Universitat Autònoma de Barcelona

Departament de Física
Grup de Física Teòrica

Quantum Effects in Brane World scenarios: moduli stabilization and the hierarchy problem

Oriol Pujolàs Boix

Universitat Autònoma de Barcelona
Institut de Física d'Altes Energies

maig del 2003 [1]

[1]Memòria presentada com a tesi doctoral per optar al grau de doctor en Ciències Físiques per la Universitat Autònoma de Barcelona

El Dr. Jaume Garriga Torres, catedràtic del Departament de Física Fonamental de la Universitat de Barcelona, i membre de l'Institut de Física d'Altes Energies,

CERTIFICA: Que la present memòria, que porta per títol "Quantum Effects in Brane World scenarios: moduli stabilization and the hierarchy problem", ha estat realitzada sota la seva direcció per l'Oriol Pujolàs Boix. Aquesta memòria constitueix la tesi doctoral d'aquest estudiant, per optar al grau de doctor en Ciències Físiques per la Universitat Autònoma de Barcelona.

Bellaterra, maig del 2003

Dr. Jaume Garriga Torres

Vull donar les gràcies en primer lloc a en Jaume Garriga, sense el qual aquesta tesi no hauria estat possible. El suport científic d'en Takahiro Tanaka i d'en Nino Flachi també m'han resultat d'un ajut inestimable. Agraeixo amb molt de gust a l'Alex Pomarol, en Xavi Montes, l'Ariel Mégevand i l'Albert Roura per les valuoses discussions per tirar endavant aquest projecte.

He d'agrair a en Dani, l'Ariel, en Santi, en Francesc i la resta de membres de l'IFAE el bon ambient que he pogut respirar durant aquest temps.

Dedico questa tesi molt especialment a la Susagna, i a la resta de la meva família; de ben segur que gaudeixen de la lectura.

Contents

Chapter 1

Introduction

The Standard Model (SM) of elementary particle physics describes the microcosmos with great precision at energies up to the electroweak (EW) scale. Several theoretical issues question the validity of this model for energies higher than this scale. One of the most intriguing puzzles is the *hierarchy problem*. In perturbation theory, the quantum corrections to the Higgs mass are sensitive to the cutoff scale, believed to be of order of the Planck or the grand unification scale. For the Higgs mass to remains of EW size at the quantum level, a fine tuning of the parameters is required. This renders the theory quite unnatural. Thus, first of all, it has to be said that the hierarchy problem does not concern the consistency but the naturality of the SM. Several approaches, including supersymmetry or technicolor, have been proposed since long ago to overcome this problem.

The Brane World (BW) scenario [1] suggests a new framework where this problem can be successfully solved. In the BW picture, the spacetime has extra dimensions but matter is confined on a four dimensional surface called *brane*. In contrast, gravity propagates through the whole *bulk* space.

The relevance of the Brane World scenario to phenomenology was first realized by Arkani-Hamed, Dimopoulos and Dvali (ADD) [1]. The key observation is that current experimental bounds on the size of extra dimensions accessible only to gravity allow for a bulk size of a fraction of a millimeter. Such a large extra space can account for the observed weakness of gravity with the fundamental cutoff of the theory M around a few TeV, hence the hierarchy problem disappears.

Randall and Sundrum (RS) [2] proposed a simple brane model in five dimensions where the 16 orders of magnitude separating the Planck and the electroweak scales are due not to the large bulk size but to its curved Anti-de Sitter (AdS) geometry.

As in KK theories, the size of the bulk is described by a four dimensional field known as the *radion*. In models of both ADD or RS type, the EW/Planck hierarchy (*the hierarchy*) is determined by the size of the bulk, *i.e.* , by the radion vacuum expectation value (vev). On the other hand, since the radion is one of the components of the higher dimensional metric, it is a massless field at tree level. In order to avoid long range scalar interactions the 'predicted' radion must acquire a large enough mass [1, 3, 4]. Thus, in the BW scenario, the hierarchy problem is equivalent to the problem of *stabilizing* the radion at a suitable vev and with a large enough mass. This is why sometimes, the stabilization of the radion is also called 'hierarchy stabilization'. An overview of the stabilization mechanisms proposed so far in the literature, is given in Sections 2.3 and 3.4.

In the context of RS type models, perhaps the most popular is the proposal made by Goldberger and Wise (GW) [5]. It introduces a classical bulk field with appropriate boundary conditions, and stabilizes the radion generating a large hierarchy without fine tuning.

Weinberg and Candelas [6] showed that the Casimir effect can stabilize extra dimensions in Kaluza Klein models. In the RS model, the possibility that quantum effects stabilize a large hierarchy was first considered in [7]. The outcome was that generic bulk fields may stabilize the interbrane distance, but a large hierarchy is not naturally obtained, *i.e.* fine tuning of the parameters is needed (and thus the hierarchy problem would be replaced by another fine tuning problem). However, the Casimir force due to a bulk gauge field naturally stabilizes the hierarchy [8], generating a sizable radion mass.

The aim of this thesis is to find out whether the Casimir energy can stabilize a large hierarchy in more general brane models. Specifically, we concentrate on three types of models. The first is a generalization of the RS model with a scalar field in the bulk. This gives rise to solutions similar to the RS but with a non-exponential warp factor. This kind of models arise *e.g.* , in compactifications to 5D of higher dimensional models such as the Hořava Witten theory [9, 10]. Aside from the radion, another scalar appears in the four dimensional effective theory (arising from the bulk scalar). Together with the radion, these light degrees of freedom are generically called *moduli*. The EW/Planck hierarchy depends on both moduli and a mechanism to stabilize them is required in order to avoid unobserved long range scalar interactions.

The second case deals with models in more than 5 dimensions with topology $M_4 \times \Sigma \times S^1/Z_2$, where Σ is some compact 'internal' space and M_4 is the four dimensional Minkowski space. We shall first consider the case when both M_4 and Σ are warped. In this case, the hierarchy originates by a combination of ADD and RS mechanisms. We propose a scenario where the 16 orders of magnitude separating the Planck and EW scales are easily obtained from such models, and the hierarchy is naturally stabilized by the Casimir force.

In the last example, we consider higher dimensional models with the same topology but with non-warped internal space, so that the space is in fact the direct product $\mathrm{AdS}_5 \times \Sigma$. We shall see that this case is very similar to the RS model.

We advance that, depending on the field content and the boundary conditions, the quantum effects stbilize the moduli in all these examples, and the hierarchy can be naturally generated in some of them.

Plan of the thesis

Chapters 2 and 3 contain introductory material. Chapter 2 presents some of the main features of ADD-type brane models, and a brief description of KK theories in Section 2.1. The RS model is discussed in Chapter 3, with special emphasis on the radion. Chapter 4 concerns the techniques used to evaluate the Casimir energy, with emphasis on the spacetimes with warped extra dimensions.

Presentation of original work begins in Section 4.4, based on [15]. The procedure to compute the effective potential using dimensional and zeta function regularization are discussed, and their equivalence is proven. Chapter 5 reviews the one loop effective potential induced by bulk fields in the RS model [7, 11, 12, 13, 14, 8].

Chapter 6 is based on [15] and deals with the 5D generalization of the RS model with a scalar in the bulk. Chapter 7 builds on [16] and is devoted to the case when the non compact M_4 and the compact Σ factors share the same warp factor. Chapter 8 is based on [17] and contains the case when only the M_4 factor is warped. At the end of each Chapter, a summary of the main ideas is included.

Conventions and notation

In this thesis we consider spacetimes topologies of the form $M_4 \times S^1/Z_2$, $M_4 \times \Sigma$ and $M \times S^1/Z_2 \times \Sigma$. In Chapters 7 and 8, D_1 and D_2 indicate the dimensions of the Minkowski and internal space Σ respectively, and the total dimension of the space is $D \equiv D_1 + D_2 + 1$. The coordinates x^μ with $\mu, \nu \dots = 0, 1, 2, 3$ span the Minkowski factor. The *internal* space Σ is covered by the coordinates X^i. We call the proper and conformal coordinates along the S^1/Z_2 orbifold y and z respectively. We refer collectively the $D_1 + D_2 + 1$ as $x^M \dots = \{x^\mu, y, X^i\}$; the coordinates on the $D_1 + D_2$ branes as $x^A \dots = \{x^\mu, X^i\}$ and the 5D bulk coordinates by $x^\alpha = \{x^\mu, y\}$.

Throughout this thesis the metric signature is $(- + + + \cdots)$. Higher dimensional quantities and tensors are labeled with some upper/lower index indicating the dimension when necessary. In the text, we use "nD" to mean "n dimensional". A hat $\hat{}$ is reserved for the 4D metric corresponding to the Einstein frame. We define the Riemann tensor as $\mathcal{R}^a{}_{bcd} = +\Gamma^a{}_{bc,d} - \cdots$ is the Riemann tensor, $\mathcal{R}_{bc} = \mathcal{R}^a{}_{bac}$ is the Ricci tensor and $\mathcal{R} = \mathcal{R}_{ab} g^{ab}$ is the curvature scalar. The extrinsic curvature is given by $\mathcal{K}_{\mu\nu} \equiv (1/2)\partial_y g_{\mu\nu}$, where $g_{\mu\nu}(y)$ is the induced metric on y-constant hypersurfaces, and $\mathcal{K} = \mathcal{K}_{\mu\nu}\, g^{\mu\nu}$.

We denote the four dimensional Planck mass by $m_P \equiv (16\pi G_N)^{-1/2} \simeq 1.7\,10^{15}$TeV, the fundamental cutoff by M, the size of the orbifold (interbrane distance) by d and of Σ by R. The quantities related to the brane with positive (negative) tension are labeled with a $+$ $(-)$ super/subscript. In all the models, the ratio of the electroweak scale $m_{EW} \simeq 246$GeV $\simeq$ $(1/4)$TeV to the Planck mass m_P is referred to as *the hierarchy*. It is denoted by h and its 'observed' value is $h \equiv (m_{EW}/m_P) \sim 10^{-16}$.

Chapter 2

The Brane World scenario

Building on earlier ideas [18, 19, 20], Arkani-Hamed, Dimopoulos and Dvali proposed [1] a scenario where space has a number of extra dimensions accessible to gravity but matter is confined on a surface called 'brane' [1], as shown in Figure 2.1. Localization of matter is a common and well known phenomenon occurring in field theoretical models with topological defects [21, 22, 23, 19, 24] (see [25] for a recent review). In the context of string theory, the branes are the extended objects on which open strings can end/begin. The open string excitations correspond to matter and gauge bosons. Thus, in string theory the localization on the branes is automatic.

This is known as the 'Brane World (BW)' scenario, and it provides a large variety of new mechanisms to obtain phenomenologically interesting features, ranging from chiral fermions, fermion mass hierarchies, low scale baryogenesis, supersymmetry breaking as well as inflation and even alternatives to inflation or to compactification. Moreover, it gives a new opportunity to address long standing puzzles of particle physics such as the cosmological constant problem or the hierarchy problem. The latter is the main focus of attention in this thesis.

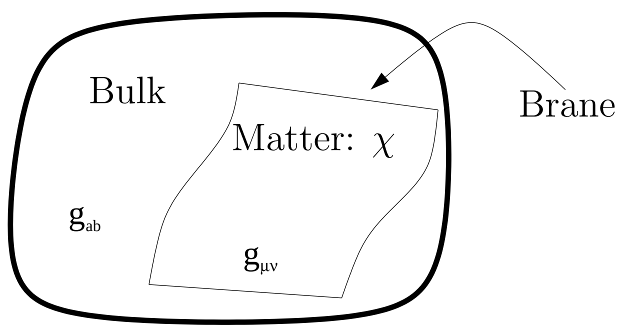


Figure 2.1: In the BW picture, gravity can propagate through the bulk, whereas matter fields χ are confined on the brane.

Weakness of gravity and the hierarchy problem

The hierarchy problem of particle physics consists on the quadratic sensitivity with the cutoff scale M in the quantum corrections to the mass of 'fundamental' scalar field. The SM Higgs field is such a scalar and plays crucial role in the Standard Model, since its vev determines the mass of all the particles. The SM is believed to be valid up to the grand unification or the quantum gravity scales, of order 10^{16}TeV in any case. To keep the Higgs mass well below the cutoff scale involves a fine tuning of the parameters in the theory order by order in perturbation theory. Technically this is not inconsistent, but renders the theory unnatural. Proposed solutions include technicolor and supersymmetry (SUSY), though current precision electroweak measurements clearly disfavor a large class of technicolor models.

Independently of the instability of the scalar masses under quantum corrections (which is ultimately model dependent and moreover no 'fundamental' scalar has been observed yet), there is another issue closely related to this and to the unification gravity with the gauge interactions. The point is that e.g. the electrostatic repulsion between two protons is 10^{40} times stronger than their gravitational attraction. It is surprising that Nature has chosen such a large hierarchy in the strength of these (so far) fundamental interactions. Of course, this is does not invalidate the SM as a theory of strong and electroweak interactions. Rather, it reinforces our expectation that the SM is an effective theory, that very accurately describes nature up to 100GeV. Beyond this scale, we expect new physics which solve some of the puzzles in the SM, and hopefully can lead to a unification of gravity with the other interactions.

In the standard picture, the weakness of gravity and the instability of the Higgs mass are linked to one another. The small strength of gravity can be traced back to a very large quantum gravity scale (the scale at which gravity becomes strongand hence quantum corrections are not negligible), $m_P \ll m_{EW}$. The coupling constant of gravity is given by $1/m_P$, and all gravitational effects at available energies E are suppressed by factors $(E/m_P) \ll 1$. Thus a large cutoff entails large quantum corrections and severe fine-tunings.

From this, it is clear that a resolution of the hierarchy problem can be accomplished by one of the following possibilities.

- Protecting the scalar mass under quantum corrections, that is, canceling the quadratic sensitivity of the scalar mass to the cutoff. This can be done by introducing some symmetry and is what *e.g.* supersymmetry does.

- Eliminating (fundamental) scalars from the spectrum. This is essentially the idea behind Technicolor [26, 27]. Unfortunately, it doesn't seem that a realistic model of technicolor can be built [28]. [1]

- Lowering down the fundamental cutoff. This can be realized in brane models with a large bulk volume [1]. Models with a warped bulk [2] generate redshift effects such that the effective cutoff is lowered as well.

The ADD mechanism explains these two issues as follows. Whereas gauge interactions are confined to a four dimensional brane, gravity can propagate along a higher dimensional 'bulk' space. As shown below, the strength of gravity decays faster $\sim 1/r^{2+n}$ up to the compactification scale R. For larger distances the behaviour is four dimensional, but in order for the two regimes to match the strength is suppressed by a power of the ratio of the fundamental scale $1/M$ to the compactification radius R. Current limits on extra dimensions accessible only to gravity allow

[1] Also, some brane models can eliminate the necessity of a Higgs field using symmetry breaking effects based on the topology of the extra dimensions [29, 30, 31, 32, 33, 34, 35, 36].

for a submillimetric bulk size $R \lesssim$ mm, which is enormous compared to any fundamental scale. Thus, in this picture gravity is weak because it is diluted in the large bulk space.

As a consequence, the fundamental scale M can be lowered down to a few TeV (for $n \geq 2$).[2] With a cutoff less one order of magnitude larger than the EW scale, no significant fine tuning has to be done when computing the quantum corrections to the Higgs mass and the model does not suffer from naturality problems.

The RS model consists of a slice of 5D anti-de Sitter AdS space bounded by two branes. Due to the nontrivial geometry of AdS, the energy scales of objects localized on one brane are exponentially redshifted respect to the other. Two extremely separated scales are generated with an interbrane distance comparable to the curvature radius and all the 5D mass scales of the same order, m_P. The large scale is the Planck scale m_P and the derived one the electroweak, $\sim$ TeV. Gravity is localized on one brane and the EW sector on the other. Since the overlap is small, gravity appears weak. On the other hand, the Higgs does not acquire large corrections because in the complete 5D theory, all the masses are of order the cutoff m_P.[3]

2.1 Extra dimensions

The history of extra dimensions in physics is long and intricate. Before the advent of general relativity, G. Nordström [40] attempted to unify gravity and electromagnetism enlarging the space dimension. In contrast with the usual KK theory, he assumed a theory with pure electromagnetism in five dimensions. Then the fifth component of the vector potential could be identified with the gravitational (scalar) potential. After the great success of Maxwell's theory to unify electrostatics, magnetism and optics in the XIX century, certainly the most natural thing was to expect that gravity was another 'form' of electrodynamics.

In 1921, the mathematician T. Kaluza made his famous proposal in the context of Einstein's theory of gravity [41]. Not much significance was given to the fifth dimension, though, until the work of O. Klein [42]. His is the suggestion that the extra dimension might be compact and small.

The generalization to non Abelian gauge theories [43, 44] had to wait until Yang Mills theories were developed and gained interest after the advent of the electroweak unification. Supergravity and the description of 'spontaneous compactification' boosted KK theories in the 1970's and 1980's. An intriguing coincidence between two independent results from supergravity and group theory raised the hope on KK theories during some time. On one hand, it was shown that supersymmetric theories with fields of spin 2 at most were possible in 11 dimensions or less only. On the other hand, Witten found [45] that the minimum dimension of a compact space with isometry group equal to the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ is seven. This pointed toward 11D supergravity, a theory with a very constrained field content. Again Witten argued that no 4D chiral fermions can be obtained compactifying such a theory on a smooth

[2] This means that in these models, string or M theoretic effects are observable at energies available at the colliders that will operate in the near future.

[3] In the AdS/CFT correspondence [37, 38, 39], the dual CFT interpretation is that the EW scale is the infrared scale at which the strongly coupled CFT is spontaneously broken. All degrees of freedom at the TeV including the Higgs boson correspond to composite fields, like hadrons in QCD. Hence, such a Higgs is described by a scalar up to an effective cutoff of order TeV.

manifold. This finally put the KK idea in a hard position to meet the observed phenomenology.

Extra dimensions have been considered until now for several reasons. For instance, superstring theories have to be formulated in more than four dimensions in order to be consistent. Recent developments in string theory [46] have brought attention to non-perturbative objects (analogous to the solitons in field theory) known as *branes*. The inclusion of non-perturbative effects has revealed a web of dualities among them. The known five different string theories together with 11 dimensional supergravity compactified on S^1/Z_2 are believed to constitute limiting cases of a more fundamental theory referred to as *M theory*.

The last twist in this history was the realization [1] that models with branes and large extra dimensions were shown to account for long standing problems in particle physics, such as the hierarchy problem. A common feature in ADD and RS brane models is that they can lead to new physics at a scales as low as some TeV. In light of these testable predictions, the KK scenario with compactification scale of order a TeV has received attention for phenomenological reasons [18, 47]. In this Chapter, we present the main features of KK theories (with no branes yet).

This Section is based on some of the abundant reviews on the topic of KK theories available in the literatire [48, 49, 25, 50, 51].

Kaluza Klein theories

Before describing how the addition of extra dimensions can lead to a unification of gravity with the other gauge interactions, I shall present the generic effective theories that can be constructed in four dimensions. Consider the simplest example of one flat compact extra dimension, parametrized by the coordinate $0 \leq y \leq 2\pi R$, so that the spacetime is $M_4 \times S^1$. The action of a massless scalar $\Phi(x^\mu, y)$ field is

$$S = \frac{1}{2}\int d^5x \; \Phi(x,y)\Box_5\Phi(x,y), \tag{2.1}$$

where $\Box_5$ is the 5D D'Alembertian, and Φ obeys periodic boundary conditions $\Phi(x, y+2\pi R) = \Phi(x,y)$. This suggests to Fourier decompose Φ as

$$\Phi(x,y) = \sum_{n=-\infty}^{+\infty} \Phi^{(n)}(x) e^{iny/R}. \tag{2.2}$$

Inserting this decomposition into the action and integrating y explicitly, one obtains

$$S = \frac{1}{2}\int d^4x \sum_{n=-\infty}^{+\infty} \Phi^{(n)}(x)\left(\Box_4 - m_n^2\right)\Phi^{(n)}(x), \tag{2.3}$$

with $m_n = n/R$. This action describes an infinite set of four dimensional fields $\Phi^{(n)}(x)$ with increasing masses, referred to as the KK tower. When the five dimensional field $\Phi(x,y)$ is massless, there is one massless mode in the tower, $\Phi^{(0)}(x)$. The mass of the remaining modes is given by the compactification scale, $\sim 1/R$. If the size R of the extra dimension is sufficiently small, the energy carried by these modes is very large and thus very difficult to excite. Therefore, it is expected that at energies below $1/R$, the higher dimensional theory is effectively well described by the zero mode only.

Now we address Kaluza's [41] original proposal. In modern parlance, the proposal is that pure Einstein gravity in five dimensions, on the background spacetime $M_4 \times S^1$ contains four dimensional Einstein gravity and electromagnetism. To see this, we begin with the action for pure 5D Einstein gravity

$$S = M^3 \int d^5x \sqrt{g}\, \mathcal{R}, \tag{2.4}$$

where M is the 5D Planck mass and we note that $M_4 \times S^1$ is indeed a solution of 5D Einstein's equations. The components of the 5D metric $g_{\alpha\beta}(x, y)$ should be now Fourier (or KK) decomposed and inserted back into (2.4). For the low energy effective theory, the massive modes are not relevant. So introducing x^μ dependence only in $g_{\alpha\beta}$ suffices.

The key issue is to identify that the action (2.4) is invariant under 5D general coordinate transformations, parametrized by 5 functions $\xi^\alpha(x, y)$. For the effective theory, only $x-$dependence is relevant, and we expect that the four $\xi^\mu(x)$ parametrize 4D general coordinate transformations among the x^μ. So, $\xi^5(x)$ can play the role of the parameter of gauge transformations, which are identified as local (x dependent) translations of the extra dimension $x^5 = y$. This suggests making the following ansatz for the metric (the 'Kaluza Klein ansatz')

$$ds^2 = e^{-2\varphi/3} \left(\hat{g}_{\mu\nu} dx^\mu dx^\nu + e^{2\varphi}(dy + A_\mu dx^\mu)^2\right), \tag{2.5}$$

where g, A and φ depend on x^μ. With this choice of the conformal factor, the metric $\hat{g}_{\mu\nu}$ corresponds to the 4D Einstein frame. Inserting this ansatz into (2.4) and integrating over x^5,one obtains

$$S_{\text{eff}} = m_P^2 \int d^4x \sqrt{\hat{g}} \left(\hat{\mathcal{R}} - \frac{1}{4} e^{2\varphi} \hat{F}^2 - \frac{2}{3} (\hat{\partial}\varphi)^2\right), \tag{2.6}$$

where $m_P^2 = 2\pi R M^3$ is the 4D Planck mass, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the hat in the two last terms indicate that the indexes are raised/lowered with $\hat{g}_{\mu\nu}$. The appearance of the electromagnetic Lagrangian is what has fascinated physicists over the years, and is usually called 'the Kaluza Klein miracle'.

Along with the electromagnetic term, we have a scalar zero mode φ, ultimately related to g_{55} which parametrizes the physical radius of the circle S^1. This is a generic feature of KK theories. In more general compactifications of the form $M_4 \times \Sigma$, the parameters describing the geometry of the internal manifold Σ, enter the 4D effective theory as scalars. Here we shall call them generically *moduli*. In KK theories, the modulus parameterizing the size of the extra space is also known as *dilaton* or the breathing mode.

We shall note that in the 4D theory (2.4), the scalar, vector and tensor fields are massless. As pointed out above, there are gauge symmetries that prevent both A_μ and $\hat{g}_{\mu\nu}$ to be massive. However, this is not the case for the modulus φ. We see from (2.4) [48] that it is a Goldstone boson associated to the *global* symmetry

$$\varphi \to \varphi + c \tag{2.7}$$

$$A_\mu \to e^{-c} A_\mu. \tag{2.8}$$

It can be seen that when massive modes in the KK tower are included, this symmetry is destroyed [48]. Hence, φ is a pseudo-goldstone boson, and we expect that quantum effects may give it a mass.

To close this introduction to KK theories, we briefly comment on the phenomenological success of KK theories. First, we note that in 5D all massive KK modes have spin 2. It can be seen that a Higgs mechanism occurs by which the tensor 'eats' the vector and the scalar, adding up to the 5 polarizations of a massive spin 2 field.[4] The gravi-electromagnetic (or *graviphoton*) field A_μ couples to the graviton KK modes (as well as other higher dimensional fields) with a charge proportional to their mass $q_n = m_n/m_P$. In this way, Klein obtained the remarkable result of charge quantization [52]. If one identifies the fundamental unit of charge $q = 1/(m_P R)$ with the electron charge, then the compactification scale $1/R$ has to be somewhat larger than the Planck mass m_P, way beyond the range of any current or foreseeable accelerator. This corresponds to a size of the extra dimensions as tiny as $R \sim 10^{-33}$ cm, which agrees with our everyday experience of (effectively) living in four spacetime dimensions.

In order to construct a realistic model, one has to include matter in the picture. In principle, zero modes of a higher dimensional fermion field are expected to build ordinary matter. Hence the first nontrivial requirement is that the Dirac operator on Σ has at least one zero mode. But it is easy to see that the charge under A_μ carried by its n-th KK mode is $q_n \sim n/(m_P R) = m_n/m_P$. So, only heavy modes are charged.

This can be avoided including higher dimensional extra space Σ. In this case, one can have a 4D effective Yang-Mills (YM) theory, with a 'gauge' group given by the isometry group of Σ [43, 53, 44, 54, 55, 56, 57]. In practice, the most usually considered manifolds are coset spaces $\Sigma = G/H$, with H a subgroup of G. When a non abelian YM theory is obtained, massless fermions can be obtained with a nonzero charge.

Another major obstacle is the chirality of fermion representations in the Standard Model, first discussed by Witten [45]. The difference between the number of fermion zero modes of the Dirac operator with left and right chiralities is given by the index of the Dirac operator. This is often (though not always) zero, hence the resulting 4D theory is non-chiral.

One way to achieve a left-right asymmetry in 4D is to couple the higher dimensional fermions to a stable non-trivial background with a Yang-Mills flux on Σ (like a magnetic monopole) as was proposed by Randjbar-Daemi, Salam and Strathdee [58, 59]. The price to pay is to introduce gauge bosons in the higher dimensional picture, which contradicts the original KK spirit of obtaining all the interactions out of gravity only.

Another way to achieve a left-right asymmetry, used in the Randall Sundrum and Hořava Witten type models is to compactify on an orbifold. This kind of space is not everywhere smooth (it is not a *manifold*), hence the theorems on the absence of chiral fermions can be evaded.

2.2 The ADD mechanism: large extra dimensions

The ADD mechanism is a generic feature of brane models with large extra dimensions. To illustrate it we shall not stick to any specific model. Instead, we just assume that the space is of the form $M_4 \times \Sigma$, with Σ a smooth compact n dimensional manifold of radius R, and a four dimensional brane (a 3-brane) is located of the bulk.

By Gauss' law in $4+n$ dimensions below the compactification radius $r \ll R$, the Newtonian

[4] In 5D all the KK modes are of spin 2, but in higher dimensions, KK tower of vector and scalars appear as well.

potential between two particles of masses m_1 and m_2

$$V_N(r) = -G^{(4+n)} \frac{m_1 m_2}{r^{1+n}},$$

where $G^{(4+n)}$ is the $4+n$ dimensional Newton's constant. So, $V_N(r)$ decreases faster than the 4D interactions localized on the brane $\sim 1/r$. For larger distances $r \gg R$, the behaviour is

$$V_N(r) \sim -\frac{G^{(4+n)}}{V_n} \frac{m_1 m_2}{r} = -G_N \frac{m_1 m_2}{r},$$

where $V_n \propto R^n$ is the volume of Σ and we have identified the *effective* 4D Newton's constant as

$$G_N \sim \frac{G^{(4+n)}}{V_n}.$$

In this picture, it is readily understood that a large bulk volume V_n renders 4D gravity very weak. One of the key observations made in [1] is that observational bounds for the size of extra dimensions change dramatically if matter can propagate along them or not. In old KK theories with extra dimensions accessible to matter, their size is constrained to be at least $R \lesssim 1/\text{TeV}$, since no signature has been observed in accelerators. If only gravity can probe them the bounds come from short distance deviations of Newton's law and are much milder $R \lesssim 0.1\text{mm}$ [60, 61]. Such large sizes can easily account for an extremely suppressed gravity.

In terms of the $4+n$ dimensional Planck mass M and the usual Planck mass $m_P^2 \equiv 1/16\pi G_N$, we have

$$m_P^2 \sim V_n M^{2+n} \sim (MR)^n M^2. \tag{2.9}$$

This explicitly shows that the bulk size has to be large compared to the fundamental cutoff length $1/M$. Then, the ADD mechanism trades the hierarchy between strengths of gravity and gauge interactions by a new hierarchy.However, a geometric interpretation of the Planck/EW hierarchy sets a new chance to understand this puzzle, and as we shall see in Section 2.3, a number of mechanisms have been proposed in order to stabilize R at a large value.

On the other hand, Eq. (2.9) also suggests a solution to the problem of the quantum instability of scalar masses. Since even macroscopic values of R are not ruled out, there is the hope that the fundamental cutoff is very low, of a few TeV. In such a case, the sensitivity to the cutoff is not a problem because it is quite close to the EW scale. If we choose $M \sim$ TeV, we can read from Eq. (2.9) what is the necessary value of the size R of the bulk in order to obtain the large effective 4D Planck mass $m_P \simeq 10^{16}\text{TeV}$,

$$R \sim 10^{32/n-16} \text{ mm}. \tag{2.10}$$

Thus, with only one extra dimension its size would be astronomical, which is clearly ruled out. However, for $n \geq 2$ the necessary sizes range from submillimetric to $\geq 1/\text{TeV}$. The case with largest extra dimensions $\sim$ mm is for only two of them. In this case, deviations from Newton's law would be expected at this scale. We shall see in next Section that a cutoff M as low as 1TeV is in fact ruled out in this case by astronomical bounds. For larger number of extra dimensions, the smallness of R makes the search for deviations of Newton's law to seem hopeless.

2.3 ADD phenomenology

In this Section we discuss some of the phenomenological consequences of the BW scenario, concentrating on the simple case with flat and large extra dimensions. Many BW scenarios have generalized a bit the picture and allowed fields other than the graviton to propagate in the bulk. For instance in models inspired in string theory, the graviton multiplet contains more fields. For illustrative purposes, we restrict to the graviton.

Light KK gravitons

As shown in Section 2.1, higher dimensional fields are effectively described as a tower of 4D fields with increasing masses. Consider for instance a simple BW scenario where the extra dimensions form a flat $n-$torus T^n. We split the metric into the background and its gravitational fluctuations $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x, y)$. From the 4D viewpoint, a collection of massive 'gravitons' $h^{(q_i)}_{\mu\nu}$ appear. Here, q_i are n integers labeling the momentum that each graviton mode carries along the extra dimension.

Their coupling to matter confined to the brane is full metric,[5]

$$\int d^4x \, T^{\mu\nu} \, h_{\mu\nu}(x, 0) = \int d^4x \, \sum_{q_i} T^{\mu\nu} \, h^{(q_i)}_{\mu\nu}, \tag{2.11}$$

where $T^{\mu\nu}$ is the energy momentum of matter on the brane Thus, all the modes couple to ordinary matter with the same, *i.e.*, gravitational strength $1/m_P$.

Their 4D masses are given by $m^2_{(q_i)} = (q_1^2 + q_2^2 + \cdots + q_n^2)/R^2$ and generically are very small in the BW scenario. For $n = 2$, the first excitations weigh $\sim$ mm^{-1} $\sim$ meV. Hence, even though the KK gravitons are very weakly coupled, they are also very light and they might have observable effects. In turn, this allows to obtain observational constraints on the cutoff scale M in the BW scenario. There are a number of different physical phenomena constraining the allowed values for M, ranging from collider and laboratory experiments to astrophysics and cosmology [3] (for reviews on this topic, see [25, 65, 66, 67, 68, 69, 70]). In some cases, the most stringent bounds are obtained for particular values of n. For illustrative purposes, here we present only the most stringent bounds arising for low dimensionalities, coming from astrophysics [3, 25, 65, 66, 67]. We address the reader to the broad literature on this subject for more details [62, 63, 71, 72, 73, 74, 75, 76, 77, 78, 79].

