^\alpha.$$

Thus $\gamma_\alpha(E)\le CM^\alpha(E)$.

For the reverse inequality we use a standard argument that we reproduce for the reader's convenience. Suppose that $M^{\alpha+\varepsilon}(E)>0$ for some $\varepsilon>0$. By Frostman's Lemma (see [M1] Theorem 8.8) there exists a measure μ supported on E such that $\mu(E)\ge CM^{\alpha+\varepsilon}(E)>0$ and $\mu(B(x,r))\le r^{\alpha+\varepsilon}$, $x\in\mathbb{R}^n$, $r>0$. Then by a change of variables we obtain

$$\left|\left(\mu*\frac{x_i}{|x|^{1+\alpha}}\right)(y)\right|\le\int\frac{d\mu(x)}{|x-y|^\alpha}=\int_0^\infty\mu\left(\{x:|x-y|^{-\alpha}\ge t\}\right)dt$$

$$=\int_0^\infty\mu(B(y,t^{-1/\alpha}))dt=\alpha\int_0^\infty\frac{\mu(B(x,r))}{r^{1+\alpha}}dr$$

$$\le\alpha\left(\int_0^{\mu(E)^{\frac{1}{\alpha+\varepsilon}}}r^{\varepsilon-1}dr+\int_{\mu(E)^{\frac{1}{\alpha+\varepsilon}}}^\infty\frac{\mu(E)}{r^{1+\alpha}}\right)=\left(\frac{\alpha}{\varepsilon}+1\right)\mu(E)^{\frac{\varepsilon}{\alpha+\varepsilon}}.$$

Using this estimate we get the desired inequality, namely

$$\gamma_\alpha(E)\ge\frac{\mu(E)}{||\mu*\frac{x_i}{|x|^{1+\alpha}}||_\infty}\ge\frac{\varepsilon}{\alpha+\varepsilon}\mu(E)^{1-\frac{\varepsilon}{\alpha+\varepsilon}}=C_\varepsilon\,\mu(E)^{\frac{\alpha}{\alpha+\varepsilon}}\ge C_\varepsilon\,M^{\alpha+\varepsilon}(E)^{\frac{\alpha}{\alpha+\varepsilon}}.\quad\square$$

Let $dim(E)$ be the Hausdorff dimension of the set E. A qualitative version of the above lemma is the following.

Corollary 1.13. *Let $E \subset \mathbb{R}^n$ be compact.*

1. *If $dim(E) > \alpha$ then $\gamma_\alpha(E) > 0$.*

2. *If $dim(E) < \alpha$ then $\gamma_\alpha(E) = 0$.*

1.4 Proof of Theorem 1.

1.4.1 Distributions that are measures.

We start by a lemma that shows that certain distributions are actually measures.

Lemma 1.14. *Let $0 < \alpha < n$, $E \subset \mathbb{R}^n$ be compact with $\mathcal{H}^\alpha(E) < \infty$ and let T be a distribution with compact support contained in E and such that $T * \frac{x_i}{|x|^{1+\alpha}}$ is bounded in $\mathbb{R}^n$, $1 \le i \le n$. Then T is a measure which is absolutely continuous with respect to the restriction of $\mathcal{H}^\alpha$ to E and has a bounded density, that is,*

$$T = h\mathcal{H}^\alpha, \quad for\ some\ h \in L^\infty(\mathcal{H}^\alpha)\ supported\ on\ E. \tag{1.5}$$

Proof. We first show that T is a measure. For this it is enough to prove that

$$| < T, f > | \le C\mathcal{H}^\alpha(E)\|f\|_\infty, \quad f \in \mathcal{C}_0^\infty. \tag{1.6}$$

Given $\epsilon > 0$ we can cover the compact set E with open balls B_j of radius r_j, $j = 1, ..., k$ such that $B_j \cap E \neq \emptyset$, $r_j < \varepsilon$ and

$$\sum_{j=1}^{k} r_j^\alpha \le 2\mathcal{H}^\alpha(E) + \varepsilon. \tag{1.7}$$

Let ψ be a function in $\mathcal{C}_0^\infty$ with spt$\psi \subset B(0,1)$ and $\int \psi(x)dx = 1$. Define

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon^n}\psi(\frac{x}{\varepsilon}).$$

To prove (1.6) we can assume without loss of generality that spt$(f) \subset \cup_j B_j$. This is so because if $\beta \in \mathcal{C}_0^\infty$, spt$(\beta) \subset \cup_j B_j$, $0 \le \beta \le 1$ and $\beta(x) = 1$ in a neighborhood of E, then $< T, f > = < T, f\beta >$ and $\|\beta f\|_\infty \le \|f\|_\infty$.

Assume that $n = 2k + 1$ (the argument for even dimensions is similar). Applying Lemma 1.11 to ψ_ϵ, using the boundedness of $T * \frac{x_i}{|x|^{1+\alpha}}$ for $1 \le i \le n$ and setting

$k_\alpha(x) = |x|^{-n+\alpha}$ we have

$$
\begin{aligned}
|< T, f * \psi_{\varepsilon}>| &\leq C \sum_{i=1}^{n} \left| < T * \frac{x_i}{|x|^{1+\alpha}}, f * \triangle^k \partial_i \psi_\epsilon * k_\alpha > \right| \\
&\leq C \sum_{i=1}^{n} \int \left| \left(f * \triangle^k \partial_i \psi_\varepsilon * k_\alpha \right)(x) \right| dx \\
&= C \sum_{i=1}^{n} \int \left| \int f(y) \left(\triangle^k \partial_i \psi_\varepsilon * k_\alpha \right)(x-y) dy \right| dx \\
&\leq C \|f\|_\infty \sum_j r_j^n \sum_{i=1}^{n} \int \left| \triangle^k \partial_i \psi_\varepsilon * k_\alpha(z) \right| dz.
\end{aligned}
\tag{1.8}
$$

We will show that

$$
\int \left| \triangle^k \partial_i \psi_\varepsilon * k_\alpha(z) \right| dz \leq C \varepsilon^{-n+\alpha} \tag{1.9}
$$

where C is a constant depending on the L^1-norm of ψ and $\triangle^k \partial_i \psi$ but not on ε.

Then using (1.7) we will have

$$
\begin{aligned}
| < T, f * \psi_\varepsilon > | &\leq C \|f\|_\infty \varepsilon^{-n+\alpha} \sum_j r_j^n \leq C \|f\|_\infty \varepsilon^{-n+\alpha} \sum_j \varepsilon^{n-\alpha} r_j^\alpha \\
&= C \|f\|_\infty \sum_j r_j^\alpha \leq C \left(\mathcal{H}^\alpha(E) + \varepsilon \right) \|f\|_\infty,
\end{aligned}
\tag{1.10}
$$

which proves (1.6) by letting $\varepsilon \to 0$.

To prove (1.9) we use Fubini's Theorem and a change of variables:

$$
\begin{aligned}
&\int \left| \left(\triangle^k \partial_i \psi_\varepsilon * k_\alpha \right)(z) \right| dz \\
&= \int \left| \int \varepsilon^{-2n} \triangle^k \partial_i \psi \left(\frac{z-x}{\varepsilon}\right) k_\alpha(x) dx \right| dz = \varepsilon^{-n+\alpha} \int \left| \left(\triangle^k \partial_i \psi * k_\alpha \right)(z) \right| dz \\
&\leq \varepsilon^{-n+\alpha} \int_{|z|\geq 2} \int_{|x|\leq 1} \frac{|\psi(x)|}{|z-x|^{2n-\alpha}} \, dx \, dz + \varepsilon^{n-\alpha} \int_{|z|\leq 2} \int_{|x|\leq 1} \frac{\left|\triangle^k \partial_i \psi(x)\right|}{|z-x|^{n-\alpha}} \, dx \, dz \\
&= \varepsilon^{-n+\alpha} \int_{|x|\leq 1} |\psi(x)| \int_{|z|\geq 2} \frac{dz}{|z-x|^{2n-\alpha}} dx + \varepsilon^{n-\alpha} \int_{|x|\leq 1} \left| \Delta^k \partial_i \psi(x) \right| \int_{|z|\leq 2} \frac{dz}{|z-x|^{n-\alpha}} dx \\
&\leq C \varepsilon^{-n+\alpha} \left(\|\psi\|_1 + \|\triangle^k \partial_i \psi\|_1 \right) = C \varepsilon^{-n+\alpha}.
\end{aligned}
$$

Let B_0 be an open ball and let $\overline{B_0}$ denote its closure. Let $\mathcal{H}^\alpha_E$ stand for the restriction of $\mathcal{H}^\alpha$ to E. If we show that

$$|\mu(B_0)| \leq C\mathcal{H}^\alpha_E\left(\overline{B_0}\right), \tag{1.11}$$

then, taking a sequence of open balls $B_0^i \downarrow \overline{B_0}$ and applying (1.11) to these balls we will have

$$\left|\mu\left(\overline{B_0}\right)\right| \leq \lim_{i\to\infty}\left|\mu\left(B_0^i\right)\right| \leq \lim_{i\to\infty} C\mathcal{H}^\alpha_E\left(\overline{B_0^i}\right) = C\mathcal{H}^\alpha_E\left(\overline{B_0}\right). \tag{1.12}$$

It is shown in [M1] (p. 271) that for $\alpha = 1$ (1.12) implies

$$|\mu(A)| \leq C\mathcal{H}^\alpha(A) \text{ for sets } A \subset E \text{ with } \mathcal{H}^\alpha(A) < \infty. \tag{1.13}$$

The argument extends verbatim to any α and thus we can take (1.13) for granted, which gives (1.5) by Radon-Nikodym's Theorem.

It remains to prove (1.11). We know that for every $\delta > 0$ there exists a compact set $K \subset E \setminus \overline{B_0}$ such that

$$\mathcal{H}^\alpha(K) > \mathcal{H}^\alpha\left(E \setminus \overline{B_0}\right) - \delta. \tag{1.14}$$

Let

$$J_1 = \{j : B_j \cap \overline{B_0} \neq \emptyset\} \quad \text{and} \quad J_2 = \{j : B_j \cap K \neq \emptyset\}.$$

Recall that the radii of the balls B_j satisfy $r_j < \varepsilon$. For and appropriate $\varepsilon > 0$ the following holds:

$$\sum_{j\in J_2} r_j^\alpha \geq 2\mathcal{H}^\alpha(K) - \delta \tag{1.15}$$

and

$$\max_j r_j < \varepsilon < dist(K, \overline{B_0})/2.$$

This last condition implies that for $j_1 \in J_1$ and $j_2 \in J_2$ we have $\overline{B_{j_1}} \cap \overline{B_{j_2}} = \emptyset$. So, using inequalities (1.7), (1.15) and (1.14),

$$\sum_{j\in J_1} r_j^\alpha \leq \sum_j r_j^\alpha - \sum_{j\in J_2} r_j^\alpha \leq 2\mathcal{H}^\alpha(E) + \varepsilon - 2\mathcal{H}^\alpha(K) + \delta$$

$$< 2\mathcal{H}^\alpha(E) + \varepsilon - 2\mathcal{H}^\alpha\left(E \setminus \overline{B_0}\right) + \delta = 2\mathcal{H}^\alpha_E\left(\overline{B_0}\right) + \varepsilon + \delta.$$

If χ_{B_0} denotes the characteristic function of the ball B_0, then

$$\mu(B_0) = <\mu, \chi_{B_0}> = <\mu, \chi_{B_0\cap E}> = \lim_{\varepsilon\to 0} <\mu, \chi_{B_0\cap E} * \psi_\varepsilon>.$$

Arguing as in (1.8), (1.9) and (1.10), we get

$$|<\mu,\chi_{B_0\cap E}*\psi_\varepsilon>|\leq C\|\chi_{B_0\cap E}\|_\infty\sum_{j\in J_1}r_j^\alpha\leq C\left(\mathcal{H}_E^\alpha\left(\overline{B_0}\right)+\varepsilon+\delta\right)$$

and letting ε and δ tend to zero we get (1.11). □

1.4.2 Symmetrization of the Riesz kernel.

The symmetrization process for the Cauchy kernel introduced in [Me] has been successfully applied in this last years to many problems of analytic capacity and L^2 boundedness of the Cauchy integral operator (see [V3], [MV] and [MMV] for example ; the survey papers [D3] and [V5] contain many other references). Given 3 distinct points in the plane, z_1, z_2 and z_3, one finds out, by an elementary computation that

$$c(z_1,z_2,z_3)^2=\sum_\sigma\frac{1}{(z_{\sigma(1)}-z_{\sigma(3)})\overline{(z_{\sigma(2)}-z_{\sigma(3)})}}\tag{1.16}$$

where the sum is taken over the six permutations of the set $\{1,2,3\}$ and $c(z_1,z_2,z_3)$ is Menger curvature, that is, the inverse of the radius of the circle through z_1, z_2 and z_3. In particular (1.16) shows that the sum on the right hand side is a non-negative quantity.

On the other hand, it has been proved in [F] that nothing similar occurs for the Riesz kernel $k_\alpha = x/|x|^{1+\alpha}$ with α integer and $1<\alpha\leq n$. In this section we show that for $0<\alpha<1$ we recover an explicit expression for the symmetrization of the Riesz kernel k_α and that the quantity one gets is also non-negative. For $\alpha>1$ the phenomenon of change of signs appears again.

For $0<\alpha<n$ the quantity

$$\sum_\sigma\frac{x_{\sigma(2)}-x_{\sigma(1)}}{|x_{\sigma(2)}-x_{\sigma(1)}|^{1+\alpha}}\frac{x_{\sigma(3)}-x_{\sigma(1)}}{|x_{\sigma(3)}-x_{\sigma(1)}|^{1+\alpha}},\tag{1.17}$$

where the sum is taken over the six permutations of the set $\{1,2,3\}$, is the obvious analog of the right hand side of (1.16) for the Riesz kernel k_α. Observe, however, that if σ is a transposition of two numbers in $\{1,2,3\}$ then the term one obtains is one of the three terms associated to the permutations $(1,2,3),\ (2,3,1),\ (3,1,2)$. Thus (1.17) is exactly

$$2\,p_\alpha(x_1,x_2,x_3),$$

where $p_\alpha(x_1,x_2,x_3)$ is defined as the sum in (1.17) taken only on the three permutations $(1,2,3),\ (2,3,1),\ (3,1,2)$.

Lemma 1.15. *Let $0<\alpha<1$, and $x_1,\ x_2,\ x_3$ three distinct points in $\mathbb{R}^n$. Then we have*

$$\frac{2-2^\alpha}{L(x_1,x_2,x_3)^{2\alpha}}\leq p_\alpha(x_1,x_2,x_3)\leq\frac{2^{1+\alpha}}{L(x_1,x_2,x_3)^{2\alpha}},\tag{1.18}$$

where $L(x_1,x_2,x_3)$ is the largest side of the triangle determined by $x_1,\ x_2$ and x_3. In particular $p_\alpha(x_1,x_2,x_3)$ is a non-negative quantity.

Proof. If $n=1$ and $x_1<x_2<x_3$, then

$$p_\alpha(x_1,x_2,x_3)=\frac{a^\alpha+b^\alpha-(a+b)^\alpha}{a^\alpha b^\alpha(a+b)^\alpha}$$

where $a=x_2-x_1$ and $b=x_3-x_2$. An elementary estimate shows that (1.18) holds in this case, even with $2^{1+\alpha}$ replaced by 2^α in the numerator of the last term.

Note that if $x_1,\ x_2,\ x_3\in\mathbb{R}^n$, one can write

$$p_\alpha(x_1,x_2,x_3)=\frac{\cos(\theta_{23})|x_2-x_3|^\alpha+\cos(\theta_{13})|x_1-x_3|^\alpha+\cos(\theta_{12})|x_1-x_2|^\alpha}{|x_1-x_2|^\alpha|x_1-x_3|^\alpha|x_2-x_3|^\alpha}$$

where θ_{ij} is the angle opposite to the side x_ix_j in the triangle determined by $x_1,\ x_2,\ x_3$. Without loss of generality we can assume that $\theta_{23},\ \theta_{13}\in[0,\pi/2]$. Denote by $l_{ij}=|x_i-x_j|$, for $i\neq j,\ i,j\in\{1,2,3\}$. We consider two different cases:

Case 1: $0\leq\theta_{12}\leq\pi/2$.

Without loss of generality suppose $l_{12}\geq l_{13}\geq l_{23}$. Then we have

$$p_\alpha(x_1,x_2,x_3)=\frac{1}{l_{12}^\alpha l_{13}^\alpha}\left(\cos(\theta_{23})+\cos(\theta_{13})\frac{l_{13}^\alpha}{l_{23}^\alpha}+\cos(\theta_{12})\frac{l_{12}^\alpha}{l_{23}^\alpha}\right)$$

$$\geq\frac{1}{l_{12}^\alpha l_{13}^\alpha}\left(\cos(\theta_{23})+\cos(\theta_{13})+\cos(\theta_{12})\right)\geq\frac{1}{l_{12}^\alpha l_{13}^\alpha}\geq\frac{2-2^\alpha}{L(x_1,x_2,x_3)^{2\alpha}}.$$

For the second inequality one argues as follows.

$$p_\alpha(x_1,x_2,x_3)=\frac{1}{l_{12}^{1+\alpha}l_{13}^{1+\alpha}}\left(\cos(\theta_{23})l_{12}l_{13}+\cos(\theta_{13})l_{12}l_{23}\frac{l_{13}^{1+\alpha}}{l_{23}^{1+\alpha}}+\cos(\theta_{12})l_{13}l_{23}\frac{l_{12}^{1+\alpha}}{l_{23}^{1+\alpha}}\right)$$

$$\leq\frac{1}{l_{12}^{1+\alpha}l_{13}^{1+\alpha}}\left(\cos(\theta_{23})l_{12}l_{13}+\cos(\theta_{13})l_{12}l_{23}\frac{l_{13}^{2}}{l_{23}^{2}}+\cos(\theta_{12})l_{13}l_{23}\frac{l_{12}^{2}}{l_{23}^{2}}\right)$$

$$=l_{12}^{1-\alpha}l_{13}^{1-\alpha}p_1(x_1,x_2,x_3)=l_{12}^{1-\alpha}l_{13}^{1-\alpha}\frac{1}{2\ R^2},$$

by (1.16), where R is the radius of the circle through x_1,x_2 and x_3 . Since clearly $l_{ij}\leq 2R$, we conclude that

$$p_\alpha(x_1,x_2,x_3)\leq\frac{2}{l_{12}^\alpha l_{13}^\alpha}\leq\frac{2^{1+\alpha}}{L(x_1,x_2,x_3)^{2\alpha}}.$$

Case 2: $\pi/2 \le \theta_{12} \le \pi$.

We start by proving the first inequality in (1.18). Note that in this case the largest side of the triangle is l_{12}. Assume without loss of generality $l_{13} \ge l_{23}$ and denote by $t = l_{13}/l_{23} \ge 1$. Write $\theta_{13} = \theta_{23} + a$, with $0 \le a \le \pi/2$. Then by the triangle inequality we have

$$\begin{aligned} p_\alpha(x_1, x_2, x_3) &= \frac{1}{l_{12}^\alpha l_{13}^\alpha}\left(\cos(\theta_{23}) + \cos(\theta_{23}+a)\frac{l_{13}^\alpha}{l_{23}^\alpha} + \cos(\theta_{12})\frac{l_{12}^\alpha}{l_{23}^\alpha}\right) \\ &\ge \frac{1}{l_{12}^\alpha l_{13}^\alpha}\left(\cos(\theta_{23}) + \cos(\theta_{23}+a)t^\alpha - \cos(2\theta_{23}+a)(1+t)^\alpha\right) \qquad (1.19) \\ &\ge \frac{1}{l_{12}^\alpha l_{13}^\alpha} f(a, \theta_{23}, t), \end{aligned}$$

where

$$f(a, y, t) = \cos(y) + \cos(y+a)t^\alpha - \cos(2y+a)(1+t)^\alpha,$$

for $0 \le 2y + a \le \pi/2$, $a \ge 0$ and $y \ge 0$.

We claim that

$$f(a, y, t) \ge f(0, y, t) \ge f(0, 0, t) \tag{1.20}$$

for $0 \le 2y + a \le \pi/2$, $a \ge 0$ and $y \ge 0$. Notice that the inequality $f(a, y, t) \ge f(0, 0, t)$ in (1.20) means that the smallest value of p_α is attained when the three points x_1, x_2, x_3 lie on a line.

If we assume that the claim is proved, then going back to (1.19) and using that $t \ge 1$ we get

$$\begin{aligned} p_\alpha(x_1, x_2, x_3) &\ge \frac{1}{l_{12}^\alpha l_{13}^\alpha} f(a, \theta_{23}, t) \ge \frac{1}{l_{12}^\alpha l_{13}^\alpha} f(0, 0, t) \\ &= \frac{1}{l_{12}^\alpha l_{13}^\alpha}\left(1 + t^\alpha - (1+t)^\alpha\right) \ge \frac{2 - 2^\alpha}{l_{12}^\alpha l_{13}^\alpha} \ge \frac{2 - 2^\alpha}{L(x_1, x_2, x_3)^{2\alpha}}. \end{aligned}$$

To prove the first inequality in (1.20), we use that for $0 \le 2y + a \le \pi/2$, $a \ge 0$ and $y \ge 0$, we have $\cos(y) - \cos(y+a) \le cos(2y) - cos(2y+a)$. Thus $\cos(y) - \cos(y+a) \le \left(1 + \frac{1}{t}\right)^\alpha (cos(2y) - cos(2y+a))$, which is $f(a, y, t) \ge f(0, y, t)$.

Finally, for each t, the function

$$f(0, y, t) = \cos(y) + \cos(y)t^\alpha - \cos(2y)(1+t)^\alpha$$

has a minimum at $y = 0$ and this proves the claim and thus the first inequality in (1.18).

We are now only left with the second inequality in (1.18) for $\theta_{12} \in [\pi/2, \pi]$. Recall that we can assume without loss of generality that $l_{23} \leq l_{13} \leq l_{12}$. We have

$$p_\alpha(x_1, x_2, x_3) = \frac{1}{l_{12}^\alpha l_{13}^\alpha} \left(\cos(\theta_{23}) + \cos(\theta_{13}) \frac{l_{13}^\alpha}{l_{23}^\alpha} - \cos(\theta_{23} + \theta_{13}) \frac{l_{12}^\alpha}{l_{23}^\alpha} \right)$$

$$\leq \frac{1}{l_{12}^\alpha l_{13}^\alpha} \left(\cos(\theta_{23}) + (\cos(\theta_{13}) - \cos(\theta_{23} + \theta_{13})) \frac{l_{13}^\alpha}{l_{23}^\alpha} \right).$$

The function $g(x) = \cos x - \cos(x+y)$ is increasing for $x,\ y,$ and $x+y$ in $[0, \pi/2]$. Thus, $g(x) \leq g(\pi/2) = \sin y$, for x, y and $x + y$ in $[0, \pi/2]$. Moreover, using that $\sin(\theta_{23})/l_{23} = \sin(\theta_{13})/l_{13}$, we get

$$p_\alpha(x_1, x_2, x_3) \leq \frac{1}{l_{12}^\alpha l_{13}^\alpha} \left(\cos(\theta_{23}) + \sin(\theta_{13}) \frac{l_{23}^{1-\alpha}}{l_{13}^{1-\alpha}} \right) \leq \frac{2}{l_{12}^\alpha l_{13}^\alpha} \leq \frac{2^{1+\alpha}}{L(x_1, x_2, x_3)^{2\alpha}},$$

which completes the proof of the lemma. □

1.4.3 The main step.

Let $0 < \alpha < n$ and suppose that μ is a measure such that $\mu(B(x, r)) \leq C_0\, r^\alpha$ for some constant C_0 and for all balls $B(x, r) \subset \mathbb{R}^n$. We will now analyze what happens in a ball $B(x, r)$ satisfying the lower density condition $\mu(B(x, r)) \geq \varepsilon\, r^\alpha$ for a given number $\varepsilon > 0$.

Lemma 1.16. *There exist constants $a \geq 1$ and $b \geq 1$ depending only on C_0 and ε such that given any ball $B_0 = B(x, r)$ satisfying $\mu(B_0) \geq \varepsilon\, r^\alpha$, there exist two balls $B_1 = B(x_1, r/a)$ and $B_2 = B(x_2, r/a)$, with $x_1,\ x_2 \in$ spt $\mu \cap B_0$, such that*

1. $|x_1 - x_2| \geq 6r/a$.
2. $\mu(B_0 \cap B_i) \geq r^\alpha/b$ *for* $i = 1, 2$.

Proof. Without loss of generality we may assume that $B_0 = B(0, 1)$. Let $a \geq 1$ and $b \geq 1$ be two constants to be chosen at the end of the construction and suppose that the lemma is not true. This means that given any pair of closed balls B_1 and B_2 of radius a^{-1} centered at spt$\mu \cap B_0$ then either

$$|x_1 - x_2| < \frac{6}{a}\,, \tag{1.21}$$

or one of the two balls, say B_i, satisfies

$$\mu(B_i \cap B_0) \leq \frac{1}{b} \,.$$

Consider the covering of $\text{spt}\mu \cap B_0$ by balls of radius a^{-1} centered at $\text{spt}\mu \cap B_0$. Apply Besicovitch's covering lemma to this covering to obtain $N = N(n)$ families $\mathcal{B}_i$ of disjoint balls such that

$$\text{spt}\mu \cap B_0 \subset \bigcup_{i=1}^{N} \bigcup_{B \in \mathcal{B}_i} B.$$

Notice that a simple estimate of the volume of the union of the balls in a given family reveals that each family contains no more than $(2a)^n$ balls. We have

$$\varepsilon \leq \mu(B_0) \leq \mu(\bigcup_{i=1}^{N} \bigcup_{B \in \mathcal{B}_i} B) \leq \sum_{i=1}^{N} \sum_{B \in \mathcal{B}_i} \mu(B \cap B_0),$$

which means there exists at least one family $\mathcal{B}_i$ such that

$$\sum_{B \in \mathcal{B}_i} \mu(B \cap B_0) \geq \frac{\varepsilon}{N}.$$

Consider the set

$$\mathcal{M} = \{B \in \mathcal{B}_i : \; \mu(B \cap B_0) > \frac{1}{b}\}.$$

Condition (1.21) implies that all balls in $\mathcal{M}$ are contained in a ball of radius $8/a$, and hence

$$\sum_{B \in \mathcal{M}} \mu(B \cap B_0) \leq C_0 \left(\frac{8}{a}\right)^{\alpha},$$

using that $\mu(B(x,r)) \leq C_0\, r^{\alpha}$ holds for any ball $B(x,r)$ in $\mathbb{R}^n$.