Let us first consider a simple process where KK gravitons could leave some signature. For instance,

$$e^+ \, e^- \to \gamma + h^{(q_i)} \tag{2.12}$$

or $e^+ \, e^- \to Z + h^{(q_i)}$, as shown in Fig. 2.2. Since each KK graviton $h^{(q_i)}$ couples very weakly to matter, no signal is left other than missing energy for an observer on the brane. We can

[5] It is interesting to note [3] that the 4D scalar and vector components of the metric, h_{ij} and $h_{i\mu}$, do not couple directly to matter [3] (see [62, 63] for a more detailed discussion). This is true when we neglect the brane fluctuations or branons, which we briefly present below. Also, this interaction violates conservation of momentum in the transverse direction [3]. This is due to the presence of the brane, breaking translational invariance along the bulk. If the brane is rigid, one can interpret that it absorbs as much momentum as necessary. It can be shown that if we allow the brane to fluctuate, emission/absorption of a graviton into the bulk is accompanied by a transfer of momentum to the branon [64]. This can be viewed as a local deformation of the brane shape.

estimate the cross section for such a process with one individual KK graviton as

$$\sigma\left(e^+\, e^- \to \gamma + \text{KKgraviton}\right) = \frac{\alpha}{m_P^2}.$$

Thus the total cross section for having certain missing (center of mass) energy E carried off the brane by KK gravitons is the sum of the cross sections to emit any KK graviton with mass less than E,

$$\sigma\left(e^+\, e^- \to \gamma + \text{missing } E\right) = \frac{\alpha}{m_P^2} N(E), \tag{2.13}$$

where $N(E)$ is the number of KK modes with mass below E. Since the mass splittings are given by $\Delta m \sim 1/R$, there are $E/\Delta m = ER$ KK modes for each orthogonal direction transverse to the brane. Thus

$$N(E) = (ER)^n = m_P^2 \frac{E^n}{M^{n+2}},$$

where we used Eq. (2.9). We see that this large multiplicity (a consequence of the large volume) cancels the suppressing factor m_P^{-2} in (2.13), to finally obtain

$$\sigma\left(e^+\, e^- \to \gamma + \text{missing } E\right) = \frac{\alpha}{E^2}\left(\frac{E}{M}\right)^{n+2}, \tag{2.14}$$

which is comparable to a typical electromagnetic process at energies close to M.

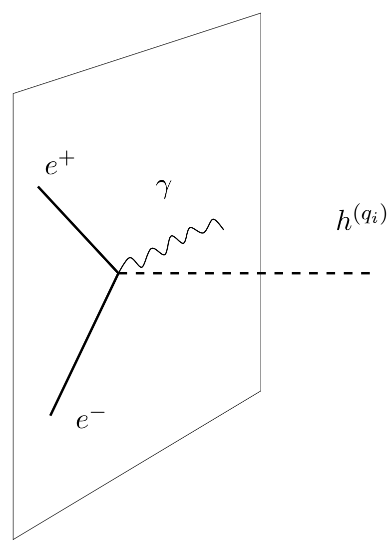


Figure 2.2:

Missing energy events in collider experiments give a lower bound for the cutoff between 1 and 2 TeV depending on the dimensionality of the bulk [3, 67]. LHC and $e^+\, e^-$ colliders at 1 TeV will be sensitive to a fundamental scale M up to several TeV, depending on n [25, 66, 67].

As mentioned above, the most stringent bounds on M arise in astrophysics, and specifically from star cooling via reactions such as (2.12). In the BW scenario, stars can cool by emitting KK gravitons into the bulk, *e.g.* through $\gamma\gamma \to 2h^{(q_i)}$, $e^+e^- \to 2h^{(q_i)}$ or $e\gamma \to eh^{(q_i)}$. From Eq. (2.14), we see that if the cutoff M is too low, the stars cool too fast. Moreover, the lowest bound on M comes from supernovae. Specifically, for SN1987A we can use the observed flux of neutrinos that reached the earth. The bounds depend on the supernova temperature ~ 30MeV, and are comprised in the ranges [80][6]

$$M \geq 9 - 60 \text{ TeV} \quad \text{for} \quad n = 2,$$
$$M \geq 0.6 - 3.8 \text{ TeV} \quad \text{for} \quad n = 3.$$

To conclude this discussion, we mention a recent paper [82, 81], where an even stronger bound is claimed to arise from neutron stars for low dimensionality of the bulk. The point is that KK gravitons around them decay into photons, electrons, positrons, and neutrinos. When these hit the neutron star they heat it. Bounds on neutron star heating then imply [82, 81]

$$M \geq 700 \text{ TeV} \quad \text{for} \quad n = 2,$$
$$M \geq 25 \text{ TeV} \quad \text{for} \quad n = 3.$$

This leads to a major problem of the ADD mechanism with only two extra dimensions. In this case, the smallest values that observations allowed for M imply that there is a 'small' hierarchy $\sim 10^2$–10^3 between the cutoff M and the EW scale ~ 300GeV. It is interesting that the deviation in the Newton's law within the ADD scenario for $n = 2$ will be tested in the near future.

We shall emphasize that the model with $n = 2$ makes a very definite, namely the change from the $1/r^2$ to the $1/r^4$ regimes in the short distance Newton's law. If the planned experiments [60, 61] do not find this transition in the nm–μm regime, the BW scenario with 2 flat extra dimensions will be ruled out.

Branons

In the BW scenario, the branes are supposed to be dynamical objects irrespective of their origin. The idea is that they might arise as topological defects in field theoretic models or as solitonic vacuum solutions in string theory. In both cases, they are described by their own dynamical degrees of freedom unless some symmetry of the underlying theory forbids it. In practice, this means that for any background with branes present, in general we should allow them to fluctuate, as illustrated in Figure 2.3.

The thin wall approximation[7] is a valid description of the extended objects for long distances, *i.e.* wavelengths large compared to the thickness $1/M$. In this approximation, if the four dimensional brane moves along a $4 + n$ dimensional bulk space of the form $M_4 \times \Sigma$, its position is fully described by a set of n functions $Y^i(x^\mu)$ determining the location of the brane in the bulk at every 4D spacetime point x^μ, as shown in Figure 2.3. Thus, the fluctuations of the brane position are described by n 4D scalar fields $Y^i(x)$.

In the context of the BW scenario, they are known as *branons* and were first discussed in [84], and further studies were carried on in [3, 85, 64, 86, 87, 88]. In the thin wall approximation,

[6]Note we have defined the fundamental cutoff scale M such that higher dimensional curvature term is $\int M^{n+2}\,\mathcal{R}_{(4+n)}$. So,

$$\frac{1}{16\pi G_N} \simeq \left(1.7210^{18}\,\text{GeV}\right)^2 = V_n M^{n+2}.$$

With this definition, the bounds on M can be obtained from those obtained in [81] rescaling their $\bar{M}_{4+n}$ by a factor $2^{-1/(n+2)}$.

[7]For a nice introduction to topological defects and their cosmological implications see [83]

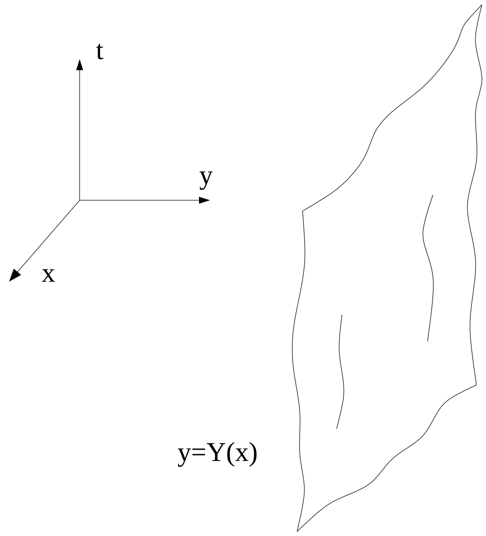


Figure 2.3:

the Nambu Goto action is a good approximation to describe the dynamics of the brane and is of the form

$$S_{\rm brane} = S_{NG} + S_{\rm matt} = -\tau \int d^4x \sqrt{g_{\rm ind}} + \int d^4x \sqrt{g_{\rm ind}}\, \mathcal{L}^{\rm matt}(\chi, g_{\rm ind}), \tag{2.15}$$

where $g^{\rm ind}$ is the induced metric on the brane, τ is its tension and we also included the Lagrangian of generic matter fields χ on the brane. If the bulk metric is flat,

$$ds^2_{\rm ind} = \left[\eta_{\mu\nu} + \delta_{ij}\partial_\mu Y^i \partial_\nu Y^j\right] dx^\mu dx^\nu \equiv g^{\rm ind}_{\mu\nu} dx^\mu dx^\nu,$$

and the NG action reads

$$S_{NG} = -\tau \int d^4x \left(1 + \frac{1}{2}\delta_{ij}\eta^{\mu\nu}\partial_\mu Y^i \partial_\nu Y^j + \mathcal{O}(\partial Y)^4\right). \tag{2.16}$$

The branons Y^i are the Goldstone bosons associated to the spontaneous breaking of translation symmetry in the bulk by the brane and are described in general by a non linear σ model with coupling constant given by the tension of the brane τ. It is clear from (2.15) that in order for their kinetic term to be positive the tension has to be positive as well. In models where the branes sit at the fixed points of an orbifold, the brane are not free to fluctuate and no branons appear [89].

In fact any explicit breaking of translation invariance (*e.g.* due to the backreaction of the brane on the bulk metric) renders this an approximate symmetry and the branons *pseudo-goldstone* bosons, with some nonzero mass.

Since the branons appear in the induced metric on the brane, their coupling to matter is of the form $\partial_\mu Y \partial Y_\nu T^{\mu\nu}$ for a flat bulk. So, their contribution to any cross section involving matter is suppressed by E^4/τ. In models with large extra dimensions, the tension of the brane is of order TeV^4, which gives a stronger coupling than the KK gravitons. In contrast with the KK gravitons, branons do not have a large multiplicity (they have no KK tower). This makes the most stringent bounds on their mass and coupling to come from collider experiments. Roughly, one obtains $m_Y \gtrsim 100\text{GeV}$ and $\tau \gtrsim (120\text{GeV})^4$. Due to their suppressed coupling to matter, it has recently been claimed [90] that they are natural candidates to Dark Matter within the BW scenario.

Radion stabilization

In the BW scenario, the modulus that parametrizes the size of the bulk can be identified with the 4D scalar field present in the Kaluza-Klein ansatz (2.5). It is generally called *the radion* and here will be denoted as φ. In the literature it is also called graviscalar or dilaton, though the latter is usually reserved to the spin 0 state present in the spectrum of fundamental strings.

Let us develop the conditions that an *efficient* stabilization mechanism has to accomplish in order for the hierarchy problem to be completely solved within models of ADD type.

First of all, the ADD mechanism relies on the assumption that the bulk volume is large compared to the fundamental scale, $\langle R\rangle \gg 1/M$. With no further justification, this is an interchange of one hierarchy (EW *vs.* Planck scales) by another (bulk size R *vs.* fundamental scale or equivalently the brane thickness $1/M$). In terms of the 4D effective theory, this means that the radion vev is large.

Second, we note that the radion is part of the higher dimensional metric. As such, it is a massless degree of freedom at tree level in the effective theory.[8] This would introduce unwanted scalar interactions unless the radion couples weakly to matter. Matter couples to gravity through the induced metric on the brane as $g^{\mu\nu}_{\text{ind}}T_{\mu\nu}$. From Eq. (2.5), one of the terms is

$$\int d^4x \, \frac{1}{m_P} \, \varphi \, T^\mu_\mu, \tag{2.17}$$

with φ canonically normalized. Thus, the radion couples with gravitational strength. Current lower bounds on the mass for scalars gravitationally coupled to matter come from short distance deviations from Newton's law and demand a millimetric mass $m_\varphi \gtrsim 1/\text{mm}$. [9]

Thus, a complete solution to the hierarchy problem requires some mechanism that explains two things. First, why the radion is stabilized at such a large vev $\langle R\rangle \gg 1/M$. Second, what gives it a mass of order $1/\text{mm} \sim$ meV.

This may be a quite non-trivial task. From Eq. (2.5), the physical radius of the extra dimensions R is related to the canonical radion field φ as $R \propto e^{\varphi/m_P}$. If the stabilizing potential $V(R)$ establishes a competition between two different powers of the radius R (and with a minimum for R large), the resulting radion mass is given by

$$m_\varphi^2 \equiv \partial_\varphi^2 V(\varphi) = \left(\frac{R}{m_P}\right)^2 \partial_R^2 V(R) \ .$$

[8] As noted in Section 2.1, in contrast with the metric spin-2 and spin-1 zero modes, there is no gauge symmetry in the 4D effective theory that protects the *radion* from getting a mass. Thus, the masslessness at tree level is expected to be an spurious effect. However, the radion mass is still expected to be *small*.

[9] It can be seen that cosmological and astrophysical bounds are milder [4].

This can be too small if the potential does not contain large enough powers [3, 91, 4][10], and the radion still may lead to observable effects.

Let us describe some of the mechanisms that have been proposed in the literature so far. In [4], several mechanisms are discussed. The attitude is to find different effects that induce an attraction between the branes at long distances (stability under expansion) and repulsion at short scales (stability under collapse). For instance, a positive cosmological constant in the bulk Λ generates a potential that scales with the volume of the bulk ΛR^n in the 4D theory and prevents the bulk from expanding [4]. On the other hand, a bulk curvature term scales as $-M^{n-2}R^{n-2}$ and a number $N_{\rm b}$ of branes with tensions τ contribute a constant term. Combining these three contributions

$$V(R) \sim \Lambda R^n - M^4(MR)^{n-2} + N_{\rm b}\tau,$$

and it is easily verified that a minimum with zero potential (effective 4D cosmological constant), $V|_{\rm min} = V'|_{\rm min} = 0$ occurs at $R \sim N_{\rm b}^{1/(n-2)}/M$. This is large if the number of branes is large, $N_{\rm b} \sim 10^{10} - 10^{20}$ and $n > 2$ [4]. Such large numbers can be obtained dynamically in a model where the branes carry some conserved charge. The idea in this mechanism is that the bulk size is not set by the interbrane distance but by the size of the brane lattice. This has been called the 'large brane number scenario', or the 'brane lattice crystallization'.

It was shown in [6] that the Casimir forces arising in higher dimensional spaces might stabilize R. The possibility that such a stabilization occurs at large values of R was also considered in [4]. The same kind of arguments as in the previous paragraph suggest that in a simple scenario with either a large or small $N_{\rm b}$, the Casimir forces cannot account for large values of the radius.

Another mechanism arises in topologically nontrivial theories [92, 91, 4, 85]. Assume for instance, that there exists a $U(1)$ gauge field in the bulk. Then, it can take monopole-type configurations, with a quantized monopole number k that is topologically conserved. This prevents the bulk from collapse. Thus, the 4D effective action for such a configuration displays a repulsive potential. As before, for large enough 'monopole' number k, the radion can be stabilized at large values. In some situations with $n = 6$ and $M \sim 10$TeV [4], the required number of monopoles is of order 1.

A mechanism to stabilize the bulk size when codimension 2 branes are present is described in [94]. Such kind of branes behave as strings in 4D, so generate deficit angles $\tau_i/2M^4$. A set of codimension 2 branes whose deficit angles add up to 2π define a compact bulk. It is easy to show that if it equals 4π, static solutions to Einstein's equations exist with a flat bulk [95, 96]. The required fine tuning of the brane tensions corresponds to usual tuning of the 4D cosmological constant. It turns out that a classical bulk scalar field in this background 'interacts' with the branes generating a logarithmic potential for the radius, and an exponentially large minimum is naturally achieved. This model requires supersymmetry in the bulk in order to suppress, for instance, the effects of a cosmological constant.

A slight variation of the ADD mechanism [97] consists in considering compact hyperbolic manifolds as the internal space. The volume of these spaces depends exponentially on their

[10] To illustrate that this intuition does not always apply, in [3], a toy model is presented where due to nonperturbative effects a scalar field in a super Yang-Mills (SYM) $SU(N) \times SU(N)$ can obtain arbitrary vev's completely uncorrelated with the curvature of the potential at the minima.

linear size, but the eigenvalues of the Laplacian are set by the curvature scale. Thus, a large volume can be easily obtained with a linear size close to the fundamental cutoff M with the key difference that the KK modes are heavy ($\sim M$). This automatically passes all cosmological and astrophysical tests.

The radion stabilization in these models is considered in [98], where it is shown that any form of higher dimensionial matter that stabilizes this kind of extra dimensions must violate the null energy condition. Thus, the vacuum energy (the Casimir effect) is a good candidate to stabilize these manifolds.

Another mechanism to obtain exponentially large bulk sizes naturally due to the self interaction of a bulk scalar field has been discussed in [99]. In this model, the radion mass is very small, $m \sim 10^{-33}$ eV. The authors argue that such a small mass is not a problem since cosmological evolution drives the coupling of the radion to matter to small enough values so that observational tests are passed. On the other hand, this light radion is considered to play a role in quintessence models [100].

2.4 Summary

We can summarize the main features of the BW scenario in the following points. First, the ADD mechanism to solve the hierarchy problem is viable with two or more extra dimensions $n \geq 2$. In other words, the branes must be of codimension[11] larger than 1. The reason is that ADD mechanism is a *large volume* effect. With only one extra dimension, it has to be of astronomical size, which is clearly ruled out. This conclusion is subject to the assumption that the bulk is flat. We shall see in Chapter 7 that the ADD mechanism can account for a large hierarchy with codimension 1 branes if the extra dimensions are warped.

The main distinctive signature of ADD phenomenology is the appearance of very light KK gravitons, with large multiplicities (a large number of them at available energies) and gravitational $1/m_P$ coupling to matter. This makes the most stringent bounds to come from astrophysics.

In order to completely solve the hierarchy problem in this scenario, the stabilization mechanism must explain (in a natural way) why the bulk size is large $\langle R \rangle \gg 1/M$ and how a (relatively small) mass $m_\varphi \gtrsim$ meV is generated for the radion.

[11] The *codimension* is the difference between the bulk and brane dimensions.

Chapter 3

The Randall Sundrum model

The Randall Sundrum (RS) model [2] consists of a simple realization of the BW scenario in five dimensions with two branes of codimension one. Specifically, consider five dimensional Einstein Hilbert action plus brane tension terms,

$$\begin{aligned} S_{RS} &= \int d^5x\sqrt{-g}\left(M^3\mathcal{R}-\Lambda\right) \\ &+ \int d^4x\sqrt{-g_+}\left(\mathcal{L}^+-\tau_+\right)+\int d^4x\sqrt{-g_-}\left(\mathcal{L}^--\tau_-\right). \end{aligned} \tag{3.1}$$

where $g^{\pm}_{\mu\nu}(x)$ are the metrics induced by the bulk metric $g_{\alpha\beta}(x,y)$ at the brane positions $y=y_{\pm}$. The tensions of each brane are $\tau_{\pm}$, and $\mathcal{L}^{\pm}$ are the Lagrangians of matter localized on them.

The warped metric[1]

$$ds^2 = a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + dy^2. \tag{3.2}$$

with $a(y)e^{-k|y|}$ is a solution of the Einstein's equations following from (3.1) as long as

$$\Lambda = -12M^3k^2. \tag{3.3}$$

The Israel matching conditions[2] are satisfied if

$$\tau_+ = -\tau_- = 12M^3k. \tag{3.4}$$

Thus, we obtain a solution if $\Lambda<0$ and we tune the parameters in the action (3.1) Λ and $\tau_{\pm}$, according to

$$\tau_{\pm}^2 = -12M^3\Lambda.$$

Due to the presence of the (gravitating) branes, the extra dimension can be rendered compact assuming that it has an S^1/Z_2 orbifold topology, as in the Hořava Witten theory [9, 10]. As illustrated in Fig. 3.1, an S^1/Z_2 orbifold is a circle with a mirror symmetric points $y\leftrightarrow -y$ identified. There are two (antipodal) points which are their own mirror images, and are called

[1] The conformal factor $e^{-2k|y|}$ in front of the Minkowski factor is known as the *warp factor*. Metrics of this form do not describe direct products of spaces, as illustrated by the metric of the unit sphere, $ds^2=d\theta^2+\sin^2\theta d\phi^2$.

[2] Upon integrating Einstein's equations in a neighborhood of the branes, one obtains the Israel matching conditions, $[\mathcal{K}_\mu{}^\nu]-\delta_\mu{}^\nu[\mathcal{K}_\rho{}^\rho]=-(1/2M^3)S_\mu{}^\nu$, where $\mathcal{K}_\mu{}^\nu$ is the extrinsic curvature, $[\mathcal{K}_\mu{}^\nu]\equiv\mathcal{K}_\mu{}^\nu(y_0+\epsilon)-\mathcal{K}_\mu{}^\nu(y_0-\epsilon)$ (with $\epsilon\to 0$) is its discontinuity across the brane, and $S_\mu{}^\nu$ is the energy momentum tensor generated by the brane.

fixed points. Hence, this orbifold can be viewed as a segment. In the RS model, the branes sit at the orbifold fixed points, corresponding to $y = 0$ and $y = d$ in our coordinate $-d \leq y \leq d$. As mentioned above, the identification of points at each side of the branes effectively prevent it to fluctuate. So, we can interpret that these branes are very rigid, namely their brane tension $\to \infty$ [4].

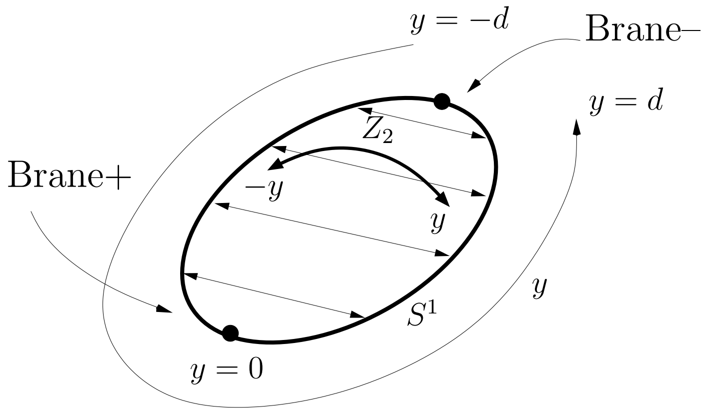


Figure 3.1:

Note that the interbrane distance d is an integration constant of the solution. Thus, it corresponds to a flat direction of the potential, *i.e.* to a massless scalar in the 4D effective theory. This 'light' degree of freedom is the same that we found in Chapter 2 and is called the *radion*. We discuss it in more detail in Section 3.3.

The line element (3.2) corresponds to a slice of 5D anti de Sitter space AdS_5 (cut along flat 4D sections and with a thickness d), which we denote by $|AdS_5|$ (see Fig. 3.2). Due to the homogeneity of AdS, we can take $y_+ = 0$ with no loss of generality, the only physically meaningful quantity is the interbrane distance $|y_+ - y_-| = d$.

3.1 The RS mechanism: redshift effect

In this Section, we describe how the RS model solves the hierarchy problem. The first issue is to find the 4D effective gravity arising from this 5D theory. To do this, we shall proceed as in Section 2.1 and introduce an appropriate KK ansatz that accounts for the low energy fluctuations of the 5D metric. In the RS case, the presence of the branes spoils the symmetry of the extra dimension under translations. Then, as seen in Section 2.1, this (gauge) symmetry is not present in the 4D theory and accordingly there is no massless *graviphoton* mode $g_{y\mu}$. Another reason for the absence of the graviphoton is the orbifold condition. If Z_2 (or mirror) symmetric points y and $-y$ are identified, all the bulk fields must be either even ($\Phi(-y) = \Phi(y)$) or odd ($\Phi(-y) = -\Phi(y)$) under Z_2[3]. The $\{\mu\nu\}$ components must be even in order that they contains a zero mode, the 4D graviton. Then $g_{y\mu}$ must be odd, which means that no zero mode arises.

Thus we only need to take into account the tensor and scalar fluctuations of $g_{\alpha\beta}$. Let us

[3]this condition can be relaxed. See *e.g.* [101, 102].

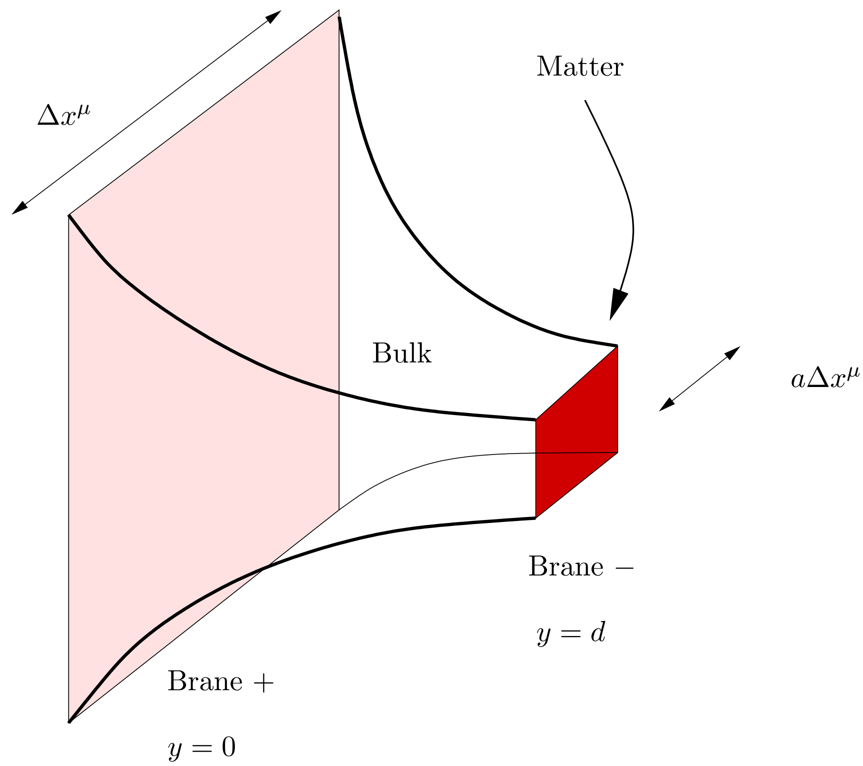


Figure 3.2: Matter is located on the negative tension brane. The same comoving distance Δx^μ represents a physical distance $a\Delta x^\mu$ (Δx^μ) on the negative (positive) tension brane .

concentrate now on the tensor perturbations. We can always parametrize $g_{\alpha\beta}$ as

$$ds^2 = a^2(y)\, \tilde{g}_{\mu\nu}(x)\, dx^\mu dx^\nu + dy^2, \tag{3.5}$$

where $\tilde{g}_{\mu\nu}$ depends on the x^μ only. The Ricci scalar corresponding to this ansatz is $\mathcal{R} = a^{-2}(y)\,\tilde{\mathcal{R}}$, where $\tilde{\mathcal{R}}$ is computed with $\tilde{g}_{\mu\nu}$. Inserting (3.5) into (3.1) and integrating the explicit dependence on y, we obtain

$$S^{(4)}_{RS} = m_P^2 \int d^4x \sqrt{\tilde{g}}\, \tilde{\mathcal{R}}, \tag{3.6}$$

where we have identified

$$m_P^2 = M^3 \int_{-d}^{d} dy\, a^2(y) = \frac{M^3}{k}\left(1 - a^2\right), \tag{3.7}$$

and we have introduced

$$a \equiv a(y_-) = e^{-kd}.$$

In contrast with KK theories and the ADD mechanism, the actual value of the 4D effective Planck mass m_P depends very marginally on d, which means that the RS mechanism is not based on a large volume effect. On the other hand, Since we do not want to introduce large numbers in the model, we assume that the curvature scale of the background ($1/k$ is the curvature radius of AdS) is somewhat below but of the order of the fundamental scale $k \lesssim M$ (*i.e.* , $\Lambda \lesssim M$). In conclusion, both scales are of order $m_P \sim 10^{16}$TeV.

The key point for the RS model to account for the Planck/EW hierarchy is to assume that ordinary (EW) matter is located on the negative tension brane , at $y = d$. Its action corresponds to the piece of (3.1) with

$$S_{\rm matt} \equiv \int d^4x \sqrt{g_-} \mathcal{L}^- \left(\chi_-, g_-\right),$$

where we have made explicit that the Lagrangian for matter depends on the induced metric on the brane. So, matter couples universally to the metric (recall that $a \equiv e^{-kd}$)

$$g^-_{\mu\nu} = a^2 \tilde{g}_{\mu\nu}.$$

Consider for instance a massive scalar field χ^-

$$S_{\rm matt} = -\frac{1}{2} \int d^4x \sqrt{g_-} \left(g_-^{\mu\nu} \partial_\mu \chi_- \partial_\nu \chi_- + m^2 \chi_-^2\right). \tag{3.8}$$

Let us write now this action in the 4D Einstein frame defined by $\tilde{g}_{\mu\nu}$ (see Eq. (3.6)). In terms of the rescaled fields $\chi \equiv a\chi_-$ (with canonical kinetic terms),

$$S_{\rm matt} = -\frac{1}{2} \int d^4x \sqrt{\tilde{g}} \left(\tilde{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + a^2 m^2 \chi^2\right). \tag{3.9}$$

Thus, the 4D physical mass of this field is exponentially smaller than in the 5D theory. This is a completely general result. Any energy scale E in the 5D theory localized on the negative tension brane corresponds to a 4D physical scale $aE = e^{-kd}E$. This is due to the warp factor in (3.2), hence a purely geometrical redshift phenomenon.

It is now clear how to solve the hierarchy problem. We only need to locate the EW sector on the negative tension brane , with physical 5D masses of order the cutoff M. From the 5D theory the sensitivity of the scalar masses with the cutoff is not a problem since it is of the same order as the masses themselves. From the 4D effective theory, these masses appear as $\sim aM$. So the ratio of the EW to the Planck scales is precisely given by

$$\frac{aM}{m_P} \sim a, \tag{3.10}$$

where we have used (3.7). Such an enormous redshift can be accomplished with

$$kd \sim 37, \tag{3.11}$$

which is certainly not a very large number. This means that the required interbrane distance d is also of order M. In contrast with the ADD mechanism where two or more dimensions in the bulk are needed, the RS mechanism is viable with one extra dimension.

As in ADD, a full solution of the hierarchy problem requires some mechanism that naturally fixes the interbrane distance to the required value. This is discussed in Sections 3.3 and 3.4. Before entering into that, in the next Section we present the KK structure of the RS model, showing a sharp contrast with KK theories or models with large extra dimensions.

3.2 Bulk fields in |AdS|

In order to present the KK spectrum arising in the RS model we shall first consider the simpler case of a generic bulk scalar field. Then, we show how the graviton and gauge boson in the bulk can be recovered as some special cases of such scalar.

Considering fields other than the graviton propagating in the bulk is a slight generalization of the BW picture (see for instance [103, 104, 105]) that has several motivations. In many BW scenarios like the RS model, the size of the extra dimensions are not large, $\mathrm{TeV}^{-1} \leq R \leq m_P^{-1}$. This opens up the phenomenologically acceptable possibility of matter fields propagating in the bulk.

Before entering into details of the KK decomposition, it will be technically convenient for later use to describe the RS background solution in an arbitrary number D of dimensions (*i.e.* , $|\mathrm{AdS}_D|$) using the *conformal* coordinate z. This is defined so that the metric takes the form

$$ds_D^2 = g^{\scriptscriptstyle D}_{\alpha\beta} dx^\alpha dx^\beta = a^2(z)\left[\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right], \tag{3.12}$$

where $\mu, \nu, \dots = 0, 1, \dots, D-1$. Up to an additive constant, $z = e^{ky}/k$, and the conformal coordinate corresponding of the branes are

$$z_+ = \frac{1}{k} \qquad \text{and} \qquad z_- = \frac{1}{ak}\ .$$

In this coordinate he warp factor takes the form (in the bulk)

$$a(z) = \frac{1}{kz}\ . \tag{3.13}$$

We can write the D dimensional D'Alembertian as

$$\Box_D = \frac{1}{\sqrt{g_{\scriptscriptstyle D}}}\partial_\alpha\left(\sqrt{g_{\scriptscriptstyle D}}\, g_{\scriptscriptstyle D}^{\alpha\beta}\partial_\beta\right) = \frac{1}{a(z)^2}\left[\partial_z z^{D-2}\partial_z z^{2-D} + \Box_0\right] \tag{3.14}$$

where $\Box_0$ is the $D-1$ dimensional flat D'Alembertian.

Scalar field

Consider a massive scalar field Φ in AdS_D [103, 12] with a non minimal coupling to the curvature ξ. Its equation of motion in the bulk is

$$\left[-\Box_D + m^2 + \xi\mathcal{R}_D\right]\Phi = 0. \tag{3.15}$$

Scalar fields can obey two types of boundary conditions, depending on whether Φ is even $\Phi(-y) = \Phi(y)$ or odd $\Phi(-y) = -\Phi(y)$.[4] In the last case, Φ must vanish on the branes, $\Phi(0) = \Phi(d) = 0$, *i.e.* the field satisfies *Dirichlet* boundary conditions. If Φ is even, it need not vanish on the branes and whether the derivatives vanish or not depend on the coupling that the field has on the brane. For example, a boundary mass term

$$\frac{1}{2}\int d^{D-1}x\sqrt{g_\pm}m_\pm\Phi^2$$

[4] *Odd* and *even* fields are also known as *twisted* and *untwisted* respectively.

induces δ function terms in the equation of motion , and give rise to *Neumann*[5] type boundary conditions of the form

$$\left(\partial_y - \frac{1}{2}m_\pm\right)\Phi\Big|_\pm = 0.$$

Here $|_\pm$ means that it should be evaluated at $y = y_\pm \pm \epsilon$, with $\epsilon \to 0$. We note that a non minimal coupling to the curvature also generates 'mass terms' on the branes, since the appearance of $|y|$ in the warp factor (3.2) induces δ function terms. This can be taken into account by adding $\pm 4(D-1)\xi$ to brane masses $m_\pm$. In the following, we assume that the brane mass parameters differ only in sign, $m_+ = -m_- \equiv m_b$.

The Ricci scalar of AdS_D is a constant given by

$$\mathcal{R}_D = -D(D-1)k^2, \tag{3.16}$$

so that a non minimal coupling to curvature (with $\xi > 0$) is equivalent to a negative $(\mathrm{mass})^2$. Using last equation, (3.14) and (3.14), Eq. (3.15) can be written as

$$\left[-z^{D-2}\partial_z z^{2-D}\partial_z - \Box_0 + \left(\left(\frac{m}{k}\right)^2 - \xi D(D-1)\right)\frac{1}{z^2}\right]\Phi = 0. \tag{3.17}$$

We can now decompose the bulk field in $D-1$ plane-wave modes as

$$\Phi(y,x) = \sum_n e^{ik_\mu^{(n)}x^\mu} f_n(z). \tag{3.18}$$

For each mode, $\Box_0$ gives a factor

$$-\eta^{\mu\nu}k_\mu^{(n)}k_\nu^{(n)} \equiv m_n^2,$$

which we identify as its physical KK mass. Then, we are left with the following eigenvalue problem

$$\left[\partial_z z^{D-2}\partial_z z^{2-D} - \left(\left(\frac{m}{k}\right)^2 - \xi D(D-1)\right)\frac{1}{z^2}\right]\Phi = m_n^2\Phi, \tag{3.19}$$

that fixes the KK mode 'wave functions' $f_n(z)$ and masses m_n when we specify the boundary conditions for Φ.

Aside from a possible zero mode, which will be discussed below, the solutions to this equation are

$$f_n(z) = \epsilon(z)(kz)^{(D-1)/2}\left[A_n J_\nu(m_n z) + B_n Y_\nu(m_n z)\right], \tag{3.20}$$

where A_n and B_n are constants, $\epsilon(z) = 1$ for even Φ and

$$\epsilon(z) = \frac{|y|}{y} = \begin{cases} -1 & \text{for } z > 1/k \\ 1 & \text{for } z < 1/k \end{cases}, \tag{3.21}$$

for odd Φ. The index ν is given by

$$\nu^2 = \frac{m^2}{k^2} - D(D-1)\xi + \frac{(D-1)^2}{4}. \tag{3.22}$$

The boundary conditions determine the KK masses m_n implicitly as the zeros of

$$F_\nu(m_n z_-) = 0, \tag{3.23}$$

[5] also known as *Robin* in the literature.

where

$$F_\nu(z) = \begin{cases} Y_\nu(az)J_\nu(z) - J_\nu(az)Y_\nu(z) & \text{for } \Phi \quad \text{odd} \,, \\ y_\nu(az)j_\nu(z) - j_\nu(az)y_\nu(z) & \text{for } \Phi \quad \text{even} \,, \end{cases} \tag{3.24}$$

and in the even case the combinations of Bessel functions arising are

$$\begin{aligned} j_\nu(z) &= zJ_{\nu-1}(z) + \varepsilon J_\nu(z) \,, \\ y_\nu(z) &= zY_{\nu-1}(z) + \varepsilon Y_\nu(z) \,, \end{aligned} \tag{3.25}$$

with

$$\varepsilon = \frac{D-1}{2} - \nu - \frac{m_b}{2} - 2(D-1)\xi. \tag{3.26}$$

Gauge field

The equation of motion for a gauge field $A_\alpha(x^\mu, z)$ are $\nabla_\alpha F^{\alpha\beta} = 0$, where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. In $|\mathrm{AdS}_5|$, and using the 'physical' gauge

$$A_5 = \partial_\mu A^\mu = 0$$

(where the equations decouple), these equations reduce to [105]

$$\left(z\partial_z z^{-1}\partial_z + \Box_0\right) A_\mu = 0. \tag{3.27}$$

Assuming that A_μ are even and that A_α has no brane mass terms $m_b = 0$, the solutions to these equation are of the form (3.18, 3.20), taking $\xi = 1/8$ and $m^2 = -k^2/2$. Thus, we obtain $\nu = 1$ and $\varepsilon = 0$ for a 5D gauge field.

This is readily understood since the equation of motion can also be written in this gauge as

$$\left(\Box_5\, \delta_\mu^\nu - \mathcal{R}_\mu^\nu\right) A_\nu = 0, \tag{3.28}$$

where $\mathcal{R}_\mu^\nu$ is the 5D Ricci tensor, with boundary conditions $\partial_z A_\mu|_\pm = 0$.

Since the Ricci tensor in the bulk is constant and proportional to the metric $\mathcal{R}_\mu^\nu = 4k^2\delta_\mu^\nu$, the components of A_μ satisfy the Klein-Gordon equation $(\Box_5 - m^2 - \xi\mathcal{R})A_\mu = 0$. The mass term is needed because the relation between the bulk part and the δ function terms for the Ricci tensor and for the Ricci scalar are different. The precise values of m and ξ that reproduce (3.28) are the ones given above.

It is clear from (3.27) that the 5D gauge field has a massless mode ($\Box_0 = 0$) satisfying the Neumann boundary conditions. This corresponds to the 4D photon, the massless mode in the KK decomposition.

The graviton

Our background has maximally symmetric foliations orthogonal to the y direction, and just like in the case of cosmological Friedmann-Robertson-Walker models, each graviton polarization contributes as a massless minimally coupled scalar field. The correspondence is straightforward at the classical level. At the quantum level, it can also be shown to hold, although this is not so straightforward to prove because careful gauge-fixing of the gravitational sector must be done.

In short, the correspondence is as follows [7]. Perturbations of the gravitational field are described by splitting the full metric into the background solution $g_{\alpha\beta}$ (3.2) plus perturbations around it, $h_{\alpha\beta}$. In the Randall-Sundrum gauge $h_{55} = h_{\mu 5} = 0$, we can express the metric fluctuations as

$$h_{\mu\nu}(x,z) = \sum_{i=1}^{5} a^2(z) h^{(i)}_{\mu\nu}(x,z), \tag{3.29}$$

where the polarization tensors $h^{(i)}$ satisfy $\partial^\mu h^{(i)}_{\mu\nu} = h^{(i)\mu}_{\ \ \mu} = 0$. The quadratic reduced action for one particular polarization becomes

$$-\int d^D x \sqrt{g}\, h^{(i)} \Box h^{(i)} + (\text{boundary term}), \tag{3.30}$$

where $\Box$ is the usual covariant scalar Laplacian associated with the 5D background metric g (3.2).

Thus in the physical gauge, metric perturbations are equivalently described by scalar fields with appropriate boundary conditions at the branes. These are determined from Israel's junction conditions plus the requirement that the metric components $h_{\mu\nu}$ are even. So, they reduce to the standard Neumann boundary conditions

$$\partial_z h^{(i)} = 0, \tag{3.31}$$

at $z = z_\pm$.

This means that we can effectively take into account the gravitons as massless minimally coupled scalar fields with no brane mass terms. In particular we obtain $\nu = 2$ and $\varepsilon = 0$.

Thus, the KK tower has masses given by

$$m_n = a k x_n,$$

where x_n are the roots of $F_2(x)$ defined in (3.24) with $\varepsilon = 0$ and in the even case. This leads to the first sharply distinguished feature in the phenomenology of the RS model. In contrast with ADD or KK theories, the KK modes (of order TeV) are much lighter than he compactification scale, $1/d \sim k \sim m_P$.

Another remarkable feature of the RS model arises in the coupling of the graviton KK modes $h^{(n)}$ with SM fields. Decomposing the metric perturbation as in (3.18),

$$h_{\mu\nu}(x,z) = \sum_n f_n(z)\, h^{(n)}_{\mu\nu}(x),$$

with f_n defined in (3.20). It is easy to verify that in the limit of large hierarchy $a \ll 1$, the coefficients in (3.20) satisfy $B_n \ll A_n$, and $A_n \simeq a\sqrt{k}/J_2(x_n) \sim a\sqrt{k}$ for low n. The interaction with matter occurs through the induced metric on the negative tension brane , [106]

$$-\int d^4x \sqrt{g_-} \frac{1}{M^{3/2}} h^{\mu\nu}(x, z_-) T^{(-)}_{\mu\nu} = -\int d^4x \left\{ \frac{1}{m_P} h^{\mu\nu}_{(0)} T_{\mu\nu} + \sum_{n=1}^{\infty} \frac{1}{\Lambda_{KK}} h^{\mu\nu}_{(n)} T_{\mu\nu} \right\} \tag{3.32}$$

with

$$\Lambda_{KK} = a\, m_P \sim \text{TeV}.$$

In (3.32), we have used that under a conformal transformation $g^{-}_{\mu\nu} = a^2 \tilde{g}_{\mu\nu}$, the energy momentum tensor transforms as $T^{(-)\,\mu}_{\nu} = a^{-4} T^{\mu}_{\nu}$ [107]. We have specialized to five dimensions, used that $f_n(z_-) \sim \sqrt{k}/a$ and (3.7).

From (3.32), we see that whereas the 4D graviton couples to matter with the ordinary gravitational strength, the coupling of the graviton KK modes is of TeV strength. Consequently, each of these modes are individually detectable in colliders at energies of order TeV.

We can trace back this property to the form of the KK wave functions $f_n(z)$, highly peaked on the negative tension brane . This is a generic feature of the RS model. The KK modes of any bulk field are peaked on the negative tension brane , so all their couplings to matter are of TeV size. The same happens with their masses. As we have seen above, these are of order $\langle a \rangle k$TeV. The reason is that these effective 4D masses result from integrating over the orbifold. Since the mode functions $f_n(z)$ are exponentially suppressed near $z = z_+$, these modes behave as particles localized on the negative tension brane . So, the same redshift effect described in Section 3.1 is expected to operate [103].

This has a further interpretation in terms of the AdS/CFT correspondence. Whereas the fields localized on the positive tension brane correspond to the fundamental degrees of freedom in the CFT, relevant at the UV cutoff scale m_P, the KK modes are dual to the CFT resonances occurring at the IR scale, and thus have $\mathcal{O}$(TeV) parameters [37, 38, 39].

On the other hand, with TeV masses and couplings for *any* bulk field, the RS scenario was soon seen to allow for SM fields propagating in the bulk [103, 104, 105].

3.3 The radion

In this Section we take into account the fluctuations of the interbrane distance, described by a 4D scalar field usually called *the radion*. As mentioned above, due to the Z_2 parity of the orbifold, there is no massless graviphoton in the RS model.

Classical effective action

A useful way to take into account the fluctuations of the radius consists in the moduli approximation approach. This consists in identifying the free parameters (the interbrane distance d) of our background solution and promote them to 4D fields. The reason is that if they are not fixed by the equation of motion , they correspond to a flat direction of the potential. Such a degree of freedom is light and thus relevant for the low energy effective theory.

In our case, we shall rewrite the RS metric in a form where d appears explicitly. What we can do is to define an angular coordinate $y = r_c\theta$, so that

$$d = \pi r_c$$

and the unperturbed RS metric is $ds^2 = e^{-2kr_c|\theta|}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2 d\theta^2$. Now we promote r_c to a 4D field $r(x)$ (so that $\langle r \rangle = r_c$). Then the perturbed RS metric is

$$ds^2 = e^{-2kr(x)|\theta|}\tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu + r^2(x)d\theta^2. \tag{3.33}$$

Here, $\tilde{g}_{\mu\nu}$ is a fluctuation about Minkowski space and is the physical graviton of the 4D effective theory (and is the massless mode in the Kaluza-Klein decomposition of $g_{\mu\nu}$).

We can insert this ansatz back to the RS action (3.1), and integrating the explicit θ dependence, the 4D effective action is given by

$$S^{(4)}_{RS} = m_P^2 \int d^4x \sqrt{\tilde{g}} \left\{ \left(1 - a^2\right) \tilde{R} - 6(\tilde{\partial} a)^2 \right\}, \tag{3.34}$$

where we have set $m_P = M^3/k$ (*cf.* Eq. (3.7)), we take $a = e^{-k\pi r}$, and $\tilde{\partial}$ means that the indexes are to be raised with $\tilde{g}_{\mu\nu}$. In terms of the canonical *radion* field

$$\varphi = fa, \qquad \text{with} \qquad f = \sqrt{12M^3/k} = \sqrt{12}\, m_P$$

this action takes the form

$$S^{(4)}_{RS} = \int d^4x \sqrt{\tilde{g}} \left\{ m_P^2 \left(1 - (\varphi/f)^2\right) \tilde{R} - \frac{1}{2}(\tilde{\partial}\varphi)^2 \right\}. \tag{3.35}$$

The 4D Einstein frame $g_{\mu\nu}$, is defined so that the graviton kinetic term is of the form $\sqrt{g}R$, with R the Ricci scalar computed with $g_{\mu\nu}$. Then, $g_{\mu\nu} = (1 - a^2)\tilde{g}_{\mu\nu}$, and for the values of the radion that solve the hierarchy $\langle a \rangle \ll 1$, the frame $\tilde{g}$ coincides with the Einstein frame with a good accuracy, and $f \simeq \sqrt{12} m_P$.

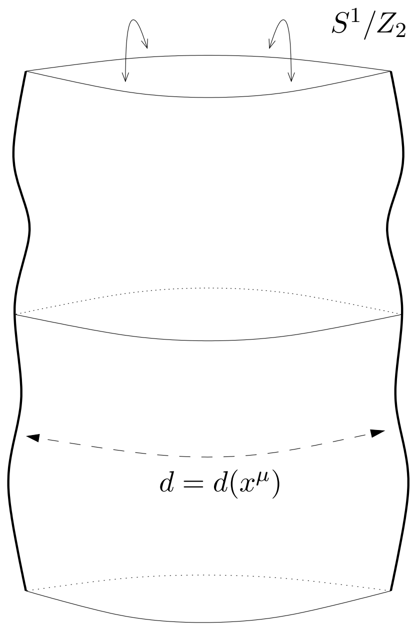


Figure 3.3: In contrast with the 'branon' fluctuations discussed in Sec. 2.3, in the RS model the brane positions do not fluctuate. The effect of perturbing the radion is to modify the interbrane distance at each point x^μ by changing the size of the orbifold S^1/Z_2.

As we shall see, Eq. (3.34) implies that the radion is coupled to matter with EW rather than gravitational strength. This is in sharp contrast with respect to ADD type brane models.

We note that in order to obtain the classical effective action for the radion, the moduli approximation [2, 103, 108] is not the only possibility. The equations of motion and couplings to matter can be obtained in the covariant approach [109, 110], with essentially equivalent results.

As pointed out in [110], the ansatz (3.33) does not solve the linearized Einstein's equations around the RS solution. Using the covariant approach, the 'KK ansatz' that solves the linearized equations was found in [110]. However, the two approaches lead to the same results when the hierarchy is large [111].

Coupling to matter

Let us now describe how the radion φ couples to matter localized on the negative tension brane . This is coupled to 5D gravity (and hence to φ) through the induced metrics on the brane,

$$S_{\rm matt} = \int d^4x \,\sqrt{g_-}\mathcal{L}^- \left(\chi_-, g^-_{\mu\nu}\right),$$

where

$$g^-_{\mu\nu} = a^2\, \tilde{g}_{\mu\nu} \ .$$

Expanding $S_{\rm matt}$ around the radion vev $a = \langle a\rangle + \delta a$, the term linear in the radion is

$$S_{\rm matt-rad} = -\int d^4x\sqrt{g_-}\,\frac{\delta a}{a}\, g^{\mu\nu}_- T^-_{\mu\nu} \equiv -\int d^4x\sqrt{\tilde{g}}\,\frac{\delta\varphi}{\varphi}\,\tilde{g}^{\mu\nu}T_{\mu\nu} \tag{3.36}$$

where the energy momentum tensor is defined as usual,

$$T^-_{\mu\nu}(\chi_-, g^-) = -\frac{2}{\sqrt{g_-}}\frac{\delta}{\delta g^{\mu\nu}_-} S^-_{\rm matt}[\chi_-, g_-].$$

With this definition, the combination $\tilde{T}^\mu_\mu = \tilde{g}^{\mu\nu}T_{\mu\nu}$ is the trace of the physical energy momentum tensor in the Einstein frame. We conclude that the canonical radion φ couples to matter on the negative tension brane with $\langle\varphi\rangle \sim$ TeV suppressed strength. This is in sharp contrast with ADD, where the radion is coupled with gravitationally suppressed strength.

Rewriting Eq. (3.36) as

$$-\int d^4x\sqrt{\tilde{g}}\,\gamma\frac{1}{v}\,\delta\varphi\,\tilde{T}^\mu_\mu \tag{3.37}$$

we see that this coupling is very similar to that of the SM neutral Higgs boson [112, 108, 113, 114], with a correction factor given by the ratio of the Higgs and radion vevs,

$$\gamma \equiv \frac{v}{\langle\varphi\rangle}. \tag{3.38}$$

Thus, a stabilization mechanism is required that gives it a mass much larger than ~ 1/mm [112]. Specifically, the limits on the Higgs mass from collider physics tell us that the radion mass must be at least of order GeV.

This derivation of the radion coupling, based on the moduli approximation (see (3.33)), does not provide the correct result for the coupling to matter on the positive tension brane . The reason is that the ansatz (3.33) assumes that the radion vev $\langle\varphi\rangle$ is constant along the extra dimension. The more accurate treatment of [110] shows that the radion wave function is peaked near the negative tension brane , and the coupling to matter on the positive tension brane is in fact Planckian [109, 108, 110].

Some immediate consequence of the coupling (3.37) is that the radion copules most strongly to h_0 and the weak gauge bosons. In contrast with the Higgs, it couples to gluons and photons directly through the trace anomaly. Thus, gluon fusion is the most important production channel in hadronic collisions, followed by ZZ and WW fusion [113, 131, 70].

Radion-Higgs mixing

As first noted in [113, 142], a non minimal coupling of the Standard Model Higgs fields to the curvature is allowed by covariance. This is of the form

$$S_{\text{mixing}} = -\int d^4x\sqrt{g_-}\,\xi\,\mathcal{R}_-\; H_-^\dagger H_- \tag{3.39}$$

where $g^-_{\mu\nu} = a^2\,\tilde{g}_{\mu\nu}$ is the induced metric on the negative tension brane and $\mathcal{R}_-$ is its (intrinsic) curvature, computed with g_-, *i.e.*

$$\mathcal{R}_- = a^{-2}\left(\tilde{\mathcal{R}} + 6a^{-1}\tilde{\Box}a\right).$$

In order to see the effect of this term, we expand it to quadratic order in the fields, taking $\varphi = \langle\varphi\rangle + \delta\varphi$ with $\langle\varphi\rangle = \langle a\rangle f$ and the rescaled Higgs field

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} v + h_0 \\ 0 \end{pmatrix},$$

where $v = 246$GeV is the Higgs' vev. This provides a Higgs-radion mixing term. Adding it to the kinetic terms for the radion (3.35) and the Higgs, one obtains in the (approximate) Einstein frame [115, 142, 113]

$$S_{\text{kinetic}} = -\frac{1}{2}\int d^4x\sqrt{\tilde{g}}\left\{\left(1+6\gamma^2\xi\right)\varphi\tilde{\Box}\varphi + m_\varphi^2\varphi^2 + h_0\left(\tilde{\Box}+m_h^2\right)h_0 + 6\xi\gamma\varphi\tilde{\Box}h_0\right\}, \tag{3.40}$$

where we have introduced $\gamma = v/\langle\varphi\rangle$, and $m_h^2 = 2\lambda v^2$ and m_φ^2 are the Higgs and radion masses before mixing.

The physical states diagonalizing the kinetic energy are combinations of φ and h_0. The resulting physical masses following from diagonalizing the mass matrix depend on γ and the 'mixing' parameter ξ. Such a mixing affects other observables and for generic values of ξ the properties of the SM Higgs are substaintially modified [116, 117, 131, 70].

According to [116], the situation where the eigenmasses are close to degenerate is disfavored unless ξ or γ are small. For example, with $m_h = 115$GeV and $\xi\gamma = 0.2$, either $m_\varphi > 234$GeV or $m_\varphi < 56$GeV, with the latter being possibly disfavored by direct searches.

The phenomenology of the RS model when matter fields are placed on the negative tension brane is completely determined by the following parameters

$$\xi,\ \langle\varphi\rangle,\ \frac{k}{m_P},\ m_\varphi\ \text{ and }\ m_h.$$

From these parameters, we can write the coupling of matter to the KK gravitons as $\Lambda_{KK} = \langle\varphi\rangle/\sqrt{12}$, and the KK masses as $m_n = (x_n/\sqrt{12})(k/m_P)\langle\varphi\rangle$, where x_n is the nth zero of the Bessel functions (3.24). For the graviton or a gauge field $x_1 \simeq 3.83$ or 2.45 respectively. Thus the first expected KK resonance (if any) would correspond to a gauge field. The radion coupling is $\Lambda_\varphi = \langle\varphi\rangle$ and, as argued below, is expected to be lighter than the gauge KK resonances (in models that do not significantly distort the RS geometry).

Bounds on radion mass and couplings

The Randall Sundrum radion is massless at tree level. Generically, any mechanism that gives it a mass m_φ without significantly distorting the RS geometry is expected to leads to small m_φ, since in the limit of vanishing backreaction, one has to recover $m_\varphi = 0$. Thus, it was soon claimed [5] that this is the lightest expected excitation in the RS model. As such, it would provide the first direct signal of such a scenario.

In generalized models where the geometry is not |AdS| such as in [89], the radion is stabilized at tree level, and m_φ can be of the same order as the KK excitations.

As with the KK modes of bulk fields, the most stringent bounds on radion mass and couplings arise from ccollider physics. The literature on radion phenomenology in the RS model is already quite vast, see for instance [142, 113, 115, 118, 119, 114, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130], and [131, 70] for recent reports. The lower bounds on the radion mass from experimental considerations can be inferred from the bounds on the Higgs mass $\gtrsim$ 100GeV. Since the radion coupling to matter is somewhat suppressed, smaller values are not ruled out. However, a precise statement is very model dependent.

The analysis of [115] from LEP limits on scalar particles with ZZ couplings indicates that a scenario with both Higgs and radion light is impossible. Rather, it is claimed that precision EW data naturally satisfied with both masses modest, $\lesssim$ 200GeV.

As mentioned above, a further complication in this analysis is introduced by a possible non minimal coupling of the Higgs $\xi\mathcal{R}|H|^2$, giving rise to radion-Higgs mixing. The authors of [115] find that when $\xi = 0$, a 'slight' preference in the RS model with φ substantially lighter than h. If $m_\varphi < 2m_h$ (*e.g.* 60 and 120 GeV) then the $h \to \varphi\varphi$ decay channel is open for $\xi \neq 0$ and it can be dominant, depending on the value of ξ. If the radion is heavy enough, then the decay $\varphi \to hh$ is dominant.

Two distinct situations arise depending on the bulk field content. If gauge fields are present in the bulk, their KK excitations also contribute to the Z mass, and this constrains $\langle\varphi\rangle$ to be few TeV. If only gravity can probe the bulk, values close to 1 TeVare allowed. As we show below, the precise value of the radion mass in the RS model depends on the actual value of $\langle\varphi\rangle$ and the curvature scale k. The precise form is given by Eq. (3.49) for the Goldberger and Wise mechanism, and (5.16) for the stabilization by Casimir energy. In both cases, the radion mass increases with $\langle\varphi\rangle$.

3.4 Stabilization mechanisms

As shown in the previous Section, in the absence of a mechanism that gives it a mass, the radion gives rise to a universal long range attractive force about 32 orders of magnitude stronger than gravity. More specifically, the effective 4D gravity experienced by observers on the negative tension brane was found [109] to be a Brans-Dicke (BD) gravity with a small BD parameter $\omega_{BD} \simeq -3/2$ if the hierarchy is large, $a \ll 1$. This is clearly inconsistent with observations, which demand $\omega_{BD} \gtrsim 3000$ if the scalar is massless. Thus, some mechanism must operate that gives it a mass. From current bounds on the Higgs mass $\gtrsim$ 100GeV and the observation that the radion couplings are similar to those of a Higgs boson, suppressed by a factor γ (see Eq. (3.38)), one concludes that m_φ should also lie in the 100GeV range [103].

From another point of view, in the cosmology of RS models, an unusual dependence of the Hubble constant with the matter energy density $H^2 \propto \rho_{(4)}^2$ was soon revealed [132, 133, 134]. This abnormal behavior may persist at late times if the extra dimension is static only because of a fine-tuned cancellation between positive and negative energy densities on the branes [134]. As shown in [135], a natural resolution to this problem can be obtained with the stabilization of the extra dimension. As a bonus, this removes the need for an unphysical correlation between energy densities on different branes. It was further shown that such a stabilization mechanism requires a non-zero pressure along the bulk direction $T_{yy} \neq 0$ [108, 136, 137, 138, 139].

Generically, to accomplish a stabilization of moduli is a quite nontrivial task because it must be compatible with the large hierarchy. This is intuitively clear, since this requires to find an effective potential $V_{eff}(a)$ with a minimum at a very small value for the *hierarchy*, a. It is easy that one has to fine-tune the parameters a lot in order for this to happen, unless the form of the potential is somehow special. Even in this case, it is not at all clear that one can succeed in generating a large enough radion mass$^2 \sim \partial_a^2 V_{eff}$.

To develop some intuition on the origin of the effective potential $V(a)_{eff}$, we shall discuss the most simple effects in the RS model, a detuning (or finite renormalization) $\delta\tau_\pm$ of the brane tensions with respect to the values for the solution (3.4). In the RS model, a renormalization of the cosmological constant has the same effects as a detuning of the brane tensions, since

$$\int d^5x\,\sqrt{g}\delta\Lambda = \int d^4x\,\frac{\delta\Lambda}{2k}\left(1-a^4\right) = \sum_\pm \int d^4x\,\sqrt{g_\pm}\,\delta\tau_\pm\,, \qquad \text{with} \qquad \delta\tau_\pm = \mp\delta\Lambda/2k.$$

From the 4D theory, a detuning of the positive tension brane $\delta\tau_+$ only shifts the effective cosmological constant by this value. However, a shift in $\delta\tau_-$ generates a potential for the radion of the form

$$\delta\tau_-\;a^4 = \delta\tau_-\;e^{-4kd}.$$

This is repulsive for $\delta\tau_- > 0$ and attractive for $\delta\tau_- < 0$. Thus, values of $\tau_\pm$ other than (3.4) generate either a non vanishing 4D cosmological constant, or a monotonic potential for the radion. As expected, both effects render the configuration unstable.

In the following, we describe a number of effects that induce some dynamics for the radion. In most cases, the resulting potential is monotonic and by itself would destabilize the model. However, we can always consider a brane tension renormalization in order to compensate for it. When the competition between both effects leads to a minimum of the potential, the radion stabilization is achieved. Then, a mechanism is *natural* if the brane tension renormalizations required in order to obtain the minimum of the potential at a realistic value, $a = 10^{-16}$, are of the order of the fundamental scale M.

Before describing the 1 loop effective potential (or equivalently the Casimir energy) in the RS, we shall overview some other mechanisms to stabilze the hierarchy available in the literature.

The Goldberger and Wise mechanism

Perhaps the most popular mechanism to stabilize naturally the radion is the proposal by Goldberger and Wise (GW) [5, 112]. They introduce a bulk scalar field Φ of mass m^2, with (m^2/k^2)

somewhat small[6]. They also consider large potentials on the positive and negative tension branes that energetically force $\Phi = \Phi_+$ on the positive tension brane and $\Phi = \Phi_-$ on the negative tension brane with $\Phi_+ \neq \Phi_-$. The bulk kinetic energy and potential of Φ depend differently on the radion. This generates a potential for a with two competing terms. The radion sits at the value where the sum of gradient and potential energies is minimized. This mechanism is perhaps somewhat *ad hoc*, but it has the virtue that a large hierarchy and an acceptable radion mass can be achieved without fine tuning.

In more detail (here we follow [39], where the holographic interpretation of this mechanism is discussed), the bulk scalar equation of motion is (neglecting the back-reaction of Φ on the metric)

$$(\Box^2 - m^2)\Phi = 0 \Rightarrow \Phi'' - \frac{3}{z}\Phi' - \frac{m^2}{z^2}\Phi = 0. \tag{3.41}$$

A trial solution of the form

$$\Phi(z) \sim (kz)^p \,, \tag{3.42}$$

gives a solution as long as p satisfies

$$p(p-4) - (m/k)^2 = 0. \tag{3.43}$$

The most general solution for Φ in the bulk is then of the form

$$\Phi(z) = A_+ \,(kz)^{p_+} + A_- \,(kz)^{p_-} \,, \tag{3.44}$$

where $A_\pm$ are constants fixed by the boundary conditions $\Phi(z = 1/k) = \Phi_+$ and $\Phi(z = 1/ak) = \Phi_-$. When m^2 is small and z_- is large (recall $z_- = 1/(ak)$) , we can approximate $p_+ = 4 + m^2/4k^2$, $p_- = -m^2/4k^2$ and

$$A_+ \simeq a^4 \left(\Phi_- - \Phi_+ a^{-m^2/4k^2}\right), \qquad A_- \simeq \Phi_+. \tag{3.45}$$

The energy stored in the Φ field

$$V(a) = \int_{1/k}^{1/ak} dz \frac{1}{(kz)^5}\left(z^2\Phi'^2 + m^2\Phi^2\right) \tag{3.46}$$

is easily computed. The leading terms for m/k and a small is

$$V(a) = k\Phi_+^2 \frac{m^2}{4k^2} + \frac{4k}{a^4}A_+^2 + \mathcal{O}\left(a^4 \frac{m^2}{4k^2}\right) = k\Phi_+^2 \frac{m^2}{4k^2} + 4k\Phi_-^2 a^4 \left(1 - \frac{\Phi_+}{\Phi_-}\, a^{-m^2/4k^2}\right)^2 + \dots \tag{3.47}$$

This potential has a minimum when $a = 0$ (non compact bulk) and when

$$a = \left(\frac{\Phi_-}{\Phi_+}\right)^{4k^2/m^2} \,. \tag{3.48}$$

Thus, an exponential hierarchy can be generated without fine tuning of $\Phi_\pm$.

The constant term in (3.47) is a contribution to the 4D cosmological constant, which can be set to zero by appropriately detuning the positive tension τ_+. A straightforward computation leads to the radion mass from the potential (3.47),

$$m_\varphi^2 = \frac{1}{f^2}\partial_a^2 V(a) \simeq \frac{1}{2}\frac{k\Phi_+^2}{f^2}\left(\frac{m}{k}\right)^4\, a^2 \,, \tag{3.49}$$

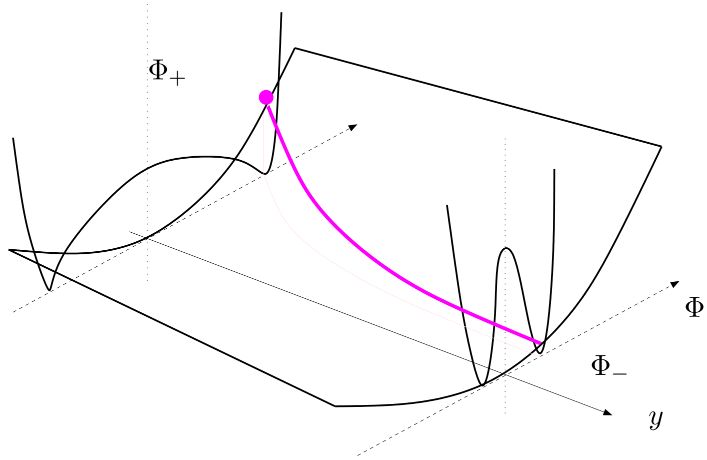


Figure 3.4: The Goldberger and Wise mechanism assumes a bulk scalar Φ with a small bulk mass and brane potentials such that the vev of Φ on the branes $\Phi_\pm$ are different. Then, Φ develops a nontrivial profile $\Phi(z)$, and the energy stored in this configuration (3.47) is minimized when the interbrane distance is given by (3.48).

which is in the MeV to GeV range for reasonable choices of the parameters.

This mechanism has been extended to a generic potential for the bulk scalar in a background close to AdS, [141, 142] in order to take into account the backreaction of Φ on the metric. One of the conclusions was that in the limit of small backreaction, the normalization of the radion kinetic term induced by the bulk field Φ is small. This is is reassuring because it implies that the strong coupling of the radion to matter is a robust prediction of the RS model.

In particular, in [89] a generalization of the GW mechanism with explicit brane potentials and an exact solution of the gravity-bulk scalar that takes into account the backreaction is given. It is explicitly shown that with generic bulk and brane potentials, only one fine tuning (corresponding to matching the 4D cosmological constant) is necessary, and the radion is stabilized. For geometries close to AdS(small backreaction) the hierarchy is easily generated. However, it can also arise for bulk spaces that strongly deviate from AdS.

In [39, 38], the GW mechanism was given an interpretation in terms of the AdS/CFT correspondence and it was claimed in that the GW mechanism is quite generic from the CFT point of view. The idea in AdS/CFT is that tree level effects in AdScorrespond to the leading 1 loop effects at large N in the (strongly coupled, large N) CFT. The GW mechanism is due to a classical bulk field. In the 4D CFT dual, this corresponds to the running of coupling constants from the UV (Plank) to the IR (weak) scales.[7] We shall also mention that this mechanism has been extended to higher form fields in [143]. Other studies concerning the GW mechanism can

[6] Classical stabilization forces due to non-trivial background configurations of a scalar field along an extra dimension were first discussed by Gell-Mann and Zwiebach [140].

[7] In passing, we point out that since the Casimir force is a 1 loop effect in the 5D AdS theory, this corresponds to a next-to-leading order effect in the CFT dual interpretation. Indeed [8], the Casimir energy is reproduced as the '1-loop' effective potential from the CFT bound states, with 1-loop corrected masses. These correction arises from the mixing with the zero mode.

be found in the literature [144, 145].