The fact that each family $\mathcal{B}_i$ contains no more than $(2a)^n$ balls implies

$$\sum_{\substack{B \in \mathcal{B}_i \\ B \notin \mathcal{M}}} \mu(B \cap B_0) \leq \frac{(2a)^n}{b}$$

and so we get

$$\varepsilon \leq N \sum_{B \in \mathcal{B}_i} \mu(B \cap B_0) \leq N \left(\frac{(2a)^n}{b} + C_0 \left(\frac{8}{a}\right)^{\alpha} \right).$$

If a and b are appropriately chosen, this inequality gives a contradiction. $\square$

Let $0 \leq \alpha < \infty$, and let μ be a positive Borel measure on $\mathbb{R}^n$. The upper α-density of μ at $x \in \mathbb{R}^n$ is defined by

$$\Theta^{*\,\alpha}(\mu, x) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{(2r)^\alpha}.$$

Theorem 1.17. *Let $0 < \alpha < 1$ and let μ be a positive Borel measure with $0 < \Theta^{*\,\alpha}(\mu, x) < \infty$ for μ almost all $x \in \mathbb{R}^n$. Then*

$$\iiint p_\alpha(x_1, x_2, x_3) d\mu(x_1) d\mu(x_2) d\mu(x_3) = +\infty.$$

Proof. Since $\Theta^{*\,\alpha}(\mu, x) < \infty$ for μ almost all $x \in \mathbb{R}^n$, there exists a compact set $K_1 \subset \mathbb{R}$ with $\mu(K_1) > 0$ and a constant $c_1 > 0$ such that $\mu(K_1 \cap B(x,r)) \leq c_1 r^\alpha$ for every ball $B(x,r) \subset \mathbb{R}^n$. It is well-known that $\Theta^{*\,\alpha}(\mu|K_1, x) = \Theta^{*\,\alpha}(\mu, x)$ for μ almost all $x \in K_1$ (see [M1] Theorems 6.2 and 6.9), whence, replacing μ by $\mu|K_1$, we can assume that $\mu(B(x,r)) \leq c_1 r^\alpha$ for $x \in \mathbb{R}^n$.

From the fact that $\Theta^{*\,\alpha}(\mu, x) > 0$ for μ almost all $x \in \mathbb{R}^n$, we deduce that there exists a compact set $K_2 \subset \mathbb{R}^n$ with $\mu(K_2) > 0$ and a constant $c_2 > 0$, such that for each $x \in K_2$ there is a sequence $r_i(x) > 0$ with $\lim_{i \to \infty} r_i(x) = 0$ and $\mu(B(x, r_i(x))) \geq c_2 r_i(x)^\alpha$. Notice that truncating the sequences of radii apropriately, we can assume that $\sup_{x \in K_2} r_i(x) \to 0$, $i \to \infty$.
By the 5-covering Theorem (see [M1] Theorem 2.1), for each $i \in \mathbb{N}$ there are disjoint balls $B_j^i = B(a_j, r_i(a_j))$, $1 \leq j \leq m_i$, such that $K_2 \subset \bigcup_{j=1}^{m_i} 5B_j^i$. Then we have

$$\mu(K_2) \leq \sum_{j=1}^{m_i} \mu(5B_j^i) \leq c_1 5^\alpha \sum_{j=1}^{m_i} r_i(a_j)^\alpha,$$

that is,

$$\sum_{j=1}^{m_i} r_i^\alpha(a_j) \geq \frac{\mu(K_2)}{5^\alpha c_1}. \tag{1.22}$$

Fix $i = 1$ and consider the disjoint balls B_j^1, for $1 \leq j \leq m_1$. For every B_j^1 we can use Lemma 1.16 twice to find three balls $B_1,\ B_2,\ B_3$ centered at *spt*$(\mu) \cap B_j^1$ enjoying the following properties: their mutual distances and their radii are comparable to $r(a_j)$; the mass $\mu(B_j^1 \cap B_l)$ is also comparable to $r(a_j)$. The comparability constants in the above statements depend only on c_1, c_2 and n. Define a set of triples by

$$S_{j,1} = (B_j^1 \cap B_1) \times (B_j^1 \cap B_2) \times (B_j^1 \cap B_3), \text{ for } 1 \leq j \leq m_1.$$

Applying Lemma 1.15 we obtain

$$
\begin{aligned}
&\iiint\limits_{(B_j^1)^3} p_\alpha(x_1,x_2,x_3)d\mu(x_1)d\mu(x_2)d\mu(x_3)\\
&\geq \iiint\limits_{S_{j,1}} p_\alpha(x_1,x_2,x_3)d\mu(x_1)d\mu(x_2)d\mu(x_3) \qquad (1.23)\\
&\geq C\iiint\limits_{S_{j,1}} \frac{1}{|x_1-x_3|^{2\alpha}}d\mu(x_1)d\mu(x_2)d\mu(x_3) \geq Cr_1(a_j)^\alpha.
\end{aligned}
$$

Set

$$A_1=\bigcup_{j=1}^{m_1} S_{j,1}\subset \bigcup_{j=1}^{m_1}(B_j^1\times B_j^1\times B_j^1),$$

$$d_j=\min\{dist(B_j^1\cap B^k, B_j^1\cap B^l): k,l\in\{1,2,3\}, k\neq l\}$$

and

$$t_1=\min_{1\leq j\leq m_1} d_j.$$

For $(x_1,x_2,x_3)\in A_1$ we then have $|x_i-x_j|>t_1$ for $i,j\in\{1,2,3\}$, $j\neq i$. Moreover, using (1.22) and (1.23)

$$
\begin{aligned}
&\iiint\limits_{A_1} p_\alpha(x_1,x_2,x_3)d\mu(x_1)d\mu(x_2)d\mu(x_3)\\
&=\sum_{j=1}^{m_1}\iiint\limits_{S_{j,1}} p_\alpha(x_1,x_2,x_3)d\mu(x_1)d\mu(x_2)d\mu(x_3)\geq C\sum_{j=1}^{m_1} r_1(a_j)^\alpha \geq C.
\end{aligned}
$$

Let q be such that

$$\sup_{x\in K_2} r_q(x)\leq \frac{t_1}{2} \qquad (1.24)$$

and consider the balls of the q-th generation, namely B_j^q, for $1\leq j\leq m_q$. Repeat the process described above replacing B_j^1 by B_j^q. We then find balls B_1, B_2 and B_3 centered at points in $\text{spt}\mu\cap B_j^q$, whose mutual distances and radii are comparable to $r_q(a_j)$ and such that $\mu(B_j^q\cap B_l)$ is also comparable to $r_q(a_j)$, $l=1,2,3$.

Set

$$S_{j,2}=(B_j^q\cap B_1)\times(B_j^q\cap B_2)\times(B_j^q\cap B_3)$$

and

$$A_2 = \bigcup_{j=1}^{m_q} S_{j,2} \subset \bigcup_{j=1}^{m_q} \left(B_j^q \times B_j^q \times B_j^q \right).$$

Hence, again by (1.24),

$$\iiint_{A_2} p_\alpha(x_1, x_2, x_3) d\mu(x_1) d\mu(x_2) d\mu(x_3) \geq C.$$

Notice that the sets of triples A_1 and A_2 are disjoint because of the definition of q. Define t_2 as we did before for t_1, so that for $(x_1, x_2, x_3) \in A_2$ one has $|x_i - x_j| > t_2$ for $i, j \in \{1, 2, 3\}$, $i \neq j$. It becomes now clear that we can inductively construct disjoint sets of triples $A_k,\ k = 1, 2, ...$ such that

$$\iiint_{A_k} p_\alpha(x_1, x_2, x_3) d\mu(x_1) d\mu(x_2) d\mu(x_3) \geq C\ , \quad k = 1, 2, ...$$

and therefore

$$\iiint p_\alpha(x_1, x_2, x_3) d\mu(x_1) d\mu(x_2) d\mu(x_3)$$

$$\geq \sum_{k=1}^{\infty} \iiint_{A_k} p_\alpha(x_1, x_2, x_3) d\mu(x_1) d\mu(x_2) d\mu(x_3) \geq \sum_{k=1}^{\infty} C = +\infty. \quad \square$$

1.4.4 End of the proof of Theorem 1.

Suppose $\gamma_\alpha(E) > 0$ for $0 < \alpha < 1$. Applying Lemma 1.14 we find a measure of the form $\nu = b\mathcal{H}^\alpha$, with $b \in L^\infty(\mathcal{H}^\alpha, E)$ such that the $\alpha-$ Riesz potential $R_\alpha(\nu) = \nu * \frac{x}{|x|^{1+\alpha}}$ is in $L^\infty(\mathbb{R}^n)$ and $\int_E b\ d\mathcal{H}^\alpha = \gamma_\alpha(E)$. We can apply now Theorem 1.5 to get a set $F \subset E$ of positive $\mathcal{H}^\alpha-$ measure such that the operator R_α is bounded in $L^2(\mathcal{H}^\alpha, F)$. On the other hand, since $\mathcal{H}^\alpha(F) < \infty$ we have $2^{-\alpha} \leq \Theta^{*\,\alpha}(\mathcal{H}^\alpha_{|F}, x) \leq 1$ for $\mathcal{H}^\alpha$ almost all $x \in \mathbb{R}^n$ (see [M1], Theorem 6.2). This means that we can apply the previous Theorem to obtain

$$\iiint p_\alpha(x_1, x_2, x_3) d\mathcal{H}^\alpha_{|F}(x_1) d\mathcal{H}^\alpha_{|F}(x_2) d\mathcal{H}^\alpha_{|F}(x_3) = +\infty.$$

This last fact contradicts the L^2- boundedness of R_α on $L^2(\mathcal{H}^\alpha, F)$ by a well-known argument that we now outline briefly (see [Me] or [MV]).

Set $\mu = \mathcal{H}^\alpha_{|F}$. Then

$$\int |\ R_{\alpha,\epsilon}(\mu)(x)\ |^2 d\mu(x) = \iiint_{T_\epsilon} R_\alpha(x - y) R_\alpha(x - z) d\mu(x) d\mu(y) d\mu(z),$$

where

$$T_\epsilon = \{(x,y,z) :| x-y |> \epsilon, | x-z |> \epsilon\}.$$

Interchanging the roles of x and y and then of x and z, and estimating the error terms in a standard way we obtain

$$\int | R_{\alpha,\epsilon}(\mu)(x) |^2 d\mu(x) = \frac{1}{3} \iiint\limits_{S_\epsilon} p_\alpha(x,y,z) d\mu(x) d\mu(y) d\mu(z) + O(\mu(F)),$$

where

$$S_\epsilon = \{(x,y,z) :| x-y |> \epsilon, | x-z |> \epsilon, | y-z |> \epsilon\}.$$

Letting $\epsilon \to 0$ we get the promised contradiction. □

1.5 Proof of Theorem 1.2.

The proof of Theorem 1.2 is a straightforward consequence of the following result.

Theorem 1.18. *Let $0 < \alpha < n$ and let $E \subset \mathbb{R}^n$ be a compact Ahlfors-David regular set of dimension α. Assume that there exists some function $h \in L^\infty(E, \mathcal{H}^\alpha)$ such that $\|R_\alpha(h\mathcal{H}^\alpha)\|_\infty \leq 1$ and $\int_E h d\mathcal{H}^\alpha \neq 0$. Then, there exists a compact Ahlfors-David regular set $E' \subset \mathbb{R}^n$ of dimension α and a function $b \in L^\infty(E', \mathcal{H}^\alpha)$ with the following properties:*

1. *$\mathcal{H}^\alpha(E') = \mathcal{H}^\alpha(E)$ and $\mathcal{H}^\alpha(E' \cap E) > 0$.*
2. *The function b is dyadic para-accretive with respect to a family of dyadic cubes that we define while constructing E'.*
3. *$R_\alpha(b\mathcal{H}^\alpha)$ belongs to dyadic $BMO(E', \mathcal{H}^\alpha)$.*

Using Theorem 1.18, one can prove Theorem 1.2 as follows.

Proof of Theorem 1.2. Let $0 < \alpha < n$ and let $E \subset \mathbb{R}^n$ be a compact Ahlfors-David regular set of dimension α. Suppose $\gamma_\alpha(E) > 0$. Then there exists a distribution S with compact support contained in E, whose α-Riesz potential $S * \frac{x}{|x|^{1+\alpha}}$ is in $L^\infty(\mathbb{R}^n)$ and $< S, 1 > \neq 0$.
By Lemma 1.14, $S = h\mathcal{H}^\alpha$ with $h \in L^\infty(E, \mathcal{H}^\alpha)$. Thus $< S, 1 >= \int_E h(x) d\mathcal{H}^\alpha(x) \neq 0$. Since all the hypothesis of Theorem 1.18 are satisfied, there exists a compact Ahlfors-David regular set $E' \subset \mathbb{R}^n$ of dimension α and a dyadic para-accretive function $b \in L^\infty(E', \mathcal{H}^\alpha)$ such that $R_\alpha(b\mathcal{H}^\alpha)$ belongs to dyadic $BMO(E', \mathcal{H}^\alpha)$. Then, by the $T(b)$-theorem (Theorem 1.9), the α-Riesz operator R_α is bounded in $L^2(E', \mathcal{H}^\alpha)$. Applying now Theorem 1.10 we conclude that α must be an integer. □

1.5.1 Proof of Theorem 1.18.

In the following lemma we construct an Ahlfors-David regular set $\widetilde{E}$ which should be viewed as a variation of E with larger $\mathcal{H}^\alpha$-measure.

Lemma 1.19. *Let $0 < \alpha < n$, $E \subset \mathbb{R}^n$ an Ahlfors-David regular set of dimension α and a_0 some (small) fixed positive constant. If there exists some function $h \in L^\infty(E)$ with $R_\alpha(h\mathcal{H}^\alpha) \in L^\infty(\mathbb{R}^n)$ and $\int_E hd\mathcal{H}^\alpha \neq 0$, then there exists a set $\widetilde{E} \subset \mathbb{R}^n$ and a function $g \in L^\infty(\widetilde{E})$ such that:*

1. *$\widetilde{E}$ is an Ahlfors-David regular set of dimension α, whose Ahlfors-David regularity constant depends only on the Ahlfors-David regularity constant of E, $\mathcal{H}^\alpha(E)$, a_0, α and n.*

2. *$d(E)/2 \leq d(\widetilde{E}) \leq d(E)$.*

3. *$\mathcal{H}^\alpha(\widetilde{E}) = a_0^{-1}\mathcal{H}^\alpha(E)$.*

4. *$\displaystyle\int_{\widetilde{E}} gd\mathcal{H}^\alpha = \int_E hd\mathcal{H}^\alpha$.*

5. *$\|g\|_\infty = a_0\|h\|_\infty$ and $\|R_\alpha(g\mathcal{H}^\alpha)\|_\infty = (2^n a_0^{-1})^{\frac{\alpha}{n-\alpha}}\|R_\alpha(h\mathcal{H}^\alpha)\|_\infty$.*

Proof. Let $\delta = \frac{1}{2M}$, where M is a big positive integer to be fixed later. Consider a partition of the unit interval $[0,1] \subset \mathbb{R}$ into smaller intervals $I_j = [(j-1)\delta, j\delta]$ of length $l(I_j) = \delta$ for $1 \leq j \leq \delta^{-1}$. Let $H = \cup_j I_{2j-1}$ be the union of the intervals l_j with odd index j. Consider now the product set $H_0 = H^n \subset \mathbb{R}^n$. If we let N stand for M^n, then H_0 is a disjoint union of cubes Q_k of side length $l(Q_k) = \delta$ for $k = 1, \cdots, N$.

Without loss of generality assume $E \subset Q$ with $Q \subset \mathbb{R}^n$ a cube of side length $l(Q) = 1$ and $d(E) = 1/2$.

We construct the set $\widetilde{E}$ as follows : inside each cube Q_k, we will put a translated $\delta-$dilation of our initial set E. The new set $\widetilde{E}$ will be the union of these translated dilations.

To be more precise, fix points $\{b_k\}_{k=1}^N \in \mathbb{R}^n$ such that the sets

$$E_k = \delta E + b_k = \{\delta x + b_k \ : \ x \in E\},$$

are contained in the cubes Q_k for $1 \leq k \leq N$. Then define the set $\widetilde{E}$ by

$$\widetilde{E} = \bigcup_{k=1}^N E_k.$$

Since

$$\mathcal{H}^\alpha(\widetilde{E}) = \sum_{k=1}^N \mathcal{H}^\alpha(E_k) = N\delta^\alpha\mathcal{H}^\alpha(E),$$

if we choose M so that

$$a_0 = \frac{2^\alpha}{M^{n-\alpha}}, \tag{1.25}$$

then we get $\mathcal{H}^\alpha(\widetilde{E}) = a_0^{-1}\mathcal{H}^\alpha(E)$.

Now we want to show that $\widetilde{E}$ is an Ahlfors-David regular set. Take $r > 0$ and $x \in \widetilde{E}$; then x belongs to some E_k.

If $r \le \delta$, then

$$\mathcal{H}^\alpha(\widetilde{E} \cap B(x,r)) = \mathcal{H}^\alpha(E_k \cap B(x,r)) = \delta^\alpha \mathcal{H}^\alpha(E \cap B(x - b_k, r/\delta))$$

$$\approx \delta^\alpha r^\alpha \delta^{-\alpha} = r^\alpha.$$

If $r > \delta$, let i be the positive integer such that $i\delta < r \le (i+1)\delta$. Then

$$\sharp\{j: \; E_j \subset \widetilde{E} \text{ and } E_j \cap B(x,r) \ne \emptyset\} \approx i^n.$$

Therefore

$$\mathcal{H}^\alpha(\widetilde{E} \cap B(x,r)) \approx i^n \delta^\alpha \mathcal{H}^\alpha(E) = (i\delta)^\alpha i^{n-\alpha} \mathcal{H}^\alpha(E). \tag{1.26}$$

Since $i \ge 1$, (1.26) gives

$$\mathcal{H}^\alpha(\widetilde{E} \cap B(x,r)) \ge C r^\alpha.$$

Using (1.25) and $i\delta \le 1$, we obtain from (1.26) the inequality

$$\mathcal{H}^\alpha(\widetilde{E} \cap B(x,r)) \le C r^\alpha \delta^{\alpha-n} = C\frac{2^n}{a_0} r^\alpha.$$

Define

$$g(x) = \frac{\delta^{-\alpha}}{N} \sum_{k=1}^N h\left(\frac{x - b_k}{\delta}\right) \chi_{E_k}(x) = a_0 \sum_{k=1}^N h\left(\frac{x - b_k}{\delta}\right) \chi_{E_k}(x).$$

Clearly, $\int_{\widetilde{E}} g d\mathcal{H}^\alpha = \int_E h d\mathcal{H}^\alpha$ and $\|g\|_\infty = a_0\|h\|_\infty$. Finally, changing variables and using (1.25), we have

$$R_\alpha(g\mathcal{H}^\alpha)(x) = a_0 \sum_{k=1}^N \int_{E_k} \frac{y - x}{|y - x|^{1+\alpha}} h\left(\frac{y - b_k}{\delta}\right) d\mathcal{H}^\alpha(y)$$

$$= a_0 \sum_{k=1}^N \int_E \frac{z - \delta^{-1}(x - b_k)}{|z - \delta^{-1}(x - b_k)|^{1+\alpha}} h(z) d\mathcal{H}^\alpha(z) = a_0 \sum_{k=1}^N R_\alpha(h\mathcal{H}^\alpha)\left(\frac{x - b_k}{\delta}\right),$$

which implies

$$\|R_\alpha(g\mathcal{H}^\alpha)\|_\infty = a_0 N \|R_\alpha(h\mathcal{H}^\alpha)\|_\infty = (2^n a_0^{-1})^{\frac{\alpha}{n-\alpha}} \|R_\alpha(h\mathcal{H}^\alpha)\|_\infty. \quad \square$$

Remark. *For a system of dyadic cubes on* $\widetilde{E}$, $\mathcal{Q}(\widetilde{E})$, *satisfying the conclusions of Theorem 1.6, take first the whole set* $\widetilde{E} = \bigcup_{k=1}^{N} E_k$, *together with each set* E_k, *and together with the dyadic systems* $\mathcal{Q}(E_k) = \mathcal{Q}(\delta E + b_k) = \{\delta Q_\beta^j + b_k : Q_\beta^j \in \mathcal{Q}(E),\ \beta \in \mathbb{N},\ j \in \mathbb{N}\}$ *for* $1 \leq k \leq N$.

Before proving Theorem 1.18, we first sketch briefly the main ideas to construct the set E' and the function b with the required properties. Some of them are already in Christ's paper [Ch2]. We will use strongly the accretivity of the given function h at the first scale, that is,

$$\int_E h d\mathcal{H}^\alpha \neq 0. \tag{1.27}$$

To construct the set E', we begin by excising from E the dyadic cubes where the function h is not dyadic para-accretive and replacing these parts essentially by translated dilations of our initial set E. Then we will also change our function h on these modified parts. If $Q \subset E$ is one of the dyadic cubes where h is not dyadic para-accretive, then we excise Q from E and we replace it with $\lambda_1 E + a_1$, for appropriate λ_1 and a_1. Instead of h, we take now $h\left(\frac{x-a_1}{\lambda_1}\right)$ on $\lambda_1 E + a_1$. Note that in this way, using the accretivity property (1.27) at the first scale we have replaced a non-accretive cube Q by an accretive cube $\lambda_1 E + a_1$. Of course when looking inside $\lambda_1 E + a_1$ non-accretivity will reappear with the cube $\lambda_1 Q + a_1$, but doing the replacement we just described has the advantage of "improving" the accretivity of h. Repeating this process for all the non-accretive cubes, we obtain new accretive cubes and the new function turns out to be dyadic-paraacretive on these new cubes. Now repeat this process on each of the sets $\lambda_1 E + a_1$. In each of these dilations, we will change again the cubes where we do not have the dyadic paraacretivity condition and replace them with other dilations $\lambda_2 E + a_2$ (note that we always replace the same cubes, modulo translations and dilations, because we are always dealing with translations and dilations of the same set E). This process is repeated indefinitely, passing to higher and higher generations. The limit set and the limit function one obtains by repeating the previous algorithm will be our set E' and function b.

Proof of Theorem 1.18: We start by describing the basic algorithm that will be iterated infinitely many times.
Let $\mathcal{Q}(E)$ be a system of dyadic cubes on E satisfying the properties 1 through 6 in Theorem 1.6. The first dyadic cube of E to examine is E itself. By hypothesis there exists a function $h \in L^\infty(E)$ such that $\int_E h d\mathcal{H}^\alpha \neq 0$. Let $\varepsilon_0 > 0$ be a sufficiently small constant to be fixed later and such that $\left|\int_E h d\mathcal{H}^\alpha\right| > \varepsilon_0 \mathcal{H}^\alpha(E)$. Then for every positive integer k, there exists at least one cube Q_γ^k satisfying $|\int_{Q_\gamma^k} h d\mathcal{H}^\alpha| > \varepsilon_0 \mathcal{H}^\alpha(Q_\gamma^k)$, since otherwise for some k

$$|\int_E h d\mathcal{H}^\alpha| = |\int_{\cup_\gamma Q_\gamma^k} h d\mathcal{H}^\alpha| \leq \varepsilon_0 \sum_\gamma \mathcal{H}^\alpha(Q_\gamma^k) = \varepsilon_0 \mathcal{H}^\alpha(E),$$

which is a contradiction.

We now run a stopping-time procedure. Let $\varepsilon > 0$ be another constant, much smaller than ε_0, to be chosen later. Take a dyadic cube $Q \in \mathcal{Q}^1(E)$ and check whether or not the condition

$$|\int_Q h d\mathcal{H}^\alpha| \leq \varepsilon \mathcal{H}^\alpha(Q), \tag{1.28}$$

is satisfied. If (1.28) holds for that cube Q and Q has more than one child, we call it a stopping time cube. If (1.28) holds but Q has only one child, then we look for the first descendent of Q with more than one child and we call it a stopping time cube. Notice that (1.28) remains true for this descendent.

If (1.28) does not hold for Q, then we examine each child of Q and repeat the above procedure. After possibly infinitely many steps and possibly passing through all generations we obtain a collection of pairwise disjoint stopping time cubes $\{P_\gamma\}$ in E. Each P_γ has at least two children and satisfies the non-accretivity condition (1.28) with Q replaced by P_γ.

Set $\|h\|_\infty = M$. Then

$$\mathcal{H}^\alpha(E \setminus \bigcup_\gamma P_\gamma) = \int_{E\setminus\bigcup_\gamma P_\gamma} d\mathcal{H}^\alpha \geq \frac{1}{M}\int_{E\setminus\bigcup_\gamma P_\gamma} |h| d\mathcal{H}^\alpha \geq \frac{1}{M}\left|\int_{E\setminus\bigcup_\gamma P_\gamma} h d\mathcal{H}^\alpha\right|$$

$$\geq \frac{1}{M}|\int_E h d\mathcal{H}^\alpha| - \frac{1}{M}\sum_\gamma |\int_{P_\gamma} h d\mathcal{H}^\alpha| > \frac{1}{M}(\varepsilon_0 \mathcal{H}^\alpha(E) - \varepsilon \sum_\gamma \mathcal{H}^\alpha(P_\gamma)).$$

Therefore

$$\sum_\gamma \mathcal{H}^\alpha(P_\gamma) \leq (1-\eta)\mathcal{H}^\alpha(E), \tag{1.29}$$

for $\eta = \dfrac{\varepsilon_0 - \varepsilon}{M - \varepsilon}$.

We want to construct a set E_1, by excising from E the union of the stopping time cubes P_γ, and replacing each child R_β of P_γ by a certain translated dilation of the set E. This is not exactly what we will do, because we want the sets replacing R_β to be separated and the measure $\mathcal{H}^\alpha$ to remain unchanged. Because of this, we will work with translated dilations of the set $\widetilde{E}$ given by Lemma 1.19.

Property 5 of Theorem 1.6 gives us a constant $0 < c < 1$, such that for each Q, there exists z_Q with $B(Q) = B(z_Q, cd(Q)) \cap E \subset Q$, $\operatorname{dist}(B(Q), E - Q) \approx d(Q)$ and $\mathcal{H}^\alpha(B(Q)) \approx \mathcal{H}^\alpha(Q)$. Moreover, given a small positive number a_0, Lemma 1.19 gives us an Ahlfors-David regular set $\widetilde{E}$ with $d(E)/2 \leq d(\widetilde{E}) \leq d(E)$ and $\mathcal{H}^\alpha(\widetilde{E}) = a_0^{-1}\mathcal{H}^\alpha(E)$. With these two facts in mind, associate to each cub Q the set E_Q, which is a translation of a certain dilation of the set $\widetilde{E}$, namely

$$E_Q = \lambda_Q \widetilde{E} + a_Q,$$

where $a_Q \in \mathbb{R}^n$ is the translation needed to locate the E_Q appropriately and $\lambda_Q > 0$ is a dilation factor. We choose them to satisfy the two constraints below:

1. $E_Q \subset B(z_Q, c\, d(Q)/2)$.

2. $\mathcal{H}^\alpha(E_Q) = \mathcal{H}^\alpha(Q)$, i.e., $\lambda_Q^\alpha = \dfrac{\mathcal{H}^\alpha(Q)}{\mathcal{H}^\alpha(\widetilde{E})}$.

The fact that properties 1 and 2 hold at the same time, is possible by choosing a suitable constant $a_0 > 0$ in Lemma 1.19, as we explain below.

Assume without loss of generality that $d(E) = 1$, and so $d(\widetilde{E}) \leq 1$. Property 1 implies $\lambda_Q \leq 2d(E_Q) < 2c\, d(Q)$. Using the second property, this is the same as to say

$$\frac{\mathcal{H}^\alpha(Q)}{\mathcal{H}^\alpha(\widetilde{E})} < Cd(Q)^\alpha. \tag{1.30}$$

Due to the Ahlfors-David regularity of E, we have $\mathcal{H}^\alpha(Q) \approx d(Q)^\alpha$. Then (1.30) tells us that to find the dilation factors λ_Q we only need to have $\mathcal{H}^\alpha(\widetilde{E}) > C^{-1}$ for some small constant C. This can be achieved by just choosing $a_0 > 0$ small enough in Lemma 1.19.

We will construct the set E_1 by excising from E the union of all these stopping time cubes P_γ, and replacing each child R_β of P_γ with the set E_{R_β} defined above. For each stopping time cube $P_\gamma = \cup_\beta R_\beta$, set $F_\gamma = \cup_\beta E_{R_\beta}$. That is, for each γ, the sets F_γ replace the stopping time cubes P_γ in the new set E_1. In other words,

$$E_1 = \left(E \setminus \bigcup_\gamma P_\gamma\right) \cup \bigcup_\gamma F_\gamma.$$

Notice that for each γ we have

$$\mathcal{H}^\alpha(F_\gamma) = \sum_\beta \mathcal{H}^\alpha(E_{R_\beta}) = \sum_\beta \mathcal{H}^\alpha(R_\beta) = \mathcal{H}^\alpha(P_\gamma), \tag{1.31}$$

so that $\mathcal{H}^\alpha(E) = \mathcal{H}^\alpha(E_1)$.

For a system of dyadic cubes $\mathcal{Q}(E_1)$ on E_1, take all dyadic cubes $Q \in \mathcal{Q}(E)$ which are not contained in any stopping time cube P_γ, together with each $F_\gamma = \cup_\beta E_{R_\beta}$ and together with the dyadic cubes of $\mathcal{Q}(E_{R_\beta})$ in each F_γ (see the remark after Lemma 1.19 for the definition of $\mathcal{Q}(E_{R_\beta})$). Namely,

$$\mathcal{Q}(E_1) = \mathcal{Q}_1(E_1) \cup \mathcal{Q}_2(E_1),$$

where $\mathcal{Q}_1(E_1) = \{(S \setminus \bigcup_{P_\gamma \subset S} P_\gamma) \cup (\bigcup_{P_\gamma \subset S} F_\gamma) : \ S \in \mathcal{Q}(E) \setminus \{P_\gamma\}\} \cup \{F_\gamma\}$ and $\mathcal{Q}_2(E_1)$ consists of the dyadic systems $\mathcal{Q}(E_{R_\beta})$ associated to the sets E_{R_β} coming from all the F_γ. Hence each F_γ is a dyadic cube in $\mathcal{Q}(E_1)$.

For future reference, note that for every cube $Q \in \mathcal{Q}(E_1)$, such that $Q \neq F_\gamma$ for all γ, there is a non-stopping time cube $Q^* \in \mathcal{Q}(E)$ uniquely associated to Q by the identity:

$$Q = (Q^* \setminus \bigcup_{P_\beta \subset Q^*} P_\beta) \cup (\bigcup_{P_\beta \subset Q^*} F_\beta). \tag{1.32}$$

One has

$$\mathcal{H}^\alpha(Q) = \mathcal{H}^\alpha(Q^*) - \sum_{P_\beta \subset Q^*} \mathcal{H}^\alpha(P_\beta) + \sum_{P_\beta \subset Q^*} \mathcal{H}^\alpha(F_\beta) = \mathcal{H}^\alpha(Q^*)$$

and

$$d(Q) \approx d(Q^*).$$

After defining the set E_1 and the system of dyadic cubes $\mathcal{Q}(E_1)$, we modify the function h on the union $\cup_\gamma F_\gamma$ to obtain a new function h_1 defined on E_1. We want h_1 to be bounded and to satisfy

$$\int_{F_\gamma} h_1 d\mathcal{H}^\alpha = \int_{P_\gamma} h d\mathcal{H}^\alpha, \quad \text{for each } \gamma. \tag{1.33}$$

Condition (1.33) does not seem to contribute to the accretivity of the new function h_1, because the cubes P_γ were chosen precisely because the mean of h on them became too small. But although our h_1 has a small mean on F_γ, as h does on P_γ, we will have a satisfactory lower bound on the integral of h_1 over each child E_{R_β} of F_γ. Is in this way that h_1 becomes "more" accretive than h.

The function h_1 is defined on E_1 by

$$h_1(x) = \begin{cases} \displaystyle\sum_\beta c_\beta g(\frac{x-a_\beta}{\lambda_\beta})\chi_{E_{R_\beta}} & \text{on } \displaystyle\bigcup_\gamma F_\gamma = \bigcup_\beta E_{R_\beta} \\ h(x) & \text{on } E \setminus \displaystyle\bigcup_\gamma P_\gamma, \end{cases}$$

where the function g is the one defined on $\widetilde{E}$ given by Lemma 1.19 (recall that $\widetilde{E}$ is a union of translated dilations of our initial set E and the function g is the composition of h with the corresponding translation and dilation), and the coefficients c_β are defined below to get the boundedness of h_1 and (1.33).

Notice first that due to property 5 and 6 of Theorem 1.6 , $E_{R_\beta} \cap E_{R_\eta} = \emptyset$, for $\beta \neq \eta$ and $E_{R_\beta} \cap (E \setminus \cup_\gamma P_\gamma) = \emptyset$, so that the function h_1 is well defined on E_1.

To define the coefficients c_β, fix P_γ and let $N_\gamma = \sharp\{\beta : \ R_\beta \text{ is a child of } P_\gamma\}$. The number of children of the dyadic cubes is in between 2 and a fixed upper bound, that is, $2 \leq N_\gamma \leq c_1$, where c_1 is some constant independent of γ.

Order the children $\{R_\beta\}$ of P_γ starting with the cube R_β with the smallest $\mathcal{H}^\alpha$-measure and ending with the cube R_β with the biggest one. Write $\{R_\beta\} = \{R_\beta^j\}_{j=1}^{N_\gamma}$, where R_β^j stands for the j-th child R_β in this ordering. We want to divide the children of P_γ into two nonempty collections I and II, each with the same number of elements (plus or minus one) in the following way:

$$I = \{\beta : \ R_\beta = R_\beta^j \text{ for } 1 \le j \le [N_\gamma/2]\},$$

$$II = \{\beta : \ R_\beta = R_\beta^j \text{ for } [N_\gamma/2] + 1 \le j \le N_\gamma\}.$$

Clearly

$$\sum_{\beta \in \, II} \mathcal{H}^\alpha(R_\beta) - \sum_{\beta \in \, I} \mathcal{H}^\alpha(R_\beta) \ge 0.$$

Let θ be $\int_{P_\gamma} h d\mathcal{H}^\alpha (|\int_{P_\gamma} h d\mathcal{H}^\alpha|)^{-1}$ if $\int_{P_\gamma} h d\mathcal{H}^\alpha \ne 0$ and let θ be 1 if $\int_{P_\gamma} h d\mathcal{H}^\alpha = 0$. Set $a = \left|\int_E h d\mathcal{H}^\alpha\right|$ and define the coefficients c_β as

$$c_\beta = \begin{cases} \theta \mathcal{H}^\alpha(\widetilde{E}) a^{-1} & \text{if } \beta \in \ I \\ \\ -\theta \mathcal{H}^\alpha(\widetilde{E}) \widetilde{c}_\beta a^{-1} & \text{if } \ \beta \in \ II, \end{cases}$$

where the $\widetilde{c}_\beta$ satisfy $\varepsilon_0 \le \widetilde{c}_\beta \le 1$ and moreover a certain constraint specified below.

Due to properties 5 and 3 of Lemma 1.19 and to the above definition of the coeficients c_β, we get the boundedness of the function h_1:

$$\|h_1\|_\infty = \max\{\|h\|_\infty, \ |c_\beta| \|g\|_\infty\} \le \max\{\|h\|_\infty, \ a^{-1} \mathcal{H}^\alpha(\widetilde{E}) a_0 \|h\|_\infty\}$$

$$= \|h\|_\infty \max\{1, \ a^{-1} \mathcal{H}^\alpha(E)\} = \|h\|_\infty \max\{1, \ \left(\frac{1}{\mathcal{H}^\alpha(E)} |\int_E h d\mathcal{H}^\alpha|\right)^{-1}\}.$$

Using property 4 of Lemma 1.19 we also obtain

$$\int\limits_{E_{R_\beta}} g(\frac{x - a_\beta}{\lambda_\beta}) d\mathcal{H}^\alpha(x) = \lambda_\beta^\alpha a = a \frac{\mathcal{H}^\alpha(R_\beta)}{\mathcal{H}^\alpha(\widetilde{E})},$$

so that integrating h_1 on F_γ we get

$$\int_{F_\gamma} h_1 d\mathcal{H}^\alpha = \int_{\cup_\beta E_{R_\beta}} h_1 d\mathcal{H}^\alpha = \sum_{\beta \in \, I} \theta \mathcal{H}^\alpha(R_\beta) - \sum_{\beta \in \, II} \theta \widetilde{c}_\beta \mathcal{H}^\alpha(R_\beta).$$

We claim that we can choose $\varepsilon_0 > 0$ sufficiently small, so that there exist numbers $\widetilde{c}_\beta$, $\varepsilon_0 \le \widetilde{c}_\beta \le 1$, such that

$$\sum_{\beta \in \, I} \mathcal{H}^\alpha(R_\beta) - \sum_{\beta \in \, II} \widetilde{c}_\beta \mathcal{H}^\alpha(R_\beta) = |\int_{P_\gamma} h d\mathcal{H}^\alpha|. \tag{1.34}$$

Once (1.34) is proved, we get the desired expression for the integral of h_1 over F_γ, namely

$$\int_{F_\gamma} h_1 d\mathcal{H}^\alpha = \int_{\bigcup_\beta E_{R_\beta}} h_1 d\mathcal{H}^\alpha = \theta(\sum_{\beta\in\, I} \mathcal{H}^\alpha(R_\beta) - \sum_{\beta\in\, II} \widetilde{c}_\beta \mathcal{H}^\alpha(R_\beta))$$

$$= \theta|\int_{P_\gamma} hd\mathcal{H}^\alpha| = \int_{P_\gamma} hd\mathcal{H}^\alpha.$$

To show (1.34), let $N_2 = \sharp\{\beta :\ \beta \in II\}$ and define

$$\widetilde{c}_\eta = \frac{1}{N_2\mathcal{H}^\alpha(R_\eta)}\left(\sum_{\beta\in I}\mathcal{H}^\alpha(R_\beta) - |\int_{P_\gamma} hd\mathcal{H}^\alpha|\right).$$

With this choice of the coefficients $\widetilde{c}_\eta$, equality (1.34) clearly holds. Thus, we only have to show that there exists $\varepsilon_0 > 0$ such that $\varepsilon_0 \le \widetilde{c}_\eta \le 1$, for all η.
The inequality $\widetilde{c}_\eta \le 1$ is equivalent to

$$\frac{1}{N_2\mathcal{H}^\alpha(R_\eta)}\sum_{\beta\in I}\mathcal{H}^\alpha(R_\beta) \le 1 + \frac{1}{N_2\mathcal{H}^\alpha(R_\eta)}|\int_{P_\gamma} hd\mathcal{H}^\alpha|.$$

Notice that by the way the indexes were ordered, for all $\eta \in II$,

$$\sum_{\beta\in I}\mathcal{H}^\alpha(R_\beta) \le N_2\mathcal{H}^\alpha(R_\eta),$$

which implies $\widetilde{c}_\eta \le 1$.

For the lower inequality, we have to choose ε_0 such that $\widetilde{c}_\eta \ge \varepsilon_0$. Recall that for P_γ the stopping time condition (1.28) holds with Q replaced by P_γ, and that the children of P_γ have comparable measure. Moreover, we know that there exists some (small) positive constant $0 < c < 1/2$, such that $\sum_{\beta\in I}\mathcal{H}^\alpha(R_\beta) \ge c\mathcal{H}^\alpha(P_\gamma)$. Then we have,

$$\widetilde{c}_\eta \ge \frac{(c-\varepsilon)\mathcal{H}^\alpha(P_\gamma)}{N_2\mathcal{H}^\alpha(R_\eta)} \ge \frac{(c-\varepsilon)\mathcal{H}^\alpha(P_\gamma)}{N_\gamma\mathcal{H}^\alpha(P_\gamma)} \ge \frac{c-\varepsilon}{c_1},$$

where $c_1 > 0$ is the upper bound for the number of children of a dyadic cube.

We have to choose ε_0 and ε, such that $c - \varepsilon \ge \varepsilon_0 c_1$ holds. This can be achieved by requiring $\varepsilon_0 c_1 \le c/2$ and $\varepsilon < \min(\varepsilon_0, c/2)$. The identity (1.34) is now proved and therefore (1.33) holds.

To construct the function h_1, we have to carry out this procedure for each stopping-time cube P_γ.

The P_γ are the cubes where the accretivity condition for h fails. The function h_1 has the advantage that although $\int_{P_\gamma} hd\mathcal{H}^\alpha = \int_{F_\gamma} h_1 d\mathcal{H}^\alpha$, we have a satisfactory lower bound on the integral over each child E_{R_β} of F_γ. This is due to the definition of the coeficients c_β :

1. If $\beta \in I$, then $|\int\limits_{E_{R_\beta}} h_1 d\mathcal{H}^\alpha| = |c_\beta| a\lambda_\beta^\alpha = \mathcal{H}^\alpha(E_{R_\beta}) \geq \varepsilon_0 \mathcal{H}^\alpha(E_{R_\beta})$.

2. If $\beta \in II$, then $|\int\limits_{E_{R_\beta}} h_1 d\mathcal{H}^\alpha| \geq |\widetilde{c}_\beta| \mathcal{H}^\alpha(E_{R_\beta}) \geq \varepsilon_0 \mathcal{H}^\alpha(E_{R_\beta})$, because $\varepsilon_0 \leq \widetilde{c}_\beta$.

This completes the description of the basic algorithm.

We begin with the pair (E, h) with E a dyadic cube and h in $L^\infty(E, \mathcal{H}^\alpha)$ satisfying $|\int_E h d\mathcal{H}^\alpha| \geq \varepsilon_0 \mathcal{H}^\alpha(E)$. The result of the algorithm is another function h_1, a collection of sets $\{F_\gamma\}$ replacing the collection $\{P_\gamma\}$ of stopping time subcubes of E and a set $E_1 = \left(E \setminus \bigcup_\gamma P_\gamma\right) \cup \bigcup_\gamma F_\gamma$ (recall that, by construction, if $P_\gamma = \cup_\beta R_\beta$, then $F_\gamma = \cup_\beta E_{R_\beta} = \cup_\beta(\lambda_\beta \widetilde{E} + a_\beta)$, for some $a_\beta \in \mathbb{R}^n$; for the definition of the set $\widetilde{E}$ see Lemma 1.19). We list now seven properties concerning h, h_1, E and E_1. The first three are straightforward consequences of the algorithm:

$$\|h_1\|_\infty \leq A_0 \|h\|_\infty, \tag{1.35}$$

where $A_0 = \max\{\left(\frac{1}{\mathcal{H}^\alpha(E)} |\int_E h d\mathcal{H}^\alpha|\right)^{-1}, 1\}$.

$$h \equiv h_1 \;\; on \;\; E \setminus \bigcup_\gamma P_\gamma \tag{1.36}$$

and

$$\int_{P_\gamma} h d\mathcal{H}^\alpha = \int_{F_\gamma} h_1 d\mathcal{H}^\alpha. \tag{1.37}$$

The fourth is

$$\mathcal{H}^\alpha(E) = \mathcal{H}^\alpha(E_1) \tag{1.38}$$

and is due to (1.31). The fifth is

$$\sum_\gamma \mathcal{H}^\alpha(F_\gamma) = \sum_\beta \mathcal{H}^\alpha(P_\gamma) \leq (1-\eta)\mathcal{H}^\alpha(E) \tag{1.39}$$

and follows from (1.31) and (1.29). The next to the last is

$$|\int_{E_{R_\beta}} h_1 d\mathcal{H}^\alpha| \geq \varepsilon_0 \mathcal{H}^\alpha(E_{R_\beta}), \tag{1.40}$$

which has been discussed before. The last one is

$$|\int_S h_1 d\mathcal{H}^\alpha| \geq \varepsilon \mathcal{H}^\alpha(S), \tag{1.41}$$

for every dyadic cube $S \in \mathcal{Q}(E_1)$ of the form

$S = \left(S^* \setminus \bigcup_{P_\gamma \subset S^*} P_\gamma\right) \cup \bigcup_{P_\gamma \subset S^*} F_\gamma$ for some non-stopping time cube $S^* \in \mathcal{Q}(E)$ (see (1.32) for the definition of the cube S^*).

To show (1.41) we remark that using (1.36) and (1.37) we have

$$\int_S h_1 d\mathcal{H}^\alpha = \int_{S^* \setminus \bigcup_\gamma P_\gamma} h d\mathcal{H}^\alpha + \sum_\gamma \int_{F_\gamma} h_1 d\mathcal{H}^\alpha$$

$$= \int_{S^* \setminus \bigcup_\gamma P_\gamma} h d\mathcal{H}^\alpha + \sum_\gamma \int_{P_\gamma} h d\mathcal{H}^\alpha = \int_{S^*} h d\mathcal{H}^\alpha,$$

which implies

$$\left|\int_S h_1 d\mathcal{H}^\alpha\right| = \left|\int_{S^*} h d\mathcal{H}^\alpha\right| > \varepsilon \mathcal{H}^\alpha(S),$$

due to the identity $\mathcal{H}^\alpha(S) = \mathcal{H}^\alpha(S^*)$ and to the fact that S^* is a non-stopping time cube.

We need to repeat this algorithm infinitely many times. Begin with the dyadic cube E itself and the function h with $\|h\|_\infty \le A_0$. Apply the algorithm to obtain cubes $\{P^1_{\gamma_1}\}_{\gamma_1}$, sets $\{F^1_{\gamma_1}\}_{\gamma_1}$, a set $E_1 = \left(E \setminus \cup_{\gamma_1} P^1_{\gamma_1}\right) \cup \bigcup_{\gamma_1} F^1_{\gamma_1}$ with a system of dyadic cubes $\mathcal{Q}(E_1)$ and a function h_1 defined on E_1, satisfying properties (1.35) through (1.41). For each β_1 the pair $(E_{R^1_{\beta_1}}, h_1\chi_{E_{R^1_{\beta_1}}})$ is an admissible input for the algorithm (recall that for each γ_1, $F^1_{\gamma_1} = \cup_{\beta_1} E_{R^1_{\beta_1}} = \cup_{\beta_1}(\lambda^1_{\beta_1}\widetilde{E} + a_{\beta_1})$). So we may apply the algorithm to each one of these pairs to obtain further cubes $\{P^2_{\gamma_2}\}_{\gamma_2}$, sets $\{F^2_{\gamma_2}\}_{\gamma_2}$, a set $E_2 = (E_1 \setminus \bigcup_{\gamma_2} P^2_{\gamma_2}) \cup \bigcup_{\gamma_2} F^2_{\gamma_2}$ (with a system of dyadic cubes $\mathcal{Q}(E_2)$) and a function h_2 defined on E_2 with properties (1.35) through (1.41) (with h and h_1 replaced by h_1 and h_2 respectively). An infinite number of repetitions produces functions h_j, collections $\{P^j_{\gamma_j}\}_{\gamma_j}$, $\{F^j_{\gamma_j}\}_{\gamma_j}$ and sets $E_j = \left(E_{j-1} \setminus \bigcup_{\gamma_j} P^j_{\gamma_j}\right) \cup \bigcup_{\gamma_j} F^j_{\gamma_j}$ endowed with systems of dyadic cubes $\mathcal{Q}(E_j)$, for every integer $j \ge 1$.

By (1.39), the sets $F^j_{\gamma_j}$ that replace the stopping time cubes $P^j_{\gamma_j}$ at the $j-$th step satisfy

$$\sum_{\gamma_j} \mathcal{H}^\alpha(F^j_{\gamma_j}) \le (1-\eta)^j \mathcal{H}^\alpha(E). \tag{1.42}$$

We want to define the limit set E' of the sequence $\{E_j\}_{j\ge1}$ and for this we will use the fact that the family of all non-empty compact subsets of $\mathbb{R}^n$ is a complete metric space with the *Hausdorff metric* ρ (see [Fe] 2.10.21),

$$\rho(A,B) = \max\{d(x,A),\ d(y,B)\ :\ x \in B,\ y \in A\}.$$

It is then enough to show that $\{E_j\}_{j\ge1}$ forms a Cauchy sequence in the metric ρ.

We claim that there exists some small constant $0 < \tau < 1$, such that

$$\rho(E_{j+1}, E_j) \le \tau\rho(E_j, E_{j-1}),\ \text{for all } j \ge 2. \tag{1.43}$$

To see (1.43), recall that $E_j = \left(E_{j-1} \setminus \bigcup_{\beta_j} R^j_{\beta_j}\right) \cup \left(\bigcup_{\beta_j} E_{R^j_{\beta_j}}\right)$, where the $R^j_{\beta_j}$ are the children of the stopping time cubes found at the j-th step of the previous construction. Thus there exists some index β_j^* such that

$$\rho(E_j, E_{j-1}) = \rho(E_{R^j_{\beta^*_j}}, R^j_{\beta^*_j}).$$

For any j and β_j, $E_{R^j_{\beta_j}}$ is a union of translations of $\lambda^j_{\beta_j}\delta E$ (see Lemma 1.19). This means that the children $R^{j+1}_{\beta_{j+1}}$ of the stopping time cubes contained in a particular translation of $\lambda^j_{\beta_j}\delta E$ are simply translations of $\lambda^j_{\beta_j}\delta R^1_{\gamma_1}$, where the $R^1_{\gamma_1}$ are the children of the stopping time cubes of E. Thus the Hausdorff distance is at each step the same, modulo the translation and dilation factors that appear when running the algorithm. Hence we have

$$\rho(E_{j+1}, E_j) = \max_{\beta_{j+1}} \lambda^{j+1}_{\beta_{j+1}}\, \rho(E_{R^j_{\beta^*_j}}, R^j_{\beta^*_j}).$$

Because of the election of a_0,

$$\left(\lambda^{j+1}_{\beta_{j+1}}\right)^\alpha = a_0 \frac{\mathcal{H}^\alpha(R^{j+1}_{\beta_{j+1}})}{\mathcal{H}^\alpha(E)} < a_0 < 1 \text{ for all } \beta_{j+1},$$

which proves claim (1.43) with $\tau = a_0^{1/\alpha}$.

Let $E' = \lim_{j\to\infty} E_j$ be the limit set of the sequence $\{E_j\}_j$ and set

$$b(x) = \lim_{j\to\infty} h_j(x),$$

which exists for almost all $x \in E'$.

Because properties (1.35), (1.37) and (1.38) hold at each step of the construction, we have

$$\|b\|_\infty \le C,$$

$$\int_{E'} b(x) d\mathcal{H}^\alpha = \int_E h(x) d\mathcal{H}^\alpha,$$

and

$$\mathcal{H}^\alpha(E) = \mathcal{H}^\alpha(E').$$

Moreover, since

$$\int_E h d\mathcal{H}^\alpha = \int_{E\setminus \cup_\gamma P_\gamma} h d\mathcal{H}^\alpha + \int_{\cup_\gamma P_\gamma} h d\mathcal{H}^\alpha \neq 0,$$

$$\int_{\cup_\gamma P_\gamma} h d\mathcal{H}^\alpha < \varepsilon \sum_\gamma \mathcal{H}^\alpha(P_\gamma) < \varepsilon \mathcal{H}^\alpha(E)$$

and

$$|\int_E h d\mathcal{H}^\alpha| \geq \varepsilon_0 \mathcal{H}^\alpha(E),$$

we get

$$|\int_{E\setminus\cup_\gamma P_\gamma} h d\mathcal{H}^\alpha| \geq (\varepsilon_0 - \varepsilon)\mathcal{H}^\alpha(E) > 0,$$

from the choice of ε_0 and ε. This tells us that $\mathcal{H}^\alpha(E \setminus \cup_\gamma P_\gamma) > 0$, and therefore that $\mathcal{H}^\alpha(E \cap E') > 0$ because of the inclusion $E \setminus \cup_\gamma P_\gamma \subset E' \cap E$.

At this point we define a class of sets $\mathcal{Q}(E')$ that eventually will be shown to be a system of dyadic cubes for E'. We distinguish two types of "cubes" Q in $\mathcal{Q}(E')$.