Other proposals

In [146, 147], a supersymmetric extension of the RS model with two pure (super) Yang–Mills sectors, one in the bulk and the other localized on the positive tension brane brane are considered. Gaugino condensation both in both STM sectors generates a potential that can naturally fix the radius and the hierarchy at a sufficiently large value, and the obtained radion mass can be large. The presence of unbroken supersymmetry up to the compactification scale is fundamental in this model in order that quantum effects do not spoil such an exponentially small potential.

A static solution to Einstein's equations is described in [148] where the branes carry matter density and pressure in addition to tension. As in the RS model, the pressures and energy densities on each brane have to be fine tuned in order to obtain the static solution. The difference is that in [148], the radion is fixed. The energy densities on the branes determine the radion stability and the precise value in such a way that a large hierarchy can only be obtained at the price of fine tuning them.

A dynamical cosmological mechanism was presented in the context of M theory [149]. The 5D effective theory of the Hořava Witten model is a scalar-tensor with some bulk and brane potentials related by supersymmetry [9, 10, 149]. This scenario is very similar to the RS model and is discussed in detail in Chapter 6. One of the main differences is that the 5D scalar introduces another light 4D scalar, and the effective 4D gravity is a 'bi-scalar-tensor' model. The authors of [149] consider the cosmological evolution of the interbrane distance and the bulk scalar field for different matter contents on each branes. There exist attractor solutions which drive the moduli fields toward values consistent with observations. This is similar to the attractor mechanism present in Brans-Dicke theories [150, 151, 152] where the cosmological evolution of the scalar drives them toward values where general relativity is recovered. The efficiency of this attractor mechanism depends on the matter content on each branes. However, from the 5D point of view, such attractors correspond to the motion of the negative tension brane toward a bulk singularity. The 4D description is expected to loose its validity in this regime.

Other recent developments on the effects of the radion mode in the cosmology of the RS model can be found in [154, 155, 156, 157, 158, 159, 160, 161, 162, 163].

Stabilization by Casimir effect

Though they are not the only possibility, potentials involving logarithms are one such type of 'special' potentials that can have a minimum at exponentially small (or large) values. On the other hand, logarithms usually appear from quantum effects. This is one of the main reasons to expect that the 1 loop effective potential might naturally stabilize the hierarchy. A more quantitative argument is as follows.

We have seen that the KK masses depend on the hierarchy $m_n(a)$ (as in any KK theory they, they depend on the size of the extra dimension). Consider the contribution from one single KK mode. From the 4D point of view, this reduces to the Coleman Weinberg potential induced by a field with mass $m_n(a)$,

$$V_{\mathit{eff}}^{(n)}(a) \sim m_n^4(a) \ln\big(m_n(a)/\mu\big). \tag{3.50}$$

This clearly shows that $\ln a$ may well arise in the effective potential from bulk fields. It is equally clear that this is not all the truth, since the sum over n of (3.50) is badly divergent. This only means that the regulator that has been used to derive (3.50) must be withdrawn after performing the sum over n. This would be unimportant if it weren't for the outcome of the final result. As we shall see below, it turns out that when summing over the whole KK tower, the logarithms sometimes disappear. Guessing whether they are present or not is quite involved, and we believe that it is worth studying.

In this thesis, we will investigate whether quantum effects provide an efficient mechanism to stabilize the hierarchy in warped brane models. In Chapter 4, we describe the methods to compute the 1-loop effective potential, or equivalently the Casimir energy. In Chapter 5, we review the 1-loop effective potential induced by bulk fields in the RS model [7, 11, 12, 13, 8]. In Chapter 6, a generalization of the RS that relaxes the exponential behaviour of the warped factor is examined. Chapters 7 and 8 deal with higher dimensional warped brane models with codimension 1 branes and a S^1/Z_2 orbifold bulk. As we shall see, in both cases a large hierarchy can be stabilized naturally.

3.5 Summary

Upon solving Einstein's equations, and taking into account the gravitational field induced by the branes in the BW scenario, non-trivial warp factors naturally arise. This has important phenomenological and theoretical implications. A redshift effect can account for the large hierarchy of scales between different branes without the need to introduce large extra dimensions. The phenomenology is quite distinct from the scenario of large radius compactification.

The KK masses are of order a TeV, and couple with TeVsuppressed (rather than Planck-suppressed) strength. This renders graviton KK excitations individually detectable in high energy accelerators at energies $\sim$ TeV. Since there is only one extra dimension and is not large, no severe bounds on the size of the fundamental scale arise from cosmology or astrophysics, the most stringent ones coming from collider physics.

A massless radion would render the effective 4D gravity of Brans-Dicke with unacceptable Brans-Dicke parameter. The coupling strength of the radion to matter is of TeV size [112], so its phenomenology is close to the Higgs boson. In particular, current bounds for EW scalars apply to the radion, hence its mass must lie on the GeV range.

In the RS scenario, the actual value of the hierarchy is given by the *redshift* factor $m_W/m_P = a = e^{-kd}$. Thus, the hierarchy problem is solved as long as some (natural) stabilization mechanism fixes the interbrane distance d somewhat larger than the fundamental scale $\langle d\rangle \sim 37/k$ and generates a radion mass $\gtrsim$ GeV.

Several mechanisms have been proposed in the literature to stabilize a large hierarchy without introducing any fine tuning. In Chapter 5, the Casimir energy arising in the RS model is reviewed. As we shall see, the contribution from a gauge field in the bulk can naturally stabilize the hierarchy and give the radion a sizable mass.

Chapter 4

Quantum effects

4.1 Casimir force

Up to date, very few macroscopic manifestations of quantum physics have been described.Among them there are superconductivity, superfluidity, the quantum Hall effect and the Casimir effect [182],[1] one of the most remarkable successes of Quantum Field Theory (QFT). It originates from the 'half quanta' of the harmonic oscillator, first introduced as long ago as in Planck's times. In QFT, the fields are an infinite set of oscillators labeled by some quantum numbers k. The nth excitation of a single oscillator k corresponds to a state with n field quanta and energy $E_n^{(k)} = \hbar w(n + 1/2)$. Thus, the state with no real quanta has a nonzero energy

$$E_0^{(k)} = \frac{\hbar w_k}{2},$$

which results in a infinite total energy for the 'vacuum' $E_{\text{Cas}} = (\hbar/2)\sum_k w_k$. In canonical quantization, one takes advantage of the ambiguity in the operator ordering in the definition of the Hamiltonian and imposes the 'normal ordering' prescription, which effectively sets the energy of this state to zero. This can be done because (in the absence of gravity) the energy can be physically defined up to an additive constant and only energy differences are relevant.

This leads to the Casimir effect, namely the dependence of the vacuum energy on the boundary conditions for the field. Casimir [182] computed the (regulated) vacuum energy of the electromagnetic field with two perfectly conducting parallel plates a distance d apart and subtracted the Minkowski contribution (with infinitely distant plates). The famous resulting attractive force is

$$F(d) = \frac{\pi^2}{240}\frac{\hbar c}{d^4}\mathcal{A},$$

where $\mathcal{A}$ is the area of the plates. Remarkably, the electron charge e does not appear in this expression, which means that this is not an effect of coupling the electromagnetic field to the material plates (which are assumed to be neutral). Rather, they attract each other due to the shift in the electromagnetic vacuum energy that they induce by changing the boundary conditions. This effect was verified experimentally quite soon [186], though precise experimental checks have awaited until recently [187].

[1] See *e.g.* [183, 184] for reviews on this topic, and [185] for a resource letter.

The term 'Casimir effect' is applied in the literature to a number of other long-range interactions, such as those between atoms and molecules (Van der Vaals), an atom and a surface (or Casimir-Polder) which initially [188] motivated Casimir's work. Here, we will use this term to refer to the force induced by the nontrivial boundary conditions.

The Lifshitz theory [189] constitutes an alternative formulation of this effect that does not rely on the quantization of the electromagnetic field. Rather, the Casimir force is obtained from considerations of the charge fluctuations within the material bodies and Van der Vaals interactions. Both points of view are complementary descriptions of the same phenomenon [185].

It is clear that the Casimir effect is not exclusive of the electromagnetic field, but occurs for any quantum field that propagates in the 'bulk'. On the other hand, the analogy between the parallel plates configuration and brane models of the RS type is straightforward. Thus, Casimir forces are expected to arise from the graviton, in the minimal models where other fields are confined to the branes.

In order to show more explicitly that this a purely quantum effect, we shall illustrate the connection between Casimir energy with the 1-loop effective potential V_{eff} with a simple example.

Let us consider a scalar field with mass m in D dimensional Minkowski space. The quantum numbers labeling the mode frequencies are the momentum $\boldsymbol{k}$, and

$$w_{\boldsymbol{k}} = \sqrt{m^2 + \boldsymbol{k}^2}. \tag{4.1}$$

Thus, using dimensional regularization $D = 4 - 2\epsilon$, the Casimir energy density is

$$\frac{E_{\text{Cas}}}{v_{D-1}} = \frac{\hbar}{2}\mu^{4-D}\int \frac{d^{D-1}k}{(2\pi)^{D-1}}\sqrt{m^2+\boldsymbol{k}^2} = -\frac{\mu^4}{2(4\pi)^{D/2}}\Gamma\left(-\frac{D}{2}\right)(m/\mu)^D \tag{4.2}$$

where μ is introduced for dimensional reasons and v_{D-1} is the volume of space. In Eq. (4.2), use has been made of

$$F(\alpha,\beta,y) = \int_0^\infty dx \frac{x^\beta}{(x^2+y^2)^\alpha} = \frac{\Gamma\left(\frac{\beta+1}{2}\right)\Gamma\left(\frac{2\alpha-\beta-1}{2}\right)}{2\Gamma\left(\alpha\right)} y^{\beta+1-2\alpha}. \tag{4.3}$$

As we show in Section 4.3, for such a scalar field, the 1-loop effective potential is, in dimensional regularization

$$V_{\mathit{eff}} = \frac{\hbar}{2}\mu^{D-4}\int \frac{d^Dk}{(2\pi)^D}\ln\left(\frac{k^\mu k_\mu + m^2}{\mu^2}\right) = \frac{\mu^{D-4}}{(4\pi)^D\Gamma\left(D/2\right)}\int_0^\infty dk k^{D-1}\ln\left(\frac{k^2+m^2}{\mu^2}\right),$$

where k^μ is the off shell 'four' momentum, and we have restored the $\hbar$ factors, see Eq. (4.19). The last integral is divergent for any 'regularized' dimension D. However, it can be defined from the formula (4.3), as

$$-\lim_{\alpha\to 0}\partial_\alpha F(\alpha, D-1, m/\mu) = -\frac{1}{2}\Gamma\left(\frac{D}{2}\right)\Gamma\left(-\frac{D}{2}\right)(m/\mu)^D,$$

where we have used that D is not an integer. Thus, we conclude that both quantities are trivially related,

$$V_{\mathit{eff}} = \frac{E_{\text{Cas}}}{v_{D-1}}.$$

In the limit $D \to 4$, (4.2) is divergent

$$V_{\mathit{eff}} = -\frac{1}{4(4\pi)^2}\,\frac{1}{\epsilon} m^4 \left\{1 - 2\epsilon \ln\left(m/\mu\right) + \mathcal{O}(\epsilon^2)\right\}, \tag{4.4}$$

where we redefined μ. The divergent part can be absorbed in the cosmological constant, and the renormalized potential is, in $D = 4$,

$$V_{\mathit{eff}}^{\mathrm{ren}} = \frac{1}{2(4\pi)^2}\, m^4 \ln\left(m/\mu\right), \tag{4.5}$$

the well known Coleman-Weinberg result.

In a curved spacetime, the Casimir energy and the effective potential may not coincide in general. However, in a wide class of spacetimes they do up to a finite term proportional to local operators [190] that can always be absorbed in a redefinition of the renormalization scale.

Another concept closely related to the Casimir energy is the vacuum expectation value of the energy momentum tensor, $\langle T_{\mu\nu}\rangle = \langle 0|T_{\mu\nu}|0\rangle$, which leads to a further equivalent 'definition' of the Casimir energy, of the form $\int \langle T_{00}\rangle$. It can be shown that if all the interactions are turned off, then this coincides with E_{Cas} [183].

In the above paragraphs, we have identified the Casimir energy with the quantum effective potential. The Casimir energy is indeed expected to be a function of the radion φ since it depends on the boundary conditions, hence on the interbrane distance. Note that the Casimir energy arises from *free* fields propagating in the bulk $\Phi(x, y)$. Here, *free* means that self couplings of Φ are neglected, but of course Φ couples to the background geometry.

Let us describe how can a dependence on the radion φ arise from the effective potential generated by the bulk field $\Phi(x, y)$ in the simple 5D KK theory. In the 4D effective theory, $\Phi(x, y)$ is decomposed as a KK tower

$$\Phi(x, y) = \sum_n \Phi_{(n)}(x)\, e^{iny/R},$$

where $R(\varphi)$ is the compactification radius. The 5D kinetic term for Φ reads

$$\int d^5x\ \Phi\Box_{(5)}\Phi = \int d^4x\ \sum_n \left\{\Phi_{(n)}\Box_{(4)}\Phi_{(n)} - \left(\frac{n}{R(\varphi)}\right)^2 \Phi_{(n)}^2\right\}.$$

The second term in the rhs gives the vertex for the interaction of the radion with each KK mode $\Phi_{(n)}$, illustrated in Fig. 4.1. Thus, the effective potential induced by the bulk field Φ corresponds to the total contribution taking into account loops of all the KK modes.

In the 4D picture, the KK modes $\Phi_{(n)}$ are heavy, hence their corresponding degrees of freedom are difficult to excite. Thus, they can be explicitly integrated out, and the effective potential encodes the all the effects that they produce in low energy dynamics.

Moduli stabilization

The use of Casimir effect to stabilize the size of the extra dimensions is not a new idea. Unwin first noted the existence of nontrivial Casimir forces in models with extra dimensions, though no explicit computation was given [164]. In the original Kaluza Klein theory $M_4 \times S^1$, Appelquist

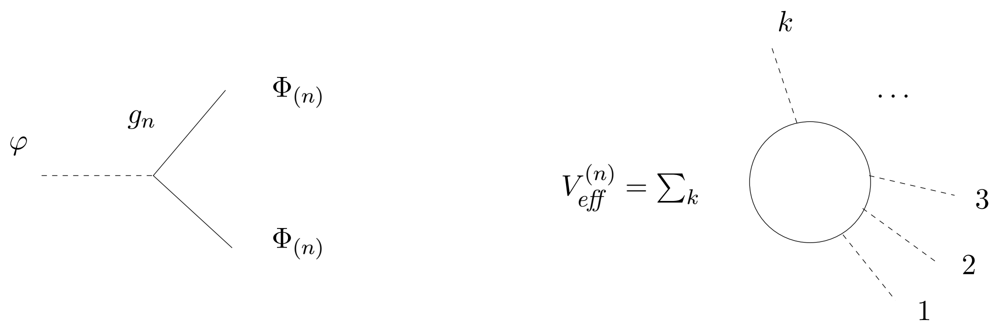


Figure 4.1: The radion φ couples to the KK modes through their masses $m_n = n/R(\varphi)$. Expanding this function around the radion vev $\langle\varphi\rangle$, one obtains for instance trilinear interactions with coupling constants $g_n \equiv \left.\frac{\partial m_n^2}{\partial\varphi}\right|_{\langle\varphi\rangle}$. The effective potential from the bulk field is the sum over the KK tower of $V_{\mathit{eff}}^{(n)}$.

and Chodos [165] performed the first quantitative computations. On purely dimensional grounds, the potential has to behave as $V \sim C/R^4$, and the constant $C > 0$. This results in an 'attractive' potential, which drives the size of the circle R toward values where the computation is expected to fail, *i.e.* $R \sim 1/m_P$. The gravitational contribution to the Casimir energy is quite involved in general. It is currently believed that gravity alone cannot generate a stabilizing potential at 1-loop for factorizable odd dimensional spacetimes of the form $M_4 \times S^n$ (see [49] and references therein).

Weinberg and Candelas [6] considered the same class of spaces where the internal dimensions form one-parameter manifold. Using symmetry arguments, the expectation value of the energy momentum tensor $\langle T_{MN}\rangle$ from (free) massless fields has to be proportional to the metric. This allows to obtain the full form of $\langle T_{MN}\rangle$ from the effective potential, which can be computed from the KK spectrum. Then, Einstein's equations with $\langle T_{MN}\rangle$ as a source are equivalent to imposing that V_{eff} has a minimum at zero. The effective potential consists of three competing terms: the Casimir energy, (a renormalization of) the cosmological constant, and the curvature terms from S^n. They found that depending on the ratio of fermion to bosonic degrees of freedom (and for a large number of them), stable solutions with R of Planckian size arise. This indicates that 1-loop self consistent stabilization of the radius due to matter fields is possible.

In the context of the BW scenario, some computations of the effective potential have been performed. The consequences of a TeV sized flat orbifold for the electroweak and supersymmetry breaking were considered in [166, 167].

Also in flat S^1/Z_2 and T^2/Z_k orbifolds, [168] computed the Casimir energy.The contributions from a bulk field with mass m together with that from a massless one can stabilize naturally the radius at $\sim 1/m$. In the 5D orbifold, kinetic terms localized at the fixed points shift the KK mass spectrum. The resulting effective potential can be evaluated, and can have minima for values of the radion somewhat larger than the fundamental length $1/M$.

As we show in detail in Chapter 5, the Casimir energy due to bulk fields can stabilize the radion in the RS model [7, 11, 12, 13, 8]. Furthermore, bulk gauge fields (or its supersymmetric partners) can stabilize a large hierarchy without fine tuning.

4.2 Conformally trivial case

Before describing the procedure to evaluate the 1-loop effective potential in general, we shall concentrate on the case of conformally coupled fields. This case is much simpler because the RS model is conformally flat and the problem is 'conformally trivial'. This allows for a complete determination of V_{eff} and of the expectation value of the energy momentum tensor $\langle T_{\alpha\beta}\rangle$.

The argument goes as follows [7]. In flat space $\langle T_{\alpha\beta}\rangle$ is traceless for conformally invariant fields. Moreover, because of the symmetries of our background, it must have the form [107]

$$\langle T^z{}_z\rangle_{flat} = (D-1)\rho_0(z), \quad \langle T^\mu{}_\nu\rangle_{flat} = -\rho_0(z)\ \delta^\mu{}_\nu.$$

By the conservation of energy-momentum, ρ_0 must be a constant, given by

$$\rho_0 = \frac{V_0}{2L} = -(-1)^F \frac{A}{2L^D},$$

where $F = 0, 1$ for bosons and fermions respectively. Here $V_0 \propto 1/L^{D-1}$ is the potential in the flat case, and we have introduced [7]

$$A \equiv -\frac{(-1)^{(D-1)/2}}{(4\pi)^{(D-1)/2}((D-1)/2)!}\pi^{D-1}\zeta_R'(1-D) > 0.$$

Now, let us consider the curved space case. Since the bulk dimension is odd, there is no anomaly [107] and the energy momentum tensor is traceless in the curved case too. This tensor is related to the flat space one by (see e.g. [107])

$$\langle T^\alpha{}_\beta\rangle_g = a^{-D}\langle T^\alpha{}_\beta\rangle_{flat}.$$

Hence, the energy density is given by

$$\rho = a^{-D}\rho_0. \tag{4.6}$$

This simple scaling with $a(z)$ allows one to obtain 1 loop self consistent warped solutions. This contribution is a source for curvature that can be added to the $T_{\alpha\beta}$ in Einstein's equations. The resulting solution takes into account the backreaction on the geometry of the vacuum energy (4.6).

The (yy) component of Einstein's equations gives

$$\left(\frac{a'}{a}\right)^2 = \frac{1}{3M^3}\left(\frac{\rho_0}{a^5} - \frac{1}{4}\Lambda\right) \tag{4.7}$$

and the Israel matching conditions,

$$\left(\frac{a'}{a}\right)_\pm = \mp\frac{1}{12M^3}\tau_\pm. \tag{4.8}$$

Here, a prime denotes differentiation with respect to y.

Equation (4.7) controls the shape of the warp factor, subject to the (boundary) conditions that the slope at $z_\pm$ is set by the brane tensions $\tau_\pm$. From (4.7), we can determine the value of the hierarchy (the ratio of warp factors) in terms of Λ and $\tau_\pm$, as

$$a = \frac{a(z_-)}{a(z_+)} = \left(\frac{\tau_+^2 + 12M^3\Lambda}{\tau_-^2 + 12M^3\Lambda}\right)^{1/5} . \tag{4.9}$$

Thus, the brane tensions need to be fine tuned in order to obtain a large hierarchy, and in the absence of other input this is not a natural mechanism. Furthermore, using the techniques developed in [137], it is easy to see that the radion mass is small.

The solution to (4.7,4.8) with $\Lambda = 0$ was found in [7]. It exists for a fermionic contribution ($\rho > 0$), and is of the form $a(z) \propto (y - y_0)^{2/5}$. The generalization to $\Lambda \neq 0$ was considered in [14, 169, 170, 139, 137]. For $\Lambda < 0$, one finds (see Fig. 4.2)

$$\begin{aligned} a(y) &\propto \cosh^{2/5}\left(\frac{5}{2}k(y - y_0)\right), \qquad \text{or} \\ a(y) &\propto \sinh^{2/5}\left(\frac{5}{2}k(y - y_0)\right), \end{aligned} \tag{4.10}$$

where y is the proper distance and $k^2 = -\Lambda/(12M^3)$. The sinh (cosh) arises for a fermion (boson) dominated vacuum energy ρ. For $\Lambda > 0$, the hyperbolic functions are replaced by trigonometric ones [139].

This opens up (for the cosh) the interesting possibility to have a solution with two positive tension branes, or even with a single brane. These models are more likely to admit a field theoretic realization than the models with negative tension branes, since topological defects arising in field theory always have positive energy densities [89]. With two positive tension branes the hierarchy problem would be solved with a large ratio of warp factors. With only one, a large volume is necessary. As in the non compact RS model with one brane, this is controlled by the curvature radius. Unfortunately, for such solutions, the Casimir energy is attractive. In order to compensate for it, we have to add a positive correction to the negative tension brane, which acts as a repulsive potential for the radion, $a(y_-)^4 \sim e^{-4kd}$. The combination of both effects has a maximum and we conclude that this solution is unstable [139]. Furthermore, from (4.9) it is clear that this case requires a fine tuning of the brane tensions in order to obtain $a \sim 10^{-16}$. These models cannot be stabilized by a bulk scalar interaction of the GW type [145, 139].

The efficiency of the Casimir force to stabilize brane models has been investigated in several generalizations of the RS model, to include de Sitter branes [171, 172, 173, 174, 175, 176, 177], This case can be of relevance for the bulk inflaton model [178]. The 1-loop corrections to the radion kinetic terms due to bulk fields was computed in [179], finding that they vanish for conformally coupled fields. In [180], the Casimir effect due to conformally coupled bulk scalar fields on conformally flat warped brane-world geometries is investigated. For discussions of the possible relevance of quantum effects in cosmological brane-world scenarios and [181].

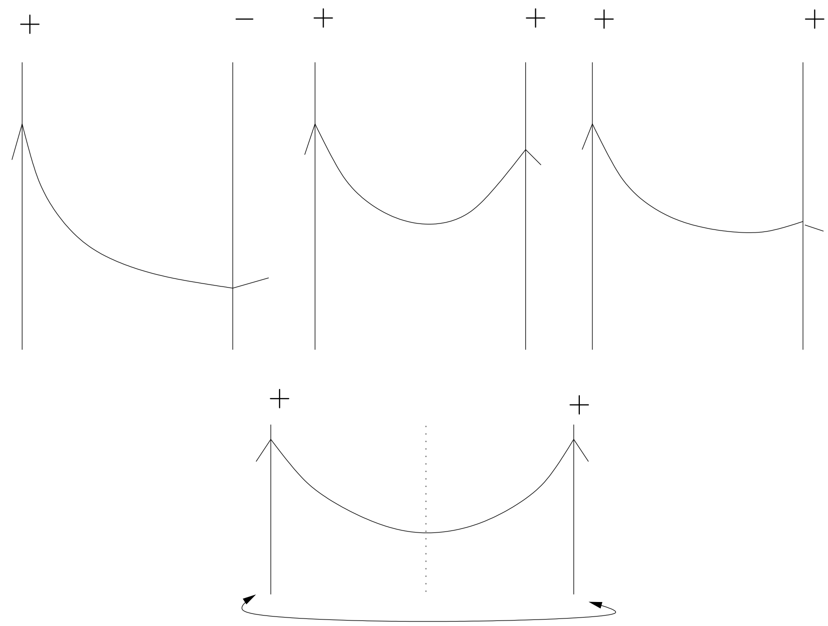


Figure 4.2: The 1 loop self-consistent warp factors $a(y)$ (4.10). From left to right, we represent the sinh solution [14] (a singularity after the negative tension brane is not shown), the cosh solution [169] with non-tuned (positive) tensions (generating a low hierarchy), and with fine tuned (positive) tensions. Below, the solution with a single brane. One of the branes has vanishing tension and sits at the minimum of $a(y)$ [136].

4.3 The effective potential $V_{\it eff}$

In this Section we present the path integral method to evaluate the 1 loop effective potential for the radion $V_{\it eff}[\varphi]$ due to a generic, non-conformal bulk scalar field $\Phi(x^\mu, y)$ [191].

The effective action $S_{\it eff}$ is defined in Euclidean time ($t = -it_E$) as the path integral

$$e^{-S_{\it eff}} \equiv \int \mathcal{D}\Phi \; e^{-\{S+S^{(\Phi)}\}}, \tag{4.11}$$

where S is the action for the model under consideration and $S^{(\Phi)}$ is a quadratic action of the form

$$S^{(\Phi)} = \frac{1}{2}\int d^5x\sqrt{g}\;\Phi\;P\;\Phi,$$

with a generic operator

$$P \equiv (\Box_g - m^2 - \xi\,\mathcal{R}). \tag{4.12}$$

The long wavelength fluctuations around the background spacetime can be taken into account considering an appropriate KK ansatz for the metric, such as (2.5) or (3.33) in the KK and the RS models respectively. This is equivalent to a zero mode truncation in the KK expansion for the metric. Upon substitution of these ansätze into the corresponding action S, we obtain the *classical* action for the perturbed background, given by Eqs. (2.6) and (3.35) in the KK and RS models. Note that, these include only a kinetic term for the radion. This is to be expected, since the radion is a *modulus* of the corresponding background solutions. Since its vev is is not fixed, it must correspond to a massless degree of freedom.

The analogous KK ansätze for the models discussed in Chapters 6 and 7 are discussed around (6.10) and (7.9). The classical action for the moduli S_b corresponds to (6.19) and (7.12). The KK ansatz for the model discussed in Chapter 8 follows the lines of the RS model.

Thus, we identify S_{eff} as an action functional for the radion (or moduli) fields,

$$e^{-S_{eff}[\varphi]} = e^{-S_b[\varphi]} \int \mathcal{D}\Phi \; e^{-S^{(\Phi)}}. \tag{4.13}$$

where S_b is a classical and purely kinetic term. We shall see below that the path integral over Φ can be formally done, since it is quadratic in Φ. Before doing this, we have to be a bit more specific about the path integral measure.

A Quantum Field Theory (QFT) is defined not just by the classical Lagrangian, but also by the measure of functional integration, $\mathcal{D}\Phi$. The latter is usually prescribed by demanding certain symmetries or invariances. For instance, for scalar fields in curved space, invariance under diffeomorphisms is an obvious requirement. If gravity is the only background field, then this requirement uniquely defines $\mathcal{D}\Phi$. We argue in Section 6.2 that whenever there is a scalar field in the background with nontrivial profile, then the functional measure is not unique, since it can be defined to be covariant with respect to any conformal frame. However, the different possible choices are related by the addition or renormalization of local operators in the action. In Sec. 4.4, we describe the behaviour of $\mathcal{D}\Phi$ under conformal transformations. In Section 6.2, we apply these results to quantify the effects of the ambiguity in choosing $\mathcal{D}\Phi$. Now, we assume that the metric is the only nontrivial field in the background, so that $\mathcal{D}\Phi$ is uniquely defined.

A 'volume' measure $\mathcal{D}\Phi$ in field space $\mathcal{F}$ can be found from a metric G_{xy} on $\mathcal{F}$, through the relation [15]:

$$\mathcal{D}\Phi = \sqrt{G} \; \prod_x d\Phi^x. \tag{4.14}$$

Here, the spacetime coordinates x and x' are considered as continuous labels for the coordinates $\Phi^x \equiv \Phi(x)$ of the infinite dimensional space $\mathcal{F}$, and G is the determinant of $G_{xx'}$. To specify $G_{xx'}$, we note that a natural definition of a scalar product in the space of field variations $\delta\Phi$ can be given in terms of the spacetime measure $d\mu(x)$, through the relation

$$\langle \delta\Phi_1, \delta\Phi_2 \rangle_\mu \equiv \int\int d\mu(x) d\mu(x') \; G_{xx'} \; \delta\Phi_1^x \; \delta\Phi_2^{x'} \equiv \int d\mu(x) \delta\Phi_1(x) \delta\Phi_2(x).$$

We denote field variations by $\delta\Phi$ just to emphasize that we are referring to elements of the tangent space. More precisely, $\delta\Phi = \delta\Phi^x e_x$, where $e_x = \partial/\partial\Phi^x$ is the coordinate basis of the tangent space at the point p which corresponds to the background solution. In a Riemannian spacetime, the invariant measure is given by

$$d\mu(x) = \sqrt{g(x)} d^5 x, \tag{4.15}$$

where g is the determinant of $g_{\alpha\beta}$. The implicit definition of $G_{xx'}$ given above is just the identity $\delta_\mu(x, x')$ with respect to $d\mu$ integration,

$$G_{xx'} = \delta_\mu(x, x') = \frac{\delta^{(n)}(x - x')}{\sqrt{g(x)}}. \tag{4.16}$$

It is convenient to express the field variations in an orthonormal basis Φ_n, with $\langle \Phi_n, \Phi_m \rangle = \delta_{nm}$, so that $\delta\Phi(x) = \sum_n c^n \Phi_n(x)$. In this basis, the components of the field variation are c^n, and the metric is just the usual delta function (the continuous or the discrete delta function depending on whether the normalization of Φ_n is continuous or discrete):

$$G_{nm} = \delta_{nm}.$$

Substituting in (4.14), we have

$$\mathcal{D}\Phi = \prod_n dc^n. \tag{4.17}$$

Given the form of $S^{(\Phi)}$, the eigenbasis of the operator P is particularly convenient. Solving the eigenvalue problem for P,

$$P\,\Phi_i = -\lambda_i \Phi_i$$

with Φ_i a complete and orthonormal basis,

$$\int d^5x \sqrt{g}\; \Phi_i^* \Phi_j = \delta_{ij}$$

then, we can expand $\Phi = \sum_i c^i \Phi_i$, with constant complex coefficients c^i. Thus, $S^{(\Phi)} = \frac{1}{2}\sum_i \lambda_i |c^i|^2$ and we see that the c^i have dimensions of length. Thus, the *dimensionless* functional measure is

$$\mathcal{D}\Phi = \prod_i \left(\frac{\mu dc^i}{\sqrt{2\pi}}\right), \tag{4.18}$$

with μ an arbitrary energy scale needed to render the measure dimensionless, and the 2π are conventional.

Now, the integral (4.13) is translated into trivial c^i integrals, and one obtains

$$\int \mathcal{D}\Phi\; e^{-S^{(\Phi)}} = \prod_i \left\{ \int \frac{\mu dc^i}{\sqrt{2\pi}}\; e^{-\frac{1}{2}\lambda_i |c^i|^2} \right\} = \prod_i (\mu \lambda_i^{-1/2}), \tag{4.19}$$

which can be formally identified as $\left[\det P/\mu^2\right]^{-1/2}$. Since the eigenvalues of P depend on the radion through the KK masses $m_n(\varphi)$, this can be viewed as a potential for the radion

$$\int d^4x \sqrt{\tilde{g}}\; V_{\mathit{eff}}(\varphi) \equiv \frac{1}{2} \ln\left[\left(\det P/\mu^2\right)(\varphi)\right] = \frac{1}{2}\sum_i \ln(\lambda_i/\mu^2) = \frac{1}{2}\mathrm{Tr}\ln(P/\mu^2).$$

Thus, the 1 loop effective potential (energy density) is [15]

$$V_{\mathit{eff}}(\varphi) \equiv \frac{1}{2\mathcal{A}} \mathrm{Tr}\ln\left(\frac{P}{\mu^2}\right), \tag{4.20}$$

with $\mathcal{A} = \int d^4x \sqrt{\tilde{g}}$ the comoving area.

With no more specifications, V_{eff} is an ill defined quantity, since such a sum over all the eigenvalues is divergent. Thus, the evaluation of V_{eff} must be accompanied with a regularization procedure. There are many ways to do this, but here we present the $\zeta-$*function* and *dimensional regularization* techniques.

4.4 Regularization of V_{eff} in warped compactifications

The direct evaluation of the determinant of P appearing in Eq. (4.20) turns out to be rather impractical, due to the complicated form of the implicit equation which defines its eigenvalues.