Type 1. For some $j \geq 1$, $Q \in \mathcal{Q}(E_j) \setminus \mathcal{Q}(E_{j-1})$ (for $j = 0$ set $E_0 = E$) and Q does not contain any stopping time cube $P^{j+1}_{\gamma_{j+1}}$ (in the $(j+1)$-th application of the algorithm). Notice that in fact $Q \in \mathcal{Q}(E_m)$ for all $m \geq j$.

Type 2. For some $j \geq 0$ we take a cube $Q^* \in \mathcal{Q}(E_j) \setminus \mathcal{Q}(E_{j-1})$ (set $E_{-1} = \emptyset$) that contains some stopping time cubes $P^{j+1}_{\gamma_{j+1}}$ but it is not one of them. This Q^* will produce a "cube" Q in $\mathcal{Q}(E')$ after infinitely many modifications corresponding to successive applications of the algorithm (exactly in the same way one gets E' from E). The first modification consists in replacing the stopping time cubes $P^{j+1}_{\gamma_{j+1}} \subset Q^*$ by the sets $F^{j+1}_{\gamma_{j+1}}$ in the way illustrated in (1.32).

Notice that for each "cube" Q in $\mathcal{Q}(E')$ there exist some index $j \geq 0$ and an associated non-stopping dyadic cube $Q^* \in \mathcal{Q}(E_j) \setminus \mathcal{Q}(E_{j-1})$ in such a way that $Q^* = Q$, if Q is a cube of the first type and Q^* is the cube involved in the definition of Q, if Q is of type2. The cubes Q and Q^* coincide only if there are no stopping time cubes contained in Q^*, because in this case, the iteration of the algorithm does not modify the cube Q^* at all.

If $Q \in \mathcal{Q}(E')$ is a cube of the second type and $Q^* \in \mathcal{Q}(E_j) \setminus \mathcal{Q}(E_{j-1})$ is the corresponding associated non-stopping time cube, then there exists a sequence of cubes $\{Q^*_k\}_{k\geq j}$ such that $Q^*_j = Q^* \in \mathcal{Q}(E_j) \setminus \mathcal{Q}(E_{j-1})$ and for all $k > j$, $Q^*_k \in \mathcal{Q}(E_k) \setminus \mathcal{Q}(E_{k-1})$ is the modification corresponding to the application of the algorithm to the cube $Q^*_{k-1} \in \mathcal{Q}(E_{k-1})$. Namely, $Q^*_k \to Q$, as $k \to \infty$, in the same way as $E_k \to E'$, when $k \to \infty$.

Set $F^0_{\gamma_0} = E$. Notice that if $Q^* \in \mathcal{Q}(E_j) \setminus \mathcal{Q}(E_{j-1})$ for some $j \geq 0$, then there exists some index γ_j such that either $Q^* \subsetneq F^j_{\gamma_j}$ or $Q^* = F^j_{\gamma_j}$ (otherwise we would have $Q^* \in \mathcal{Q}(E_{j-1})$).

Observe that the definition of the "cubes" $\mathcal{Q}(E')$ immediately yields the properties of a true family of dyadic cubes listed in Theorem 1.6.

By construction we have

$$d(Q) \approx d(Q^*). \tag{1.44}$$

We claim that

$$\mathcal{H}^\alpha(Q) = \mathcal{H}^\alpha(Q^*) \tag{1.45}$$

and

$$\int_Q b d\mathcal{H}^\alpha = \int_{Q^*} h_j d\mathcal{H}^\alpha. \tag{1.46}$$

To show this we distinguish two cases:

1. If Q is of type 1, then $Q = Q^*$ and $b = h_j$ (set $h_0 = h$). Hence (1.45) and (1.46) hold obviously.

2. If Q is of type 2, then, although the identity $Q = Q^*$ does not hold because the cube Q^* has been modified when running the algorithm, the claim is still true because properties (1.31) and (1.37) at each step imply (1.45) and (1.46) respectively.

Notice that the sequence of cubes $\{Q_k^*\}_{k\geq j}$ approaching Q when $k \to \infty$ also fulfil $d(Q_k^*) \approx d(Q)$ and $\mathcal{H}^\alpha(Q_k^*) = \mathcal{H}^\alpha(Q)$ for all $k \geq j$.

We still have to prove the Ahlfors-David regularity of the set E', that is, that there exists some constant $C > 0$, with

$$C^{-1} r^\alpha \leq \mathcal{H}^\alpha(E' \cap B(x,r)) \leq C r^\alpha, \tag{1.47}$$

for every $x \in E'$ and for $0 < r \leq d(E')$.

We show first the Ahlfors-David regularity of the sets $\{E_n\}_{n\geq 0}$ converging to E'. We prove it by induction on n. The case $n = 0$ ($E_0 = E$) is true by hypothesis.

Let $x \in E_1$ and $0 < r \leq d(E_1)$. Recall that $E_1 = (E_0 \setminus \bigcup_{\gamma_1} P^1_{\gamma_1}) \cup \bigcup_{\gamma_1} F^1_{\gamma_1}$. Notice that for any $y \in E_1$, there exists a decreasing sequence of cubes $\{Q^j(y)\}_j$ in $\mathcal{Q}(E_1)$ such that for every index j, $y \in Q^j(y)$ and $Q^j(y) \in \mathcal{Q}^j(E_1)$, that is, $Q^j(y)$ is a cube of generation j in $\mathcal{Q}(E_1)$. For each $y \in E_1 \cap B(x,r)$, let $Q^k(y)$ be the smallest dyadic cube in this sequence not contained in $E_1 \cap B(x,r)$. Then for some positive constant C,

$$r \leq d(Q^k(y)) \leq C d(Q^{k+1}(y)) \leq Cr,$$

because $Q^k(y) \nsubseteq B(x,r)$ and $Q^{k+1}(y) \subset B(x,r)$. Hence,

$$\mathcal{H}^\alpha(E' \cap B(x,r)) \geq \mathcal{H}^\alpha(Q^{k+1}(y)) = \mathcal{H}^\alpha(Q^{k+1}(y)^*) \approx d(Q^{k+1}(y)^*)^\alpha \approx r^\alpha.$$

because $\mathcal{H}^\alpha(Q^{k+1}(y)^*) = \mathcal{H}^\alpha(Q^{k+1}(y))$, $Q^{k+1}(y)^* \in \mathcal{Q}(E)$ and E is an Ahlfors-David regular set.

For the reverse inequality, observe that if $x \in F^1_{\gamma_1}$ for some index γ_1 and $r < d(P^1_{\gamma_1})$, then due to the Ahlfors-David regularity of $F^1_{\gamma_1}$ we have

$$\mathcal{H}^\alpha(E_1 \cap B(x,r)) = \mathcal{H}^\alpha(F_{\gamma_1} \cap B(x,r)) \approx r^\alpha.$$

Otherwise, for each $y \in E_1 \cap B(x,r)$ let $Q^{k_y}_y \in \mathcal{Q}^{k_y}(E_1)$ stand for the smallest dyadic cube in $\mathcal{Q}(E_1)$ containing y and not contained in $B(x,2r)$. Then $r \approx d(Q^{k_y}_y)$ as before. The maximal cubes $\{Q_l\}_l$ in this family form a disjoint covering of $E_1 \cap B(x,r)$ with $d(Q_l) \approx r$. Consider the associated dyadic cubes $Q^*_l \in \mathcal{Q}(E)$ defined in (1.32). Then $\mathcal{H}^\alpha(Q^*_l) = \mathcal{H}^\alpha(Q_l)$ and $d(Q^*_l) \approx d(Q_l) \approx r$.

Notice that due to Lemma 19 in [Ch2], there exists some constant $N_0 < \infty$ such that for every bounded subset F of an Ahlfors-David regular set E, the number of disjoint dyadic cubes intersecting F, with diameter greater or comparable to the diameter of F, is at most N_0 (N_0 depends only on the Ahlfors-David regularity constant of E).

The cubes Q^*_l are disjoint and cover the bounded subset $E \cap B(x,r)$ of E (the fact that $r \geq d(P_{\gamma_1})$ implies that $E \cap B(x,r) \neq \emptyset$), thus we have at most N_0 of such cubes. Then

$$\mathcal{H}^\alpha(E_1 \cap B(x,r)) \leq \sum_l \mathcal{H}^\alpha(Q_l) = \sum_l \mathcal{H}^\alpha(Q^*_l) \leq C N_0 r^\alpha,$$

because the cubes Q^* are contained in the Ahlfors-David regular set E and hence $\mathcal{H}^\alpha(Q^*_l) \approx d(Q^*_l)^\alpha \approx r^\alpha$ for all l.

The induction hypothesis will be to assume the Ahlfors-David regularity of E_{n-1}, with Ahlfors-David regularity constant depending only on that of E.

Let $x \in E_n$ and $0 < r \leq d(E_n)$. Recall that $E_n = (E_{n-1} \setminus \bigcup_{\gamma_n} P^n_{\gamma_n}) \cup \bigcup_{\gamma_n} F^n_{\gamma_n}$. To Prove the Ahlfors-David regularity of E_n, argue like in the case $n = 1$, but replacing E_1 by E_n, E by E_{n-1} (which is Ahlfors-David regular due to the induction hypothesis), $F^1_{\gamma_1} \in \mathcal{Q}(E_1)$ by $F^n_{\gamma_n} \in \mathcal{Q}(E_n)$ and the cubes Q^*_l and $(Q^{k+1}(y))^*$ in $\mathcal{Q}(E)$ ($y \in E_1 \cap B(x,r)$) by $(Q_l)^*_{n-1}$ and $(Q^{k+1}(y))^*_{n-1}$ in $\mathcal{Q}(E_{n-1})$ ($y \in E_n \cap B(x,r)$) respectively.

We turn now to the proof of (1.47). The argument is very similar to what we have just done with the sets E_n. The main difference lies on the fact that in working with the cubes Q^*_l, we will be forced to jump over several E_n, instead of remaining in the previous one.

Let $x \in E'$ and $0 < r \leq d(E')$. For $y \in E' \cap B(x,r)$, let $Q^{k_y}(y) \in \mathcal{Q}^{k_y}(E')$ be the smallest dyadic cube in $\mathcal{Q}(E')$ containing y and not contained in $B(x,r)$, then $d(Q^{k_y}(y)) \approx r$. Then using (1.45), (1.44) and the fact that for some m, $(Q^{k_y+1}(y))^* \in \mathcal{Q}(E_m)$, with E_m an Ahlfors-David regular set, we get

$$\mathcal{H}^\alpha(E' \cap B(x,r)) \geq \mathcal{H}^\alpha(Q^{k_y+1}(y)) = \mathcal{H}^\alpha((Q^{k_y+1}(y))^*) \approx d((Q^{k_y+1}(y))^*)^\alpha \approx r^\alpha.$$

Thus the lower inequality in (1.47) holds.

We are left now with the upper inequality in (1.47). For each $y \in E' \cap B(x,r)$ we let $Q_y^{k_y}$ stand for the smallest dyadic cube in $\mathcal{Q}(E')$ containing y and not contained in $B(x,2r)$. Then $r \approx d(Q_y^{k_y})$ as before. The maximal cubes $\{Q_l\}_l$ in this family form a disjoint covering of $E' \cap B(x,r)$, with $d(Q_l) \approx r$.

We claim now that there are finitely many Q_l. To show this, we assume that there are at least N of such cubes and we show that N is bounded above by a constant N_0 depending only on the Ahlfors-David regularity constant of E. Let $\{Q_l^*\}_{l=1}^N$ be the associated non-stopping time cubes defined before. Then for each l, $1 \leq l \leq N$, there exists some index $j(l)$ such that $Q_l^* \in \mathcal{Q}(E_{j(l)})$. We know that each cube Q_l is the limit set of the sequence $\{(Q_l)_k^*\}_{k \geq j(l)}$, with $(Q_l)_k^* \in \mathcal{Q}(E_k) \setminus \mathcal{Q}(E_{k-1})$ for all $k \geq j(l)$ (recall that, by definition, $(Q_l)_{j(l)}^* = Q_l^* \in \mathcal{Q}(E_{j(l)})$). Since we are dealing with only N cubes Q_l, for some sufficiently big index m , there exists some index $k(l) \geq j(l)$ such that $(Q_l)_{k(l)}^* \in \mathcal{Q}(E_m)$ and $d((Q_l)_{k(l)}^*) \approx r$, for all $1 \leq l \leq N$. Recalling that E_m is an Ahlfors-David regular set with Ahlfors-David regularity constant depending only on those of E and using Lemma 19 in [Ch2], as before, we conclude that there at most N_0 of these dyadic cubes $(Q_l)_{k(l)}^*$. Thus using that the cubes $(Q_l)_{k(l)}^* \in \mathcal{Q}(E_m)$ satisfy $\mathcal{H}^\alpha(Q_l) = \mathcal{H}^\alpha((Q_l)_{k(l)}^*)$ and $d((Q_k)_{k(l)}^*) \approx r$, the Ahlfors-regularity of the set E_m gives us

$$\mathcal{H}^\alpha(E' \cap B(x,r)) \leq \sum_l \mathcal{H}^\alpha(Q_l)) = \sum_l \mathcal{H}^\alpha((Q_l)_{k(l)}^*) \leq C N_0 r^\alpha.$$

Our function b is dyadic para-accretive by construction. Take some dyadic cube $Q \in \mathcal{Q}(E')$ and consider the uniquely associated non-stopping time cube $Q^* \in \mathcal{Q}(E_j) \setminus \mathcal{Q}(E_{j-1})$, $j \geq 0$. We distinguish the following two cases :

1. (a) If $Q^* = F_{\gamma_j}^j$ for some $j \neq 0$, then although we do not have

$$\int_Q b d\mathcal{H}^\alpha \geq \varepsilon \mathcal{H}^\alpha(Q),$$

the function is still dyadic para-accretive, namely, the above inequality holds replacing the cube $Q \in \mathcal{Q}(E')$ by any child $R_\eta \subset Q$. Notice first that if

$$Q^* = F_{\gamma_j}^j = \bigcup_\beta E_{R_\beta} = \bigcup_\beta (\lambda_\beta \widetilde{E} + a_\beta) \in \mathcal{Q}(E_j)$$

and R is a child of Q, then clearly for some β

$$R^* = E_{R_\beta} = \lambda_\beta \widetilde{E} + a_\beta \subset Q^*.$$

Then, using (1.46), (1.40) and (1.45) we get

$$\int_{R_\beta} b d\mathcal{H}^\alpha = \int_{R_\beta^*} h_j d\mathcal{H}^\alpha = \int_{E_{R_\beta}} h_j d\mathcal{H}^\alpha \geq \varepsilon_0 \mathcal{H}^\alpha(E_{R_\beta}) = \varepsilon_0 \mathcal{H}^\alpha(R_\beta).$$

(b) If $Q^* = E$, then $Q = E'$. Thus using (1.46), the accretivity of h at the highest scale and (1.45), we obtain

$$\int_Q b d\mathcal{H}^\alpha = \int_{E'} b d\mathcal{H}^\alpha = \int_E h d\mathcal{H}^\alpha \geq \varepsilon_0 \mathcal{H}^\alpha(E) = \varepsilon_0 \mathcal{H}^\alpha(E').$$

2. If Q^* is strictly contained in some $F^j_{\gamma_j}$, then by (1.46), (1.41) and (1.45),

$$\int_Q b d\mathcal{H}^\alpha = \int_{Q^*} h_j d\mathcal{H}^\alpha \geq \varepsilon \mathcal{H}^\alpha(Q^*) = \varepsilon \mathcal{H}^\alpha(Q).$$

To complete the proof we only need to show that $R_\alpha(b\mathcal{H}^\alpha)$ is a dyadic $BMO(E')$ function. In what follows, to simplify the notation we will set $T(f) = R_\alpha(f\mathcal{H}^\alpha)$ for $f \in L^1(E', \mathcal{H}^\alpha)$.

We claim that, since the function $b \in L^\infty(E')$, it is enough to show the following L^1-inequality:

$$\|T(b\chi_Q)\|_{L^1(Q)} \leq C\mathcal{H}^\alpha(Q). \tag{1.48}$$

for every $Q \in \mathcal{Q}(E')$.

Suppose (1.48) holds for every $Q \in \mathcal{Q}(E')$ and let $2Q = \{x \in E' : \mathrm{dist}(x, Q) \leq Ad(Q)\}$, for some positive constant A. As a consequence of the "small boundary condition" for the dyadic cubes (see property 6 in Theorem 1.6) we have

$$\|T(b\chi_{2Q\setminus Q})\|_{L^1(Q)} \leq C\mathcal{H}^\alpha(Q)$$

(see the bound for the second integral in (1.58) below). The standard estimates for the Calderón-Zygmund operators show that

$$\|T(b\chi_{(2Q)^c})(x) - T(b\chi_{(2Q)^c})(x_0)\|_{L^1(Q)} \leq C\mathcal{H}^\alpha(Q), \tag{1.49}$$

where x_0 is a fixed point in Q. This implies that

$$\begin{aligned}&\int_Q |T(b)(x) - T(b\chi_{(2Q)^c})(x_0)| d\mathcal{H}^\alpha(x) \leq \int_Q |T(b\chi_Q)(x)| d\mathcal{H}^\alpha(x) \\ &+ \int_Q |T(b\chi_{2Q\setminus Q})(x)| d\mathcal{H}^\alpha(x) + \int_Q |T(b\chi_{(2Q)^c})(x) - T(b\chi_{(2Q)^c})(x_0)| d\mathcal{H}^\alpha(x) \\ &\leq C\mathcal{H}^\alpha(Q),\end{aligned}$$

which proves the claim.

To see (1.48), let $Q \in \mathcal{Q}(E')$ be some dyadic cube of E' and let Q^* be the uniquely associated dyadic cube in $\mathcal{Q}(E_j) \setminus \mathcal{Q}(E_{j-1})$ defined before (recall that the index j is determined by Q). Then the cube Q is the limit set of the sequence $\{Q_k^*\}_{k\geq j}$, with $Q_k^* \in \mathcal{Q}(E_k) \setminus \mathcal{Q}(E_{k-1})$.

Notice that the definition of b implies that we can write

$$b\chi_Q = h_j\chi_{Q^*} + \sum_{k=j+1}^{\infty} \sum_{P_{\gamma_k}^k \in Q_{k-1}^*} \left(h_k \chi_{F_{\gamma_k}^k} - h_{k-1}\chi_{P_{\gamma_k}^k} \right).$$

Then applying T we obtain,

$$T(b\chi_Q) = T(h_j\chi_{Q^*}) + \sum_{k=j+1}^{\infty} \sum_{P_{\gamma_k}^k \in Q_{k-1}^*} T\left(h_k \chi_{F_{\gamma_k}^k} - h_{k-1}\chi_{P_{\gamma_k}^k} \right).$$

To show (1.48), we only need to prove the next three inequalities:

$$\|T(h_j\chi_{Q^*})\|_{L^1(Q)} \leq C\mathcal{H}^\alpha(Q) \tag{1.50}$$

and

$$\|T(h_k\chi_{F_{\gamma_k}^k}) - T(h_{k-1}\chi_{P_{\gamma_k}^k})\|_{L^1(F_{\gamma_k}^k)} \leq C\mathcal{H}^\alpha(F_{\gamma_k}^k) \tag{1.51}$$

$$\|T(h_k\chi_{F_{\gamma_k}^k}) - T(h_{k-1}\chi_{P_{\gamma_k}^k})\|_{L^1(Q\setminus F_{\gamma_k}^k)} \leq C\mathcal{H}^\alpha(F_{\gamma_k}^k) \tag{1.52}$$

for all $k > j$ such that $P_{\gamma_k}^k \in Q_{k-1}^*$.

Notice that if $Q = Q^*$, then $b = h_j$ and (1.50) implies

$$\int_Q |T(b\chi_Q)|d\mathcal{H}^\alpha = \int_Q |T(h_j\chi_{Q^*})|d\mathcal{H}^\alpha \leq C\mathcal{H}^\alpha(Q),$$

which is (1.48).

Otherwise $Q \neq Q^*$ and if properties (1.50), (1.51) and (1.52) hold, using (1.42) and (1.45) at each step of the algorithm we get

$$\|T(b\chi_Q)\|_{L^1(Q)} \leq C\mathcal{H}^\alpha(Q) + C\sum_{k=j+1}^{\infty} \sum_{P_{\gamma_k}^k \in Q_{k-1}^*} \mathcal{H}^\alpha(F_{\gamma_k}^k)$$

$$\leq C\mathcal{H}^\alpha(Q) + C\sum_{k=1}^{\infty}(1-\eta)^k\mathcal{H}^\alpha(Q) \leq C\mathcal{H}^\alpha(Q),$$

which is (1.48).

Thus we are only left with the task of proving (1.50), (1.51) and (1.52). We first show (1.50). Recall that given any cube $Q \in \mathcal{Q}(E')$, we have an associated cube $Q^* \in \mathcal{Q}(E_j) \setminus \mathcal{Q}(E_{j-1})$ (for some $j \geq 0$ depending on Q) and either $Q^* = F_{\gamma_j}^j$ or $Q^* \subset F_{\gamma_j}^j$ for some index γ_j.

1. (a) If $Q^* = F^j_{\gamma_j}$ for some $j \neq 0$, then recall that $F^j_{\gamma_j} = \bigcup_{\beta_j} E_{R^j_{\beta_j}}$ and for $x \in E_{R^j_{\beta_j}} = \lambda^j_{\beta_j}\tilde{E} + a_{\beta_j}$,

$$h_j(x) = \sum_{\beta_j} c_{\beta_j} g(\frac{x - a_{\beta_j}}{\lambda_{\beta_j}})\chi_{E_{R^j_{\beta_j}}}.$$

Thus

$$T(h_j\chi_{F^j_{\gamma_j}})(x) = \sum_{\beta_j} c_{\beta_j} \int_{E_{R^j_{\beta_j}}} \frac{y-x}{|y-x|^{1+\alpha}} g\left(\frac{y-a_{\beta_j}}{\lambda^j_{\beta_j}}\right) d\mathcal{H}^\alpha(y)$$

$$= \sum_{\beta_j} c_{\beta_j} \int_{\tilde{E}} \frac{z-(x-a_{\beta_j})/\lambda^j_{\beta_j}}{|z-(x-a_{\beta_j})/\lambda^j_{\beta_j}|^{1+\alpha}} g(z)d\mathcal{H}^\alpha(z) = \sum_{\beta_j} c_{\beta_j} T(g)\left(\frac{x-a_{\beta_j}}{\lambda^j_{\beta_j}}\right).$$

Now, using that the number of children $E_{R^j_{\beta_j}}$ of $F^j_{\gamma_j}$ is between 2 and a fixed upper bound , that the constants c_{β_j} are uniformly bounded from above and that $\|T(g)\|_\infty \leq C$ (see Lemma 1.19), we obtain

$$\|T(h_j\chi_{F^j_{\gamma_j}})\|_\infty \leq C. \tag{1.53}$$

This $L^\infty-$inequality implies that

$$\|T(h_j\chi_{Q^*})\|_{L^1(Q)} = \|T(h_j\chi_{F^j_{\gamma_j}})\|_{L^1(Q)} \leq C\mathcal{H}^\alpha(Q),$$

which is (1.50) in this case.

(b) If $Q^* = E$, then $Q = E'$ and by hypothesis

$$\|T(h_0\chi_{Q^*})\|_\infty = \|T(h)\|_\infty \leq C.$$

Thus

$$\|T(h_0\chi_{Q^*})\|_{L^1(Q)} = \|T(h)\|_{L^1(E')} \leq C\mathcal{H}^\alpha(E').$$

2. If $Q^* \subset F^j_{\gamma_j}$, set $f_j = h_j\chi_{F^j_{\gamma_j}}$ and $g_j = h_j\chi_{F^j_{\gamma_j}\setminus 2Q^*}$ (where $2Q^* = \{x \in F^j_{\gamma_j} : dist(x, Q^*) \leq Ad(Q^*)\}$). Then one has a BMO estimate for $T(g_j)$ restricted to Q^*, namely, there exists some constant c, depending on g_j and Q^*, such that

$$\|T(g_j) - c\|_{L^1(Q^*)} \leq C\mathcal{H}^\alpha(Q^*), \tag{1.54}$$

(something similar was done before (1.49) to show that (1.48) suffices for the BMO bound). And using the small boundary condition (see property 6 in Theorem 1.6) and (1.45) we have

$$\|T(h_j \chi_{2Q^* \setminus Q^*})\|_{L^1(Q^*)} \leq C\mathcal{H}^\alpha(Q^*) = C\mathcal{H}^\alpha(Q) \tag{1.55}$$

(see the bound for the second integral in (1.58) below).

Thus writing

$$\int_{Q^*} T(h_j \chi_{Q^*}) d\mathcal{H}^\alpha$$

$$= \int_{Q^*} T(f_j) d\mathcal{H}^\alpha - \int_{Q^*} T(h_j \chi_{2Q^* \setminus Q^*}) d\mathcal{H}^\alpha - \int_{Q^*} (T(g_j) - c)\, d\mathcal{H}^\alpha - c\mathcal{H}^\alpha(Q^*),$$

to show (1.50) it suffices to find an upper bound for $|c|$ independent of Q^*. To get such a bound consider the integral over Q^* of the product of $h_j \chi_{Q^*}$ with $T(h_j \chi_{Q^*})$. On the one hand, it is zero by antisymmetry. On the other hand, writing $T(h_j \chi_{Q^*}) = T(f_j) - T(g_j) - T(h_j \chi_{2Q^* \setminus Q^*})$, it equals to $\int_{Q^*} h_j T(f_j) d\mathcal{H}^\alpha - \int_{Q^*} h_j (T(g_j) - c)\, d\mathcal{H}^\alpha - c \int_{Q^*} h_j d\mathcal{H}^\alpha - \int_{Q^*} T(h_j \chi_{2Q^* \setminus Q^*})$. Hence due to (1.53), (1.54), (1.55) and $\|h_j\|_\infty \leq C$ (see (1.35) replacing h_1 by h_j), we get

$$|c| \left| \int_{Q^*} h_j d\mathcal{H}^\alpha \right| \leq \left| \int_{Q^*} h_j T(f_j)\ d\mathcal{H}^\alpha \right| + \left| \int_{Q^*} h_j \left(T(g_j) - c\right)\ d\mathcal{H}^\alpha \right| \leq C\mathcal{H}^\alpha(Q^*).$$

The upper bound on $|c|$ is obtained by using that $Q^* \in \mathcal{Q}(E_j)$ is not a stopping time cube, namely, $\left| \int_{Q^*} h_j d\mathcal{H}^\alpha \right| > \varepsilon \mathcal{H}^\alpha(Q^*)$. Thus (1.50) is proved.

Inequality (1.51) is proved by estimating each of the terms $\|T(h_k \chi_{F^k_{\gamma_k}})\|_{L^1(F^k_{\gamma_k})}$ and $\|T(h_{k-1} \chi_{P^k_{\gamma_k}}\|_{L^1(F^k_{\gamma_k})}$. Replacing h_j and $F^j_{\gamma_j}$ by h_k and $F^k_{\gamma_k}$ in (1.53), we get $\|T(h_k \chi_{F^k_{\gamma_k}})\|_{L^1(F^k_{\gamma_k})} \leq C\mathcal{H}^\alpha(F^k_{\gamma_k})$. To estimate the second term, recall first that $P^k_{\gamma_k} \subset Q^*_{k-1} \in \mathcal{Q}(E_{k-1}) \setminus \mathcal{Q}(E_{k-2})$, thus for some index γ_{k-1}, we have $P^k_{\gamma_k} \subset F^{k-1}_{\gamma_{k-1}}$. Since $F^{k-1}_{\gamma_{k-1}}$ is a finite union of $E_{R^{k-1}_{\beta_{k-1}}}$ and $P^k_{\gamma_k}$ is a stopping time cube, $P^k_{\gamma_k} \subset E_{R^{k-1}_{\beta_{k-1}}}$ for some β_{k-1} . Thus the definition of h_{k-1} implies that

$$h_{k-1}(x) \chi_{P^k_{\gamma_k}}(x) = c_{\beta_{k-1}} g\left(\frac{x - a_{\beta_{k-1}}}{\lambda^{k-1}_{\beta_{k-1}}}\right) \chi_{P^k_{\gamma_k}}(x),$$

for some index β_{k-1}.

Let $h_{k-1} \chi_{P^k_{\gamma_k}} = f_{k-1} + g_{k-1}$, where $f_{k-1} = h_{k-1} \chi_{F^{k-1}_{\gamma_{k-1}}}$ and $g_{k-1} = h_{k-1} \chi_{F^{k-1}_{\gamma_{k-1}} \setminus P^k_{\gamma_k}}$. Using (1.53) (with h_j and $F^j_{\gamma_j}$ replaced by h_{k-1} and $F^{k-1}_{\gamma_{k-1}}$ respectively) we obtain

$$\|T(h_{k-1} \chi_{P^k_{\gamma_k}})\|_{L^1(F^k_{\gamma_k})} \leq C\mathcal{H}^\alpha(F^k_{\gamma_k}) + \|T(g_{k-1})\|_{L^1(F^k_{\gamma_k})}. \tag{1.56}$$

Hence we only have to estimate this second integral. To do this, let $z^k = z^k_{\gamma_k} \in P^k_{\gamma_k}$ be the point in $P^k_{\gamma_k}$ given by property 5 in Theorem 1.6, $B_k = B(z^k, cd(P^k_{\gamma_k}))$ and $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$. We know that $P^k_{\gamma_k} \subset F^{k-1}_{\gamma_{k-1}}$ for some index γ_{k-1} and each set $F^{k-1}_{\gamma_{k-1}}$ is Ahlfors-David regular, then by the upper inequality in the Ahlfors-David condition, the fact that $\|h_{k-1}\|_\infty \leq C$ for all k and for some positive constant c_0 we have

$$\frac{1}{\mathcal{L}^n(B_k)} \int_{B_k} \left| T(h_{k-1}\chi_{P^k_{\gamma_k}})(y) \right| d\mathcal{L}^n(y) \leq \frac{C}{d(P^k_{\gamma_k})^n} \int_{P^k_{\gamma_k}} \int_{B(x,c_0 d(P^k_{\gamma_k}))} \frac{d\mathcal{L}^n(y)}{|x-y|^\alpha d\mathcal{H}^\alpha(x)}$$

$$= \frac{C}{d(P^k_{\gamma_k})^n} \int_{P^k_{\gamma_k}} \int_{B(0,c_0 d(P^k_{\gamma_k}))} \frac{d\mathcal{L}^n(z)}{|z|^\alpha} d\mathcal{H}^\alpha(x) = C \frac{\mathcal{H}^\alpha(P^k_{\gamma_k})}{d(P^k_{\gamma_k})^n} \int_{B(0,c_0 d(P^k_{\gamma_k}))} \frac{d\mathcal{L}^n(z)}{|z|^\alpha} \leq C,$$

with C depending only on c_0, n, α, the uniform upper bound for $\|h_{k-1}\|_\infty$ and the upper constant in the Ahlfors-David condition. The fact that this mean value integral is finite, implies that there exists some point $y_k \in B_k$ such that

$$\left| T(h_{k-1}\chi_{P^k_{\gamma_k}})(y_k) \right| \leq C.$$

Then using (1.53) (with h_j and $F^j_{\gamma_j}$ replaced by h_{k-1} and $F^{k-1}_{\gamma_{k-1}}$ respectively),

$$|T(g_{k-1})(y_k)| \leq \|T(f_{k-1})\|_\infty + \left| T(h_{k-1}\chi_{P^k_{\gamma_k}})(y_k) \right| \leq C.$$

Thus

$$\|T(g_{k-1})\|_{L^1(F^k_{\gamma_k})} \leq \|T(g_{k-1}) - T(g_{k-1})(y_k)\|_{L^1(F^k_{\gamma_k})} + C\mathcal{H}^\alpha(F^k_{\gamma_k}) \leq C\mathcal{H}^\alpha(F^k_{\gamma_k}),$$

which is the right estimate for the second integral in (1.56) and shows inequality (1.51). In this last step we use the same idea as in the proof of (1.57) below.

To show (1.52) we will adapt an argument from [Ch2] (see Lemma 18) that uses standard estimates for the Calderón-Zygmund operators, the identities $\int_{P^k_{\gamma_k}} h_{k-1} d\mathcal{H}^\alpha = \int_{F^k_{\gamma_k}} h_k d\mathcal{H}^\alpha$ and the inequalities $\|h_k\|_\infty \leq C$ for all k. Let $z^k = z^k_{\gamma_k} \in P^k_{\gamma_k}$ be the point in $P^k_{\gamma_k}$ given by property 5 in Theorem 1.6. Then we can write

$$T(h_{k-1}\chi_{P^k_{\gamma_k}} - h_k\chi_{F^k_{\gamma_k}})(y) = \int_{P^k_{\gamma_k}} h_{k-1}(x) \left(\frac{x-y}{|x-y|^{1+\alpha}} - \frac{z^k-y}{|z^k-y|^{1+\alpha}} \right) d\mathcal{H}^\alpha(x)$$

$$+ \int_{F^k_{\gamma_k}} h_k(x) \left(\frac{z^k-y}{|z^k-y|^{1+\alpha}} - \frac{x-y}{|x-y|^{1+\alpha}} \right) d\mathcal{H}^\alpha(x).$$

Now $\operatorname{dist}(z^k, Q \setminus P^k_{\gamma_k}) \geq cd(P^k_{\gamma_k})$. When $x \in F^k_{\gamma_k}$ and $y \in Q \setminus F^k_{\gamma_k}$, then $|y-x| \geq cd(P^k_{\gamma_k})$. Therefore the standard estimates for the Calderón-Zygmund kernels give

$$\begin{aligned}
&\int_{Q\setminus F^k_{\gamma_k}} \int_{F^k_{\gamma_k}} \left| \frac{z^k-y}{|z^k-y|^{1+\alpha}} - \frac{x-y}{|x-y|^{1+\alpha}} \right| d\mathcal{H}^\alpha(x) d\mathcal{H}^\alpha(y) \\
&\leq C \int_{F^k_{\gamma_k}} \sum_{j=1}^{\infty} \int_{\{2^{j-1}cd(P^k_{\gamma_k}) \leq |y-x| \leq 2^j cd(P^k_{\gamma_k})\}} \frac{|x-z^k|}{|x-y|^{1+\alpha}} d\mathcal{H}^\alpha(y) d\mathcal{H}^\alpha(x) \\
&\leq C \int_{F^k_{\gamma_k}} \sum_{j=1}^{\infty} \frac{\mathcal{H}^\alpha(\{2^{j-1}cd(P^k_{\gamma_k}) \leq |y-x| \leq 2^j cd(P^k_{\gamma_k})\})}{\left(2^{j-1}cd(P^k_{\gamma_k})\right)^{1+\alpha}} d(P^k_{\gamma_k}) d\mathcal{H}^\alpha(x) \\
&\leq C \int_{F^k_{\gamma_k}} \sum_{j=1}^{\infty} 2^{-j} d\mathcal{H}^\alpha(x) \leq C\mathcal{H}^\alpha(F^k_{\gamma_k}).
\end{aligned} \tag{1.57}$$

Since $(Q \setminus F^k_{\gamma_k}) \cap P^k_{\gamma_k} = \emptyset$,

$$\begin{aligned}
&\int_{Q\setminus F^k_{\gamma_k}} \int_{P^k_{\gamma_k}} \left| \frac{z^k-y}{|z^k-y|^{1+\alpha}} - \frac{x-y}{|x-y|^{1+\alpha}} \right| d\mathcal{H}^\alpha(x) d\mathcal{H}^\alpha(y) \\
&= \int_{(Q\setminus F^k_{\gamma_k})\setminus 2P^k_{\gamma_k}} \int_{P^k_{\gamma_k}} \left| \frac{z^k-y}{|z^k-y|^{1+\alpha}} - \frac{x-y}{|x-y|^{1+\alpha}} \right| d\mathcal{H}^\alpha(x) d\mathcal{H}^\alpha(y) \\
&+ \int_{(Q\setminus F^k_{\gamma_k})\cap 2P^k_{\gamma_k}\setminus P^k_{\gamma_k}} \int_{P^k_{\gamma_k}} \left| \frac{z^k-y}{|z^k-y|^{1+\alpha}} - \frac{x-y}{|x-y|^{1+\alpha}} \right| d\mathcal{H}^\alpha(x) d\mathcal{H}^\alpha(y).
\end{aligned} \tag{1.58}$$

The first integral in (1.58) may be estimated in the same way as (1.57). Thus we get

$$\int_{(Q\setminus F^k_{\gamma_k})\setminus 2P^k_{\gamma_k}} \int_{P^k_{\gamma_k}} \left| \frac{z^k-y}{|z^k-y|^{1+\alpha}} - \frac{x-y}{|x-y|^{1+\alpha}} \right| d\mathcal{H}^\alpha(x) d\mathcal{H}^\alpha(y) \leq C\mathcal{H}^\alpha(P^k_{\gamma_k}) = \mathcal{H}^\alpha(F^k_{\gamma_k}).$$

For the second integral in (1.58), let $j \in \mathbb{Z}$ and define the set

$$A_j = \{x \in P^k_{\gamma_k} : \ 2^{j-1}d(P^k_{\gamma_k}) < \operatorname{dist}(x, 2P^k_{\gamma_k} \setminus P^k_{\gamma_k}) \leq 2^j d(P^k_{\gamma_k})\}.$$

Now, for $x \in A_j$, let $F_i(x) = \{y \in 2P^k_{\gamma_k} \setminus P^k_{\gamma_k} : 2^{i-1}d(P^k_{\gamma_k}) < |x-y| \leq 2^i d(P^k_{\gamma_k})\}$. Then we have

$$\int_{2P^k_{\gamma_k}\setminus P^k_{\gamma_k}} \left|\frac{x-y}{|x-y|^{1+\alpha}}\right| d\mathcal{H}^\alpha(y) = \sum_{i=j}^{1} \int_{F_i(x)} \left|\frac{x-y}{|x-y|^{1+\alpha}}\right| d\mathcal{H}^\alpha(y)$$

$$\leq \sum_{i=j}^{1} \int_{F_i(x)} \frac{1}{|x-y|^\alpha} d\mathcal{H}^\alpha(y) \leq \sum_{i=j}^{1} \frac{C\left(2^i d(P^k_{\gamma_k})\right)^\alpha}{\left(2^{i-1}d(P^k_{\gamma_k})\right)^\alpha}$$

$$\leq C\sum_{i=j}^{1} 1 \leq C(1+|j|).$$

Summing over j and using the “small boundary” condition stated in property 6 of Theorem 1.6 gives

$$\int_{P^k_{\gamma_k}} \int_{2P^k_{\gamma_k}\setminus P^k_{\gamma_k}} \left|\frac{x-y}{|x-y|^{1+\alpha}}\right| d\mathcal{H}^\alpha(y) d\mathcal{H}^\alpha(x) \leq C \sum_{j=-\infty}^{0} (1+|j|) \int_{A_j} d\mathcal{H}^\alpha(x)$$

$$= C \sum_{j=-\infty}^{0} (1+|j|)\mathcal{H}^\alpha(A_j) \leq C \sum_{j=-\infty}^{0} (1+|j|) b_1 2^{\eta j} \mathcal{H}^\alpha(P^k_{\gamma_k}) \leq C\mathcal{H}^\alpha(P^k_{\gamma_k}).$$

Moreover,

$$\int_{P^k_{\gamma_k}} \int_{2P^k_{\gamma_k}\setminus P^k_{\gamma_k}} \left|\frac{z^k-y}{|z^k-y|^{1+\alpha}}\right| d\mathcal{H}^\alpha(y) d\mathcal{H}^\alpha(x) \leq \frac{\mathcal{H}^\alpha(P^k_{\gamma_k})\mathcal{H}^\alpha(2P^k_{\gamma_k}\setminus P^k_{\gamma_k})}{(cd(P^k_{\gamma_k}))^\alpha} \leq C\mathcal{H}^\alpha(P^k_{\gamma_k}).$$

Therefore we have

$$\int_{2P^k_{\gamma_k}\setminus P^k_{\gamma_k}} \int_{P^k_{\gamma_k}} \left|\frac{z^k-y}{|z^k-y|^{1+\alpha}} - \frac{x-y}{|x-y|^{1+\alpha}}\right| d\mathcal{H}^\alpha(y) \leq C\mathcal{H}^\alpha(P^k_{\gamma_k}),$$

and so we are done with the estimate of the second integral in (1.58) and we get

$$\int_{Q\setminus F^k_{\gamma_k}} \int_{P^k_{\gamma_k}} \left|\frac{z^k-y}{|z^k-y|^{1+\alpha}} - \frac{x-y}{|x-y|^{1+\alpha}}\right| d\mathcal{H}^\alpha(y) \leq C\mathcal{H}^\alpha(P^k_{\gamma_k}).$$

Hence using that $\|h_k\|_\infty \leq C$ for all k, we finally get the L^1-inequality in (1.51). $\square$

Chapter 2

Signed Riesz capacities and Wolff potentials.

2.1 Introduction.

In this paper we study the capacity γ_α associated to the signed vector valued Riesz kernels $k_\alpha(x) = \frac{x}{|x|^{1+\alpha}}$, $0 < \alpha < n$, in $\mathbb{R}^n$. If $K \subset \mathbb{R}^n$ is compact one sets

$$\gamma_\alpha(K) = \sup | < T, 1 > |$$

where the supremum is taken over all distributions T supported on K such that $T * \frac{x_i}{|x|^{1+\alpha}}$ is a function in $L^\infty(\mathbb{R}^n)$ and $\|T * \frac{x_i}{|x|^{1+\alpha}}\|_\infty \leq 1$, for $1 \leq i \leq n$. For $n = 2$ and $\alpha = 1$ this is basically analytic capacity (see [T2]), and for $\alpha = n-1$ and any $n \geq 2$, γ_{n-1} is Lipschitz harmonic capacity (see [Par], [MP] and [V1]).

In [P1] one discovered the fact that if $0 < \alpha < 1$, then a compact set of finite α-dimensional Hausdorff measure has zero γ_α capacity. This is in strong contrast with the situation for integer α, in which α-dimensional smooth hypersurfaces have positive γ_α capacity. The case of non-integer $\alpha > 1$ is not completely understood, although it was shown in [P1] that for Ahlfors-David regular sets the result mentioned above for $0 < \alpha < 1$ still holds.

In this paper we establish the equivalence between γ_α, $0 < \alpha < 1$, and one of the well-known Riesz capacities of non-linear potential theory (see [AH], Chapter 1, p. 38). The Riesz capacity $C_{s,p}$ of a compact set $K \subset \mathbb{R}^n$, $1 < p < \infty$, $0 < sp \leq n$, is defined by

$$C_{s,p}(K) = \inf\{\|\varphi\|_p^p : \ \varphi * \frac{1}{|x|^{n-s}} \geq 1 \ \text{ on } K\},$$

where the infimum is taken over all compactly supported infinitely differentiable functions on $\mathbb{R}^n$. The capacity $C_{s,p}$ can be described by means of Wolff potentials. The Wolff potential of a positive Radon measure μ is defined by

$$W^\mu(x) = W^\mu_{s,p}(x) = \int_0^\infty \left(\frac{\mu(B(x,r))}{r^{n-sp}}\right)^{p'-1} \frac{dr}{r}, \ \ x \in \mathbb{R}^n,$$

where $p' = p/(p-1)$ is the exponent conjugate to p.

The Wolff energy of μ is

$$E(\mu) = E_{s,p}(\mu) = \int_{\mathbb{R}^n} W^\mu(x) d\mu(x).$$

By Wolff's inequality ([AH], Theorem 4.5.4, p.110) one has

$$C^{-1} \sup_\mu \frac{1}{E_{s,p}(\mu)^{p-1}} \leq C_{s,p}(K) \leq C \sup_\mu \frac{1}{E_{s,p}(\mu)^{p-1}},$$

where C is a positive constant depending only on s, p and n, and the supremum is taken over the probability measures μ supported on K.

The main result of this paper is the following.

Theorem. *For each compact set $K \subset \mathbb{R}^n$ and for $0 < \alpha < 1$ we have*

$$C^{-1} C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}(K) \leq \gamma_\alpha(K) \leq C\, C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}(K), \tag{2.1}$$

where C is a positive constant depending only on α and n.

Since it is well-known that $C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}$ vanishes on sets of finite α-dimensional Hausdorff measure (see [AH], Theorem 5.1.9, p.134), the same applies to γ_α. Thus we recover one of the main results of [P1]. On the other hand, $C_{s,p}$ is a subadditive set function (almost by definition, see [AH], p.26), and consequently, γ_α is semiadditive for $0 < \alpha < 1$, that is, given compact sets K_1 and K_2,

$$\gamma_\alpha(K_1 \cup K_2) \leq C \left\{ \gamma_\alpha(K_1) + \gamma_\alpha(K_2) \right\}, \tag{2.2}$$

for some constant C depending only on α and n. In fact γ_α is countably semiadditive. For $\alpha = 1$ and $n = 2$ inequality (2.2) is still true and is a remarkable result obtained in [T2]. For $\alpha = n-1$ and any n (2.2) has been shown very recently in [Vo].

Another interesting consequence of the Theorem is that γ_α is a bilipschitz invariant. This means that if $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a bilipschitz homeomorphism of $\mathbb{R}^n$, namely,

$$L^{-1}|x-y| \leq |\phi(x) - \phi(y)| \leq L|x-y|,$$

for $x,\ y \in \mathbb{R}^n$ and for some constant $L > 0$, then for compact sets K one has

$$C^{-1}\gamma_\alpha(K) \leq \gamma_\alpha(\phi(K)) \leq C\gamma_\alpha(K),$$

where C depends only on L, α and n.

The bilipschitz invariance of the analytic capacity γ has been recently proved by X. Tolsa (see [T4]). The result for a big class of Cantor sets was proved before by Garnett and Verdera (see [GV]).

Our proof of the Theorem rests on two steps. The first one is the analogue for $0 < \alpha < 1$ of the main result in [T2], namely, the equivalence between γ_α and $\gamma_{\alpha,+}$. For a compact set $K \subset \mathbb{R}^n$, the positive γ_α capacity is defined by

$$\gamma_{\alpha,+}(K) = \sup \mu(K)$$

where the supremum is taken over those positive Radon measures μ supported on K such that $\frac{x_i}{|x|^{1+\alpha}} * \mu$ is in $L^\infty(\mathbb{R}^n)$ and $\left\| \frac{x_i}{|x|^{1+\alpha}} * \mu \right\|_\infty \leq 1$, for $1 \leq i \leq n$.

Theorem 2.1. *For each compact set $K \subset \mathbb{R}^n$ and $0 < \alpha < 1$, we have*

$$C^{-1}\gamma_{\alpha,+}(K) \leq \gamma_\alpha(K) \leq C\gamma_{\alpha,+}(K),$$

where C is some positive constant depending only on α and n.

We claim that Theorem 2.1 can be proved by adapting the scheme of the proof of Theorem 1.1 in [T2] and the adjustments introduced in [T3] to prove Theorem 7.1 there. This is explained in some detail in section 2.2.2. When analyzing the argument used in [T2] one realizes that it is based on two main technical ingredients : the non-negativity of the quantity obtained when symmetrizing the kernel, which was proved in [P1] for the Riesz kernel k_α with $0 < \alpha < 1$, and the possibility of localizing the signed $\alpha-$Riesz potential, which is proved in section 3 for $0 < \alpha < n$. When the localization lemma is available then there is no obstruction in adapting Lemma 7.2 (part h)) in [T3]. Once Theorem 2.1 is at our disposal we need to relate $\gamma_{\alpha,+}$ to $C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}$ and this is the second step in the proof of the Theorem, which is discussed in section 4.

The plan of the paper is the following. Section 2 contains some preliminary definitions and results that will be used throughout the article. In section 3 we prove a localization theorem for the signed Riesz potentials. In section 4 we prove the main Theorem.

Constants independent of the relevant parameters are denoted by C and may be different at each occurrence. The notation $A \approx B$ means, as it is usual, that for some constant C one has $C^{-1}B \leq A \leq CB$.

2.2 Preliminaries.

2.2.1 Simmetrization of Riesz kernels.

The symmetrization process for the Cauchy kernel introduced in [Me] has been successfully applied in these last years to many problems of analytic capacity and L^2 boundedness of the Cauchy integral operator (see [MV], [MMV] for example; the survey [D3] and the book [Pa2] contain many other interesting references). Given 3 distinct points in the plane, z_1, z_2 and z_3, one finds out, by an elementary computation that

$$c(z_1, z_2, z_3)^2 = \sum_\sigma \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}} \tag{2.3}$$

where the sum is taken over the six permutations of the set $\{1,2,3\}$ and $c(z_1,z_2,z_3)$ is Menger curvature, that is, the inverse of the radius of the circle through z_1, z_2 and z_3. In particular (2.3) shows that the sum on the right hand side is a non-negative quantity.

It can be shown that for $0<\alpha<1$ the symmetrization of the Riesz kernel $k_\alpha(x) = x/|x|^{1+\alpha}$, gives also a positive quantity. On the other hand, for $1<\alpha<n$, the phenomenon of change of signs appears when symmetrizing the kernel k_α, as one can easily check.

For $0<\alpha<n$ the quantity

$$\sum_\sigma \frac{x_{\sigma(2)}-x_{\sigma(1)}}{|x_{\sigma(2)}-x_{\sigma(1)}|^{1+\alpha}} \frac{x_{\sigma(3)}-x_{\sigma(1)}}{|x_{\sigma(3)}-x_{\sigma(1)}|^{1+\alpha}}, \tag{2.4}$$

where the sum is taken over the six permutations of the set $\{1,2,3\}$, is the obvious analogue of the right hand side of (2.3) for the Riesz kernel k_α. Notice that (2.4) is exactly

$$2\, p_\alpha(x_1,x_2,x_3),$$

where $p_\alpha(x_1,x_2,x_3)$ is defined as the sum in (2.4) taken only on the three permutations $(1,2,3),\ (2,3,1),\ (3,1,2)$.

In the following lemma we state the explicit description that was found in [P1] for the symmetrization of the Riesz kernel k_α, for $0<\alpha<1$.

Lemma 2.2. *Let $0<\alpha<1$, and $x_1,\ x_2,\ x_3$ three distinct points in $\mathbb{R}^n$. Then we have*

$$\frac{2-2^\alpha}{L(x_1,x_2,x_3)^{2\alpha}} \le p_\alpha(x_1,x_2,x_3) \le \frac{2^{1+\alpha}}{L(x_1,x_2,x_3)^{2\alpha}},$$

where $L(x_1,x_2,x_3)$ is the largest side of the triangle determined by $x_1,\ x_2$ and x_3. In particular $p_\alpha(x_1,x_2,x_3)$ is a positive quantity.

The relationship between the quantity $p_\alpha(x,y,z)$ and the L^2 estimates of the operator with kernel k_α is as follows. Take a positive finite Radon measure μ in $\mathbb{R}^n$ which satisfies the growth condition $\mu(B(x,r))\le r^\alpha$, $x\in\mathbb{R}^n$, $r>0$. Given $\varepsilon>0$, set

$$R_{\alpha,\varepsilon}(\mu)(x) = \int_{|y-x|>\varepsilon} k_\alpha(y-x)d\mu(y).$$

Then (see in [MV] or [Pa2] the argument for $\alpha=1$)

$$\left|\int |R_{\alpha,\varepsilon}(\mu)(x)|^2\, d\mu(x) - \frac{1}{3}p_{\alpha,\varepsilon}(\mu)\right| \le C\|\mu\|,$$

where C is a constant depending only on α and n, and

$$p_{\alpha,\varepsilon}(\mu) = \iiint_{S_\varepsilon} p_\alpha(x,y,z)d\mu(x)d\mu(y)d\mu(z),$$

with

$$S_\varepsilon = \{(x, y, z) : \ |x - y| > \varepsilon, \ |x - z| > \varepsilon \text{ and } |y - z| > \varepsilon\}.$$

Thus

$$p_\alpha(\mu) \le 3 \sup_\varepsilon \int |R_{\alpha,\varepsilon}(\mu)(x)|^2 \, d\mu(x) + C\|\mu\|, \tag{2.5}$$

where

$$p_\alpha(\mu) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p_\alpha(x, y, z) d\mu(x) d\mu(y) d\mu(z).$$

2.2.2 The scheme of the proof of Theorem 2.1.

In this section we give an outline of the arguments involved in the proof of Theorem 2.1. The proof uses an induction argument on scales, analogous to the one in [MTV] and [T2]. The main idea is to show, by induction, that

$$\gamma_{\alpha,+}(E \cap Q) \approx \gamma_\alpha(E \cap Q)$$

for squares Q of any size.