The kind of geometries considered here are conformally flat[2]. Specifically, we shall be interested in metrics of the form

$$ds^2 = a^2(z)\left[dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right] \qquad \text{with} \quad a(z) = \left(\frac{z}{z_0}\right)^\beta . \tag{4.21}$$

The RS model corresponds to $\beta = -1$, and in Chapter 6 we consider other values of β.

The eigenvalues of the conformally related operator P_0 can be easily obtained and in fact are simply related to the masses of the KK modes. Thus, we can evaluate $\log\det P$ by relating it to $\log\det P_0$ and computing the latter. The connection between them is not trivial but can be derived from the properties of the Laplacian operator under conformal transformations. For completeness, and in order to illustrate practical methods for calculating the effective potential, we shall consider dimensional regularization and zeta function regularization. Both methods will lead to identical results.

Conformal transformations

Following [7, 15], we introduce a one-parameter family of metrics $g^\theta_{\alpha\beta}$ which interpolate between a fictitious flat spacetime and the original metric $g_{\alpha\beta}$

$$g^\theta_{\alpha\beta} = \Omega^2_\theta\, g_{\alpha\beta}, \tag{4.22}$$

where θ parametrizes a path in the space of conformal factors. For definiteness we shall restrict attention to conformal factors $\Omega_\theta(z)$ of the form

$$\Omega_\theta(z) = \left(\frac{z}{z_0}\right)^{\beta\,(\theta-1)}. \tag{4.23}$$

With this choice, $\theta = 0$ represents flat space and $\theta = 1$ corresponds to the original metric (4.21).

It is convenient to define the operator P_θ associated with the metric $g^\theta_{\alpha\beta}$ by

$$\Omega_\theta^{(D-2)/2} P_\theta \Omega_\theta^{(2-D)/2} = \Omega_\theta^{-2} P, \tag{4.24}$$

where P was introduced in Eq. (4.12). This operator can be written in covariant form as [7, 15]

$$P_\theta = -(\Box_\theta + E_\theta),$$

where

$$E_\theta = \left(\frac{D-2}{2}\right)\,\Box_\theta \ln\Omega_\theta - \left(\frac{D-2}{2}\right)^2 g_\theta^{\alpha\beta}\;\partial_\alpha \ln\Omega_\theta\;\partial_b \ln\Omega_\theta + \Omega_\theta^{-2}E,$$

and $\Box_\theta$ is the covariant D'Alembertian corresponding to $g^\theta_{\alpha\beta}$.

The operator $P_0 \equiv P_{\theta=0}$ is the wave operator for the KK modes which one would use in a four-dimensional description. The Lorentzian equation of motion $P\Phi = 0$ can be written as

$$P_0\Phi_0 = 0,$$

[2] or conformal to a space with flat non compact directions.

where

$$P_0 = -\Box_{D-1} + \hat{M}^2(z). \tag{4.25}$$

Here $\Box_{D-1}$ is the flat space D'Alembertian along the branes, and

$$\hat{M}^2 \equiv -\partial_z^2 - E_0,$$

is the Schrödinger operator whose eigenvalues are commonly referred to as the KK masses m_n:

$$\hat{M}^2(z)\Phi_{0,n}(z) = m_n^2\Phi_{0,n}(z). \tag{4.26}$$

The interesting feature of (4.25) is that it separates into a four-dimensional part and a z dependent part. A mode of the form $\Phi_0 = e^{ik_\mu x^\mu}\Phi_{0,n}$ will solve the equation of motion (4.25) provided that the dispersion relation

$$k_\mu k^\mu + m_n^2 = 0,$$

is satisfied, and hence modes labeled by n behave as four-dimensional massive particles. Technically, the advantage of working with P_0 is that its (Euclidean) eigenvalues $\lambda_{n,k} = k_\mu k^\mu + m_n^2$ separate as a sum of a four-dimensional part plus the eigenvalue of the Schrödinger problem in the fifth direction.

Introducing the conformally transformed field $\Phi_\theta \equiv \Omega_\theta^{(2-D)/2}\Phi$, the action for the scalar field can be expressed as

$$S[\Phi] = \frac{1}{2}\int d^D x\sqrt{g_\theta}\ \Phi_\theta\ P_\theta\ \Phi_\theta. \tag{4.27}$$

In terms of $g^\theta_{\alpha\beta}$ the field Φ_θ has a perfectly canonical and covariant kinetic term.

Thus, the same arguments which lead to (4.16) can now be used in order to find the natural line element in field space associated with the spacetime measure $d\mu_\theta(x)$:

$$d\mathcal{S}_\theta^2 = \int\int d\mu_\theta(x)d\mu_\theta(y)G^\theta_{xy}\ d\Phi^x_\theta\ d\Phi^y_\theta\ + ...\ = \int d^D x\sqrt{g_\theta(x)}\ (d\Phi_\theta(x))^2 + ...$$

Here, the ellipsis denote the omitted terms which correspond to variations of other fields in the theory (in particular, these include the variations of the gravitational field and the dilaton). Let us compare this line element with the one considered above

$$d\mathcal{S}^2 = \int\int d\mu(x)d\mu(y)G_{xy}\ d\Phi^x\ d\Phi^y\ + ... = \int d^D x\sqrt{g(x)}\ (d\Phi(x))^2 + ...$$

For field variations where Φ changes but the rest of the fields (metric, dilaton, etc.) are constant, we have

$$d\mathcal{S}_\theta^2 = \int d^D x\ \Omega_\theta^2\sqrt{g(x)}\ (d\Phi(x))^2 \neq d\mathcal{S}^2 \qquad (g,\phi,... = const.; d\Phi^x \neq 0), \tag{4.28}$$

and therefore $d\mathcal{S}_\theta^2 \neq d\mathcal{S}^2$ in general. Of course, the corresponding measures of integration will also be different. In the basis $\{\Phi_{\theta n}\}$ which is orthonormal with respect $d\mu_\theta$, the field variation can be expanded as $\delta\Phi_\theta(x) = \sum_n c_\theta^n\Phi_{\theta n}$, and the new measure takes the form

$$(\mathcal{D}\Phi)_\theta = \prod_n dc_\theta^n. \tag{4.29}$$

Using $\Phi_{\theta m} = \Omega_\theta^{-D/2}\Phi_m$, it is straightforward to show that $c^m = M^m_n c^n_\theta$, where $M^m_n = \langle \Phi_m, \Omega_\theta^{-1}\Phi_n\rangle_\mu \equiv (\Omega_\theta^{-1})^m_n$. Hence the two measures (4.17) and (4.29) are related by [15]

$$\mathcal{D}\Phi = J_\theta \; (\mathcal{D}\Phi)_\theta, \tag{4.30}$$

where the Jacobian is formally given by

$$J_\theta = \det(\Omega_\theta^{-1}) = \exp\left[-\mathrm{Tr}\ln\Omega_\theta\right]. \tag{4.31}$$

In the last step we have used the formal definition of the L_2 trace:[3]

$$\mathrm{Tr}[\mathcal{O}] = \sum_m \int d^D x \; g^{1/2} \; \Phi_m(\mathcal{O}\Phi_m) = \sum_m \int d^D x \; {g_\theta}^{1/2} \; \Phi_{\theta m}(\mathcal{O}\Phi_{\theta m}).$$

The trace is well defined if the diagonal matrix elements of the operator $\mathcal{O}$ decay sufficiently fast at large momenta. Unfortunately, the diagonal matrix elements of $\ln\Omega_\theta$ do not decay at all at large m, and so the trace is ill defined unless we introduce a regulator. We will address this question below, where we will explicitly define what we mean by J_θ.

Perhaps we should add, for the sake of clarity, that the difference between the line elements $d\mathcal{S}^2$ and $d\mathcal{S}^2_\theta$, and consequently the difference between the associated measures, *is not due to field redefinitions*. Both objects are different, but since they are defined geometrically, they are both invariant under field redefinitions (in the same sense that any line element is invariant under coordinate transformations). Rather, the relation (4.31) expresses the well known conformal anomaly. The measure is not invariant under conformal transformations because these do not correspond to a change of coordinates in field space $\mathcal{F}$. They correspond to a change of the spacetime metric and consequently to a change of the metric on $\mathcal{F}$. This sort of ambiguity does not arise when we consider scalar fields in flat space. Consider an action with a general kinetic term of the form:

$$S = \int d^D x \; G_{AB}(\phi^C)\eta^{\alpha\beta}\partial_\alpha\phi^A\partial_b\phi^B + ...$$

A natural line element in field space can be obtained by "stripping off" the flat metric $\eta^{\alpha\beta}$ and replacing the partial derivatives with differentials of the fields:

$$d\mathcal{S}^2 = \int d^D x \; G_{AB}(\phi^C)d\phi^A d\phi^B.$$

This procedure cannot be transported into a curved space, because in erasing the factor $g^{\alpha\beta}$ from the kinetic term, it makes a difference what exactly we have chosen to call the metric of spacetime: $g_{\alpha\beta}$ or $g^\theta_{\alpha\beta}$ (this is, by the way, the reason why the factor of Ω^2_θ appears in (4.28)). Usually, flat space definitions can be generalized to curved space through the principle of general covariance: objects should be defined geometrically, and they should reduce to their flat space definition when the spacetime metric is flat. The question is, however, which object should be considered to play the role of spacetime metric, so that we know when to call it flat. Physically,

[3] The definition of the trace is robust, in the sense that it is independent on the metric one uses in order to define the orthonormal basis, as long as the corresponding measures are in the same L_2 class. This will be the case, for instance, if the metrics are related by a conformal factor which is bounded above and below on the manifold.

too, one should expect that a preferred spacetime metric should play a role in regularizing and renormalizing the theory. Suppose that we attempt to regularize with a physical cut-off, so that all degrees of freedom beyond a certain scale are ignored. In our background (which is conformally flat) a constant physical cut-off scale corresponds to a different co-moving scales at different places. The relation between physical and co-moving scale is of course given by the metric, and therefore it can make a difference which one we use.

The contribution of the field Φ to the renormalized effective potential V_θ per unit co-moving volume parallel to the branes is given by:

$$\exp\left[-\mathcal{A}(V_\theta + V_\theta^{div})\right] \equiv \int (\mathcal{D}\Phi)_\theta \; e^{-S[\Phi]} = (\det P_\theta)^{-1/2}, \tag{4.32}$$

where $\mathcal{A}$ is the co-moving volume under consideration and we have used (4.27) and the measure (4.29) to express the gaussian integral as a determinant. The term V_θ^{div} is a local counterterm which, in dimensional regularization, needs to be subtracted from the regularized effective potential (its explicit form will be given in the coming Sections). In zeta function regularization, the left hand side is already finite, and V_θ^{div} is unnecessary (it corresponds to a finite renormalization of couplings). Eq. (4.32) can be written as

$$V_\theta \equiv \frac{1}{2\mathcal{A}} \ln(\det P_\theta) - V_\theta^{div}. \tag{4.33}$$

Eq. (4.30) suggests the notation [15]

$$V_\theta = \frac{1}{2\mathcal{A}} \ln(\det P) - V_\theta^{div} + \frac{1}{\mathcal{A}} \ln J_\theta. \tag{4.34}$$

The reader should be aware, however, that the definition of the Jacobian in Eq. (4.31) is only formal because the trace in the r.h.s. of this equation is ill defined. For that reason, it is not clear that Eq. (4.34) would hold with the definition (4.31), after substituting determinants by traces and applying any kind of regularization to the formally divergent traces. To avoid misinterpretations, in the discussions that follow we shall take J_θ to be defined by Eq. (4.34), that is [15]

$$\ln J_\theta \equiv \frac{1}{2}\left[\ln(\det P_\theta) - \ln(\det P)\right], \tag{4.35}$$

where the expression in the right hand side is to be calculated in some regularization scheme. Note that $\log J_\theta$ is usually called the *cocycle* function in the literature [192, 173].

The way the θ dependence of V_θ arises is very different in different regularization schemes. In Eq. (4.34), the determinant of P is independent of θ (we recall that this operator corresponds to the choice $\Omega_\theta = 1$). In dimensional regularization, $\ln J_\theta$ vanishes, but the divergent term V_θ^{div} which is subtracted from $\ln(\det P)$ depends on the choice of physical metric $g^\theta_{\alpha\beta}$. On the other hand, in zeta function regularization, $\ln(\det P)$ is finite and V^{div} does not play a role (in any case, any finite renormalization does not introduce a dependence in θ). Rather, in this case, the dependence on θ comes from $\ln J_\theta$, which does not vanish in this regularization scheme. In both cases, the θ dependence of V_θ is the same.

Dimensional regularization

A naive reduction to flat four-dimensional space suggests that the effective potential can be obtained as a sum over the KK tower:

$$V^D = \mu^\epsilon \sum_n \frac{1}{2} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \ln\left(\frac{k^2 + m_n^2(\varphi, D)}{\mu^2}\right). \tag{4.36}$$

Here $D = 4 + 1 - \epsilon$ is the dimension of spacetime, and we have added $(-\epsilon)$ dimensions parallel to the brane. The renormalized effective potential should then be given by an expression of the form

$$V(\varphi) = V^D - V^{div}, \tag{4.37}$$

and the question is what to use for the divergent subtraction V^{div}. Since Eq. (4.36) is similar to an ordinary effective potential in 4-dimensional flat space[4], one might imagine that V can be obtained from V^D just by dropping the pole term, proportional to $1/\epsilon$; but this is not true for warped compactifications

$$V(\varphi) \neq V^D - (\text{pole term}).$$

The point is that the theory is five dimensional and the spacetime is curved, and this fact must be taken into account in the process of renormalization.

Rather than proceeding heuristically from (4.37), we must take the definition of the effective potential Eq. (4.33) as our starting point, where it is understood that the formally divergent trace must be regularized and renormalized. In order to identify the divergent quantity to be subtracted, we shall use standard heat kernel expansion techniques. Let us introduce the dimensionally regularized expressions [193]

$$V_\theta^D \equiv \frac{\mu^\epsilon}{2\mathcal{A}} \mathrm{Tr} \ln\left(\frac{P_\theta(D)}{\mu^2}\right) = -\frac{\mu^\epsilon}{2\mathcal{A}} \lim_{s\to 0} \partial_s \zeta_\theta(s, D), \tag{4.38}$$

where

$$\zeta_\theta(s, D) = \mathrm{Tr}\left[\left(\frac{P_\theta(D)}{\mu^2}\right)^{-s}\right] = \frac{2\mu^{2s}}{\Gamma(s)} \int_0^\infty \frac{d\xi}{\xi} \xi^{2s}\, \mathrm{Tr}\left[e^{-\xi^2 P_\theta(D)}\right]. \tag{4.39}$$

It should be noted that the operator P_θ is positive and therefore the integral is well behaved at large ξ.

As is well known, the regularized potential V_θ^D contains a pole divergence in the limit $D \to 5$. To see that this is the case, one introduces the asymptotic expansion of the trace for small ξ [194, 195, 196],

$$\mathrm{Tr}\left[f\ e^{-\xi^2 P_\theta(D)}\right] \sim \sum_{n=0}^\infty \xi^{n-D} a_{n/2}^D(f, P_\theta), \tag{4.40}$$

where $a_{n/2}^D$ are the so-called generalized Seeley-De Witt coefficients. In (4.40) we have introduced the arbitrary smearing function $f(x)$. This is unnecessary for the present discussion, but it will be useful later on. For $n \leq 5$ their explicit form is known for a wide class of covariant operators, which includes our P_θ. They are finite and can be constructed from local invariants (terms

[4] It should be mentioned also that each KK contribution in Eq.(4.36) is not just like a flat space contribution, because in warped compactifications the KK masses $m_n(\varphi, D)$ depend on the number of external dimensions parallel to the brane.

constructed from the metric, the mass term E_θ and the smearing function f), integrated over spacetime. For even n, they receive contributions from the bulk and from the branes, whereas for odd n they are made out of invariants on the boundary branes only.

For definiteness, let us focus on the simplest case of a Dirichlet scalar field, satisfying

$$\Phi(z_\pm) = 0. \tag{4.41}$$

We can use the results found in [197, 198, 199, 200, 202] to compute the Seeley-De Witt coefficients for a Dirichlet field with a bulk operator $P = -(\Box + E)$. The lowest order ones for odd n are given in Appendix A.

As mentioned above, the integral (4.39) is well behaved for large ξ. For small ξ, the integral is convergent for $2s > D$, as can be seen from the asymptotic expansion (4.40). In the end, we have to consider the limit $s \to 0$, and so we must keep track of divergences which may arise in this limit. For this purpose, it is convenient to separate the integral into a small ξ region, with $\xi < \Lambda$, and a large ξ region with $\xi > \Lambda$, where Λ is some arbitrary cut-off. Substituting (4.40) into (4.39), we can explicitly perform the integration in the small ξ region for $2s > D$. This gives [7, 15]

$$\zeta(s, D) \sim 2\frac{\mu^{2s}}{\Gamma(s)} \left\{ \sum_{n=0}^{\infty} \frac{\Lambda^{n-D+2s}}{n - D + 2s} a^D_{n/2}(P_\theta) + \int_\Lambda^\infty \frac{d\xi}{\xi} \xi^{2s} \mathrm{Tr}\left[e^{-\xi^2 P_\theta(D)}\right] \right\}, \tag{4.42}$$

where we have used the standard notation

$$a^D_{n/2}(P_\theta) = a^D_{n/2}(f = 1, P_\theta).$$

Equation (4.42) is known as the *Mittag Leffler* expansion for the ζ function. This representation will be discussed in App. C in detail.

The second term in curly brackets in Eq. (4.42) is perfectly finite for all values of s. Analytically continuing and taking the derivative with respect to s at $s = 0$, we have

$$\zeta'(0, D) \sim \sum_{n=0}^{\infty} \frac{2\Lambda^{n-D}}{n - D} a^D_{n/2}(P_\theta) + \text{finite}, \tag{4.43}$$

where the last term is just twice the integral in (4.42) evaluated at $s = 0$. Introducing the regulator $\epsilon = 5 - D$, the ultraviolet divergent part of V^D_θ is thus given by

$$V^{div}_\theta = -\frac{1}{\epsilon \mathcal{A}} a^D_{5/2}(P_\theta). \tag{4.44}$$

The divergence is removed by renormalizing the couplings in front of the invariants which make up the coefficient $a^D_{5/2}$, and so this infinite term can be dropped. The renormalized effective potential of our interest is therefore given by

$$V_\theta = \lim_{D\to 5} \left[V^D_\theta - V^{div}_\theta\right]. \tag{4.45}$$

To proceed, we need to calculate V^D_θ, which in principle requires calculating a trace which involves the eigenvalues of P_θ, and as mentioned above, these are not related in any simple way to the KK masses.

However, it turns out that the dimensionally regularized V_θ^D is independent of θ when D is not an integer. The dependence of V_θ^D on θ can be found in the following way. First we note that

$$\partial_\theta \mathrm{Tr}\left[e^{-\xi^2 P_\theta}\right] = \mathrm{Tr}\left[2\xi^2 f_\theta(x)\Omega_\theta^{-2} P e^{-\xi^2 P_\theta}\right] = -\xi\partial_\xi \mathrm{Tr}\left[f_\theta(x) e^{-\xi^2 P_\theta}\right], \tag{4.46}$$

where we have introduced

$$f_\theta \equiv \partial_\theta \ln \Omega_\theta,$$

and the cyclic property of the trace was used. The above relation enables us to find the dependence of V_θ^D on the conformal factor:

$$\partial_\theta \lim_{s\to 0} \partial_s \zeta_\theta(s, D) = \lim_{s\to 0} \partial_s \frac{2\mu^{2s}}{\Gamma(s)} \int_0^\infty d\xi \xi^{2s} \partial_\xi \mathrm{Tr}\left[-f_\theta e^{-\xi^2 P_\theta}\right]. \tag{4.47}$$

As with the expansion (4.42) we may again introduce the regulator Λ and separate the integral into a large ξ part with $\xi > \Lambda$, which is finite and a small ξ part with $\xi < \Lambda$ which contains the divergent ultraviolet behaviour. Assuming that $2s > D$ and integrating by parts, the resulting integrals in the small ξ region can be performed explicitly and we have

$$\partial_\theta \lim_{s\to 0} \partial_s \zeta_\theta(s, D) \sim \lim_{s\to 0} \partial_s \frac{4s\mu^{2s}}{\Gamma(s)} \left[\sum_{n=0}^{\infty} \frac{\Lambda^{n-D+2s}}{n-D+2s} a^D_{n/2}(f_\theta, P_\theta) + \text{finite}\right]. \tag{4.48}$$

As before, the last term just indicates the integral in the large ξ region. Provided that D is not an integer, all terms in square brackets remain finite at small s, and so the right hand side of (4.48) vanishes. Hence, we find that [15]

$$\partial_\theta V_\theta^D = 0, \quad (D \neq \text{integer}). \tag{4.49}$$

In other words, the dimensionally regularized determinant of P_θ coincides with the dimensionally regularized determinant of P_0, and we have

$$V_\theta^D = V_0^D \equiv V^D \equiv \sum_n \mu^\epsilon \frac{1}{2} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \ln\left(\frac{k^2 + m_n^2(\varphi, D)}{\mu^2}\right), \qquad (D \neq \text{integer}). \tag{4.50}$$

As was anticipated, we find that $\ln J_\theta$ vanishes in the dimensional regularization presented here in the sense given in (4.35).

Finally, from (4.45) and (4.50), the renormalized effective potential is given by

$$V_\theta(\varphi) = \lim_{D\to 5}\left[V^D - \frac{1}{(D-5)} \frac{1}{\mathcal{A}}\, a^D_{5/2}(P_\theta)\right], \tag{4.51}$$

where the Seeley-De Witt coefficient $a^D_{5/2}$ is given in (A.7) with $f = 1$. The above equation bears the ambiguity in the choice of integration measure in the second term in square brackets. Different values of θ give different results. If we take $g_{\alpha\beta}$ as the preferred metric, then we should use $\theta = 1$, whereas if we take $g^{(s)}_{\alpha\beta}$ as the preferred metric, we should use $\theta = -1/\beta$. As we shall see in the next subsection, when we set $D = 5$ the coefficient $a_{5/2}(P_\theta)$ is also independent of θ. Hence, the pole term in the second term in (4.51) is independent of θ, as it should, in order to cancel the pole in V^D. However, the finite part does depend on the choice of θ.

The right hand side of (4.51) is ready for explicit evaluation, which is deferred to the next Chapters. We shall now turn our attention to the equivalent method of zeta function regularization.

Zeta function regularization

The method of zeta function regularization exploits the fact that the formal expression for the effective potential (4.38) is finite if the limit $D \to 5$ is taken before the limit $s \to 0$. This can be seen from Eq. (4.42), where the term with $n = 5$ is finite if we set $D = 5$ before taking the derivative with respect to s and setting $s \to 0$. Clearly, the change in the order of the limits simply removes the divergent term V^{div} given in (4.44) and it reproduces the results obtained by the method of dimensional regularization (up to finite renormalization terms which are proportional to the geometric invariant $a^{D=5}_{5/2}(P_\theta)$).

In zeta function regularization we define [7, 15]

$$V_\theta \equiv -\frac{1}{2\mathcal{A}} \ln(\det P_\theta) \equiv -\frac{1}{2\mathcal{A}} \lim_{s\to 0} \partial_s \zeta_\theta(s), \tag{4.52}$$

where $\zeta_\theta(s) \equiv \zeta_\theta(s, D = 5)$ [see Eq. (4.39)]. As in the case of dimensional regularization, it is more convenient to calculate V_0 than V_θ since the eigenvalues of P_0 are related to the spectrum of KK masses. An important difference with dimensional regularization is that

$$-2\mathcal{A}\partial_\theta V_\theta = \partial_\theta \zeta'_\theta(0) = 2a_{5/2}(f_\theta, P_\theta) \neq 0,$$

a result which we already encountered in Ref. [7] (see also [197, 198]). This can be seen from (4.48). If we set $D = 5$ from the very beginning, the term with $n = 5$ in Eq. (4.48) is linear in s, and its derivative with respect to s does not vanish in the limit $s \to 0$. Here, and in what follows, we use the notation

$$a_{n/2} \equiv \lim_{D\to 5} a^D_{n/2}.$$

Integrating along the conformal path parameterized by θ, we can relate the effective potential per unit comoving volume V_θ, with the "flat space" effective potential V_0 as [7, 15]

$$V_\theta = V_0 - \frac{1}{\mathcal{A}} \int_0^\theta d\theta' \, a_{5/2}(f_{\theta'}, P_{\theta'}). \tag{4.53}$$

The general expression for $a_{5/2}(f_\theta, P_\theta)$ which applies to our case has been derived by Kirsten [197, 202]. In Ref. [7] we evaluated the integral in (4.53) for the Randall-Sundrum case, in order to obtain V_1 from V_0. Here we shall present an alternative expression for this integral which does not require the knowledge of $a_{5/2}(f_\theta, P_\theta)$, but only the knowledge of $a^D_{5/2}(P_\theta)$ for dimension $D = 5 - \epsilon$. This will also illustrate the relation between the method of zeta function regularization and the method of dimensional regularization.

The relation between the zeta function representation of the functional determinants for conformally related operators (4.53) was obtained independently in [7], but it had previously been derived by Dowker and Apps [192]. In particular, the function

$$\int_0^\theta d\theta' a_{5/2}(f_{\theta'}, P_{\theta'})$$

is usually called the *cocycle* function in the literature.

Equivalence between ζ function and dimensional regularizations

From the asymptotic expansion of the first and the last terms in Eq. (4.46), we have [201]

$$\partial_\theta a^D_{n/2}(P_\theta) = (D-n) a^D_{n/2}(f_\theta, P_\theta). \tag{4.54}$$

Integrating over θ, we get

$$(D-5)\int_0^\theta a^D_{5/2}(f_{\theta'}, P_{\theta'}) d\theta' = a^D_{5/2}(P_\theta) - a^D_{5/2}(P_0). \tag{4.55}$$

Writing $D = 5-\epsilon$, we have

$$V_\theta - V_0 = -\frac{1}{\mathcal{A}}\int_0^\theta a_{5/2}(f_{\theta'}, P_{\theta'}) d\theta' = \frac{1}{\epsilon\mathcal{A}}\left[a^D_{5/2}(P_\theta) - a^D_{5/2}(P_0)\right]. \tag{4.56}$$

Note that, from (4.44) and (4.51), the previous equation can also be written as

$$V^D \equiv V_\theta + V_\theta^{div} = V_0 + V_0^{div}. \tag{4.57}$$

This equation simply expresses the fact that the dimensionally regularized V^D is independent of the conformal parameter θ, as we had shown in the previous subsection [see e.g. Eq. (4.49)].

From (4.54), with $D = n = 5$, one finds that the coefficient $a_{5/2}(P_\theta)$ is conformally invariant [201], and therefore

$$a_{5/2}(P_\theta) = a_{5/2}(P_0). \tag{4.58}$$

Substituting this into (4.56), we obtain [15]

$$\int_0^\theta a_{5/2}(f_{\theta'}, P_{\theta'}) d\theta' = \frac{d}{dD} a^D_{5/2}(P_\theta)\Big|_{D=5} - \frac{d}{dD} a^D_{5/2}(P_0)\Big|_{D=5}. \tag{4.59}$$

Thus, the integral in (4.53) can be evaluated in two different ways. One is by using the explicit expression of $a_{5/2}(f,P)$ given by Kirsten [197, 202]. The other is by taking the derivative of the coefficients $a^D_{5/2}(P_\theta)$, given in (A.7) with $f=1$, with respect to the dimension. [Note that the terms which are linear in derivatives of f, which we have just indicated symbolically in (A.7), disappear when f is a constant].

4.5 Summary

The Casimir energy, $E_{Cas} = (1/2)\sum \hbar w$, and the 1 loop effective potential $V_{\mathit{eff}}^{1\text{loop}}$ induced by the KK modes of bulk fields are physically equivalent. The evaluation of $V_{\mathit{eff}}^{1\text{loop}}$ reduces to the computation of $\log\det P = \mathrm{Tr}\log P$ (see Eq. (4.20)), where P is the Laplacian operator appearing in the field's action. This is usually performed by finding and summing over the Laplacian's eigenvalues.

In a warped spacetime, it is not straightforward to find the eigenvalues of P. For a conformally flat space, $\det P$ can be related to the conformally related operator in flat space $\det P_0$, see Eq. (4.53). The relationship involves the so-called *cocycle* function and can be expressed in terms of geometric invariants.

We have shown the precise connection between $\log\det P$ and $\log\det P_0$ using both zeta function and dimensional regularization, and obtained two independent expressions for the cocycle function (Eq. (4.56)). The equivalence between them stands from the well known properties under conformal transformations of the Seeley-DeWitt coefficients.

For practical purposes, this provides two procedures to compute V_{eff}. The dimensionally regularized potential V^{reg} does not depend on the conformal frame. This allows us to choose the easiest one to compute it, which is the frame with flat metric because the eigenvalues split into the KK form $k_\mu k^\mu + m_n^2$. The renormalized potential is then obtained by subtracting the divergences proportional to covariant operators in the original frame.

Using zeta function regularization, no divergences have to be subtracted since the regularized potential is already finite. The potential in the physical frame is related to that in flat space by the cocycle function.

Chapter 5

Radion effective Potential in the Randall Sundrum model

In this Chapter we review the 1-loop effective potential for the radion induced by bulk fields in the RS model. This was first computed by us in [7] using zeta function regularization for the graviton and for conformally or minimally coupled scalar fields. It was subsequently computed using dimensional regularization in [11, 12, 13]. The correct expression for the bulk gauge boson contribution was obtained in [8], where it was realized that in this case it stabilizes naturally the hierarchy.

5.1 Generic fields

In dimensional regularization (see Chapter 4), the one loop effective potential can be expressed as the sum over the contributions of each mode,

$$V_{RS}^{\rm reg} = \sum_n \mu^{2\epsilon}\frac{1}{2}\int \frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}} \ln\left(\frac{k^2+m_n^2}{\mu^2}\right) = -\frac{\mu^{2\epsilon}}{2(4\pi)^2}\Gamma(s){\sum_n}' m_n^{-2s} \ , \tag{5.1}$$

where we introduced $s=-2+\epsilon$ and the prime in the sum means that the zero mass mode is excluded. As usual, μ is a renormalization scale introduced for dimensional reasons. The sum over the KK masses (defined in (3.23) has been done in [7, 11, 12, 13].

Without going into details, the resulting regularized potential is

$$V_{RS}^{\rm reg} = -\frac{k^4}{32\pi^2}(k/\mu)^{-2\epsilon}\left\{-d_4\frac{1}{\epsilon}\left(1+a^{4-2\epsilon}\right)+c_1+a^4c_2-2a^4\mathcal{V}(a)\right\}, \tag{5.2}$$

where the coefficients c_1 and c_2 do not depend on a and the coefficient d_4 depends on the mass and non-mininal coupling of Φ, and is defined in Appendix D. In (5.2), for Dirichlet boundary conditions

$$\mathcal{V}(a) = \int_0^\infty t^3 \ \ln\left(1-\frac{k_\nu(t)i_\nu(at)}{k_\nu(at)i_\nu(t)}\right), \tag{5.3}$$

and for Neumann boundary conditions, we have to replace I_ν and K_ν with

$$i_\nu(z) \quad = \quad zI_{\nu-1}(z)+\varepsilon I_\nu(z) \ ,$$

$$k_\nu(z) \;=\; -zK_{\nu-1}(z) + \varepsilon K_\nu(z)\ . \tag{5.4}$$

Equation (5.2) is infinite for $\epsilon \to 0$, but the divergence is proportional to 5D geometrical terms. Indeed, a renormalization of the brane tension by an amount $\delta\tau_\pm$ result in terms of the form (in $(4-2\epsilon)+1$ dimensions, and with $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$)

$$\int d^{4-2\epsilon}x\ \left\{\delta\tau_+ + \delta\tau_-\, a^{4-2\epsilon}\right\}. \tag{5.5}$$

Thus, the divergence in (5.2) can be absorbed in the brane tensions, and the same happens with the constants $c_{1,2}$. We are left with the *arbitrary* finite part of $\delta\tau_\pm$, which allows us to write the renormalized potential as [7]

$$V_{\mathit{eff}}^{RS}(a) = \frac{k^4}{16\pi^2}\left\{\gamma_+ + \gamma_- a^4 + a^4\mathcal{V}(a)\right\}, \tag{5.6}$$

where $\gamma_\pm$ are the finite renormalizations of brane tensions in units of k^4. As such, they cannot be determined from our calculation. They are expected to be of the order 1. One of them will have to be fine tuned in order to cancel the 4D effective cosmological constant.

In order to illustrate this, we shall specialize this result to a massless conformally coupled fields in 5D, with $m = 0$ and $\xi = 3/16$. In this case, from (5.3) with $\nu = 1/2$ or integrating (4.6) over z, one can see that the potential has a simple expression

$$\mathcal{V}(a) = 2\int a^5(z)\rho\, dz = -(-1)^F k^4 \frac{Aa^4}{(1-a)^4}, \tag{5.7}$$

where $A \approx 2.46\cdot 10^{-3}$ and $F = 0, 1$ for bosons and fermions. Adding the brane tension renormalization terms, we find [7]

$$V_{\mathit{eff}}^{RS}(a) = -(-1)^F k^4\left[\gamma_+ + \gamma_- a^4 + \frac{Aa^4}{(1-a)^4}\right]. \tag{5.8}$$

The *renormalization conditions* that we shall impose on $\gamma_\pm$ are on one hand that the radion a is fixed to the 'observed' value. In the RS scenario, this means that $\langle a\rangle = 10^{-16}$, in order to really solve the hierarchy problem. Technically this translates into the existence of a minimum at that precise value of a. On the other hand, we have to impose that the effective 4D cosmological constant vanishes, since we have assumed that the 4D slices are Minkowski. In fact, the observed 4D effective cosmological constant is millimetric. Thus, we can summarize the renormalization conditions as

$$\partial_a V_{\mathit{eff}}^{RS}|_{\langle a\rangle} = 0 \qquad \text{and} \qquad V_{\mathit{eff}}^{RS}(\langle a\rangle) \simeq 10^{-122} m_P^4. \tag{5.9}$$

From these conditions, the unknown coefficients $\gamma_\pm$ can be found, and then the mass of the radion is calculable.