The starting point in the proof of Theorem 1.1 in [T2] is the construction of a special family of cubes $\{Q_j\}_{j=1}^N$ that cover E and satisfy

$$\gamma_{\alpha,+}(\cup_{j=1}^N Q_j) \le C\gamma_{\alpha,+}(E)$$

and

$$\sum_{j=1}^N \gamma_{\alpha,+}(3Q_j \cap E) \le C\gamma_{\alpha,+}(E).$$

The construction of these cubes works without difficulty in the same way as in [T2] for $0 < \alpha < 1$, because we have non-negativity of the quantity obtained when symmetrizing the Riesz kernel (see Lemma 2.2 above).

From the definition of the capacity γ_α, it follows that there exists a distribution T_0 supported on E such that

1. $\gamma_\alpha(E) \ge \dfrac{1}{2} |\langle T_0, 1 \rangle|$,

2. $\|T_0 * \dfrac{x_i}{|x|^{1+\alpha}}\|_\infty \le 1, \ \ 1 \le i \le n.$

Consider now a family of infinitely differentiable functions $\{\varphi_j\}_{j=1}^N$ such that each φ_j is compactly supported on $2Q_j$, $0 \leq \varphi_j \leq 1$, $\|\partial^s \varphi_j\|_\infty \leq \dfrac{C}{\ell(Q_j)^{|s|}}$, $0 \leq |s| \leq n$, and $\sum_{j=1}^N \varphi_j = 1$ on $\cup_{j=1}^N Q_j$. At this point we need an inequality of the type

$$\|\varphi_j T_0 * \frac{x_i}{|x|^{1+\alpha}}\|_\infty \leq C$$

for $1 \leq i \leq n$, $1 \leq j \leq N$ and $0 < \alpha < n$. This will be proved in section 2.3. Then, by definition of γ_α , we will obtain that

$$|\langle \varphi_j T_0, 1\rangle| \leq C\gamma_\alpha(2Q_j \cap E). \tag{2.6}$$

for $1 \leq j \leq N$.

Inequality (2.6) is used later on in the proof in order to construct a bounded function b to which a suitable variant of the $T(b)$ theorem will be applied. There is still one more difficulty in applying the Nazarov, Treil and Volberg $T(b)$-type theorem one needs, namely, finding a substitute for what they call the suppressed operators. It was already explained in [P1] that there are at least two versions of such operators for the Riesz kernels that work appropriately.

2.3 Localization of Riesz potentials.

One of the ingredients of the proof of Theorem 1.1 in [T2] is the localization of the Cauchy potential. The localization method for the Cauchy potential, $T * 1/z$, developed by A.G. Vitushkin for rational approximation was adapted in [Par] to localize the potential $T * x/|x|^n$ and used in problems of $\mathcal{C}^1$-harmonic approximation.
In this section we will be concerned with the localization of the vector valued $\alpha-$Riesz potentials $T * x/|x|^{1+\alpha}$, $0 < \alpha < 1$.

Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$. For $s = (s_1, ..., s_n)$, $0 \leq s_i \in \mathbb{Z}$, we set $x^s = x_1^{s_1} \cdots\ x_n^{s_n}$, $s! = s_1! \cdots\ s_n!$, $|s| = s_1 + s_2 + \cdots + s_n$, $\partial^s = \partial^{s_1}/\partial x_1^{s_1} \cdots\ \partial^{s_n}/\partial x_n^{s_n}$, $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ and $\partial_j = \partial/\partial x_j$, $1 \leq j \leq n$. In what follows, given a cube $Q \subset \mathbb{R}^n$, φ_Q will denote an infinitely differentiable function supported on $2Q$ and such that $\|\partial^s \varphi_Q\|_\infty \leq C_s \ell(Q)^{-|s|}$, $0 \leq |s| \leq n$.

We state now the following localization lemma.

Lemma 2.3. *Let $0 < \alpha < 1$ and let T be a compactly supported distribution such that $T * \frac{x_i}{|x|^{1+\alpha}}$ is a bounded measurable function for $1 \leq i \leq n$. Then there exists some constant $C = C(n, \alpha) > 0$ such that*

$$\|\varphi_Q T * \frac{x_i}{|x|^{1+\alpha}}\|_\infty \leq C \sup_{1\leq i\leq n} \|T * \frac{x_i}{|x|^{1+\alpha}}\|_\infty.$$

Proof. Set $K^i(y) = \dfrac{y_i}{|y|^{1+\alpha}}$ and for some fixed point x, set

$$K_x^i(y) = \frac{x_i - y_i}{|x-y|^{1+\alpha}}.$$

We assume first that n is odd and of the form $n = 2k+1$. We distinguish two cases:

Case 1: $x \in (3Q)^c$. Set $g(y) = \varphi_Q(y) K_x^i(y)$. Lemma 11 in [P1] (see Lemma 1.11 in the first Chapter of this dissertation) tells us that

$$g(x) = c_{n,\alpha} \sum_{j=1}^{n} \left(\Delta^k \partial_j g * \frac{1}{|y|^{n-\alpha}} * K^j \right)(x), \tag{2.7}$$

for some constant $c_{n,\alpha}$ depending only on n and α. Thus

$$\left(\varphi_Q T * K^i\right)(x) =< T, \varphi_Q K_x^i >= c_{n,\alpha} \sum_{j=1}^{n} < T * K^j, \Delta^k \partial_j g * \frac{1}{|y|^{n-\alpha}} >,$$

and so

$$\begin{aligned} \left(\varphi_Q T * K^i\right)(x) &= \sum_{j=1}^{n} c_{n,\alpha} \int_{(3Q)^c} (T * K^j)(z) \left(\Delta^k \partial_j g * \frac{1}{|y|^{n-\alpha}} \right)(z) dz \\ &+ \sum_{j=1}^{n} c_{n,\alpha} \int_{3Q} (T * K^j)(z) \left(\Delta^k \partial_j g * \frac{1}{|y|^{n-\alpha}} \right)(z) dz = A + B. \end{aligned} \tag{2.8}$$

To deal with A we use that $T * K^j$ is a bounded function. Notice that for $x \in (3Q)^c$ and $y \in 2Q$ we have

$$|g(y)| \le \frac{C \|\varphi_Q\|_\infty}{\ell(Q)^\alpha}.$$

Let Q_0 stand for the unit cube centered at 0. Moving $\Delta^k \partial_j$ from g to $\frac{1}{|y|^{n-\alpha}}$ and making the obvious change of variables one gets

$$\begin{aligned} |A| &\le C \sup_{1 \le i \le n} \|T * K^i\|_\infty \frac{\|\varphi_Q\|_\infty}{l(Q)^\alpha} \int_{(3Q)^c} \int_{2Q} \frac{dydz}{|z-y|^{2n-\alpha}} \\ &\le C \sup_{1 \le i \le n} \|T * K^i\|_\infty \int_{(3Q_0)^c} \int_{2Q_0} \frac{dydz}{|z-y|^{2n-\alpha}} \le C \sup_{1 \le i \le n} \|T * K^i\|_\infty. \end{aligned}$$

Let's now turn our attention to B. Recall that we have

$$\Delta^k(hg) = \sum_{i_1,\dots,i_k=1}^{n} \sum_{l_1,\dots,l_k=0}^{2} \binom{2}{l_1} \cdots \binom{2}{l_k} \partial^{l_1\dots l_k}_{i_1\dots i_k} h \, \partial^{2-l_1\dots 2-l_k}_{i_1\dots i_k} g, \tag{2.9}$$

where $\partial^{l_1\dots l_k}_{i_1\dots i_k} = (\partial_{i_1})^{l_1}\dots(\partial_{i_k})^{l_k}$.

Since

$$\Delta^k(\partial_j g) = \Delta^k\left(K^i_x \, \partial_j \varphi_Q\right) + \Delta^k\left(\varphi_Q \, \partial_j K^i_x\right),$$

we have

$$\begin{aligned} B &\le C \sup_{1\le i\le n} \|T * K^i\|_\infty \int_{3Q} \left| \left(\Delta^k \left(K^i_x \, \partial_j \varphi_Q\right) * \frac{1}{|y|^{n-\alpha}} \right)(z) \right| dz \\ &+ C \sup_{1\le i\le n} \|T * K^i\|_\infty \int_{3Q} \left| \left(\Delta^k \left(\varphi_Q \, \partial_j K^i_x\right) * \frac{1}{|y|^{n-\alpha}} \right)(z) \right| dz \\ &= C \sup_{1\le i\le n} \|T * \frac{x_i}{|x|^{1+\alpha}}\|_\infty \left(B_1 + B_2\right). \end{aligned} \tag{2.10}$$

Then using (2.9), $\|\partial^s \varphi_Q\|_\infty \le C_s \ell(Q)^{-|s|}$, $|s| \ge 0$, the fact that $|x-y| \ge \ell(Q)$, $y \in 2Q$, and changing variables, we get

$$\begin{aligned} B_1 &\le \sum_{i_1,\dots,i_k=1}^{n} \sum_{l_1,\dots,l_k=0}^{2} \frac{C}{\ell(Q)^{l_1+\dots+l_k+1}} \int_{3Q}\int_{2Q} \frac{dz\,dy}{|z-y|^{n-\alpha}|x-y|^{\alpha+2-l_1+\dots+2-l_k}} \\ &\le \frac{C}{\ell(Q)^{n+\alpha}} \int_{3Q}\int_{2Q} \frac{dz\,dy}{|z-y|^{n-\alpha}} = \frac{C\ell(Q)^{2n}}{\ell(Q)^{n+\alpha+n-\alpha}} \int_{3Q_0}\int_{2Q_0} \frac{dz\,dy}{|z-y|^{n-\alpha}} \\ &\le C. \end{aligned}$$

Arguing similarly we obtain $B_2 \le C$ and hence we conclude that

$$A + B \le C \sup_{1\le i\le n} \|T * K^i\|_\infty.$$

Case 2: $x \in 3Q$. Without loss of generality assume $x = 0$. Now the function $g(y) = -\varphi_Q(y)K^i(y)$ may not be smooth, but (2.7) still holds in the distributions sense. Writing $f(z) = \left(T * \frac{1}{|x|^{1-\alpha}}\right)(z)$ one gets

$$
\begin{aligned}
&\left(\varphi_Q T * K^i\right)(0) = < T, g >= c_{n,\alpha} \sum_{j=1}^{n} < T * K^j, \Delta^k \partial_j g * \frac{1}{|y|^{n-\alpha}} > \\
&= C \sum_{j=1}^{n} < \partial_j \left(f - f(0)\right), \Delta^k \partial_j g * \frac{1}{|y|^{n-\alpha}} >
\end{aligned}
$$

We claim now that integrating by parts gives

$$
\begin{aligned}
&\sum_{j=1}^{n} < \partial_j \left(f - f(0)\right), \Delta^k \partial_j g * \frac{1}{|y|^{n-\alpha}} > \\
&=< f - f(0), \Delta^{k+1} g * \frac{1}{|y|^{n-\alpha}} > + O\left(\sup_{1\le i\le n} \|T * K^i\|_\infty\right).
\end{aligned}
\tag{2.11}
$$

We postpone the proof of (2.11) and we continue with the argument. If (2.11) holds, then we can write

$$
\begin{aligned}
&\left|\left(\varphi_Q T * K^i\right)(0)\right| \le C \left|\int_{(3Q)^c} \left(f(z) - f(0)\right)\left(\Delta^{k+1} g * \frac{1}{|y|^{n-\alpha}}\right)(z) dz\right| \\
&+ C \left|\int_{3Q} \left(f(z) - f(0)\right)\left(\Delta^{k+1} g * \frac{1}{|y|^{n-\alpha}}\right)(z) dz\right| + C \sup_{1\le i\le n} \|T * K^i\|_\infty.
\end{aligned}
$$

Set

$$
A = \int_{(3Q)^c} \left(f(z) - f(0)\right)\left(\Delta^{k+1} g * \frac{1}{|y|^{n-\alpha}}\right)(z) dz
$$

and

$$
B = \int_{3Q} \left(f(z) - f(0)\right)\left(\Delta^{k+1} g * \frac{1}{|y|^{n-\alpha}}\right)(z) dz.
$$

Using the boundedness of the function $T*K^j = \partial_j f$, Fubini and changing variables we obtain

$$
\begin{aligned}
|A| &\le C \sup_{1\le i\le n} \|T * K^i\|_\infty \sum_{j=1}^n \int_{(3Q)^c} |z| \int_{2Q} \frac{|g(y)|}{|z-y|^{2n+1-\alpha}} dydz \\
&\le C \sup_{1\le i\le n} \|T * K^i\|_\infty \|\varphi_Q\|_\infty \sum_{j=1}^n \int_{(3Q)^c} \int_{2Q} \frac{|z-y|+|y|}{|y|^\alpha |z-y|^{2n+1-\alpha}} dydz \\
&\le C \sup_{1\le i\le n} \|T * K^i\|_\infty \sum_{j=1}^n \int_{2Q} \frac{1}{|y|^\alpha} \int_{(3Q)^c} \frac{dz}{|z-y|^{2n-\alpha}} dy \\
&+ C \sup_{1\le i\le n} \|T * K^i\|_\infty \sum_{j=1}^n \ell(Q) \int_{2Q} \frac{1}{|y|^\alpha} \int_{(3Q)^c} \frac{dz}{|z-y|^{2n+1-\alpha}} dy \\
&\le C \sup_{1\le i\le n} \|T * K^i\|_\infty .
\end{aligned}
$$

For the term B, write

$$
\begin{aligned}
|B| &\le \left| \int_{3Q} (f(z)-f(0)) \left(\Delta^{k+1} g * \frac{1}{|y|^{n-\alpha}} \right)(z) dz \right| \\
&\le C \left| \int_{3Q} \sum_{|r|+|s|=n+1} (f(z)-f(0)) \left(\left(\partial^r \varphi_Q \partial^s K^i\right) * \frac{1}{|y|^{n-\alpha}} \right)(z) dz \right| ,
\end{aligned}
$$

where the last sum is over those multi-indexes r and s that appear in distributing between φ_Q and K^i the $n+1$ derivatives coming from Δ^{k+1}. We will now divide the above sum in two parts, the first one containing the indexes $|r| \ge 2$ and the second one the rest of them. In order to be able to estimate the integral of this second part, which is the worse, we will have to subtract a Taylor polinomial of φ_Q of order one. Let

$$
R(y) = \varphi_Q(y) - \sum_{|m|=0}^{1} \partial^m \varphi_Q(0) y^m .
$$

Then

$$
\begin{aligned}
|B| &\leq C \sum_{|r|\geq 2} \int_{3Q} |f(z)-f(0)| \int_{2Q} \frac{dydz}{\ell(Q)^{|r|}|y|^{\alpha+n+1-|r|}|z-y|^{n-\alpha}} \\
&+C \int_{3Q} |f(z)-f(0)| \left| \sum_{\substack{|r|+|s|=n+1 \\ |r|\leq 1}} \int \frac{\partial^r R(y) \partial^s K^i(y)}{|z-y|^{n-\alpha}} dy \right| dz \\
&+C \left| \int_{3Q} (f(z)-f(0)) \sum_{|m|=0}^{1} \partial^m \varphi(0) \left(y^m \Delta^{k+1} K^i * \frac{1}{|y|^{n-\alpha}} \right)(z) dz \right| \qquad (2.12) \\
&+C \sup_{|m|=1} |\partial^m \varphi_Q(0)| \left| \int_{3Q} (f(z)-f(0)) \sum_{|s|=n} \left(\partial^s K^i * \frac{1}{|y|^{n-\alpha}} \right)(z) dz \right| \\
&= B_1 + B_2 + B_3 + B_4.
\end{aligned}
$$

Notice that if $|r| \geq 2$, since $0 < \alpha < 1$, we have $\alpha + n + 1 - |r| \leq \alpha + n - 1 < n$. Hence using the boundedness of the function $T * K^i$, $1 \leq i \leq n$, B_1 is finite and by homogeneity independent of $\ell(Q)$. Thus,

$$
B_1 \leq C \sup_{1\leq i\leq n} \|T * K^i\|_\infty.
$$

We deal now with B_2. Write

$$
\begin{aligned}
B_2 &= C \int_{3Q} |f(z)-f(0)| \left| \sum_{\substack{|r|+|s|=n+1 \\ |r|\leq 1}} \int_{4Q} \frac{\partial^r R(y) \partial^s K^i(y)}{|z-y|^{n-\alpha}} dy \right| dz \\
&+C \int_{3Q} |f(z)-f(0)| \left| \sum_{\substack{|r|+|s|=n+1 \\ |r|\leq 1}} \int_{(4Q)^c} \frac{\partial^r R(y) \partial^s K^i(y)}{|z-y|^{n-\alpha}} dy \right| dz = B_{21} + B_{22}.
\end{aligned}
$$

For the integral over $4Q$, we have to use the Taylor expansion to get integrability. Estimating first the term with $|r| = 1$ and then the term with $|r| = 0$ we get

$$B_{21} \leq C \sup_{1\leq i\leq n} \|T * K^i\|_\infty \ell(Q) \int_{3Q}\int_{4Q} \frac{|y|}{\ell(Q)^2|y|^{\alpha+n}|z-y|^{n-\alpha}} dydz$$

$$+C \sup_{1\leq i\leq n} \|T * K^i\|_\infty \ell(Q) \int_{3Q}\int_{4Q} \frac{|y|^2}{\ell(Q)^2|y|^{\alpha+n+1}|z-y|^{n-\alpha}} dydz.$$

Then by homogeneity,

$$B_{21} \leq C \sup_{1\leq i\leq n} \|T * K^i\|_\infty.$$

For the integral over $(4Q)^c$, we do not apply Taylor; we just estimate term by term. For $|r|=0$ we have that

$$\left|R(y)\partial^s K^i(y)\right| \leq \frac{C|y|}{\ell(Q)|y|^{n+\alpha+1}} = \frac{C}{\ell(Q)|y|^{\alpha+n}}.$$

For $|r|=1$ the term $|\partial^r R(y)\partial^s K^i(y)|$ can be estimated by $C\ell(Q)^{-1}|y|^{-\alpha-n}$, because now $|s|=n$. Therefore

$$B_{22} \leq C \sup_{1\leq i\leq n} \|T * K^i\|_\infty \ell(Q) \int_{3Q}\int_{(4Q)^c} \frac{dy}{\ell(Q)|y|^{\alpha+n}|z-y|^{n-\alpha}} dz$$

$$\leq C \sup_{1\leq i\leq n} \|T * K^i\|_\infty.$$

For B_3, separate the terms according to whether $|m|=0$ of $|m|=1$ as follows:

$$B_3 = \left|\int_{3Q} (f(z)-f(0))\,\varphi_Q(0)\left(\Delta^{k+1}K^i * \frac{1}{|y|^{n-\alpha}}\right)(z)dz\right|$$

$$+\left|\int_{3Q} (f(z)-f(0)) \sum_{|m|=1} \partial^m\varphi_Q(0)\left(y^m\Delta^{k+1}K^i * \frac{1}{|y|^{n-\alpha}}\right)(z)dz\right|$$

$$= B_{31} + B_{32}.$$

Now we treat the term B_{31}. Taking Fourier transforms on the convolution $\Delta^{k+1}K^i * \frac{1}{|y|^{n-\alpha}}$ we obtain for an appropriate constant C,

$$\left(\Delta^{k+1}\widehat{K^i * \frac{1}{|y|^{n-\alpha}}}\right)(\xi) = C\xi_i.$$

Thus

$$\left(\Delta^{k+1}K^i * \frac{1}{|y|^{n-\alpha}}\right)(z) = C\partial_i\delta(z).$$

Hence,

$$B_{31} \leq C\|\varphi_Q\|_\infty \sup_{1\leq i\leq n}\|T * K^i\|_\infty.$$

For B_{32}, we take also Fourier transforms on $y^m\Delta^{k+1}K^i * \frac{1}{|y|^{n-\alpha}}$, $|m| = 1$. Then

$$\widehat{\left(y^m\Delta^{k+1}K^i * \frac{1}{|y|^{n-\alpha}}\right)}(\xi) = C\partial^m\left(\frac{|\xi|^{2k+2}\xi_i}{|\xi|^{n+1-\alpha}}\right)\frac{1}{|\xi|^\alpha} = C\delta_{m,m_i} + C\frac{\xi^m\xi_i}{|\xi|^2}.$$

where m_i is the multi-index with all entries equal to 0 except the $i-$th entry which is 1; δ_{m,m_i} equals one when $m = m_i$ and zero otherwise. Hence

$$\left(y^m\Delta^{k+1}K^i * \frac{1}{|y|^{n-\alpha}}\right)(z) = C\delta_{m,m_i}\delta + C\frac{z^m z_i}{|z|^{n+2}}$$

and since $|m| = 1$,

$$B_{32} \leq \frac{C}{\ell(Q)}\sup_{1\leq i\leq n}\|T * K^i\|_\infty \int_{3Q}\frac{dz}{|z|^{n-1}} \leq C\sup_{\leq i\leq n}\|T * K^i\|_\infty.$$

Now we are left with term B_4. Taking Fourier transforms on the convolution $\partial^s K^i * \frac{1}{|y|^{n-\alpha}}$, we obtain

$$\widehat{\left(\partial^s K^i * \frac{1}{|y|^{n-\alpha}}\right)}(\xi) = C\xi^s\frac{\xi_i}{|\xi|^{n+1-\alpha}}\frac{1}{|\xi|^\alpha} = C\frac{\xi^s\xi_i}{|\xi|^{n+1}}.$$

Hence

$$\left(\partial^s K^i * \frac{1}{|y|^{n-\alpha}}\right)(z) = C\frac{z^s z_i}{|z|^{2|s|+1}}.$$

Recall that in B_4 we had $|s| = n$. Thus by homogeneity

$$B_4 \leq C\sup_{1\leq i\leq n}\|T * K^i\|_\infty\frac{1}{\ell(Q)}\int_{3Q}\frac{dz}{|z|^{n-1}} \leq C\sup_{1\leq i\leq n}\|T * K^i\|_\infty.$$

We still have to show claim (2.11). Let $1 \leq j \leq n$ and put

$$\omega_j = (-1)^{j-1}dy_1 \wedge \cdots \wedge \widehat{dy_j} \wedge \cdots \wedge dy_n.$$

Then Stokes Theorem gives

$$\sum_{j=1}^{n} < \partial_j \left(f - f(0) \right), \Delta^k \partial_j g * \frac{1}{|y|^{n-\alpha}} >$$

$$= - < f - f(0), \Delta^{k+1} g * \frac{1}{|y|^{n-\alpha}} >$$

$$+ \sum_{j=1}^{n} \lim_{\varepsilon \to 0} \int_{|y|=\varepsilon^{-1}} \left(f(y) - f(0) \right) \left(g * \Delta^k \partial_j \frac{1}{|y|^{n-\alpha}} \right)(y)\, \omega_j$$

$$- \sum_{j=1}^{n} \lim_{\varepsilon \to 0} \int_{|y|=\varepsilon} \left(f(y) - f(0) \right) \left(\Delta^k \partial_j g * \frac{1}{|y|^{n-\alpha}} \right)(y)\, \omega_j.$$

The first integral converges to 0 when $\varepsilon \to 0$ as $\varepsilon^{n-\alpha}$, thus we are only left with the second one. For $1 \leq j \leq n$ and for a suitable constant C we can write (recall that for some constant C depending on n and α, $\frac{C}{|y|^{n-\alpha}} = \Delta \left(\frac{1}{|y|^{n-\alpha-2}} \right)$)

$$\int_{|y|=\varepsilon} \left(f(y) - f(0) \right) \left(\Delta^k \partial_j g * \frac{1}{|y|^{n-\alpha}} \right)(y)\, \omega_j$$

$$= C \sum_{l=0}^{n} \int_{|y|=\varepsilon} \left(f(y) - f(0) \right) \left(\Delta^k \partial_j \partial_l g * \frac{y_l}{|y|^{n-\alpha}} \right)(y)\, \omega_j.$$

Notice that when looking at the above integral, the worst case one has is when all the derivatives $\Delta^k \partial_j \partial_l$ of the product $g = -\varphi_Q K^i$ are on the kernel K^i. We will only be concerned with this case. For the other cases argue like in (2.12). Recall that $R(y) = \varphi_Q(y) - \sum_{|m|=0}^{1} \partial^m \varphi_Q(0) y^m$. To get integrability we will have to use this Taylor expansion. Then for $1 \leq j \leq n$ we have

$$\int_{|y|=\varepsilon}(f(y)-f(0))\left(\sum_l \varphi_Q\Delta^k\partial_j\partial_l K^i * \frac{y_l}{|y|^{n-\alpha}}\right)(y)\,\omega_j$$

$$= \int_{|y|=\varepsilon}(f(y)-f(0))\left(\sum_l R\,\Delta^k\partial_j\partial_l K^i * \frac{y_l}{|y|^{n-\alpha}}\right)(y)\,\omega_j$$

$$+\varphi_Q(0)\int_{|y|=\varepsilon}(f(y)-f(0))\left(\sum_l \Delta^k\partial_j\partial_l K^i * \frac{y_l}{|y|^{n-\alpha}}\right)(y)\,\omega_j$$

$$+\sum_{|m|=1}\partial^m\varphi_Q(0)\int_{|y|=\varepsilon}(f(y)-f(0))\sum_l\left(y^m\Delta^k\partial_j\partial_l K^i * \frac{y_l}{|y|^{n-\alpha}}\right)(y)\,\omega_j$$

$$= A_1 + A_2 + A_3.$$

We will now show that A_1 and A_3 converge to zero when $\varepsilon\to 0$ and that A_2 is bounded above by $C\sup_{1\le i\le n}\|T*K^i\|_\infty$.

$$A_1 = \int_{|y|=\varepsilon}(f(y)-f(0))\int_{3Q}R(z)\sum_l \Delta^k\partial_j\partial_l K^i(z)\frac{y_l-z_l}{|y-z|^{n-\alpha}}dz\,\omega_j +$$

$$+\int_{|y|=\varepsilon}(f(y)-f(0))\int_{(3Q)^c}R(z)\sum_l \Delta^k\partial_j\partial_l K^i(z)\frac{y_l-z_l}{|y-z|^{n-\alpha}}dz\,\omega_j$$

$$= A_{11} + A_{12}.$$

We deal first with A_{11}. Notice that the Taylor expansion appearing in R kills part of the singularity of $\Delta^k\partial_j\partial_l K^i$ and makes the product $R\,\Delta^k\partial_j\partial K^i$ a locally integrable function. Thus using the boundedness of $T*K^j$ we get

$$|A_{11}| \le C\varepsilon\int_{|y|=\varepsilon}\int_{3Q}\frac{dz}{|z|^{n-1+\alpha}|z-y|^{n-1-\alpha}}|\,\omega_j| \to 0,\ \text{when } \varepsilon\to 0.$$

Moreover,

$$|A_{12}| \le C\varepsilon\int_{|y|=\varepsilon}\int_{(3Q)^c}\frac{dz}{|z|^{n+\alpha}|z-y|^{n-1-\alpha}}|\omega_j| \to 0,\ \text{when } \varepsilon\to 0.$$

Thus A_1 tends to zero with ε.

To estimate A_2, take Fourier transforms on $\sum_l \Delta^k\partial_j\partial_l K^i * \frac{y_l}{|y|^{n-\alpha}}$. Then for an appropriate constant C one has

$$\left(\widehat{\sum_l \Delta^k \partial_j \partial_l K^i * \frac{y_l}{|y|^{n-\alpha}}}\right)(\xi) = C \sum_l |\xi|^{2k} \xi_j \xi_l \frac{\xi_i}{|\xi|^{n+1-\alpha}} \frac{\xi_l}{|\xi|^{2+\alpha}} = C \frac{\xi_i \xi_j}{|\xi|^2}.$$

Thus

$$\sum_l \left(\Delta^k \partial_j \partial_l K^i * \frac{y_l}{|y|^{n-\alpha}}\right)(y) = C \frac{y_i y_j}{|y|^{n+2}}.$$

Hence

$$\begin{aligned} |A_2| &= \left| C\varphi_Q(0) \int_{|y|=\varepsilon} (f(y) - f(0)) \frac{y_i y_j}{|y|^{n+2}} \omega_j \right| \\ &\leq C \sup_{1\leq i\leq n} \|T * K^i\|_\infty \varepsilon^{1-n} \int_{|y|=\varepsilon} |\omega_j| = C \sup_{1\leq i\leq n} \|T * K^i\|_\infty \end{aligned}$$

For the last term, taking Fourier transform of $\sum_l y^m \Delta^k \partial_j \partial_l K^i * \frac{y_l}{|y|^{n-\alpha}}$, we get for a suitable constant C

$$\begin{aligned} &\left(\widehat{\sum_l y^m \Delta^k \partial_j \partial_l K^i * \frac{y_l}{|y|^{n-\alpha}}}\right)(\xi) \\ &= C \sum_l \partial^m \left(\frac{|\xi|^{2k} \xi_j \xi_l \xi_i}{|\xi|^{n+1-\alpha}}\right) \frac{\xi_l}{|\xi|^{\alpha+2}} = C \sum_l \partial^m \left(\frac{\xi_j \xi_l \xi_i}{|\xi|^{2-\alpha}}\right) \frac{\xi_l}{|\xi|^{\alpha+2}} \\ &= C \sum_l \left(\delta_{m,m_j} \frac{\xi_i \xi_l}{|\xi|^{2-\alpha}} + \delta_{m,m_i} \frac{\xi_j \xi_l}{|\xi|^{2-\alpha}} + \delta_{m,m_l} \frac{\xi_i \xi_j}{|\xi|^{2-\alpha}} + \frac{\xi_j \xi_i \xi_l \xi^m}{|\xi|^{4-\alpha}}\right) \frac{\xi_l}{|\xi|^{\alpha+2}} \\ &= C \left(\delta_{m,m_j} \frac{\xi_i}{|\xi|^2} + \delta_{m,m_i} \frac{\xi_j}{|\xi|^2} + \frac{\xi^m \xi_j \xi_i}{|\xi|^4} + \sum_l \delta_{m,m_l} \frac{\xi_i \xi_j \xi_l}{|\xi|^4}\right). \end{aligned}$$

Hence

$$\begin{aligned} &\left(\sum_l y^m \Delta^k \partial_j \partial_l K^i * \frac{y_l}{|y|^{n-\alpha}}\right)(y) \\ &= C \left(\delta_{m,m_j} \frac{y_i}{|y|^n} + \delta_{m,m_i} \frac{y_j}{|y|^n} + \frac{y^m y_j y_i}{|y|^{n+2}} + \sum_l \delta_{m,m_l} \frac{y_i y_j y_l}{|y|^{n+2}}\right) \end{aligned}$$

and since $|m| = 1$,

$$|A_3| \leq C \int_{|y|=\varepsilon} \frac{|f(y) - f(0)|}{|y|^{n-1}} |\omega_j| \leq C\varepsilon^{2-n} \int_{|y|=\varepsilon} |\omega_j| = C\varepsilon \to 0, \text{ when } \varepsilon \to 0,$$

which proves claim (2.11).

For even n one argues similarly using the corresponding formula in Lemma 11 in [P1] (see Lemma 1.11 in the first Chapter of this dissertation). $\square$

Remark. This localization Lemma holds also more generally for $0 < \alpha < n$. The first case in the above proof, namely $x \in (3Q)^c$, applies to any $0 < \alpha < n$. For the second case, $x \in 3Q$, the proof works similarly modifying appropriately the representation formula appearing in Lemma 11 in [P1] (see Lemma 1.11 in the first Chapter of this dissertation).

2.4 Proof of the Theorem.

Let μ be a positive Radon measure and $0 < \alpha < 1$. For $x \in \mathbb{R}^n$, set

$$p^2_\alpha(\mu)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p_\alpha(x, y, z) d\mu(y) d\mu(z),$$

$$M_\alpha \mu(x) = \sup_{r>0} \frac{\mu(B(x, r))}{r^\alpha}$$

and

$$U^\mu_\alpha(x) = M_\alpha \mu(x) + p_\alpha(\mu)(x).$$

Observe that $p_\alpha(\mu) = \int_{\mathbb{R}^n} p^2_\alpha(\mu)(x) d\mu(x)$. U^μ_α is the analogue of the potential introduced in [V4]. The energy associated to this potential is

$$E_\alpha(\mu) = \int_{\mathbb{R}^n} U^\mu_\alpha(x) d\mu(x).$$

Lemma 2.4. *For each compact set $K \subset \mathbb{R}^n$ and $0 < \alpha < 1$ we have*

$$\gamma_{\alpha,+}(K) \approx \sup_\nu \frac{1}{E_\alpha(\nu)},$$

where the supremum is taken over the probability measures ν supported on K.

Proof . Take a positive Radon measure μ supported on K such that $\left| \left(\frac{x_i}{|x|^{1+\alpha}} * \mu \right)(x) \right| \leq 1$ for almost all $x \in \mathbb{R}^n$, $1 \leq i \leq n$. We claim that

$$\mu(B(x, r)) \leq C r^\alpha, \quad x \in \mathbb{R}^n, \quad r > 0.$$

To prove the claim take an infinitely differentiable function φ, supported on $B(x, 2r)$ such that $\varphi = 1$ on $B(x, r)$, and $\|\partial^s \varphi\|_\infty \leq C_s r^{-|s|}$, $|s| \geq 0$. Assume first that n is odd and of the form $n = 2k+1$. Then, by Lemma 11 in [P1] (Lemma 1.11 in the first Chapter),

$$\mu(B(x,r)) \leq \int \varphi d\mu = c_{n,\alpha} \int \left(\sum_{i=1}^{n} \Delta^k \partial_i \varphi * \frac{1}{|x|^{n-\alpha}} * \frac{x_i}{|x|^{1+\alpha}} \right)(y) d\mu(y)$$

$$= -c_{n,\alpha} \sum_{i=1}^{n} \int \left(\mu * \frac{x_i}{|x|^{1+\alpha}} \right)(y) \left(\Delta^k \partial_i \varphi * \frac{1}{|x|^{n-\alpha}} \right)(y) dy$$

$$\leq C \sum_{i=1}^{n} \int_{B(x,3r)} \left| \left(\Delta^k \partial_i \varphi * \frac{1}{|x|^{n-\alpha}} \right)(y) \right| dy + C \int_{\mathbb{R}^n \setminus B(x,3r)} \left| \left(\Delta^k \partial_i \varphi * \frac{1}{|x|^{n-\alpha}} \right)(y) \right| dy$$

Arguing as in Lemma 2.3 we get that the last two integrals can be estimated by Cr^α.

If n is even we use the corresponding representation formula in Lemma 11 of [P1] (Lemma 1.11 in the first Chapter).

On the other hand, it can be easily shown that

$$|R_{\alpha,\varepsilon}(\mu)(x)| \leq C, \quad x \in \mathbb{R}^n, \ \varepsilon > 0,$$

and so, by (2.5), we obtain

$$p_\alpha(\mu) \leq C\|\mu\|.$$

By Schwartz inequality

$$E_\alpha(\mu) \leq C\|\mu\| + \|\mu\|^{1/2} p_\alpha(\mu)^{1/2} \leq C\|\mu\|.$$

Set $\nu = \mu/\|\mu\|$, so that

$$E_\alpha(\nu) = \frac{E_\alpha(\mu)}{\|\mu\|^2} \leq \frac{C}{\|\mu\|},$$

and consequently

$$\gamma_{\alpha,+}(K) \leq C \sup_\nu \frac{1}{E_\alpha(\nu)}.$$

The reverse inequality is proved as in [V4] and involves the $T(1)$-Theorem for non-doubling measures. $\square$

Lemma 2.5. *For each positive finite Radon measure μ on $\mathbb{R}^n$ we have*

$$p_\alpha(\mu) \approx E_{\frac{2}{3}(n-\alpha),\frac{3}{2}}(\mu) = \int_{\mathbb{R}^n} \int_0^\infty \left(\frac{\mu(B(x,r))}{r^\alpha}\right)^2 \frac{dr}{r} d\mu(x).$$

Proof. Suppose that

$$\int_{\mathbb{R}^n} \int_0^\infty \left(\frac{\mu(B(x,r))}{r^\alpha}\right)^2 \frac{dr}{r} d\mu(x) < \infty$$

and set $G = \{(x_1, x_2, x_3) : |x_1 - x_2| \le |x_1 - x_3| \le |x_2 - x_3|\}$. Using Lemma 2.2 and Riemann-Stieltjes integration, we obtain

$$\begin{aligned}
p_\alpha(\mu) &= 3\iiint_G p_\alpha(x_1,x_2,x_3)d\mu(x_1)d\mu(x_2)d\mu(x_3) \\
&\approx \iint \int_{B(x_3,|x_2-x_3|)} |x_2-x_3|^{-2\alpha} d\mu(x_1)d\mu(x_2)d\mu(x_3) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\mu(B(x_3,|x_2-x_3|))}{|x_2-x_3|^{2\alpha}} d\mu(x_2)d\mu(x_3) \\
&= \int_{\mathbb{R}^n}\int_0^\infty \frac{\mu(B(x_3,r))}{r^{2\alpha}} d\mu(B(x_3,r))d\mu(x_3).
\end{aligned} \tag{2.13}$$

Notice that

$$\lim_{r\to\infty} \left(\frac{\mu(B(x,r))}{r^\alpha}\right)^2 \le \lim_{r\to\infty} \left(\frac{\mu(\mathbb{R}^n)}{r^\alpha}\right)^2 = 0. \tag{2.14}$$

Moreover,

$$\int_\rho^{2\rho} \left(\frac{\mu(B(x,r))}{r^\alpha}\right)^2 \frac{dr}{r} \ge \mu(B(x,\rho))^2 \int_\rho^{2\rho} \frac{dr}{r^{2\alpha+1}} = C\left(\frac{\mu(B(x,\rho))}{\rho^\alpha}\right)^2.$$

Thus

$$\lim_{r\to 0} \left(\frac{\mu(B(x,r))}{r^\alpha}\right)^2 = 0. \tag{2.15}$$

Integration by parts in the last integral of (2.13), together with (2.14) and (2.15), show that

$$p_\alpha(\mu) \approx \int_{\mathbb{R}^n} \int_0^\infty \left(\frac{\mu(B(x,r))}{r^\alpha}\right)^2 \frac{dr}{r} d\mu(x).$$

Suppose now that $p_\alpha(\mu) < \infty$. We claim that we can assume that

$$\liminf_{r\to 0} \frac{\mu(B(x,r))}{r^\alpha} = 0, \quad \text{for } \mu\text{-almost all } \ x \in \mathbb{R}^n. \tag{2.16}$$

If (2.16) holds, then integrating by parts in the last integral of (2.13) one can deduce that

$$p_\alpha(\mu) \approx \int_{\mathbb{R}^n} \int_0^\infty \left(\frac{\mu(B(x,r))}{r^\alpha}\right)^2 \frac{dr}{r} d\mu(x),$$

and in this case we are done.

Otherwise there exists a μ-measurable set F such that $\mu(F) > 0$ and

$$\liminf_{r\to 0} \frac{\mu(B(x,r))}{r^\alpha} > 0, \quad x \in F.$$

Shrinking F we can assume that

$$\liminf_{r\to 0} \frac{\mu(B(x,r))}{r^\alpha} > a > 0, \quad x \in F.$$

By Egorov we can find $r_0 > 0$ and a μ-measurable subset G of F such that $\mu(G) > 0$ and

$$\mu(B(x,r)) > \frac{a}{2}\, r^\alpha, \ x \in G \ \text{ and } r \leq r_0. \tag{2.17}$$

From (2.13) we get, applying (2.17) twice,

$$\begin{aligned}
p_\alpha(\mu) &\approx \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\mu(B(x_3,|x_2-x_3|))}{|x_2-x_3|^{2\alpha}} d\mu(x_2)d\mu(x_3) \\
&\geq \int_G \int_{B(x_3,r_0)} \frac{\mu(B(x_3,|x_2-x_3|))}{|x_2-x_3|^{2\alpha}} d\mu(x_2)d\mu(x_3) \\
&\geq \frac{a}{2} \int_G \int_{B(x_3,r_0)} \frac{d\mu(x_2)d\mu(x_3)}{|x_2-x_3|^{\alpha}} \\
&= \frac{a}{2} \int_G \int_0^\infty \mu(\{x_2 \in B(x_3,r_0) : \ |x_2-x_3|^{-\alpha} \geq t\}) dt d\mu(x_3) \\
&\geq \frac{a\alpha}{2} \int_G \int_0^{r_0} \frac{\mu(B(x_3,r))}{r^{1+\alpha}} dr d\mu(x_3) \\
&\geq \frac{a^2\alpha}{2} \int_G \int_0^{r_0} \frac{dr}{r} = +\infty,
\end{aligned}$$

which is a contradiction.

□

Remark. In Theorem 2.2 of [M2] it is shown that for any finite Borel measure in $\mathbb{C}$, one has the following inequality,

$$\int_{\mathbb{C}}\int_{\mathbb{C}}\int_{\mathbb{C}} c^2(x_1,x_2,x_3)d\mu(x_1)d\mu(x_2)d\mu(x_3) \le C\int_{\mathbb{C}}\int_0^\infty \frac{\mu(B(x,r))^2}{r^2}\frac{dr}{r}d\mu(x). \quad (2.18)$$

On the other hand, for $\alpha = 1$, there is no general lower inequality like the one in Lemma 2.2. Although we have

$$c(x_1,x_2,x_3) \le \frac{2}{|x_2 - x_3|},$$

the reverse inequality may fail very badly. Thus the reverse inequality in (2.18) does not hold for general measures μ. However, see Theorem 2.3 in [M2] where a related result is shown when the measure μ is the Hausdorff measure related to some measure function h, restricted to some Cantor sets.

We turn now to the proof of the main Theorem.

Proof of the Theorem. We deal first with the inequality

$$C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}(K) \le C\gamma_{\alpha_+}(K). \quad (2.19)$$

Assume that for a probability measure μ supported on K we have

$$E_{\frac{2}{3}(n-\alpha),\frac{3}{2}}(\mu) = \int_{\mathbb{R}^n}\int_0^\infty \left(\frac{\mu(B(x,r))}{r^\alpha}\right)^2\frac{dr}{r}d\mu(x) \equiv E < \infty.$$

Then by Chebyshev, for each $t > 0$,

$$\mu\{x \in K : \int_0^\infty \left(\frac{\mu(B(x,r))}{r^\alpha}\right)^2\frac{dr}{r} > t\} \le \frac{E}{t}.$$

Taking $t = 2E$, we obtain a compact set $F \subset K$ such that

$$\int_0^\infty \left(\frac{\mu(B(x,r))}{r^\alpha}\right)^2\frac{dr}{r} \le 2E, \quad x \in F,$$

and

$$\mu(F) \ge \frac{1}{3}.$$

If we set $\nu = \mu_{|F}/\mu(F)$, then for some positive constant C depending on α,

$$C\left(\frac{\nu(B(x,\rho))}{\rho^\alpha}\right)^2 \le \int_\rho^{2\rho}\left(\frac{\nu(B(x,r))}{r^\alpha}\right)^2\frac{dr}{r} \le 18E, \quad x \in F. \quad (2.20)$$

To see that ν satisfies the α-growth condition, notice that if $x \notin F$ and $B(x,r) \cap F = \emptyset$, then $\nu(B(x,r)) = 0$, and if there is some $\xi \in F \cap B(x,r)$, then due to (2.20)

$$\nu(B(x,r)) \leq \nu(B(\xi, 2r)) \leq C r^{\alpha} \sqrt{E}.$$

Hence we have

$$M_{\alpha}\nu(x) \leq C\sqrt{E}, \; x \in \mathbb{R}^n.$$

Then by Lemma 2.5 and Schwartz inequality we get

$$E_{\alpha}(\nu) = \int_{\mathbb{R}^n} U_{\alpha}^{\nu}(x) d\nu(x) \leq C\sqrt{E} + p_{\alpha}(\nu)^{1/2} \leq C\sqrt{E}.$$

Thus, by Lemma 2.4, we obtain

$$E^{-1/2} \leq C E_{\alpha}(\nu)^{-1} \leq C \gamma_{\alpha,+}(K),$$

which implies (2.19).

To see the reverse inequality, let μ be a probability measure supported on K such that

$$E_{\alpha}(\mu) = \int_{\mathbb{R}^n} U_{\alpha}^{\mu}(x) d\mu(x) < \infty.$$

Since

$$E_{\alpha}(\mu) \geq \int p_{\alpha}(\mu)(x) d\mu(x),$$

as before, by Chebyshev,

$$\mu\{x \in K : \; p_{\alpha}(\mu)(x) > t\} \leq \frac{E_{\alpha}(\mu)}{t}, \; t > 0.$$

Taking $t = 2E_{\alpha}(\mu)$ we find a compact set $F \subset K$ such that

$$p_{\alpha}(\mu)(x) \leq 2E_{\alpha}(\mu), \quad \text{for } x \in F,$$

and

$$\mu(F) \geq \frac{1}{3}.$$

Set $\nu = \mu_{|F}/\mu(F)$. Then

$$p_{\alpha}(\nu) = \int_F p_{\alpha}^2(\nu)(x) d\nu(x) \leq 36 E_{\alpha}(\mu)^2,$$

and so, by Lemma 2.5

$$E_{\alpha}(\mu)^{-1} \leq 6 p_{\alpha}(\nu)^{-1/2} \approx E_{\frac{2}{3}(n-\alpha), \frac{3}{2}}(\nu)^{-1/2} \leq C_{\frac{2}{3}(n-\alpha), \frac{3}{2}}(K),$$

which ends the proof of the Theorem. $\square$

Chapter 3

Sets with vanishing signed Riesz capacity.

3.1 Introduction.

The aim of this paper is to understand a bit more the capacity γ_α associated to the signed vector valued Riesz kernel $x/|x|^{1+\alpha}$ in $\mathbb{R}^n$, for non-integer indexes $1 < \alpha < n$. In general, given $0 < \alpha < n$ and a compact set $E \subset \mathbb{R}^n$, γ_α is defined as follows,

$$\gamma_\alpha(E) = \sup |< T, 1 >|, \tag{3.1}$$

where the supremum is taken over all real distributions T supported on E such that for $1 \leq i \leq n$, the i-th signed α-Riesz potential $T * \frac{x_i}{|x|^{1+\alpha}}$ of T is a function in $L^\infty(\mathbb{R}^n)$ and $\|T * \frac{x_i}{|x|^{1+\alpha}}\|_\infty \leq 1$.

Due to the result in [T2], the capacity γ_1 in $\mathbb{R}^2$, is comparable to analytic capacity. Given a dimension $n \geq 2$, the capacity γ_{n-1} is called Lipschitz harmonic capacity (see [Par], [MP], [V1] and [Vo]). When we consider non-integer indexes α, the following is known: in [MPV] one shows that for $0 < \alpha < 1$, the capacity γ_α is equivalent to the Riesz capacity $C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}$ of non-linear potential theory (see [AH], chapter 1). From this characterization, one deduces that for $0 < \alpha < 1$, γ_α countably semiadditive and bilipschitz invariant (see [T2] and [T4] for these remarkable results in the case of analytic capacity). Also, either from the description found in [MPV] or from one of the main results in [P1], one can deduce that for $0 < \alpha < 1$ γ_α vanishes on compact sets of finite $\alpha-$dimensional Hausdorff measure.

When one considers integer indexes α, then it is known that compact subsets of α-dimensional smooth surfaces have positive γ_α capacity (see [MP], for the result in the case $\alpha = n - 1$). For non-integer $\alpha > 1$, the capacity γ_α is not understood at all. In [P1] it is shown that in this case γ_α vanishes on $\alpha-$dimensional Ahlfors-David regular sets.

Recall that a closed subset E of $\mathbb{R}^n$ is said to be Ahlfors-David regular of dimension d if it has, locally, finite and positive d-dimensional Hausdorff measure in a uniform way:

$$C^{-1}r^d \leq \mathcal{H}^d(E\cap B(x,r)) \leq Cr^d, \text{ for } x \in E, \;\; r \leq d(E),$$

where $B(x,r)$ is the open ball centered at x of radius r and $d(E)$ is the diameter of E.

Notice that if E is a compact Ahlfors-David regular set of dimension α, then $\mathcal{H}^\alpha(E) < \infty$.

In this paper we take one little step more towards the understanding of these capacities for non-integer indexes $\alpha > 1$. We shall extend the result mentioned above for Ahlfors-David regular sets, to sets having some density condition, namely we will prove the following

Theorem. *Let $0 < \alpha < n$, $\alpha \notin \mathbb{Z}$ and let $E \subset \mathbb{R}^n$ be a compact set with $\mathcal{H}^\alpha(E) < \infty$, such that for all $x \in E$,*

$$0 < \theta_*^\alpha(x,E) \leq \theta^{*\alpha}(x,E) < \infty.$$

Then $\gamma_\alpha(E) = 0$.

Recall that the quantities $\theta_*^\alpha(x,E)$ and $\theta^{*\alpha}(x,E)$ are the lower and upper densities of E at x. They are defined by

$$\theta_*^\alpha(x,E) = \liminf_{r\to 0}\frac{\mathcal{H}^\alpha(E\cap B(x,r))}{r^\alpha}$$

and

$$\theta^{*\alpha}(x,E) = \limsup_{r\to 0}\frac{\mathcal{H}^\alpha(E\cap B(x,r))}{r^\alpha}.$$

The proof of the Theorem uses an adaptation of a result of Pajot (see [Pa1]) on coverings by Ahlfors-David regular sets. In order to prove our result, we also need to study a positive version of γ_α , denoted by $\gamma_{\alpha,+}$. For $0 < \alpha < n$, the capacity $\gamma_{\alpha,+}$ is defined as γ_α, but the supremum in (3.1) is only taken over positive measures instead of over distributions. We will show that for $0 < \alpha < n$ $\gamma_{\alpha,+}$ is countably semiadditive, which will be used in the proof of the Theorem.

The proofs of the results from [P1] mentioned above are both based on the same fact. When we are in the Ahlfors-David regular case, it is shown in [Vi] that for $\alpha \notin \mathbb{Z}$ there are no Ahlfors-David regular sets where the α-Riesz operator is bounded in L^2. We do not known how to prove this result for general sets with finite non-integer α-Hausdorff measure. However, when $0 < \alpha < 1$, it is shown in [P1] that the signed α-Riesz operator is also unbounded in L^2 on sets with finite α-dimensional Hausdorff measure. Using this covering result from [Pa1], we will reduce the proof of our Theorem to the Ahlfors-David regular case.

Throughout all the paper, the letter C will stand for an absolute constant that may change at different occurrences.

If $A(X)$ and $B(X)$ are two quantities depending on the same variable (or variables) X, we will say that $A(X) \approx B(X)$ if there exists $C \geq 1$ independent of X such that $C^{-1}A(X) \leq B(X) \leq CB(X)$ for every X.

The plan of the paper is the following. Section 2 contains some preliminary definitions and results that will be used throughout the paper. The semiadditivity of the capacity $\gamma_{\alpha,+}$, for $0 < \alpha < n$, is also proved in this section. In section 3 we prove the main Theorem.

3.2 Preliminaries.

3.2.1 L^2-boundedness of Calderón-Zygmund operators.

A function $K(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) : x = y\}$ is called a Calderón-Zygmund kernel if the following holds:

1. $|K(x, y)| \leq C|x - y|^{-\alpha}$ for some $0 < \alpha < n$ (α not necessarily integer) and some positive constant $C < \infty$.

2. There exists $0 < \varepsilon \leq 1$ such that for some constant $0 < C < \infty$,

$$|K(x, y) - K(x_0, y)| + |K(y, x) - K(y, x_0)| \leq C\frac{|x - x_0|^\varepsilon}{|x - y|^{\alpha+\varepsilon}},$$

   if $|x - x_0| \leq |x - y|/2$.

Let μ be a Radon measure on $\mathbb{R}^n$. Then the Calderón-Zygmund operator T associated to the kernel K and the measure μ is formally defined as

$$Tf(x) = T(f\mu)(x) = \int K(x, y)f(y)d\mu(y).$$

This integral may not converge for many functions f, because for $x = y$ the kernel K may have a singularity. For this reason, we introduce the truncated operators T_ε, $\varepsilon > 0$:

$$T_\varepsilon f(x) = T_\varepsilon(f\mu)(x) = \int_{|x-y|>\varepsilon} K(x, y)f(y)d\mu(y).$$

We say that the singular integral operator T is bounded in $L^2(\mu)$ if the operators T_ε are bounded in $L^2(\mu)$ uniformly in ε.