We obtain

$$\gamma_- = -A(1-\langle a\rangle)^{-5}, \qquad \text{and} \qquad \gamma_+ = -\gamma_-\langle a\rangle^5. \tag{5.10}$$

The value of the negative tension brane γ_- is what determines the hierarchy $\langle a\rangle$. In the limit $\langle a\rangle \ll 1$, this requires

$$\langle a\rangle \simeq \frac{1}{5}\left(\frac{\gamma_-}{A} - 1\right),$$

which clearly is a large fine tuning of γ_-. The positive tension brane γ_+ is also fine tuned, but this is corresponds to the usual fine tuning of the (effective) cosmological constant, about which the RS model does not say anything.

We can now calculate the mass of the radion field m_a^2. For $\langle a \rangle \ll 1$ we have [7]

$$m_\varphi^2 = \frac{d^2}{d\varphi^2} V_{\mathit{eff}}^{RS} \sim -(-1)^F \langle a \rangle^3 \frac{k^4}{m_P^2} \sim \langle a \rangle \mathrm{TeV}^2 \tag{5.11}$$

So if we solve the hierarchy problem geometrically with $\langle a \rangle$ of order TeV$/m_{pl}$, then the contribution to the radion (mass)^2 (5.11) from a bulk fermion field is positive and of order of some keV. As discussed in the previous Sections, such a small mass is in conflict with observations.

In the general case, one finds that unless $\nu = 0$ or 1, the function $\mathcal{V}(a)$ behaves as $a^{2\nu}$ (for $\varepsilon \neq 0$) or as $a^{2\nu-2}$ (for $\varepsilon = 0$) [8]. It is straightforward to see that as in the case with conformally coupled fields, the brane tensions need to be fine tuned in order to obtain $a \ll 1$. The contribution to the radion mass in this case is smaller [7],

$$m_\varphi^2 \sim a^4 \frac{k^4}{f^2} \sim \text{ meV}^2,$$

far below the observational bounds.

In conclusion, if we consider only generic scalar fields or the graviton in the bulk, we must accept the existence of another stabilization mechanism.[1] In any case, this shows that the quantum corrections due to bulk scalar are small in the RS model.

The impossibility to obtain an efficient potential in these cases can be traced back to the absence of logarithms $\ln a$. In this respect, it is worth noting a remarkable coincidence that has occurred in the computation of $V_{\mathit{eff}}^{RS}(a)$. The geometrical terms (5.5) contain a divergent part $\propto \delta\tau + +a^4\delta\tau_-$ and a finite part $\propto \ln a$. This is reminiscent of the Coleman-Weinberg potential discussed in the beginning of this Section. However, in order to cancel the divergent part of (5.2), the logarithmic term is inevitably canceled also. The only 'hope' to obtain a logarithmic dependence is in the *nonlocal* part $\mathcal{V}(a)$. As we shall see in Chapters 6 and 7, this does not happen in general. Rather, it has to be interpreted as a peculiarity of the RS model due to the maximal symmetry of the bulk AdS$_5$ space [7].

5.2 Gauge field

The contribution from a 5D gauge field can be obtained from that of a scalar field (5.6) with the replacements $\nu \to 1$ and $\varepsilon \to 0$. We discuss it separately here due to the qualitatively different behaviour of the potential in this case.

The SUSY partners of the gauge boson in AdS[105, 8] are a (Dirac) fermion field (the *gaugino*) with bulk mass $m = k/2$, and a real scalar with mass $m^2 = -4k^2$. The fermion has an index of the Bessel functions $\nu = 1$, whereas the scalar has $\nu = 0$ and and Dirichlet boundary conditions. On the other hand, a 5D hypermultiplet with a fermion mass $m = k/2$ contains a scalar with

[1] Of course, if $\langle a \rangle \sim 1$ then the radion mass (5.11) would be very large, but then we must look for a different solution to the hierarchy problem.

$m^2 = -4k^2$, and $\nu = 1$. By supersymmetry, all these cases contribute the same to V_{eff}^{RS}, except for an overall factor taking into account the statistics and the number of degrees of freedom.[2]

Expanding the expression for $\mathcal{V}(a)$ for $\nu \to 1$ and $\varepsilon \to 0$, one obtains [8]

$$V_{eff} \simeq (-1)^F \frac{k^4}{16\pi} \left\{ \gamma_+ + \gamma_- a^4 - g_* \beta \frac{a^4}{\ln a} \right\}, \tag{5.12}$$

where $F = 0$ or 1 for the gauge boson or the gaugino (its SUSY counterpart), g_* is the number of physical polarizations and

$$\beta \equiv \int_0^\infty dt\, t^3 \, \frac{K_1(t)}{I_1(t)} \simeq -1.005 \tag{5.13}$$

This potential has an extremum at

$$\ln a \simeq g_* \beta / \gamma_-, \tag{5.14}$$

and gives the radion a mass [8],

$$m_\varphi^2 \simeq (-1)^{F+1} \frac{g_* \beta}{(2\pi \ln a)^2} \frac{a^2 k^4}{f^2}. \tag{5.15}$$

It follows from Eq. (5.14) that an exponentially large hierarchy can be naturally obtained, *i.e.* with γ_- of order one. This corresponds to a renormalization of the negative tension by an amount comparable to the cutoff scale (the value in background), which is certainly a natural value. It is also transparent that the contribution from the gauge boson to the potential has a maximum at this point. Thus, it is the gaugino that contributes to stabilize the radion. With another choice of boundary conditions, the situation can be the converse. For example, with Dirichlet boundary conditions on the negative tension brane and Neumann boundary conditions on the positive tension brane , the gauge boson induces a positive m_φ^2 [8].

With one bulk gaugino, $g_* = 4$, the resulting radion mass is

$$m_\varphi \simeq \frac{1}{6\pi \ln\left(\langle\varphi\rangle / f\right)} \left(\frac{k}{m_P}\right)^2 \langle\varphi\rangle, \tag{5.16}$$

which is about 100 MeV for $k/m_P = 0.1$ and $\langle\varphi\rangle = 3$TeV.

Thus, the potential induced by a bulk gauge field qualitatively differs from that of a generic scalar or the graviton in that it is much larger and the contribution to the radion mass is around the phenomenologically acceptable GeV. This has an interpretation in terms of the dual CFT representation of the RS model [8]. The idea is that the 4D photon propagator is renormalized by the coupling to the CFT states. This is encoded in the running of the coupling constant $g(\mu)$. At energies of order the resonance masses ak, the effective coupling $g(ak)$ is given by

$$\frac{1}{g^2(ak)} = \frac{1}{g^2(k)} + cN \ln\left(k/(ak)\right) \simeq -cN \ln(a),$$

in the large N approximation. The dual of the gauge boson KK modes in the RS models are spin 1 resonances in the CFT, at the IR (TeV) scale. The photon can mix with them, giving a mass

[2]We will not be so interested in the scalars, since these specific values for their masses are unstable under quantum corrections, and they inevitably involve some fine tuning.

correction (in large N) of order $\delta m_{KK}^2 \sim N g^2 (ak)(ak)^2 \simeq (ka)^2/\ln a$. Thus, the contribution from these states to the effective potential is expected to be of the form

$$m_{_{KK}}^4 \ln\left(m_{_{KK}}/\mu\right) \sim k^4 a^4 \left(1 + \frac{c}{\ln a}\right),$$

which is precisely the form obtained in this case, *cfr.* Eq. (5.12).

In summary, the 1-loop effective potential induced by a generic bulk field is given by Eqns. (5.6) and (5.3) [7]. The contribution from a bulk gauge field Eq. (5.12) [8] naturally stabilizes hierarchy and gives the radion a sizable mass.

Chapter 6

Moduli effective potential in warped compactifications

In this Chapter we present a 5D scalar tensor model with exponential potentials for the scalar both in the bulk and on the branes. This kind of models arise in compactifications of higher dimensional models with the additional extra dimensions forming a Ricci flat space, such as the Hořava Witten model [9].

These models contain solutions to Einstein's equations similar to the RS model but with a power law warp factor $a(y) \propto y^q$. We will take the parameters of the model such that we can consider any positive power of the proper distance along the bulk. In the RS model, the warp factor is exponential $a(y) = e^{-ky}$ and the curvature scale k is Planckian. This makes the *large* bulk volume to be really not very large compared to the curvature (or the cutoff) scale $d \sim 37/k$.

Models with power law warp factors are interesting in that besides the redshift effect, they can exploit the large bulk volume effect. As in the RS model, the hierarchy is essentially generated by the ratio of warp factors on both branes, which gives $(m_{EW}/m_P) \sim (y_-/y_+)^q$, where $y_\pm$ are the positions of the branes. Associated to both $y_\pm$, there are *moduli* fields analogous the *radion*, which corresponds to a specific combination of them.

If the Casimir effect can stabilize naturally the moduli at $y_- \sim 1/m_P$ and $y_+ \sim$ mm, then very low powers of q are enough to generate the necessary hierarchy, $\sim 10^{-16}$. In such a case, warp factors not as steep as in the RS model also solve naturally the hierarchy problem. This represents a nice combination of ADD and RS mechanisms: the hierarchy arises purely from a *redshift* effect, but a *large* interbrane distance is required in order to 'accumulate' enough redshift along the bulk (the dilution of gravity along the extra dimension does not play an important role here). This model also shares with the ADD scenario the feature that the EW/Planck hierarchy is a power (not an exponential) of the radius.

As we shall see, the Casimir energy from bulk fields (of any spin) can indeed generate naturally a large interbrane distance. However, if the power q is too small, the generated moduli massess are too small as well. In fact, one of the moduli fields is quite strongly coupled to matter. This makes the stabilization of this modulus more difficult, since a large mass is required.

The presence of a bulk scalar with nontrivial profile in these backgrounds raises a technical issue in the derivation of the effective potential that results in a not uniquely defined path integral measure. This translates in an ambiguous result in the potential. However, the ambiguity is proportional to a finite local term, which breaks a global symmetry present in the model at tree level. Thus, this is an anomalous symmetry (closely connected to the conformal anomaly), and its breaking is interpreted to be responsible for stabilizing the moduli $y_\pm$. The precise value of the constant in front of the anomalous term can only be obtained from the underlying theory. The higher dimensional models considered in Chapter 7 reduce upon compactification to the model considered here with some specific values of q. In these cases, the path integral measure is unique, and there is no ambiguity in the result. A complete discussion on this point is deferred to Chapter 7, since contributions from the additional extra dimensions can play a significant role.

This Chapter is organized as follows. In Section 6.1 we present the model, its symmetries and the background solution that we shall study. Also, the properties of the moduli fields and the dependence of the hierarchy upon them are described. In Section 6.2 the computation of the effective potential induced by generic bulk fields is described. In Section 6.3, the efficiency of this potential as a stabilization mechanism for the moduli $y_\pm$ is investigated.

6.1 Scalar-tensor 5D model

The classical background we shall consider in this Chapter is a 5-dimensional spacetime with a nontrivial background scalar field ϕ, which we shall refer to as the "dilaton". The fifth dimension is compactified on a Z_2 orbifold with two branes at the fixed points of the Z_2 symmetry. The action for the background fields is given by

$$\begin{aligned} S_b &= \frac{1}{16\pi G_5}\int d^5x\,\sqrt{g}\left(\mathcal{R}-\frac{1}{2}(\partial\phi)^2-\Lambda e^{c\,\phi}\right) \\ &- \sigma_+\int d^4x\sqrt{g_+}\,e^{c\,\phi/2}-\sigma_-\int d^4x\sqrt{g_-}\,e^{c\,\phi/2}, \end{aligned} \tag{6.1}$$

where $\mathcal{R}$ is the curvature scalar and the fundamental cutoff M is given by the 5D gravitational coupling constant as $16\pi G_5 = M^{-3}$. We have denoted the induced metrics on the positive and negative tension branes by $g^+_{\mu\nu}$ and $g^-_{\mu\nu}$, respectively. To find a solution to the equations of motion, we make an ansatz where the 4-dimensional metric is flat,

$$ds^2 = dy^2 + a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu, \tag{6.2}$$

with a x^μ-independent scalar field $\phi = \phi(y)$. The positive and negative branes are placed at $y = y_+$ and y_-, respectively. Under these assumptions, the equations of motion for $(a(y), \phi(y))$ in the bulk become

$$\begin{aligned} \left(\frac{\dot a}{a}\right)^2 &= \frac{1}{12}\left(\frac{1}{2}\dot\phi^2 - U(\phi)\right), \\ \ddot\phi + 4\,\frac{\dot a}{a}\,\dot\phi &= U'(\phi), \end{aligned} \tag{6.3}$$

where $U(\phi) \equiv \Lambda e^{c\,\phi}$, a dot represents differentiation with respect to y and a prime represents differentiation with respect to ϕ.

As shown in [203], there is a solution of Eqs. (6.3) for any value of c given by

$$\phi = -\sqrt{6q}\,\ln(y/y_0),$$

$$a(y) \;=\; (y/y_0)^q\,, \tag{6.4}$$

where

$$q = \frac{2}{3c^2}, \qquad y_0 = \sqrt{\frac{3q(1-4q)}{\Lambda}}. \tag{6.5}$$

(Constant rescalings of the warp factor are of course allowed, but unless otherwise stated, we shall take the convention that $a(y) = 1$ at $y = y_0$.) Assuming $y_- < y_+$, the boundary conditions which follow from Z_2-symmetry imposed on both branes are given by

$$\dot\phi_\pm = \mp\frac{c}{4}\sigma_\pm e^{(c/2)\phi_\pm}, \tag{6.6}$$

$$6\,\frac{\dot a}{a}\Big|_\pm = \pm 8\pi G_5 \sigma_\pm e^{(c/2)\phi_\pm}, \tag{6.7}$$

and they are satisfied if $\sigma_\pm$ are tuned to

$$\sigma_\pm = \pm\frac{1}{16\pi G_5}\sqrt{\frac{48q\Lambda}{1-4q}}. \tag{6.8}$$

In the absence of the branes, the spacetime (6.4) contains a singularity at $y = 0$. Since we are considering the range between y_- and y_+, this singularity is of course innocuous. Our spacetime consists of two copies of the slice comprised between y_- and y_+, which are glued together at the branes. Hence, the 5-th dimension is topologically an S^1/Z_2 orbifold.

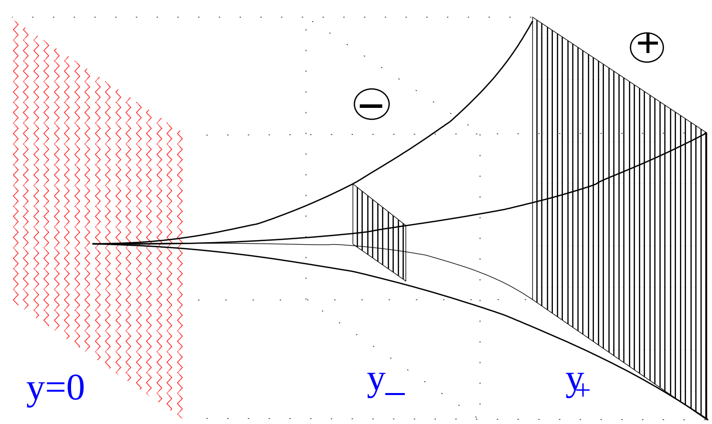


Figure 6.1: With a power law warp factor $a(y) = (y/y_0)^q$, the space is singular at $y = 0$. The solution (6.4) consist of the slice comprised between the positive tension ($y = y_+$) and negative tension ($y = y_-$) branes. Two copies of this 5D space are glued by the branes and the orbifold Z_2 identification is imposed.

For $q = 1/6$ this solution is precisely the M-theory heterotic brane model of Ref. [10]. On the other hand, the RS case, where the bulk is AdS and there is no scalar field, can be obtained by taking the limit $q \to \infty$ and $y_0 \to \infty$ simultaneously, while its ratio is kept fixed,

$$\ell = \lim_{q\to\infty} \frac{y_0}{q} = \sqrt{\frac{-12}{\Lambda}}. \tag{6.9}$$

Defining $y \equiv y_0 + y^*$, we find that in the limit the warp factor becomes an exponential

$$\lim_{q\to\infty} a = e^{y^*/\ell},$$

which corresponds to AdS space with curvature radius equal to ℓ.

Moduli fields

For fixed value of the coupling c, the solution given above contains only two physically meaningful free parameters, which are the locations of the branes y_-, and y_+. This leads to the existence of the corresponding moduli, which are massless scalar fields from the 4-dimensional point of view. In addition to these moduli, the massless sector also contains the graviton zero mode. To account for it, we generalize our metric ansatz (6.2) by promoting $\eta_{\mu\nu}$ to an arbitrary four-dimensional metric:

$$ds^2 = dy^2 + a^2(y)\tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu. \tag{6.10}$$

It is easy to show that $\mathcal{R} = a^{-2}\tilde{\mathcal{R}} + \mathcal{R}^{(0)}$, where $\mathcal{R}^{(0)}$ is the background Ricci scalar in five dimensions and $\tilde{\mathcal{R}}$ is the four-dimensional one. For constant values of the metric and moduli, we have a solution of the equations of motion whose action vanishes. Hence, only the terms which depend on derivatives of the metric or derivatives of the moduli will survive after the five-dimensional integration. This fact can be used in order to simplify the derivation of the effective action for the moduli, since all terms without any derivatives can be dropped.

In the bulk, all terms will cancel except for the one which is proportional to the four-dimensional Ricci scalar, $\tilde{\mathcal{R}}$. Let us consider the contribution from the branes. The metric induced on the branes is given by

$$g^{\pm}_{\mu\nu} = a^2_{\pm}[\tilde{g}_{\mu\nu} + a^{-2}_{\pm}\partial_\mu y_\pm \partial_\nu y_\pm].$$

Here, and in what follows, the subindices $\pm$ mean that the quantity is evaluated at the *perturbed* brane location. The brane tension terms in the action contain the determinant

$$\sqrt{-g_\pm} = a^4_\pm \sqrt{-\tilde{g}} \left[1 + \frac{1}{2a^2_\pm}\,(\tilde{\partial} y_\pm)^2\right] + \cdots,$$

which induces kinetic terms for the moduli. Here, the tilde on the kinetic term indicates that the derivatives are contracted with the metric $\tilde{g}$. In fact, the five dimensional Ricci tensor $\mathcal{R}$ contains second derivatives of the metric and therefore it is singular on the brane, giving a finite contribution to the action. To handle this contribution it is simplest to introduce fiducial boundaries in the neighborhood of the branes, were we add (back to back) pairs of Gibbons-Hawking boundary terms. These have the form

$$\frac{1}{8\pi G_5}\int d^4x\sqrt{g_\pm}\ \mathcal{K}_\pm, \tag{6.11}$$

where $\mathcal{K}_\pm$ is the trace of the extrinsic curvature of the fiducial boundary near each one of the branes. The action is separated into two parts. The first one is a "bulk" part, consisting of an integral over two copies of an open set which excludes the branes, $(y_- + \epsilon) < y < (y_+ - \epsilon)$, plus terms of the form (6.11) at the boundaries $y = y_- + \epsilon$ and $y = y_+ - \epsilon$. Then there is a "brane" part, which includes an integral of the action over the infinitesimal open sets of thickness ϵ around the branes, supplemented with terms of the form (6.11) at the boundaries of these open sets (these "brane" boundary terms sit back to back with the ones used in order to bound the "bulk", and have opposite sign relative to them, since the normal to the fiducial boundary has opposite sign on each side of the boundary. Thus the total effect of the boundary terms is to add zero to the action). As is well known, through integration by parts the boundary terms absorb the second derivatives in the Einstein term $\mathcal{R}$. Hence, the singular contribution from the gravity kinetic term on the brane disappears, and in the limit $\epsilon \to 0$, the only contribution to the action from the interval of width 2ϵ around the branes (together with the added boundary terms), is from the brane tension term itself, but not from the gravity kinetic term. On the other hand, the bulk contribution has to be supplemented with the boundary terms (6.11).

The boundary terms can be evaluated as follows. Consider, for instance the hypersurface which is located at $y = y_+(x^\sigma)$. In terms of the new coordinate $\hat{y} = y - y_+(x^\sigma)$, the brane is at $\hat{y} = 0$, and the metric can be written as

$$ds^2 = N^2 d\hat{y}^2 + g^+_{\mu\nu}(N^\mu d\hat{y} + dx^\mu)(N^\nu d\hat{y} + dx^\nu), \tag{6.12}$$

where we have introduced the lapse function $N^2 = 1 - g^+_{\mu\nu} N^\mu N^\nu$, and the shift vector $N^\mu = g^{+\mu\nu} y_{,\nu}$. Then, the trace of the extrinsic curvature is given by

$$\begin{aligned} \int dx^4 \sqrt{-g_+}\, \mathcal{K}_+ &= \int dx^4 \frac{\sqrt{-g_+}}{2N} \left[g^{+\mu\nu} \partial_{\hat{y}} g^+_{\mu\nu} - 2N^\mu{}_{|\mu} \right] \\ &= \int dx^4 \frac{\sqrt{g_+}}{2N} \left[\frac{\dot{a}}{a}\, g^{+\mu\nu} \tilde{g}_{\mu\nu} - \frac{N^\mu \partial_\mu N^2}{N^2} \right] \\ &\approx 4 \int \left(\frac{\dot{a}}{a} \right)_+ a_+^4 \sqrt{-\tilde{g}} \left[1 + \frac{3}{4a_+^2} (\tilde{\partial} y_+)^2 \right], \end{aligned} \tag{6.13}$$

where the vertical line means a covariant differentiation with respect to the induced metric $g^+_{\mu\nu}$. We neglected the terms which are higher order in in derivatives of the modulus y_+. As mentioned before, the subindices $\pm$ mean that the quantities are evaluated at the *actual* position of the brane, and the expression is in fact nonperturbative in the positions $y_\pm$ themselves (although not in the derivatives).

Substituting the previous expressions into the action (6.1), with the addition of the extrinsic curvature terms, and using the background equations of motion we find

$$\begin{aligned} S_b = \int d^4x \sqrt{-\tilde{g}} \Bigg[& \left(2 \int_{y_-}^{y_+} dy\, a^2 \right) \frac{1}{16\pi G_5} \tilde{\mathcal{R}} \\ + & \frac{1}{2} \sigma_+ e^{(c/2)\phi_+} a_+^2\, (\tilde{\partial} y_+)^2 + \frac{1}{2} \sigma_- e^{(c/2)\phi_-} a_-^2\, (\tilde{\partial} y_-)^2 \Bigg]. \end{aligned} \tag{6.14}$$

This can be rewritten as

$$S_b = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \left\{ (\psi_+^2 - \psi_-^2)\, \tilde{\mathcal{R}} + \frac{6q}{q + 1/2} \left[(\tilde{\partial}\psi_+)^2 - (\tilde{\partial}\psi_-)^2 \right] \right\}. \tag{6.15}$$

Here we have introduced

$$\psi_\pm \equiv \left(\frac{y_\pm}{y_0}\right)^{q+1/2},$$

and the four dimensional Newton's constant G given by

$$G = \left(q + \frac{1}{2}\right)\frac{G_5}{y_0}. \tag{6.16}$$

The modulus corresponding to the positive tension brane has a kinetic term with the "wrong" sign. However, this does not necessarily signal an instability, because it is written in a Brans-Dicke frame. One may go to the Einstein frame by a conformal transformation. It is convenient to introduce the new moduli ψ and ψ through [204]

$$\begin{aligned} \psi_+ &= \psi \cosh\varphi, \\ \psi_- &= \psi \sinh\varphi, \end{aligned} \tag{6.17}$$

and to define the new metric

$$\hat{g}_{\mu\nu} = \psi^2 \tilde{g}_{\mu\nu}. \tag{6.18}$$

It is then straightforward to show that $\psi^2\ \sqrt{\tilde{g}}\ \tilde{\mathcal{R}} = \sqrt{-\hat{g}}\ [\hat{R} + 6\ \psi^{-2}\ (\hat{\partial}\psi)^2]$. Substituting into the background action (6.15), we have

$$S_b = \frac{1}{16\pi G}\int d^4x\sqrt{-\hat{g}}\left\{\hat{\mathcal{R}} - \frac{6}{1+2q}\frac{(\hat{\partial}\psi)^2}{\psi^2} - \frac{12q}{1+2q}\ (\hat{\partial}\varphi)^2\right\}. \tag{6.19}$$

Therefore, both moduli have positive kinetic terms in the Einstein frame. At the classical level, the moduli are massless, but as we shall see in the following Sections, a potential term of the form

$$\delta S = -\int d^4x\ V(\psi,\varphi) \equiv -\int d^4x\sqrt{-\hat{g}}\ \hat{V}(\psi,\varphi), \tag{6.20}$$

is generated at one loop, which should be added to (6.19).

In the RS limit $q \to \infty$ [see Eq. (6.9)] the kinetic term for the ψ modulus disappears. This is to be expected, because the bulk is the maximally symmetric AdS space. In this case only the relative position of the branes $y_+ - y_-$ is physically meaningful and the other modulus can be gauged away (see also [177] for a recent discussion of this case).

In the flat space limit $q \to 0$, it is the φ modulus that decouples. This is also to be expected from the expression of the moduli $\psi_\pm$. For $q = 0$, the *radion* modulus has to be identified with ψ, since $\psi^2 = \psi_+^2 - \psi_-^2 \propto (y_+ - y_-)$, the only meaningful distance.

We find the curious property that from $q = 0$ to $q \to \infty$, the radius modulus transmutes from ψ to φ. In the intermediate cases, we will refer to φ as the radion, since this is the modulus with lightest mass.

Scaling symmetry

Since we are interested in the effective potential for the moduli, it is perhaps pertinent to start by asking why these fields are massless at the classical level. The reason is that under the global transformation

$$g_{\alpha\beta} \to T^2 g_{\alpha\beta}, \tag{6.21}$$

$$\phi \to \phi - (2/c) \ln T, \tag{6.22}$$

the action (6.1) scales by a constant factor

$$S_b \to T^3 S_b.$$

Here $g_{\alpha\beta}$ is the metric appearing in the action (6.1). Acting on a solution with one brane, the transformation simply moves the brane to a different location. Hence, all brane locations are allowed, from which the masslessness of the moduli follows. However, we should hasten to add that this is just a global scaling symmetry which need not survive quantum corrections.

It is interesting to observe that by means of a conformal transformation, we may construct a new metric $g^{(s)}_{\alpha\beta}$ which is invariant under the scaling symmetry

$$g^{(s)}_{\alpha\beta} = e^{c\phi} g_{\alpha\beta}. \tag{6.23}$$

In terms of this new metric the action takes the form

$$\begin{aligned} S_b &= \frac{1}{16\pi G_5} \int d^5x \sqrt{g^{(s)}} e^{-3c\phi/2} \left(\mathcal{R}^{(s)} - [(1/2) - 3c^2](\partial^{(s)}\phi)^2 - \Lambda \right) \\ &- \sigma_+ \int d^4x \sqrt{g^{(s)}_+}\, e^{-3c\,\phi/2} - \sigma_- \int d^4x \sqrt{g^{(s)}_-}\, e^{-3c\,\phi/2}. \end{aligned} \tag{6.24}$$

Now, the symmetry is a mere shift in ϕ. Moreover, with our background solutions for $g_{\alpha\beta}$ and ϕ, the metric $g^{(s)}_{\alpha\beta}$ is just AdS, as can be easily shown from (6.23) and (6.4).

Relation to higher dimensional models

For certain discrete values of c, the action (6.24) can be obtained from dimensional reduction of $(5+n)$ dimensional pure gravity with a cosmological term Λ, where the additional n dimensions are toroidal [16]. In this case, the factor $e^{-3c\phi/2}$ is the overall scale of the internal n-dimensional volume, and the value of c is given by $c^2 = 2/3q$, with

$$q = \frac{3+n}{n}.$$

It is not surprising, then, that the metric $g^{(s)}_{\alpha\beta}$ in (6.24) corresponds to the "external" components of a $5+n$ dimensional anti de Sitter space, since the starting point is in fact pure gravity in $(5+n)$ dimensions with a negative cosmological term. The calculation of quantum corrections in this higher dimensional space, and its relation with the calculation of quantum corrections in the effective 5D theory which we consider in this Chapter, is reported in Sec. 7.5

The hierarchy

As mentioned in the introduction, one of the motivations for studying brane-world scenarios has been the search for a geometric origin of the hierarchy between the effective Planck scale m_p and the electroweak scale. In the 5D description, all matter fields are assumed to have masses which are close to the cut-off scale of the theory $M \equiv G_5^{-1/3}$. And yet, with the help of an exponential warp factor (as in the RS model) it is easy to generate a hierarchy of the order of $m_p/m \sim 10^{16}$. Here m is the effective mass of fields which live on the negative tension brane, as "seen" in the effective four dimensional description [2]. In this subsection we shall review this mechanism, including the case of a warp factor with arbitrary power q, since there are some minor differences with the RS case.

The effective four-dimensional Planck mass is given by

$$m_p^2 = \frac{2}{1+2q} M^3 y_+ \left[1 - \left(\frac{y_-}{y_+}\right)^{2q+1}\right], \tag{6.25}$$

where M is the 5-dimensional Planck mass, as can be seen from Eqs. (6.15) and (6.16)], where, without loss of generality, we have taken $y_+ = y_0$. Here, and for the rest of this Section, we shall follow standard practice and refer all physical quantities to the measurements of clocks and rods located on the positive tension brane.

Let us now consider the mass scales of fields which live on the branes. We expect these fields to couple not only to the metric, but also to the background dilaton ϕ. There are many possible forms for this coupling, but it seems reasonable to restrict attention to those which respect the scaling symmetry (6.21-6.22). For a free scalar field Ψ which lives on the negative tension brane, and whose mass parameter is comparable to the cut-off scale, the action takes the form

$$S_\Psi = -\frac{1}{2} \int \sqrt{g_-^{(s)}} F^2(\phi) [g_-^{(s)\mu\nu} \partial_\mu \Psi \partial_\nu \Psi + \alpha M^2 \Psi^2]. \tag{6.26}$$

Here we have introduced a fudge factor α to allow for an intrinsic mass which is slightly lower than the cut-off scale. The function $F(\phi)$ can be reabsorbed in a redefinition of Ψ, and thus the relevant warp which determines the hierarchy between mass scales on the positive and in the negative tension branes is the one corresponding to the metric $g^{(s)}$. Our field will be perceived from the point of view of the 4D effective theory as having a mass squared of order

$$m^2 \sim \alpha M^2 \left(\frac{y_-}{y_+}\right)^{2q-2}. \tag{6.27}$$

Notice that there are two different factors which determine the hierarchy between m and m_p. The first one is the warp factor $(y_-/y_+)^{q-1}$ appearing in Eq. (6.27), which "redshifts" the mass scales of particles on the negative tension brane (except for $q < 1$, in which case the particles on the negative tension brane appear to be heavier than those on the positive tension brane). This generates the hierarchy in the RS model. The second one is the possibly large volume of the internal space, which enhances the effective Planck scale with respect to the cut-off scale [see Eq. (6.25)]. This generates the hierarchy in the ADD model [1] with large extra dimensions. Considering both effects, the hierarchy h is given by

$$h^2 = \frac{m^2}{m_p^2} \sim \alpha \, \frac{1+2q}{2} \frac{1}{M y_+} \left(\frac{y_-}{y_+}\right)^{2q-2}.$$

It is known that without a warp factor, it is not possible to generate the desired hierarchy from a single extra dimension, since its size would have to be astronomical. An interesting question is what is the minimum value of the exponent q which would be sufficient in order to generate a ratio $m/m_p \sim 10^{-16}$. The best we can do is to take the curvature scale (y_+/q) slightly below the millimeter scale,

$$(y_+/q) \lesssim m_p (\text{TeV})^{-2} \sim \text{mm}. \tag{6.28}$$

in order to pass the short distance tests on deviations from Newton's law. On the other hand, we also need

$$(y_-/q) \gtrsim M^{-1}, \tag{6.29}$$

since for smaller values of y_- the curvature becomes comparable to the cutoff scale M and the theory cannot be trusted [1]. Substituting in (6.25) we have $M^3 \gtrsim m_p(\mathrm{TeV})^2$ and $(y_-/y_+) \gtrsim (m_p/\mathrm{TeV})^{-4/3}$, which leads to

$$10^{-32} \sim \frac{m^2}{m_p^2} \gtrsim \alpha \left(\frac{\mathrm{TeV}}{m_p}\right)^{\frac{4(2q-1)}{3}} . \tag{6.30}$$

Hence, a warp factor with exponent $q \geq 5/4$ may account for the observed hierarchy with a single extra dimension, but it appears that this cannot be done for lower values of q. [2] In particular, the Heterotic M-theory model, with $q = 1/6$, does not seem to allow for such possibility.