The maximal operator T^* is defined as

$$T^*f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|.$$

Let $0 < \alpha < n$ and consider the Calderón-Zygmund operator R_α associated to the antisymmetric vector valued Riesz kernel $x/|x|^{1+\alpha}$.

For the proof of our Theorem a deep result of Nazarov, Treil and Volberg will be needed (see [NTV3]). They prove it for the Cauchy transform. The modifications needed to use their result for the operators R_α are explained in [P1]. In this way one obtains the following $T(b)$-Theorem for the $\alpha-$Riesz transform R_α:

Theorem 3.1. *Let μ be a positive measure on $\mathbb{R}^n$ such that $\limsup\limits_{r\to 0} \frac{\mu(B(x,r))}{r^\alpha} < +\infty$ for μ almost all x and b an $L^\infty(\mu)$ function such that $|\int b d\mu| = \gamma_\alpha$. Assume that $R^*_\alpha\, b\,(x) < +\infty$ for μ almost all x. Then there is a set F with $\mu(F) \geq \frac{\gamma_\alpha}{4}$ such that the α-Riesz transform R_α is bounded in $L^2(\mu_{|F})$.*

3.2.2 The capacities $\gamma_{\alpha,+}$ and $\gamma_{\alpha,2}$.

Recall that the capacity $\gamma_{\alpha,+}$ of a compact set $E \subset \mathbb{R}^n$ is a variant of γ_α defined by

$$\gamma_{\alpha,+}(E) = \sup\,\{\mu(E)\},$$

where the supremum is taken over those positive Radon measures μ supported on E and such that for all $1 \leq i \leq n$, the i-th α-Riesz potential $\mu * \frac{x_i}{|x|^{1+\alpha}}$ of μ is a function in $L^\infty(\mathbb{R}^n)$ with $\sup_{1\leq i\leq n} \|\mu * \frac{x_i}{|x|^{1+\alpha}}\|_\infty \leq 1$. We clearly have $\gamma_{\alpha,+}(E) \leq \gamma_\alpha(E)$.

We define now an L^2-version of the capacity $\gamma_{\alpha,+}$. For a compact set $E \subset \mathbb{R}^n$, set

$$\gamma_{\alpha,2}(E) = \sup\,\{\mu(E)\},$$

where the supremum is taken over the positive Radon measures μ supported on E with growth $\mu(B(x,r)) \leq r^\alpha$ for $x \in spt(\mu)$ and $r > 0$, and such that for $1 \leq i \leq n$, the α-Riesz transform R^i_α is bounded on $L^2(\mu)$ with L^2-norm smaller than 1.

We show now that these two capacities are comparable.

Lemma 3.2. *For $E \subset \mathbb{R}^n$, $\gamma_{\alpha,+}(E) \approx \gamma_{\alpha,2}(E)$.*

For the proof of Lemma 3.2, we need the following result (see lemma 4.2 in [MP]) that tells us how to dualize a weak type $(1,1)-$inequality for several linear operators. The result is a modification of Theorem 23 in [Ch1] (see also [U]).

Let X be a locally compact Hausdorff space and denote $\mathcal{M}(X)$, the space of all finite signed Radon measures on X equipped with the total variation norm. For any $T : \mathcal{M}(X) \to \mathcal{C}(X)$ bounded and linear, denote by $T^t : \mathcal{M}(X) \to \mathcal{C}(X)$ its transpose, that is:

$$\int (T\nu_1) d\nu_2 = \int (T^t \nu_2) d\nu_1 \text{ for } \nu_1,\ \nu_2 \in \mathcal{M}(X).$$

Lemma 3.3. *[MP] Let μ be a positive Radon measure on a locally compact Hausdorff space X and let $T_i : \mathcal{M}(X) \to C(X)$, $1 \leq i \leq n$, be bounded linear operators. Suppose that every T_i^t is of weak type $(1,1)$ with respect to μ, that is there exists a constant $A < \infty$ such that*

$$\mu(\{x : |T_i^t \nu(x)| > t\}) \leq At^{-1}\|\nu\|$$

for $1 \leq i \leq n$, $t > 0$ and $\nu \in \mathcal{M}(X)$. Then for $\tau > 0$ and any borel set $E \subset X$ with $0 < \mu(E) < \infty$ there exists $h : X \to [0,1]$ in $L^\infty(\mu)$, satisfying $h(x) = 0$ for $x \in X \setminus E$,

$$\int_E h d\mu \geq \mu(E)/2 \quad \text{and} \quad \|T_i(hd\mu)\|_\infty \leq (n+\tau)A \quad \text{for} \quad 1 \leq i \leq n.$$

Proof of Lemma 3.2. We have to prove that for some positive constants a and b,

$$a\gamma_{\alpha,+}(E) \leq \gamma_{\alpha,2}(E) \leq b\gamma_{\alpha,+}(E). \tag{3.2}$$

For the second inequality in (3.2), let σ be a positive measure supported on E, such that $\sigma(B(x,r)) \leq r^\alpha$ for $x \in spt(\sigma)$ and $r > 0$, R_α^i is bounded on $L^2(\sigma)$ with operator norm smaller than 1, $1 \leq i \leq n$, and $\sigma(E) \geq \frac{\gamma_{\alpha,2}(E)}{2}$.

From the L^2-boundedness of each R_α^i, we get that each R_α^i, $1 \leq i \leq n$, is of weak type $(1,1)$ with respect to the measure σ. This follows from standard Calderón-Zygmund theory if the measure is doubling, and by an argument found in [NTV2] in the general case.

We would like to dualize this weak type $(1,1)$ inequality applying Lemma 3.3. Unfortunately, Lemma 3.3 does not apply to the truncated operators $(R_\alpha^i)_\varepsilon$, because they do not map $\mathcal{M}(E)$ to $\mathcal{C}(E)$. This difficulty can be overcome by using the following regularized operators. For $\varepsilon > 0$ and $1 \leq i \leq n$, define

$$R_{i,\varepsilon}^\psi \nu(x) = \int \psi\left(\frac{x-y}{\varepsilon}\right) \frac{x_i - y_i}{|x-y|^{1+\alpha}} d\nu(y),$$

for Radon measures ν on $\mathbb{R}^n$, and for $f \in L^1(\sigma)$,

$$R_{i,\varepsilon}^\psi (f\sigma)(x) = \int \psi\left(\frac{x-y}{\varepsilon}\right) \frac{x_i - y_i}{|x-y|^{1+\alpha}} f(y) d\sigma(y),$$

where $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ is some radial function on $\mathbb{R}^n$ with $0 \leq \psi \leq 1$, $\psi = 0$ on $B(0,1/2)$ and $\psi = 1$ on $\mathbb{R}^n \setminus B(0,1)$.

Set $R_{i,\varepsilon} = (R_\alpha^i)_\varepsilon$. Notice that for $\varepsilon > 0$ and $x \in \mathbb{R}^n$ we have

$$|R_{i,\varepsilon}^\psi \nu(x) - R_{i,\varepsilon}\nu(x)| \leq C\widetilde{M}_\sigma \nu(x), \tag{3.3}$$

where

$$\widetilde{M}_\sigma \nu(x) = \sup_{r>0} \frac{\nu(B(x,r))}{\sigma(B(x,3r))}$$

is the *modified maximal operator* introduced in [NTV2]. Notice that if the measure σ is doubling, then $\widetilde{M}_\sigma \nu \approx M_\sigma \nu$, with constants depending only on those involved in the doubling condition. Here

$$M_\sigma \nu(x) = \sup_{r>0} \frac{\nu(B(x,r))}{\sigma(B(x,r))}$$

is the *centered Hardy Littlewood maximal operator.*

Lemma 3 in [NTV2] says that the operator $\widetilde{M}_\sigma \nu$ satisfies a weak (1,1)-inequality with respect to σ,

$$\sigma(\{x \in E : \ \widetilde{M}_\sigma \nu(x) > t\}) \leq C t^{-1} \|\sigma\| \ , \text{ for } \nu \in \mathcal{M}(E). \tag{3.4}$$

It follows from (3.4) and (3.3) that if $R_{i,\varepsilon}$ satisfies a weak type $(1,1)-$ inequality so does $R^{\psi}_{i,\varepsilon}$ and vice versa. The advantage of the $R^{\psi}_{i,\varepsilon}$ is that they do map $\mathcal{M}(E)$ to $\mathcal{C}(E)$, so we may apply Lemma 3.3 to them instead of to $R_{i,\varepsilon}$. Observe that $(R^{\psi}_{i,\varepsilon})^t = -R^{\psi}_{i,\varepsilon}$. Thus for any compact set K in E with $0 < \sigma(K) < \infty$, we can find for each $\varepsilon > 0$ a function h_ε supported on K and satisfying

$$0 \leq h_\varepsilon(x) \leq 1 \text{ for all } x, \tag{3.5}$$

$$\int_K h_\varepsilon d\sigma \geq \sigma(K)/2$$

and

$$\|R^{\psi}_{i,\varepsilon}(h_\varepsilon \sigma)\|_{L^\infty(K)} \leq 2nA. \tag{3.6}$$

In view of (3.3), (3.5), (3.6) and using the growth condition $\sigma(B(x,r)) \leq C_0 r^\alpha$ for $x \in spt(\sigma)$ and $r > 0$, we have $\|R_{i,\varepsilon}(h_\varepsilon \sigma)\|_{L^\infty(K)} \leq C$. But we also want $R_{i,\varepsilon}(h_\varepsilon \sigma)$ to be bounded outside of K.

We claim now that for all $\eta > \varepsilon$, we have $\|R_{i,\eta}(h_\varepsilon \sigma)\|_{L^\infty(K)} \leq C$. To see the claim, let first $\varepsilon \leq \eta \leq 2\varepsilon$. Then using (3.3), (3.5), (3.6) and the growth condition for σ, we have

$$\|R_{i,\eta}(h_\varepsilon \sigma)\|_{L^\infty(K)} \leq \|R_{i,\eta}(h_\varepsilon \sigma) - R_{i,\varepsilon}(h_\varepsilon \sigma)\|_{L^\infty(K)} + \|R_{i,\varepsilon}(h_\varepsilon \sigma)\|_{L^\infty(K)} \leq C.$$

If $\eta > 2\varepsilon$, then $R_{i,\eta} = (R^{\psi}_{i,\varepsilon})_\eta$. Using (3.5) and (3.6) , Cotlar's inequality (see [NTV2]) implies that the maximal operator $(R^{\psi}_{i,\varepsilon})^*(h_\varepsilon \sigma)$ is uniformly bounded on K. Hence for all $\eta > 2\varepsilon$,

$$\|R_{i,\eta}(h_\varepsilon \sigma)\|_{L^\infty(K)} = \|(R^{\psi}_{i,\varepsilon})_\eta(h_\varepsilon \sigma)\|_{L^\infty(K)} \leq \|(R^{\psi}_{i,\varepsilon})^*(h_\varepsilon \sigma)\|_{L^\infty(K)} \leq C.$$

Thus the $R_{i,\eta}(h_\varepsilon \sigma)$ are uniformly bounded on ε and η.

Let $\{\varepsilon_j\}_j$ be an arbitrary sequence tending monotonically to 0 and let h be a weak-star limit of some subsequence of $\{h_{\varepsilon_j}\}$ in $L^\infty(K)$; by passing to some subsequence we may assume that $h_{\varepsilon_j} \to h$ in the weak-star topology. Then h is supported on K, $0 \le h \le 1$, $\int h d\sigma \ge C\sigma(K)$ and $\|R_{i,\eta}(h\sigma)\|_{L^\infty(K)} \le C$ uniformly in η.

If we can prove that $\|R_{i,\varepsilon}(h\sigma)\|_{L^\infty(K^c)} \le C$, then we are done with the lower inequality in (3.2) because $\mu = h\sigma$ is an admissible measure for $\gamma_{\alpha,+}$ and so we have

$$\gamma_{\alpha,+}(E) \ge \int_E h d\sigma \ge C\sigma(E) \ge C\gamma_{\alpha,2}(E).$$

Consider any $x \in \mathbb{R}^n \setminus K$, set $d = \text{dist}(x, K)$ and choose $y \in K$ so that $d = |x-y|$. Fix $\varepsilon > 0$ and distinguish the following three cases,

1. If $\varepsilon \ge 4d$, then

$$|R_{i,\varepsilon}(h\sigma)(x)| \le |R_{i,\varepsilon}(h\sigma)(x) - R_{i,\varepsilon}(h\sigma)(y)| + \|R_{i,\varepsilon}(h\sigma)\|_{L^\infty(K)}$$

and

$$\begin{aligned}
&|R_{i,\varepsilon}(h\sigma)(x) - R_{i,\varepsilon}(h\sigma)(y)| \\
&\le \left| \int_{\{w:\, |w-x|>\varepsilon,\, |w-y|>\varepsilon\}} h(w) \left(\frac{x_i - w_i}{|x-w|^{1+\alpha}} - \frac{y_i - w_i}{|y-w|^{1+\alpha}} \right) d\sigma(w) \right| \\
&+ \left| \int_{\{w:\, |w-y|\le\varepsilon,\, |w-x|>\varepsilon\}} h(w) \frac{x_i - w_i}{|x-w|^{1+\alpha}} d\sigma(w) \right| \\
&+ \left| \int_{\{w:\, |w-x|\le\varepsilon,\, |w-y|>\varepsilon\}} h(w) \frac{y_i - w_i}{|y-w|^{1+\alpha}} d\sigma(w) \right| = A + B + C.
\end{aligned}$$

To deal with A, note that $|y-w| > \varepsilon \ge 4d = 4|x-y| \ge 2|x-y|$. Hence using the standard estimates for the Calderón-Zygmund kernels, $0 \le h \le 1$ and the $\alpha-$growth of σ we get

$$\begin{aligned}
A &\le C \sum_{j=0}^{\infty} \int_{\{w:\, 2^j\varepsilon \le |y-w| \le 2^{j+1}\varepsilon\}} \frac{|x-y|}{|y-w|^{1+\alpha}} |h(w)| d\sigma(w) \\
&\le Cd \sum_{j=0}^{\infty} \frac{1}{(2^j\varepsilon)^{1+\alpha}} \int_{\{|y-w|\le 2^{j+1}\varepsilon\}} |h(w)| d\sigma(w) \\
&\le C\frac{d}{\varepsilon} \sup_{r>0} \frac{1}{r^\alpha} \int_{|y-w|<r} |h(w)| d\sigma(w) \sum_{j=1}^{\infty} 2^{-j} \le C.
\end{aligned}$$

For the term B,

$$B \le \frac{1}{\varepsilon^\alpha} \int_{|w-y|\le\varepsilon} |h(w)| d\sigma(w) \le C.$$

Term C is treated in the same way as term B but interchanging the roles of x and y.

2. If $d/2 \leq \varepsilon < 4d$, then

$$|R_{i,\varepsilon}(h\sigma)(x)| \leq |R_{i,4d}(h\sigma)(x)| \; + \; |R_{i,\varepsilon}(h\sigma)(x) - R_{i,4d}(h\sigma)(x)|$$

$$\leq C + C \sup_{r>0} \frac{1}{r^\alpha} \int_{B(y,r)} |h(w)| d\sigma(w) \leq C,$$

by using the previous case to bound $|R_{i,4d}(h\sigma)(x)|$ and the $\alpha-$growth condition on σ and $0 \leq h \leq 1$ to bound the difference $|R_{i,\varepsilon}(h\sigma)(x) - R_{i,4d}(h\sigma)(x)|$.

3. If $\varepsilon < d/2$, then $R_{i,\varepsilon}(h\sigma)(x) = R_{i,d/2}(h\sigma)(x)$, which leads us to the second case.

For the first inequality in (3.2), let σ be a positive measure supported on E such that $\sigma(E) \geq \frac{\gamma_{\alpha,+}(E)}{2}$ and $\|\sigma * \frac{x_i}{|x|^{1+\alpha}}\|_\infty \leq 1$, $1 \leq i \leq n$.

To see that σ is admissible for $\gamma_{\alpha,2}$, we check first that it satisfies the growth condition $\sigma(B(x,r)) \leq Cr^\alpha$. Take an infinitely differentiable function φ, supported on $B(x,2r)$ such that $\varphi = 1$ on $B(x,r)$, and $\|\partial^s \varphi\|_\infty \leq C_s r^{-|s|}$, $|s| \geq 0$. Here $s = (s_1, ...s_n)$, with $0 \leq s_i \in \mathbb{Z}$, $|s| = s_1 + s_2 + ... + s_n$ and $\partial^s = (\partial/\partial x_i)^{s_1} ... (\partial/\partial x_n)^{s_n}$. Assume first that n is odd and of the form $n = 2k+1$. Then, by Lemma 11 in [P1] (see Lemma 1.11 in the first Chapter of this dissertation),

$$\sigma(B(x,r)) \leq \int \varphi d\sigma = c_{n,\alpha} \int \left(\sum_{i=1}^n \Delta^k \partial_i \varphi * \frac{1}{|x|^{n-\alpha}} * \frac{x_i}{|x|^{1+\alpha}} \right)(y) d\sigma(y)$$

$$= -c_{n,\alpha} \sum_{i=1}^n \int \left(\sigma * \frac{x_i}{|x|^{1+\alpha}} \right)(y) \left(\Delta^k \partial_i \varphi * \frac{1}{|x|^{n-\alpha}} \right)(y) dy$$

$$\leq C \sum_{i=1}^n \int_{B(x,3r)} \left| \left(\Delta^k \partial_i \varphi * \frac{1}{|x|^{n-\alpha}} \right)(y) \right| dy + C \int_{\mathbb{R}^n \setminus B(x,3r)} \left| \left(\Delta^k \partial_i \varphi * \frac{1}{|x|^{n-\alpha}} \right)(y) \right| dy$$

Arguing as in Lemma 12 in [P1] (see Lemma 1.12 in the first Chapter of this dissertation) we get that the last two integrals can be estimated by Cr^α.

When n is even we use the corresponding representation formula in Lemma 11 of [P1] (Lemma 1.11 in the first Chapter).

We are left now with the L^2-boundedness of the α-Riesz transform R^i_α for $i = 1, \cdots, n$. By assumption $\|\sigma * \frac{x_i}{|x|^{1+\alpha}}\|_\infty \leq 1$, for $1 \leq i \leq n$. In particular this implies that we can apply the $T(1)$ theorem (see Theorem 3.1 with $b \equiv 1$) and so we get the L^2-boundedness of R^i_α for $1 \leq i \leq n$. This means that σ is admissible for $\gamma_{\alpha,2}$. Thus

$$\gamma_{\alpha,2}(E) \geq C\sigma(E) \geq C\gamma_{\alpha,+}(E).$$

□

From this lemma, we can deduce the semiadditivity of the capacity $\gamma_{\alpha,+}$. In fact, $\gamma_{\alpha,+}$ is countably semiadditive.

Corollary 3.4. *Let $E \subset \mathbb{R}^n$ be compact. Let E_i, $i \geq 1$, be Borel sets such that $E = \bigcup_{i=1}^{\infty} E_i$. Then*

$$\gamma_{\alpha,+}(E) \leq C \sum_{i=1}^{\infty} \gamma_{\alpha,+}(E_i),$$

where C is some absolute constant.

Proof. Let μ be the extremal measure for $\gamma_{\alpha,2}(E)$. Then due to Lemma 3.2 and using that the measures $\mu_{|E_i}$ are admissible for the capacity $\gamma_{\alpha,2}(E_i)$,

$$\gamma_{\alpha,+}(E) \approx \gamma_{\alpha,2}(E) = \mu(E) = \mu(\cup_i E_i) \leq \sum_i \mu(E_i)$$

$$\leq C \sum_i \gamma_{\alpha,2}(E_i) \approx \sum_i \gamma_{\alpha,+}(E_i).$$

□

3.3 Proof of the Theorem.

We need the following result inspired from a theorem of H. Pajot. (see Proposition 4.4 in [Pa1]) The result of H. Pajot, says that with some density condition, every compact set of $\mathbb{R}^n$ with finite $\mathcal{H}^\alpha$-measure can be covered by a countable union of $\alpha-$dimensional Ahlfors-David regular sets.

He proves the result for sets in $\mathbb{R}^n$ of integer dimension α. With some minor changes in his proof, the same result holds also for sets in $\mathbb{R}^n$ of non-integer dimension α with $0 < \alpha < n$, that is

Theorem 3.5. *Let $E \subset \mathbb{R}^n$ be a compact set with $\mathcal{H}^\alpha(E) < \infty$, such that for all $x \in E$*

$$0 < \theta_*^\alpha(x, E) \leq \theta^{*\alpha}(x, E) < \infty.$$

Then,

$$E \subset \bigcup_{i=1}^{\infty} E_i$$

and for all $i \in \mathbb{N}$, E_i is a compact Ahlfors-David regular set of dimension α.

Now we turn to the proof of the main Theorem.

Proof of the Theorem. Suppose $\gamma_\alpha(E) > 0$. Applying Lemma 8 in [P1] we find a measure of the form $\nu = b\mathcal{H}^\alpha$, with $b \in L^\infty(\mathcal{H}^\alpha, E)$ such that the signed $\alpha-$ Riesz potential $R_\alpha(\nu) = \nu * \frac{x}{|x|^{1+\alpha}}$ is in $L^\infty(\mathbb{R}^n)$ and $\int_E b \, d\mathcal{H}^\alpha = \gamma_\alpha(E)$. We can apply now Theorem 3.1 to get a set $F \subset E$ of positive $\mathcal{H}^\alpha-$measure such that the operator R_α is bounded on $L^2(\mathcal{H}^\alpha, F)$. This implies that $\gamma_{\alpha,2}(E) > 0$. Due to Lemma 3.2 $\gamma_{\alpha,+}(E) > 0$. Notice that our set E satisfies the hypothesis of Theorem 3.5, then there exist compact Ahlfors-David regular sets E_i of dimension α such that

$$E \subset \bigcup_{i=1}^{\infty} E_i.$$

The semiadditivity of the capacity $\gamma_{\alpha,+}$, stated in Corollary 3.4 implies then that

$$0 < \gamma_{\alpha,+}(E) \leq C \sum_i \gamma_{\alpha,+}(E_i).$$

Thus at least one of the Ahlfors-David regular sets E_i, say E_k, has $\gamma_{\alpha,+}(E_k) > 0$. Then for this set E_k we have

$$0 < \gamma_{\alpha,+}(E_k) \leq \gamma_\alpha(E_k).$$

Applying now Theorem 2 in [P1] to the Ahlfors-David regular set E_k, we get that α must be an integer. $\square$

Chapter 4

Open problems

In this final chapter we will state some open problems related to the topics studied in this dissertation. These questions are quite recent.

Problem 1. Can we extend Theorem B to general compact sets $E \subset \mathbb{R}^n$, namely, is it true that if E has finite α-Hausdorff measure for some non-integer index $1 < \alpha < n$, then $\gamma_\alpha(E) = 0$?

Problem 2. Is it true that the vector valued Riesz kernels $x/|x|^{1+\alpha}$ in $\mathbb{R}^n$, $1 < \alpha < n$, $\alpha \notin \mathbb{Z}$, are unbounded on $L^2(\mathcal{H}^\alpha)$ on sets of positive finite $\mathcal{H}^\alpha$-measure?

This result holds for α-dimensional Ahlfors-David regular sets (see [Vi]). If the answer to Problem 2 is yes, then the same happens with Problem 1.

Problem 3. Is there an absolute constant C such that

$$\gamma_\alpha(E) \leq C\gamma_{\alpha,+}(E),$$

for compact sets in $\mathbb{R}^n$ and $1 < \alpha < n$?

The result for analytic capacity is proved in [T2] and adapting the method the result has been obtained for Lipschitz harmonic capacity γ_{n-1} and for γ_α, $0 < \alpha < 1$ (see [Vo] and [MPV] respectively).

Problem 4. Is the capacity γ_α for $1 < \alpha < n$ semiadditive? That is, is there a constant C depending on α and n such that

$$\gamma_\alpha(E \cup F) \leq C\left(\gamma_\alpha(E) + \gamma_\alpha(F)\right),$$

for compact sets E and F in $\mathbb{R}^n$?

A positive answer to Problem 3 would imply automatically the semiadditivity of γ_α for $1 < \alpha < n$ because the capacity $\gamma_{\alpha,+}$ turns out to be semiadditive.

Problem 5. Consider Cantor sets $E_n(\lambda) \subset \mathbb{R}^n$ for non-increasing sequences $\{\lambda_j\}_{j=1}^{\infty}$ with $0 < \lambda_j < 1/2$. Without loss of generality assume that $\lambda_0 = 1$. Set $\sigma_k = \Pi_{j=0}^{k} \lambda_j$. Let E_0 be a closed interval of length σ_0, and let E_1 be the set obtained by removing an open interval of length $\sigma_0 - 2\sigma_1$ in the middle, so that E_1 consists of two closed intervals of length σ_1. Then remove an interval of lenght $\sigma_1 - 2\sigma_2$ in the middle of each of these intervals, to obtain E_2 consisting of 2^2 intervals of length σ_2. Continuing like this we obtain after k steps a set E_k consisting of 2^k intervals of lenght σ_k. Denote the Cartesian product of n copies of E_k by $E_k^{(n)}$, and set

$$E_n(\lambda) = \bigcap_{n=0}^{\infty} E_k^{(n)}.$$

Then $E_n(\lambda)$ is called the Cantor set corresponding to the sequence $\{\lambda_n\}_{n=0}^{\infty}$.

Is it true that for $1 < \alpha < n$,

$$\gamma_\alpha(E_n(\lambda)) \approx \left(\sum_{k=0}^{\infty} \left(\frac{2^{-kn}}{\sigma_k^{\alpha}} \right)^2 \right)^{-1/2} ? \tag{4.1}$$

The result holds for $\alpha = 1$ and $n = 2$ (see [MTV]). For $0 < \alpha < 1$ and any n (4.1) also holds because for the capacity $C_{\frac{2}{3}(n-\alpha), \frac{3}{2}}$ it is true (see [AH], p. 143-146) and these two capacities are equivalent (see [MPV]).

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