Coupling to matter

As mentioned above, matter fields can be coupled to gravity in many different ways because of the presence of the nontrivial scalar in the bulk. A quadratic action for matter fields $\Psi^\pm$ localized on the branes[3] can always be written in the form

$$S^\pm_{\mathrm{matt}} = \int d^4x \sqrt{\overline{g}_\pm} F^2(\phi_\pm) \mathcal{L}^\pm(\Psi^\pm, \overline{g}^\pm_{\mu\nu}), \tag{6.31}$$

with metrics $\overline{g}^\pm_{\mu\nu} = \omega^2_\pm g^\pm_{\mu\nu}$, and some $\phi-$dependent conformal factors $\omega_\pm$. We will concentrate on the case when $\omega_\pm = \omega(\phi_\pm)$. As mentioned before, specially interesting is the case when matter is coupled to gravity preserving the scaling symmetry (6.21,6.22). This corresponds to $\overline{g} = g^{(s)}$, or $\omega(\phi) = e^{c\phi/2}$.

Equation (6.19) displays the *decoupled* degrees of freedom in the 4D Einstein frame, $\hat{g}$, φ and ψ. The moduli variables with *canonical* kinetic terms are

$$\widehat{\psi} = 2\sqrt{\frac{3}{1+2q}}\, m_P \, \ln\psi, \qquad \text{and} \tag{6.32}$$

$$\widehat{\varphi} = 2\sqrt{\frac{6q}{1+2q}}\, m_P \, \varphi, \tag{6.33}$$

Thus, the couplings of small fluctuation of *e.g.* the radion modulus $\delta\varphi$ to matter are given by

$$S^\pm_{\mathrm{matt}-\varphi} = \int d^4x \, \delta\varphi \, \frac{\delta S^\pm_{\mathrm{matt}}}{\delta\varphi}\bigg|_{\psi,\hat{g}=\mathrm{const}} . \tag{6.34}$$

[1] One should also bear in mind that the scale M might itself be a "derived" quantity, as it happens for instance with the Planck mass in theories with additional large extra dimensions. In this case, the true cut-off scale may well be below M. Hence, the lower bound (6.29) on y_- should just be considered a necessary condition for the low energy description not to break down.

[2] Except, of course, by giving up the assumption that the Lagrangian of matter on the branes should scale in the same way as the rest of the classical action [see the discussion around Eq. (6.26)]. If we allow any coupling of ϕ to the mass term for Φ, then any hierarchy can be easily generated for any value of q.

[3] Most of the models considered in this thesis assume that ordinary matter is localized on the negative tension brane . For completeness, in this subsection we investigate the coupling to possible forms of matter on the positive tension brane as well.

Bearing in mind that the dependence on the moduli in (6.31) is through $\psi_\pm$ (see Eq. (6.17)), $\phi_\pm = -(4/c(2q+1))\ln\psi_\pm$, $a_\pm = \psi_\pm^{2q/(2q+1)}$ and

$$\overline{g}^{\pm}_{\mu\nu} = \omega^2(\psi_\pm)\frac{a_\pm^2}{\psi^2}\hat{g}_{\mu\nu}\ ,$$

the total functional derivative is readily evaluated and one finds

$$S^{\pm}_{\mathrm{matt}-\varphi} = -\int d^4x\,\sqrt{\hat{g}}\,\frac{(\tanh\varphi)^{\pm1}}{q+1/2}\left\{\frac{4}{c}\frac{F'}{F}\Big|_{\pm}\hat{\mathcal{L}}^{\pm} + \left(q - \frac{2}{c}\frac{\omega'}{\omega}\Big|_{\pm}\right)\,\hat{T}^{\pm}\right\}\,\delta\varphi\ . \tag{6.35}$$

Here, we have defined

$$\sqrt{\hat{g}}\,\hat{T}^{\pm} \equiv \sqrt{\hat{g}}\;\hat{g}^{\mu\nu}\;\hat{T}^{\pm}_{\mu\nu} = \sqrt{g_\pm}\;g_\pm^{\mu\nu}\;T^{\pm}_{\mu\nu},$$

and the energy momentum tensor is, as usual

$$T^{\pm}_{\mu\nu} = -\frac{2}{\sqrt{g_\pm}}\,\frac{\delta S^{\pm}_{\mathrm{matt}}}{\delta g^{\mu\nu}_{\pm}}$$

Also, we have defined $\sqrt{\hat{g}}\hat{\mathcal{L}}^{\pm} \equiv \sqrt{g_\pm}F^2(\phi_\pm)\mathcal{L}^{\pm}$. We note that the coupling to the matter Lagrangians $\mathcal{L}^{\pm}$ in Eq. (6.35) is due to the presence of the F term in (6.31), which prevents matter to couple universally to the metric. For simplicity, we shall consider here the case where F is constant. In the model considered in Chapter 7, the moduli couple in a similar way. There F is given and is not constant.

Specializing to $\omega = e^{c\phi/2}$, the coupling can be written in terms of the canonical radion $\widehat{\varphi}$ as

$$S^{\pm}_{\mathrm{matt}-\varphi} = -\int d^4x\,\sqrt{\hat{g}}\,\frac{1}{\Lambda^{\pm}_{\varphi}}\,\hat{T}^{\pm}\,\delta\widehat{\varphi}\ , \tag{6.36}$$

with a coupling strength

$$\Lambda^{\pm}_{\varphi} = \frac{\sqrt{6q(2q+1)}}{q-1}\,(\tanh\varphi)^{\mp1}\,m_P\ . \tag{6.37}$$

This is a remarkable result. Assuming that $y_- \sim 1/M$, we can write the hierarchy h in terms of the radion,

$$h \sim (\tanh\varphi)^{(q+1/2)/(q-1/2)}\ . \tag{6.38}$$

Thus, for not very steep warp factors, the radion is very strongly coupled to matter on the negative tension brane (needless to say, its coupling to matter localized at $y = y_+$ is tremendously suppressed). Numerically, we have $\Lambda^{-}_{\varphi} \sim 10$ eV, 100 MeV and 10GeV for $q = 2$, $q = 5$ and $q = 10$ respectively. Thus, the stabilization mechanism must provide a radion mass much larger than in the RS model for low q. The divergence in Λ_φ for $q = 1$ and the zero at $q = 0$ are somewhat surprising, since we expect the φ modulus to decouple in the flat space limit, not for $q = 1$. This is due to having introduced the conformal factor $\omega = e^{c\phi/2}$ in the matter Lagrangians. With the choice $\omega = 1$, φ indeed decouples (recall that $\tanh\varphi \sim \tanh\left(\widehat{\varphi}/\sqrt{q}\right)$ in this limit). We have to add that in any case, such small values of q are not very interesting since a large enough hierarchy cannot be accommodated.

Let us derive the coupling to matter of the ψ modulus. As before, a perturbation of this modulus $\delta\psi$ according to

$$S^{\pm}_{\text{matt}-\psi} = -\int d^4x\, \delta\psi\, \frac{\delta S^{\pm}_{\text{matt}}}{\delta\psi}\Big|_{\varphi,\hat{g}=\text{const}} \tag{6.39}$$

$$= -\int d^4x\, \sqrt{\hat{g}}\, \frac{1}{q+1/2} \left\{ \frac{4}{c}\frac{F'}{F}\Big|_{\pm} \hat{\mathcal{L}}^{\pm} - \left(\frac{1}{2} + \frac{2}{c}\frac{\omega'}{\omega}\Big|_{\pm} \right) \hat{T}^{\pm} \right\} \delta\psi\, . \tag{6.40}$$

Taking a constant F and $\omega = e^{c\phi/2}$, we obtain an interaction Lagrangian analogous to (6.36) with coupling strength

$$\Lambda^{\pm}_{\psi} = 4\sqrt{(2q+1)/3}\, m_P\ .$$

Thus, a millimetric mass for ψ is enough to pass all observational bounds.

6.2 Effective potential induced by generic bulk fields

In this Section we resume the framework for computing the contribution to the 1-loop effective potential from a scalar field Φ propagating in the bulk, described in Section 4.4. We shall consider a generic mass term, which may include couplings to the curvature of spacetime as well as couplings to the background dilaton ϕ. As in the RS model, the contribution from bulk fields with other spins can be obtained from the result for the scalar upon identifying the suitable mass and nonminimal coupling. The effective potential for the moduli $y_{\pm}$ will be defined as usual in terms of a Gaussian path integral around the background solution. Before presenting the actual calculation, however, a digression on the choice of the measure of integration will be useful.

Specification of the functional measure

As mentioned in Section 4.4, the measure of functional integration needs to be specified somehow in QFT. When there are fields other than gravity with a nontrivial profile (such as the dilaton ϕ), then there is a wide class of possible choices, related to each other by dilaton dependent conformal transformations. All choices within this class are equally good from the point of view of diffeomorphism invariance. However, at the quantum level they are inequivalent due to the well known conformal anomaly. Nevertheless, all the possible choices differ in a finite renormalization of local operator, as we shall see below.

To be definite, let us concentrate in the simple case of a bulk scalar field Φ with canonical kinetic term. The (Euclidean) action for this field is given by

$$S[\Phi] = \frac{1}{2}\int d^D x \sqrt{g}\, \Phi P \Phi\, , \tag{6.41}$$

where we have introduced the covariant operator

$$P = -(\Box_g + E).$$

Here $\Box_g$ is the D'Alembertian operator associated with the metric $g_{\alpha\beta}$, and $E = E[g_{\alpha\beta}, \phi]$ is a generic "mass" term. Typically, this takes the form $E = -m^2 - \xi\mathcal{R}_g$, where m is a constant mass, $\mathcal{R}_g$ is the curvature scalar and ξ is an arbitrary coupling. Throughout this Section we shall leave E unspecified.

As we explain in Chapter 4, the volume measure in field space

$$\mathcal{D}\Phi = \sqrt{G}\ \prod_x d\Phi^x. \tag{6.42}$$

depends on the conformal frame,

$$G_{xy} = \delta_\mu(x,y) = \frac{\delta^{(n)}(x-y)}{\sqrt{g(x)}}. \tag{6.43}$$

It should be clear from the previous discussion that the definition of $\mathcal{D}\Phi$ is associated with a natural definition of $d\mu(x)$. However, in the problem under consideration in this Chapter, the choice of $d\mu$ is not unique. In our case, there is a nontrivial dilaton field ϕ, and we can consider a whole class of spacetime measures of the form

$$d\mu_\theta(x) = \sqrt{g_\theta}\ d^D x = \Omega_\theta^D(\phi)\sqrt{g}\ d^D x,$$

which correspond to conformally related metrics

$$g^\theta_{\alpha\beta} = \Omega^2_\theta g_{\alpha\beta},$$

as introduced in Section 4.4.

In the presence of a dilaton, the coupling to gravity is not universal and it is not clear which one of these metrics should be considered more physical.

Since we have a classical scaling symmetry in the gravity and dilaton sector, one could argue that $g^{(s)}_{\alpha\beta}$, which is invariant under scaling (see Section 6.1), is the preferred physical metric.[4] However, even in this case the divergent part of the effective potential will not respect the scaling symmetry, and consequently we need to introduce counterterms with the "wrong" scaling behaviour. Hence, in what follows, we shall take the conservative attitude that the measure is determined in the context of a more fundamental theory (from which our 5-D effective action is derived), and we shall formally consider on equal footing all choices associated with metrics in the conformal class of $g_{\alpha\beta}$, including of course $g^{(s)}_{\alpha\beta}$. As we shall see, the difference between these choices amounts to the addition of local terms in the effective potential.

The way the θ dependence of V_θ arises is very different in different regularization schemes. In Eq. (4.34), the determinant of P is independent of θ (we recall that this operator corresponds to the choice $\Omega_\theta = 1$). In dimensional regularization, $\ln J_\theta$ vanishes, but the divergent term V^{div}_θ which is subtracted from $\ln(\det P)$ depends on the choice of physical metric $g^\theta_{\alpha\beta}$. On the other hand, in zeta function regularization, $\ln(\det P)$ is finite and V^{div} does not play a role (in any case, any finite renormalization does not introduce a dependence in θ). Rather, in this case, the dependence on θ comes from $\ln J_\theta$, which does not vanish in this regularization scheme. In both cases, the θ dependence of V_θ is the same.

As we shall see, this dependence can be cast in the form of local operators on the branes, and therefore the ambiguity in the choice of the integration measure can also be understood as

[4]Note, in particular, that the overall scaling factor of the action (6.1) under (6.21) and (6.22) depends on the spacetime dimension, and hence the symmetry itself is different when we change the dimension. By contrast, the scaling of (6.24) remains the same in any dimension.

modification of the classical action. It should be noted, however, that the local operators which result from a shift in θ have different form than the terms arising from the usual shift in the renormalization constant μ which inevitably crops up in the regularized traces. In the cases we shall consider, the latter will take the form $K^4(y_\pm)$, where K denotes terms which behave like the extrinsic curvature of the branes at the positions $y_\pm$. On the other hand, the θ-dependent terms behave as $K^4(y_\pm)\phi(y_\pm)$. Since $K(y)$ behaves like the inverse of y whereas $\phi(y)$ behaves logarithmically with y, these terms will give rise to Coleman-Weinberg type potentials for the moduli.

For definiteness we shall restrict attention to conformal factors $\Omega_\theta(\phi)$ which have an exponential dependence on the dilaton:

$$\Omega_\theta(z) = e^{(1-\theta)\phi/3c} = \left(\frac{z}{z_0}\right)^{\beta\,(\theta-1)}. \tag{6.44}$$

With this choice, $\theta = 0$ represents flat space and $\theta = 1$ corresponds to the Einstein frame metric (6.47). For $\theta = -1/\beta$, the metric $g^{\theta}_{\alpha\beta}$ coincides with the metric $g^{(s)}_{\alpha\beta}$ introduced in Section 6.1, which is invariant under the scaling transformation (as mentioned before, this metric corresponds to a five dimensional AdS space, with curvature radius given by z_0).

This allows to identify the θ dependence of the potential. In the sense of (4.35), the difference between the effective potential computed with path integral measures covariant with respect to the frames labeled by θ and $\theta = 1$, is

$$\ln J_\theta = \frac{1}{3c\mathcal{A}} \int_\theta^1 d\theta'\, a_{5/2}(\phi, P_{\theta'}), \tag{6.45}$$

where we have used $f_\theta = \partial_\theta \ln \Omega_\theta = (1-\theta)\phi/3c$. Clearly, the effect of this factor is just adding to the classical action local terms expressed solely in terms of ϕ and the metric, such as $\sqrt{g_\pm}\,\phi_\pm\,\mathcal{K}^4_\pm$. The dependence of these terms is different from the change which results from a rescaling of the renormalization parameter μ. This corresponds to a shift in the coefficient of local terms proportional to $a_{5/2}(P)$.

Explicit evaluation

For simplicity we shall restrict attention to the case of massless fields with arbitrary coupling to the curvature:

$$E = -\xi \mathcal{R}_g,$$

and with Dirichlet boundary conditions. Here we shall use the method of dimensional regularization. Zeta function regularization is discussed in Appendix B.

It is convenient to introduce the conformal coordinate

$$z \equiv \left| \int \frac{dy}{a(y)} \right| = \frac{y_0}{|1-q|} \left(\frac{y}{y_0}\right)^{1-q}, \tag{6.46}$$

so that we may rewrite the metric as

$$ds^2 = a^2(z)\big(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\big), \qquad a(z) = (z/z_0)^\beta, \tag{6.47}$$

where

$$\beta = \frac{q}{1-q}, \qquad z_0 = \frac{y_0}{|1-q|}. \tag{6.48}$$

Here we should mention that the direction of increasing z does not coincide with the direction of increasing y when $q > 1$.

The Ricci scalar computed from the metric (6.47) in D dimensions is

$$\mathcal{R} = (D-1)\,\beta\,(1-(D-3)\beta)\,\frac{1}{z^2a^2(z)} = (D-1)\,q\,(1-(D-2)\,q)\,\frac{1}{y^2}. \tag{6.49}$$

Note that this expression reduces to the RS expression Eq. (3.16), with $1/z_0$ playing the role of k. Also, for any other value of β, the Ricci scalar does not depend on any scale, showing the no-scale nature of the models considered. Moreover, the appearence of a singularity at $y = 0$ is manifest.

The eigenmodes of the conformally related operator in flat space P_0 (4.26) are given by

$$\hat{\Phi}_n \equiv z^{1/2}\big(A_1 J_\nu(m_n z) + A_2 Y_\nu(m_n z)\big).$$

The index of the Bessel functions is given by

$$\nu(D) = \frac{1}{2}\sqrt{1-4(D-1)\beta[(D-2)\beta-2](\xi-\xi_c(D))}, \tag{6.50}$$

where

$$\xi_c(D) = \frac{1}{4}\frac{D-2}{D-1},$$

is the conformal coupling in dimension D. Imposing the boundary conditions (4.41) on both branes, we obtain the equation that defines implicitly the discrete spectrum of m_n,

$$F(\tilde{m}_n) = J_\nu(\tilde{m}_n\eta)Y_\nu(\tilde{m}_n) - Y_\nu(\tilde{m}_n\eta)J_\nu(\tilde{m}_n) = 0, \tag{6.51}$$

where we have defined

$$\tilde{m}_n = m_n z_-, \qquad \eta = \frac{z_+}{z_-}. \tag{6.52}$$

The zeros of F are all real if ν is real and η is positive. Since η is positive, the reality condition of ν guarantees that all the KK masses are real. This provides a constraint for the possible values of ξ depending on q,

$$\begin{aligned} \xi &\geq \frac{-(1-4q)^2}{16q(2-5q)}, \qquad q \leq 2/5, \\ \xi &\leq \frac{-(1-4q)^2}{16q(2-5q)}, \qquad q \geq 2/5, \end{aligned} \tag{6.53}$$

where we have used $D = 5$. Note that the values of ξ comprised between the minimal and the conformal coupling are allowed for any value of q.

In Chapter 4, we concluded that the renormalized expression for the effective potential is

$$V_\theta(\varphi) = \lim_{D\to 5}\left[V^D - \frac{1}{(D-5)}\frac{1}{\mathcal{A}}\,a^D_{5/2}(P_\theta)\right]. \tag{6.54}$$

Consider first V^D, given in Eq. (4.36). Performing the momentum integrations, we obtain

$$V^D = -\frac{1}{2(4\pi)^2}(4\pi\mu^2)^{\epsilon/2}\frac{1}{z_-^{4-\epsilon}}\Gamma(-2+\epsilon/2)\tilde{\zeta}(\epsilon-4), \tag{6.55}$$

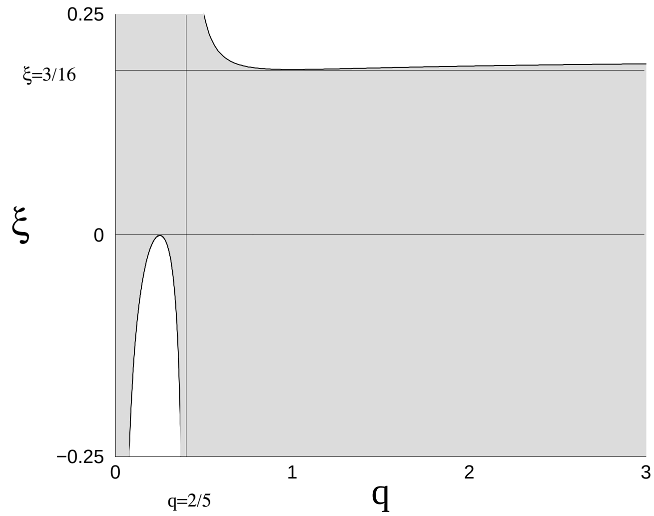


Figure 6.2: The shaded region corresponds to the allowed values in the (q –ξ) plane for a Dirichlet massless scalar field according to Eq. (6.53). Note that the range of ξ comprised between the minimal and the conformal coupling is allowed for any value of q

where we have defined [7]

$$\tilde{\zeta}(s) = \sum_n \tilde{m}_n^{-s} = \frac{s}{2\pi i} \int_{\mathcal{C}} t^{-1-s} \ln F(t)\; dt. \tag{6.56}$$

This regularized expression for the effective potential is finite when the real part of ϵ is sufficiently large. In the last equation we have used that $F(t)$ has only simple zeros which are along the real axis. The closed contour of integration $\mathcal{C}$ runs along the imaginary axis, from $t = +i\infty$ to $t = -i\infty$, skipping the origin through an infinitesimal path which crosses the positive real axis, and the contour is closed at infinity also through positive real infinity.

Now the problem reduces to the computation of $\tilde{\zeta}$, which can be done in the same way as in the case discussed in [7]. Skipping the detailed derivation, we simply give the final result:

$$\tilde{\zeta}(-4+\epsilon) = -2d_4\,(1+\eta^{-4+\epsilon}) - 2\epsilon(\eta\, a)^{-2}\Big(\mathcal{I}_K + \mathcal{I}_I a^4 + a^4\mathcal{V}(a)\Big) + O(\epsilon^2), \tag{6.57}$$

where

$$d_4 = \frac{1}{128}(13 - 56\nu^2 + 16\nu^4). \tag{6.58}$$

Here we have introduced

$$a \equiv \frac{z_<}{z_>} = \begin{cases} 1/\eta, & \text{for} \quad \text{q} < 1, \\ \eta, & \text{for} \quad \text{q} > 1, \end{cases}$$

to express the result for both $q > 1$ and $q < 1$ cases simultaneously, where $z_>$ and $z_<$ are the largest and the smallest of z_+ and z_-, respectively. Note that $a < 1$.

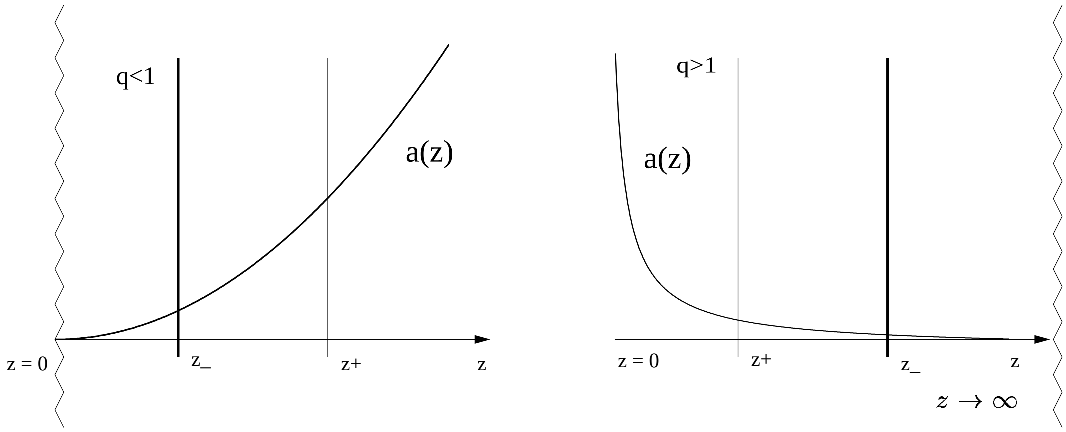


Figure 6.3: For $q < 1$, in conformal coordinate, the singularity sits at $z = 0$, and the coordinate of the negative tension brane is smaller than the coordinate for the positive tension one, $\eta = z_+/z_- > 1$. For $q > 1$, the singularity is placed at $z \to \infty$ and $z_- > z_+$, so $\eta < 1$.

The constant coefficients $\mathcal{I}_K(\nu)$, $\mathcal{I}_I(\nu)$ are calculable in principle, however their actual value is irrelevant for the present computation (see [15] for an explicit evaluation). Finally, $\mathcal{V}(a)$ is defined by (*cfr.* Eq. (5.3))

$$\mathcal{V}(a) = \int_0^\infty d\rho\, \rho^3 \ln\left(1 - \frac{I_\nu(a\rho)}{K_\nu(a\rho)}\frac{K_\nu(\rho)}{I_\nu(\rho)}\right). \tag{6.59}$$

This provides the following expression for V^D

$$V^D = \frac{1}{(4\pi)^2}\left[\left\{\left(\frac{1}{\epsilon} + \frac{3}{4} - \frac{\gamma}{2} + \frac{1}{2}\ln(4\pi\mu^2 z_0^2)\right)d_4 + d_4'\right\}\left(\frac{1}{z_-^4} + \frac{1}{z_+^4}\right)\right. \tag{6.60}$$

$$\left. + \; d_4\left(\frac{1}{z_-^4}\ln\left(\frac{z_-}{z_0}\right) + \frac{1}{z_+^4}\ln\left(\frac{z_+}{z_0}\right)\right) + \frac{\mathcal{I}_K}{z_<^4} + \frac{\mathcal{I}_I}{z_>^4} + \frac{\mathcal{V}(a)}{z_>^4}\right] + \mathcal{O}(\epsilon), \tag{6.61}$$

where we have introduced the conventions $d_4 \equiv d_4(D = 5)$ and $d_4' \equiv \partial d_4(D)/\partial D|_{D=5}$, since d_4 depends on the dimension through (6.50).

The next step is to subtract the divergent contribution. The Seeley-De Witt coefficient $a^D_{5/2}(P_\theta)$ can be computed from (A.7) for a generic dimension D, so we can expand the second term in the r.h.s of Eq. (4.51) as

$$\frac{1}{(D-5)}\frac{1}{\mathcal{A}}a^D_{5/2}(P_\theta) = \frac{1}{(4\pi)^2}\left[\frac{1}{\epsilon}d_4\left(\frac{1}{z_-^4} + \frac{1}{z_+^4}\right) - \beta\theta\, d_4\left(\frac{1}{z_-^4}\ln\left(\frac{z_-}{z_0}\right) + \frac{1}{z_+^4}\ln\left(\frac{z_+}{z_0}\right)\right) - \delta(\theta)\left(\frac{1}{z_-^4} + \frac{1}{z_+^4}\right)\right], \tag{6.62}$$

where $\delta(\theta)$ is a constant whose precise value will be unimportant.[5]

[5] For $\theta = 1$, we find $\delta = (-\beta/3072)\{432 - 1936\beta - 900\beta^2 + 2257\beta^3 - 1152(2+7(-1+\beta)\beta(1+3\beta))\xi + 3072\beta(-2+3\beta)(-2+7\beta)\xi^2\}$.

The divergent parts of the two terms in Eq. (4.51) cancel, and the finite result for the effective potential per unit comoving volume is given by

$$V_\theta = \frac{1}{(4\pi)^2}\left[\frac{\mathcal{I}_K}{z_<^4} + \frac{\mathcal{I}_I}{z_>^4} + \frac{\mathcal{V}(a)}{z_>^4}\right] + \frac{d_4(\beta\theta+1)}{(4\pi)^2}\left[\frac{1}{z_+^4}\ln\left(\frac{z_+}{z_0}\right) + \frac{1}{z_-^4}\ln\left(\frac{z_-}{z_0}\right)\right] \qquad (6.63)$$

$$+\frac{d_4}{(4\pi)^2}\left[\frac{1}{z_+^4}+\frac{1}{z_-^4}\right]\ln(\mu z_0).$$

Here, we have eliminated some terms through redefinition of μ.

As in the case of conformal fields, we must also allow for finite renormalization of the couplings in front of the invariants which make up the coefficient $a_{5/2}$. These have the general dependence of the form $a_\pm^4 \mathcal{K}_\pm^4 \sim z_\pm^{-4}$. With these additions, we finally obtain:

$$V_\theta(z_+, z_-) = \frac{d_4(\beta\theta+1)}{(4\pi)^2}\left[\frac{\ln(\mu_1 z_+)}{z_+^4} + \frac{\ln(\mu_2 z_-)}{z_-^4}\right] + \frac{1}{(4\pi)^2}\int_0^\infty dx\, x^3 \ln\left[1 - \frac{I_\nu(xz_<)}{I_\nu(xz_>)}\frac{K_\nu(xz_>)}{K_\nu(xz_<)}\right], \qquad (6.64)$$

where μ_1 and μ_2 are renormalization constants, and β and d_4 are given in equations (6.48) and (6.58). This is the main result of this Section. In the limit of small separation between the branes, $(1-a) \ll 1$, the integral $\mathcal{V}$ behaves like $(1-a)^{-4}$ (see Appendix C), and the logarithmic terms can be neglected. In this limit, the potential behaves like the one for the conformally coupled case, given in (5.7):

$$V_\theta(z_+, z_-) \sim -\frac{A}{|z_+ - z_-|^4}. \qquad (6.65)$$

For $a \ll 1$, when the branes are well separated, the integral $\mathcal{V}$ behaves like $a^{2\nu}$ and becomes negligible in the limit of small a (except in the special case when ν is very close to 0). In this case, we have

$$V_\theta(z_+, z_-) \sim \frac{d_4(\beta\theta+1)}{(4\pi)^2}\left[\frac{\ln(\mu_1 z_+)}{z_+^4} + \frac{\ln(\mu_2 z_-)}{z_-^4}\right] + O\left[\left(\frac{z_<}{z_>}\right)^{2\nu}\right]. \qquad (6.66)$$

Due to the presence of the logarithmic terms, it is in principle possible to adjust the parameters μ_1, μ_2 so that there are convenient extrema for the moduli z_+ and z_-. We shall comment on this point in the concluding Section.

Application to the Hořava Witten model

Here we shall consider an example of a contribution to the moduli effective potential in the model of Lukas *et al.* [10, 9], which may be relevant for the Ekpyrotic universe [204]. As mentioned above, this corresponds to the case $q = 1/6$. In principle, the fields in the action for this model do not have the canonical form, since in addition to the coupling to the metric they have unusual couplings to the dilaton ϕ. Nevertheless, they can be studied along the lines of the previous Sections. For instance, the Heterotic M-theory model of [10, 9] contains a scalar $\Psi(x^\alpha)$ in the universal hypermultiplet whose vev is zero in the background solution, and whose action is given by

$$S^{(\Psi)} = -\int \sqrt{g}\, d^5x\, \frac{1}{2} e^{-\phi}(\partial\Psi)^2. \qquad (6.67)$$

This contains a kinetic term only, but it has a non minimal form. However, changing to a new variable,

$$\zeta = e^{-\phi/2}\Psi,$$

we rewrite the action (6.67) as

$$\begin{aligned} S^{(\Psi)} &= -\int \sqrt{-g}\, d^5x\, \frac{1}{2}\Big(\partial\zeta + \frac{1}{2}\zeta\,\partial\phi\Big)^2 \\ &= -\int \sqrt{-g}\, d^5x\, \frac{1}{2}\Big(\partial\zeta^2 - \frac{1}{2}\left(\Box\phi - \frac{1}{2}\left(\partial\phi\right)^2\right)\zeta^2\Big) + \text{boundaryterms} \\ &= -\int a^4\, d^4x\, dy\, \frac{1}{2}\zeta\,\Big(-\Box - \frac{5}{12}\,\frac{1}{y^2}\Big)\,\zeta, \end{aligned} \tag{6.68}$$

where the indices are contracted with the five dimensional $g_{\alpha\beta}$ metric, and we have integrated by parts in the first equality. For the Dirichlet case, the boundary terms 'generated' are not relevant since still the field ζ vanishes there. The last equation shows that the potential term present in terms of ζ mimics a non minimal coupling to the curvature $-\xi\mathcal{R}\,\zeta^2$. Since the Ricci scalar for this background is

$$\mathcal{R} = \frac{7}{9}\,\frac{1}{y^2},$$

we conclude that the equivalent effective non minimal coupling of ζ is $\xi = 15/28$. Note that this point lies in the allowed region of values in the $(q\ -\xi)$ plane defined by Eq. (6.53), which corresponds to an index of the Bessel functions $\nu = 4/5$.

Hence, the contribution to the moduli effective potential induced by the field Ψ is given by (6.64) with $\nu = 4/5$. In this case we have $\beta = 1/5$ and $d_4 = -10179/80000$. There are, of course, many other contributions, corresponding to all bosonic and fermionic degrees of freedom. It turns out, however, that all bosonic contributions take a form similar to the one of the field Ψ.If supersymmetry is unbroken, then these contributions are canceled by the contributions from the fermionic degrees of freedom. But if the degeneracy between bosons and fermions is broken a la Scherk-Schwartz, for instance, then we expect that the resulting effective potential will be qualitatively similar to the one given by (6.64). A detailed study of this model is left for future research.

6.3 Moduli stabilization

In the limit of large interbrane separation, the potential (6.64) assumes a "Coleman-Weinberg" form for each one of the moduli,

$$V(y_+, y_-) \approx \sum_{i=\pm} a^4(y_i)\left\{\alpha K^4(y_i)\ln\left[\frac{K(y_i)}{\mu_i}\right] + \delta\sigma_i\right\}. \tag{6.69}$$

Here, we have expressed the potential in terms of the "curvature scale"

$$K(y) = \frac{q}{y},$$

so that $K^4(y_i)$ behaves like a generic geometric operator of dimension 4 on the brane [such as the fourth power of the extrinsic curvature, or any of the operators in the integrand of Eq. (A.7)].

Working with K_i instead of z_i has the advantage of relating directly to physical quantities, and hence it is easier to control whether we are in the range where the effective theory should be trusted or not. In particular, we should not allow K_i to be bigger than the cut-off scale of the theory. The constant α in (6.69) is given by

$$\alpha = \frac{(1-\theta)q - 1}{(4\pi)^2}\,(1-q^{-1})^4 \sum \beta_4^{(\Phi)}, \tag{6.70}$$

where we sum over the contributions from all bulk fields Φ. The numerical coefficients $\beta_4^{(\Phi)}$ are given by Eq. (6.58). The value of θ depends on the choice of integration measure in the path integral which defines the effective potential (see Sections 6.2 and 4.4). If we adopt the measure associated to the Einstein frame metric $g_{\alpha\beta}$ which enters our original action functional (6.1), then we should take $\theta = 1$. However, this is not the only possible choice, as we have repeatedly emphasized. The classical action has a scaling symmetry which transforms both $g_{\alpha\beta}$ and the background scalar field ϕ. Using a conformal transformation which involves the scalar field, we may construct a new metric $g^{(s)}_{\alpha\beta}$ which does not transform under scaling. If we adopt the measure which corresponds to this new metric, then we should take $\theta = 1 - 1/q$. With this particular choice of θ the coefficient α vanishes and the logarithmic terms in (6.69) disappear. Nevertheless, it is far from clear that this is indeed a preferred choice[6] Here we take the attitude that the parameter θ is unknown, and that it should be fixed by a more fundamental theory of which (6.1) is just a low energy limit.

The renormalization constants μ_i in (6.69) can be estimated by looking at the "renormalized coefficient" of the geometric terms of dimension 4 on the brane $c_i(K) = \alpha \ln(K/\mu_i)$. In the absence of fine-tuning, the $c_i(K)$ are expected to be of order one near the cut-off scale $K \sim M$, where M^{-3} is basically the five-dimensional Newton's constant. Hence, we expect

$$\mu_i \sim M e^{-c_i/\alpha}, \tag{6.71}$$

where $c_i = c_i(M) \sim 1$. In (6.69), we have also allowed for finite renormalization of local operators on each one of the branes. These operators are collectively denoted by $\delta\sigma_i$. In order to ensure that the effective potential V does not severely distort the background solution, this correction to the brane tension must be much smaller than the effective tension of the brane in the classical background solution. From the Darmois-Israel matching conditions, this effective tension is of order $M^3 K_i$. Hence we require

$$\delta\sigma_i \ll M^3 K_i \ll M^4. \tag{6.72}$$

In Section 6.2, we considered contributions to the effective potential from massless bulk fields. These may have an arbitrary coupling to the curvature scalar of the standard form $\xi\mathcal{R}\Phi^2$, or certain couplings to the background scalar field, such as the ones occurring in the Heterotic M-theory model considered in Section 6.2. However, if the model contains massive bulk fields,

[6] In particular, as mentioned in Section 6.1, for certain values of the model parameters our action can be obtained by dimensional reduction of a 5+n dimensional Einstein-Hilbert action with a cosmological term. Since in the higher dimensional theory only gravity is present, there is only one possible choice for the metric and the issue of the measure does not arise. The analysis of these models is performed in Chapter 7 [16] and indicates that the logarithmic terms do indeed arise in the limit when the radius of the additional dimensions are much smaller than the interbrane separation. For this case the relevant value of the parameter is $\theta = 0$.

of mass m, then we expect terms proportional to m^2K^2 in the effective potential. Even without massive bulk fields, we may expect the presence of lower dimensional worldsheet operators of the form M^3K, M^2K^2 and MK^3, due to cubic, quadratic and linear divergences in the effective theory. Hence, we may expect that $\delta\sigma_i$ has an expansion of the form

$$\delta\sigma_i(K_i) \sim \Lambda_i^4 + \gamma_{1i}M^3K_i + \gamma_{2i}M^2K_i^2 + \gamma_{3i}MK_i^3 + \gamma_{4i}K_i^4 + \mathcal{O}(K_i^5), \tag{6.73}$$

where $K_i \ll M$, $\Lambda_i \ll M$ and $\gamma_{1i} \ll 1$ in order to satisfy (6.72). For completeness, the above expansion includes the term proportional to K_i^4. It should be understood that this term is only present in the particular case $\theta = 1 - 1/q$ (corresponding to the scale invariant metric), since for other values of θ we assume that it is reabsorbed in a redefinition of μ_i [see Eq. (6.71)].

The local terms may in principle stabilize the moduli at convenient locations. Note that this effect is due to the warp factor and vanishes in flat space (where the coefficients d_4 vanish). The effect also vanishes accidentally in the RS case, because the curvature scale $K(y)$ is constant. The position of the minima are determined by $\partial_{y_i}V = 0$. This leads to the conditions

$$\delta\sigma_i = \frac{\alpha}{q}K_i^4\left[(1-q)\ln\left(\frac{K_i}{\mu_i}\right) + \frac{1}{4}\right] + K_i\frac{\delta\sigma_i'}{4q}, \tag{6.74}$$

where the prime on $\delta\sigma_i$ indicates derivative with respect to K_i. Also, we must require that the minima occur at an acceptable value of the effective cosmological constant. Using the condition (6.74), we can write the value of the potential at the minimum as

$$V_{min} = \frac{K_+^4}{4q}\sum_{i=\pm}\left(\frac{K_i}{K_+}\right)^{4(1-q)}\left\{4\alpha\ln\left(\frac{K_i}{\mu_i}\right) + \alpha + K_i^{-3}\delta\sigma_i'\right\} \lesssim 10^{-122}m_p^4. \tag{6.75}$$

The latter condition will require one fine tuning amongst the parameters in (6.73).

An interesting question is whether the effective potential (6.69) can generate a large hierarchy and at the same time give sizable masses to the moduli. As discussed in Section 6.1, the hierarchy is given by

$$h^2 = \frac{m^2}{m_p^2} \sim \frac{K_+}{M}\left(\frac{K_+}{K_-}\right)^{2q-2}, \tag{6.76}$$

where $m \sim$ TeV is the mass of the particles which live on the negative tension brane, as perceived by the observers on the positive tension brane. Consistency with Newton's law at short distances requires $K_+ \gtrsim (\text{TeV})^2/m_p \sim (\text{ mm})^{-1}$, and consistency of perturbative analysis requires $K_- \lesssim M$. With these constraints, the observed hierarchy $h \sim \exp(-37)$ can only be accommodated for $q \gtrsim 5/4$. To proceed, we shall distinguish two different cases.

Case a:

This is the generic case, where the coefficients γ_{1i}, γ_{2i} and γ_{3i} in the expansion of $\delta\sigma_i(K)$ are not too suppressed. In this case, the logarithmic terms in the effective potential are in fact subdominant, and the minima of the effective potential are determined by $4q\delta\sigma_i \approx K_i\delta\sigma_i'$.

The present discussion applies also to the special case where $\theta = 1 - 1/q$ (corresponding to the measure associated with the scale invariant metric $g_{\alpha\beta}^{(s)}$), so that no logarithmic terms are present in the effective potential. Note that terms of the form $\gamma_{4i}K_i^4$ and $\gamma_{1i}M^3K_i$ in the expansion of $\delta\sigma_i(K)$ [see (6.73)] cannot be avoided. The first is necessary in order to renormalize the effective potential, and the second is already present at the tree level, so it just corresponds to a shift in the existing parameters in the classical action.

Quite generically, this will lead to stabilization of the moduli near (or slightly below) the cut-off scale $K_i = \lambda_i M$, with $\lambda_i \sim 1$. Hence we have

$$h^2 \sim \exp[2(q-1)\ln(\lambda_+/\lambda_-)].$$

Since the logarithm is of order one, an acceptable hierarchy can be generated provided that $q \gtrsim 10$. This is "close" to the RS limit $q \to \infty$. In this case, $m_p \sim M$. On the positive tension brane the parameter Λ_+ has to be fine tuned so that the effective cosmological constant is 122 orders of magnitude smaller than the Planck scale. A straightforward calculation shows that the physical mass eigenvalues for the moduli $\psi_\pm$ in the present case are given by

$$m_+^2 \sim q^{-2} m_p^{-2} K_+^4 \lesssim m_p^2, \qquad m_-^2 \sim q^{-1} h^2 m_p^{-2} K_-^4 \lesssim h^2 m_p^2.$$

Thus, the lightest radion has a mass comparable to the TeV scale.

Case b:

This corresponds to the case where almost all of the operators in (6.73) are either extremely suppressed or completely absent, due perhaps to some symmetry. In particular, we shall concentrate on the possibility that

$$\delta\sigma_i = \gamma_{1i} M^3 K_i,$$

since an operator of this form is already present in the classical action (6.1), and it is the only one in the expansion (6.73) which is allowed by the scaling symmetry. In this case, and assuming for simplicity that the negative tension brane is near the cut-off scale $K_- \sim M$, we can rewrite (6.75) as

$$V_{min} \sim \frac{3\alpha K_+^4}{(4q-1)} \left\{ \left(\ln\left(K_+/\mu_+\right) + \frac{1}{3} \right) + h^{8(q-1)/(2q-1)} \left(\ln\left(K_-/\mu_-\right) + \frac{1}{3} \right) \right\}$$

Here we are assuming that $\theta \neq 1 - 1/q$ (so that $\alpha \neq 0$), since the alternative case was already discussed in the previous subsection. For $q > 1$, the first term dominates and the condition of a nearly vanishing cosmological constant forces $K_+ \approx \mu_+ e^{-1/3}$. A fine-tuning of Λ_+ will be necessary in order to satisfy the condition (6.74) for such value of K_+. The hierarchy is given by

$$h^2 \sim \left(\frac{\mu_+}{M}\right)^{2q-1} \sim \exp\left[-(2q-1)\alpha^{-1} c_+\right],$$

where μ_+ is given by (6.71). Since the effective coupling α can be rather small, a large hierarchy may be obtained even for moderate $q \gtrsim 1$. A straightforward calculation shows that at the minima of the effective potential (6.69) $\partial^2_{\psi_+} V = 12\alpha(1+2q)^{-2} a_+^4 K_+^4 \psi_+^{-2}$, and $\partial^2_{\psi_-} V \sim \alpha q^{-1} a_-^4 K_-^4 \psi_-^{-2}$. Hence, we find that the physical masses for the moduli fields ψ_+ and ψ_- which appear in (6.15) are given by

$$m_+^2 \sim \alpha q^{-2} h^{12/(2q-1)}\, m_p^2, \qquad m_-^2 \sim \alpha q^{-1} h^{2+4/(2q-1)} m_p^2.$$

Associated with the eigenvalue m_+ there is a Brans-Dicke (BD) field with[7] BD parameter $\omega_{BD} = -3q/(1+2q)$, coupled to matter with Planckian suppression (see Eq. (6.39)). Therefore, we must require $m_+ \gtrsim (mm)^{-1}$, which in turn requires $q > 2$. A stronger constraint on q comes from the eigenvalue m_-, since the corresponding field is coupled to ordinary matter with TeV strength or even higher. The mass of this field cannot be too far below the TeV, otherwise it would have been seen in accelerators. This requires q to be rather large $q \gtrsim 10$.

6.4 Discussion

We have studied a class of warped brane-world compactifications, with a power law warp factor of the form $a(y) = (y/y_0)^q$ and a dilaton with profile $\phi \propto \ln(y/y_0)$. Here y is the proper distance in the extra dimension. In general, there are two different moduli $y_\pm$ corresponding to the location of the branes. (in the RS limit, $q \to \infty$, a combination of these moduli becomes pure gauge).

In general, the effective potential induced by massless bulk fields with arbitrary curvature coupling is given by (6.64). In the limit when the branes are very close to each other, it behaves like $V \propto a^4|y_+ - y_-|^{-4}$, corresponding to the usual Casimir interaction in flat space. Perhaps more interesting is the moduli dependence due to local operators induced on the branes, which are the dominant terms in $V(y_+, y_-)$ when the branes are widely separated. Such operators break a scaling symmetry of the classical action, which we discussed in Section 6.1, but nevertheless are needed in order to cancel the divergences in the effective potential. If we denote by $K(y_i) = q/y_i$ the extrinsic curvature of the brane at the location $y = y_i$ $(i = \pm)$, a renormalization of the brane tension parameters $\sigma_\pm$ in the classical action (6.1) induces terms proportional to $a(y_i)^4 K_i$ in the effective potential. These terms scale like the rest of the classical action under the global transformation (6.21-6.22). On the other hand, the divergences in the effective potential, proportional to the coefficient $a_{5/2}(P)$, require world-sheet counterterms which are proportional to $a(y_i)^4 K^4(y_i)$. These have the wrong scaling behaviour [they simply do not change under (6.21-6.22)] and hence they act as stabilizers for the moduli.

In addition, there are terms proportional to $a(y_i)^4 K^4(y_i)\phi(y_i)$. The coefficient in front of the latter terms depends on the choice of the measure in the path integral. Different choices are possible, which are related amongst each other by dilaton dependent conformal transformations. Because of the conformal anomaly, different choices are inequivalent, but they are simply related by the addition of world-sheet operators to the action. These are given by the r.h.s. of (6.45). Since ϕ behaves logarithmically, these terms have the form of Coleman-Weinberg type potentials for the moduli y_i, and they can also act as stabilizers for the moduli.

To conclude, we find that worldsheet operators induced on the brane at one loop easily stabilize the moduli in brane-world scenarios with warped compactifications, and give them sizable masses. If the warp factor is sufficiently steep, $q \gtrsim 10$, then this stabilization naturally generates a large hierarchy, as in the Randall-Sundrum model. In this case, the mass of the lightest modulus is somewhat below the TeV scale. This feature is in common with the Goldberger and Wise

[7] Here we are considering the situation where the mass of ψ_- is much larger than the mass of ψ_+, and where the visible matter is on the negative tension brane. In this case, since $y_- = const.$, visible matter is universally coupled to the metric $g_{\mu\nu}$, and the BD parameter corresponding to ψ_+ can be read off from (6.15).

mechanism [5] for the stabilization of the radion in the RS model. For $q \lesssim 10$, the stabilization is also possible, but if we also demand that the hierarchy $h \sim 10^{-16}$ is generated geometrically, then the resulting masses for the moduli would be too low.

Chapter 7

Moduli stabilization in higher dimensional brane models

The Randall Sundrum model has opened up a very interesting framework for model building in particle physics, with possible cosmological implications. This scenario, however, is just the simplest amongst a large class of higher dimensional warped geometries which deserve fuller exploration. In this connection, one expects that more general internal spaces to contribute non trivially to the Casimir energy and in this Chapter we shall take a step in this direction. Our aim is twofold. On one hand, the consideration of more general spacetimes may provide interesting extensions of the RS mechanism for the geometric origin of the hierarchy. On the other, quantum effects in such scenarios can be qualitatively different, providing new ways of stabilizing the radion which do not necessarily rely on the peculiar behaviour of bulk gauge fields.

Generically, we expect that the behaviour of the effective potential for the "moduli" should be qualitatively different once we go beyond the RS scenario. This is indicated (even in five dimensions) by the models with a scalar field discussed in Chapter 6 [15].

Aside from the non-local Casimir interaction between the branes which we mentioned above, local terms which are induced by quantum effects may stabilize the moduli when we consider warped brane worlds where the bulk is different from AdS. In the RS model both the branes and the bulk space-time are maximally symmetric and thus any possible counter-term amounts to a renormalization of the brane tensions. However, this is not true in general. An explicit example is given in the previous Chapter [15], where a class of 5D models with power law warp factors is considered. In this case, the global symmetry which is responsible for the masslessness of the moduli at the classical level is anomalous. Thus, the effective potential develops terms which do not scale appropriately under the global symmetry and which therefore act as stabilizers for the moduli. Some of the 5 dimensional models considered in Chapter 6 [15] can be obtained by dimensional reduction of $5 + D_2$-dimensional models, and in this Chapter we shall focus on a class of higher dimensional models which includes those.

Specifically, we shall consider spaces with line element given by

$$ds^2_{(D)} = e^{2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2\rho(y)}\, R^2\gamma_{ij}dX^i dX^j + dy^2 \ , \qquad (7.1)$$

The coordinates x^μ parametrize four dimensional Minkowski space M_4 and the coordinates

X^i cover a D_2-dimensional compact internal manifold Σ. We locate two $D-1 \equiv (4+D_2)-$dimensional branes at the fixed points of the orbifolded dimension labeled by y. Such a metric is found as a solution of a Ddimensional system of gravity plus certain 'matter' fields. Depending on the field content, different warpings can arise. For instance, Gregory showed in [205] that a six dimensional global string solution exists for negative cosmological constant, with $\sigma(y) = -k|y|$ and $\rho(y) = constant$. Gherghetta and Shaposhnikov [206] constructed the metric solution of a six dimensional local string-like defect with $\sigma(y) = \rho(y) = -k|y|$. Generalizations of this models with more extra dimensions with $\Sigma = S^{D_2}$ were considered in [207, 208] using bulk scalar fields with a hedgehog configuration.

The authors of [209] include Yang-Mills (YM) fields with appropriately chosen gauge group, instead. They find a series of solutions classified in terms of the Ricci flatness of the manifold Σ: when the internal space is Ricci flat (for example a D_2-dimensional torus or a Calabi-Yau space), one obtains warp factors which can generically be expressed as sums of exponentials if there is no YM flux; in particular, when the bulk cosmological constant is negative, a specially simple solution with both warp factors equal to the RS one exists (including the case of higher dimensional AdS space). Turning on some YM flux can relax the condition of the Ricci flatness of the internal manifold. In this case, they find a solution where along the Minkowski direction the warp is *à la* RS, whereas along the curved manifold it is constant. This gives the interesting combination of a higher dimensional theory which is a hybrid between an ordinary Kaluza-Klein theory and the RS model. In passing, we note that the phenomenology of such a scenario has been recently considered in [210, 211].

7.1 Model

We are interested in the quantum effective action arising from a quantized bulk field on the background space-time specified by the metric (7.1). We consider two branes of codimension one with the topology of $M_4 \times \Sigma$, where the manifold Σ is taken to be Einstein and compact. The branes sit at the orbifold fixed points and the Z_2 symmetry is imposed on the solutions.

As we have already mentioned in the introduction, such solutions have been obtained in [209] for Ricci flat internal manifolds. However, as also shown in [207, 208], more general solutions with $\Sigma = S^{D_2}$ can be found by introducing additional matter content coming from a scalar field with a hedgehog configuration. In the following, we show how to obtain such a solution.

The specific space-time that we consider in this Chapter corresponds to the case when the two warp factors are equal and exponential, $\sigma(y) = \rho(y) = -k|y|$. The model consists of an G invariant non-linear sigma model parametrized by a set of bulk scalar fields ϕ^a together with a standard bulk gravity sector, and two boundary-branes. This is described by the action

$$\begin{aligned} S &= \int d^D x \sqrt{g_{(D)}} \left\{ M^{D-2} \mathcal{R}_{(D)} - \Lambda - \partial_M \phi^{a\dagger} \partial^M \phi^a - \lambda(\phi^{a\dagger}\phi^a - v^2) \right\} \\ &- \int d^{D-1}x \sqrt{g_{(D-1)+}}\, \tau_+ - \int d^{D-1}x \sqrt{g_{(D-1)-}}\, \tau_- . \end{aligned} \tag{7.2}$$

Our notation the following. The higher dimensional bulk indices are $M, N, \ldots$ and run over μ, i, y; the $(4+D_2) \equiv D-1$ dimensional brane indices are $A, B, \ldots$ and run over μ, i; $g^{(D)}_{MN}$ is the bulk metric and $g_{AB}^{(D-1)\pm}$ are the induced metrics on the branes. Finally, $\tau_\pm$ are the brane tensions, and M is the higher dimensional fundamental Planck mass.

Let us look more closely at the structure of the scalar fields. The equation of motion for the scalars can be written as usual:

$$\Box\phi^a = -\lambda\phi^a. \tag{7.3}$$

The role of non-dynamical auxiliary field λ is to impose the constraint

$$\phi^{a\dagger}\phi^a - v^2 = 0.$$

Differentiating this constraint twice, we can rewrite Eq. (7.3) as follows:

$$\Box\phi^a = -\left(\frac{\partial_M\phi^b\partial^M\phi^{b\dagger}}{v^2}\right)\phi^a. \tag{7.4}$$

The previous equation allows hedgehog solutions for ϕ^a for suitable choices of the group G. Moreover, they have a constant profile along the orbifold and satisfy

$$\Delta_\gamma\phi^a = -L^2\phi^a, \qquad \text{and} \qquad \partial_M\phi^{a\dagger}\partial^M\phi^a = e^{-2\rho}\frac{L^2v^2}{R^2} \tag{7.5}$$

where L is a 'winding number', and Δ_γ is the Laplacian obtained from γ_{ij}.

The Einstein equations for such hedgehog configurations have been studied in [206, 207, 208], where solutions of the type (7.1) with $\sigma(y) = \rho(y) = -ky$ have been found, with

$$k = \sqrt{-4M^{2-D}\Lambda/(D-1)(D-2)}, \tag{7.6}$$

where $\Lambda < 0$. In order to obtain the space-time described previously we take two copies of a slice of this D dimensional space comprised between y_+ and y_-, corresponding to the brane locations. The two copies are glued together there. Along with the identification $y - y_\pm \to 2y_\pm - y$, this gives the topology of an S^1/Z_2 orbifold in the y direction.

In order for this to be a solution of our model (7.2), the brane tensions have to satisfy

$$\tau_\pm = \pm4\sqrt{-(D-2)M^{D-2}\Lambda/(D-1)} = \pm4(D-2)M^{D-2}k, \tag{7.7}$$

as a result of the junction conditions at the branes. Besides (7.6), the Einstein equations in the bulk relate the hedgehog parameters to the curvature of the internal manifold Σ as

$$v^2 = \frac{2D_2C}{L^2}M^{D-2}. \tag{7.8}$$

Here, since Σ is homogeneous, the dimensionless constant C is defined through $\mathcal{R}_{ij}^{(\gamma)} = C\gamma_{ij}$, and $\mathcal{R}_{ij}^{(\gamma)}$ is the Ricci tensor computed out of γ_{ij}.

Associated to the sigma model scalars a number of Nambu Goldstone modes will be present. However, we shall assume that these couple to matter only through gravity, so that their effects are negligible.

Moduli

One interesting feature of this ansatz is that the parameter R, describing the volume of Σ, does not appear in the equations of motion even in the case of a curved internal space. Moreover, the positions of both branes are free at the classical level. They correspond to flat directions in the action and thus are the relevant degrees of freedom at low energies. In the moduli approximation, which we shall follow here, they are promoted to four dimensional scalar fields.

One crucial difference of these solutions with respect to the RS model is that they are not homogeneous along the orbifold, even in the case when Σ is a torus[1]. This is due to the compactness of Σ. In contrast with the RS model, the positions of both branes are physically meaningful.

However, it is clear that a scaling of R is equivalent to a shift in the positions of the branes $y_\pm$. Therefore, they are not independent. Rather, only two moduli are needed. Since we will use several combinations of the moduli along this Chapter, we summarize them briefly now: $a_\pm \equiv e^{-ky_\pm}$, the physical radii of Σ at the branes $R_\pm = a_\pm R$, the corresponding dimensionless values $r_\pm = a_\pm kR$, and $a \equiv e^{-k(y_- - y_+)} = a_-/a_+$.

In addition to the moduli, the massless sector also contains the graviton zero mode. To take it into account, we perturb the background solution (7.1) as follows

$$ds^2 = dy^2 + e^{2\sigma(y)}\left[\tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu + R^2\gamma_{ij}dX^i dX^j\right]. \tag{7.9}$$

Substituting this metric back into the action (7.2) we obtain the kinetic term for $\tilde{g}$ coming from the bulk part (see [15]). The kinetic terms for the moduli $y_\pm$ come from the boundary terms. A computation analogous to that in [15] gives

$$S_{(4)} = -m_P^2 \int d^4x\sqrt{-\tilde{g}}\left\{\left[\psi_+^2 - \psi_-^2\right]\ \tilde{\mathcal{R}} - 4\frac{D-2}{D-3}\left[(\tilde{\partial}\psi_+)^2 - (\tilde{\partial}\psi_-)^2\right]\right\}, \tag{7.10}$$

where $\psi_\pm^2 = a_\pm^{D-3} = e^{-(D-3)ky_\pm}$, and the effective four dimensional Planck mass is given by

$$m_P^2 = \frac{2}{D-3}v_\Sigma R^{D_2}M^{D-2}/k\ , \qquad \text{with} \qquad v_\Sigma = \int_\Sigma \sqrt{\gamma}d^{D_2}X. \tag{7.11}$$

We note that the moduli $\psi_\pm$ are Brans-Dicke (BD) fields and in the frame defined by $\tilde{g}_{\mu\nu}$, the kinetic term for ψ_+ has the wrong sign. Introducing the new variables ψ and φ [15, 204],

$$\psi_+ = \psi\,\cosh\varphi \quad \text{and} \quad \psi_- = \psi\,\sinh\varphi,$$

the Einstein frame is given by $\hat{g}_{\mu\nu} = \psi^2\tilde{g}_{\mu\nu}$. In this frame the action (7.10) takes the form

$$S_{(4)} = -m_P^2 \int d^4x\sqrt{-\hat{g}}\left\{\hat{\mathcal{R}} + 2\frac{D_2}{D_2+2}(\hat{\partial}\ln\psi)^2 + 4\frac{D_2+3}{D_2+2}(\hat{\partial}\varphi)^2\right\}, \tag{7.12}$$

and now the kinetic terms are both positive definite. Moreover, we note that the modulus ψ decouples in the limit $D_2 \to 0$, as expected, since this case corresponds to the usual RS model, where only one modulus is present.

[1] In this case, the solution corresponds to a toroidal compactification of a higher dimensional AdS space.

We are assuming that the ($(D-1)$ dimensional) matter fields $\chi_{\pm}^{(D-1)}$ are localized on each brane and so they couple universally to the corresponding induced metrics $g_{AB}^{(D-1)\,\pm}$ (recall $A, B, \dots = \mu, i$)

$$\begin{aligned} S^{\text{matt}} &= \sum_{\pm} \int d^{D-1}x \sqrt{-g^{(D-1)\pm}}\, \mathcal{L}^{\pm}\left(\chi^{\pm}_{(D-1)}, g_{AB}^{(D-1)\,\pm}\right) \\ &= \int d^4x \sum_{\pm} \sqrt{-g_{\pm}}\, a_{\pm}^{D_2} \mathcal{L}^{\pm}\left(\chi^{\pm}, g_{\mu\nu}^{\pm}\right). \end{aligned} \tag{7.13}$$

Here, we have kept the $\Sigma-$zero modes only, and integrated out the X dependence, the Σ volume factor has been absorbed by the four dimensional matter fields χ and couplings, and the four dimensional induced metrics are the (μ, ν) components of the $(D-1)$ dimensional ones $g_{\mu\nu}^{\pm} = g_{\mu\nu}^{(D-1)\,\pm}$.

A repeated use of the chain rule leads to the interaction of the moduli with matter given by

$$\begin{aligned} S^{\text{mod}-\text{matt}} = \int d^4x \sqrt{-\hat{g}} \Bigg\{ &- \frac{D_2}{D_2+2} \sum_{\pm} \left[\hat{T}_{\pm} - 2\hat{\mathcal{L}}_{\pm}\right] \delta \ln \psi \\ &+ \frac{2}{D_2+2} \sum_{\pm} a^{\pm(D_2+2)/2} \left[\hat{T}_{\pm} + D_2 \hat{\mathcal{L}}_{\pm}\right] \delta\varphi \Bigg\}, \end{aligned} \tag{7.14}$$

where $\hat{T}_{\pm}$ and $\hat{\mathcal{L}}_{\pm}$ are defined according to $\sqrt{-g_{\pm}}\, T_{\pm} = \sqrt{-\hat{g}}\, \hat{T}_{\pm}$, and $\sqrt{-g_{\pm}} a_{\pm}^{D_2}\, \mathcal{L}_{\pm} = \sqrt{-\hat{g}}\, \hat{\mathcal{L}}_{\pm}$. The coupling of the moduli to the Lagrangian is entirely due to the dimensions along Σ being warped, and is a generic prediction of models with a nontrivial warp factor for the extra dimensions. In fact, the 'radion' modulus φ is coupled to matter through $(\hat{T} + D_2 \hat{\mathcal{L}})_{\pm}$, which coincides with the trace of the $D-1$ dimensional energy momentum tensor. Moreover, this shows that the modulus ψ decouples from matter in the RS limit $D_2 \to 0$, as it should.

Defining the canonical fields

$$\widehat{\psi} = 2\sqrt{\frac{D_2}{D_2+2}}\, m_P\, \delta \ln \psi, \qquad \text{and} \qquad \widehat{\varphi} = 2\sqrt{2\frac{D_2+3}{D_2+2}}\, m_P\, \delta\varphi,$$

we obtain the equations of motion for the moduli

$$\begin{aligned} \hat{\Box}\widehat{\psi} &= \frac{1}{2}\sqrt{\frac{D_2}{D_2+2}}\, \frac{1}{m_P} \left[\hat{T}_+ - 2\hat{\mathcal{L}}_+ + \hat{T}_- - 2\hat{\mathcal{L}}_-\right] \\ \hat{\Box}\widehat{\varphi} &= -\frac{1}{\sqrt{2(D_2+3)(D_2+2)}\, m_P} \left[a^{(D_2+2)/2}\left(\hat{T}_+ + D_2\hat{\mathcal{L}}_+\right) + a^{-(D_2+2)/2}\left(\hat{T}_- + D_2\hat{\mathcal{L}}_-\right)\right]. \end{aligned} \tag{7.15}$$

As we explain in Sec. 7.2, we are interested in the case of $a \ll 1$ in order to have a substantial redshift effect arising from the warp factors. Unless otherwise stated, we shall set $\langle a_+ \rangle = 1$, so that, with a good accuracy, $a_- \simeq a$, $\psi \simeq \psi_+ \simeq 1$ and $\varphi \simeq \psi_- \ll 1$.

Thus, from (7.15) we can read off the couplings to the two types of matter:[2] $\widehat{\psi}$ couples to the matter at either brane $\chi_{\pm}$, with a strength $\sim 1/m_P$. As for φ, the coupling to χ_- is quite large, of order $a^{-(D_2+2)/2}/m_P$, and to χ_+ is even smaller than Planckian, $\sim a^{(D_2+2)/2}/m_P$.

[2]In the following, we will consider only matter located on the negative tension brane. Here we just consider other possible forms of matter at $y = y_+$ for the sake of generality.

Kaluza-Klein Reduction

Before the evaluation of the one-loop effective action, we turn now to the reduction in KK modes of a bulk scalar field living in the space-time described in the previous Section.

The idea is very simple: by performing a Kaluza-Klein reduction of the higher dimensional scalar field theory from D (with $D = D_1 + D_2 + 1$) down to D_1 dimensions, we obtain an equivalent lower dimensional theory consisting of an infinite number of massive Kaluza-Klein modes. Specifically, the Kaluza-Klein reduction is performed by expanding the higher dimensional scalar field in terms of a complete and orthogonal set of modes and then integrating out the dependence on the extra dimensions. The masses turn out to be quantized according to some eigenvalue problem and depend on the details of the space-time, the nature of the internal manifold and on the bulk (higher dimensional) scalar field. The one-loop effective action can then be evaluated by re-summing the contribution of each one of the modes.

Typically in Kaluza-Klein theory the mass eigenvalues are found explicitly and the subsequent evaluation of the sum over the modes does not present particular difficulties. However, in the case of warped space-times the main difference is that the orbifold nature of the extra dimension complicates the mass eigenvalues, which are expressed in terms of a transcendental equation and thus cannot be found explicitly.

In the present Section we will carry out the first step of the computation, namely the Kaluza-Klein reduction of the bulk scalar field. We will consider the most general case of a massive non-minimally coupled scalar field and assume that Σ, a compact manifold.

The bulk scalar field $\Phi(X, x, y)$ obeys the following equation of motion:

$$\left[-\Box_{(D)} + m^2 + \xi \mathcal{R}_{(D)}\right] \Phi = 0 \ , \tag{7.16}$$

where $\mathcal{R}_{(D)}$ is the higher dimensional curvature and $\Box_{(D)}$ is the D'Alembertian, both computed from the metric (7.1).

Using the explicit expression for the metric tensor, we can disentangle, in equation (7.16), the dependence on the internal manifold from the Minkowskian one. A straightforward calculation gives:

$$\Big[-e^{-2\sigma}\Box - e^{-2\rho}\frac{1}{R^2}\Delta_{(\gamma)} - e^{-\tau}\partial_y e^{\tau}\partial_y \quad + \\ +m^2 + \xi e^{-2\rho}\frac{1}{R^2}\mathcal{R}^{(\gamma)} - \xi F(y) \quad \Big] \quad \Phi = 0 \ , \tag{7.17}$$

where $\Delta_{(\gamma)}$ is the Laplacian related to γ_{ij}, $\Box$ is the D_1 dimensional flat D'Alembertian, and

$$\begin{aligned} F(y) &= 2\tau''(y) + \tau'(y)^2 + D_1\sigma'(y)^2 + D_2\rho'(y)^2 \ , \\ \tau(y) &= D_1\sigma(y) + D_2\rho(y) \ . \end{aligned} \tag{7.18}$$

We now expand the field $\Phi(x, X, y)$ in terms of a complete set of modes carrying a momentum along the orbifold and Σ directions labeled by indexes n and l respectively,

$$\Phi(x, X, y) = \sum_{l,n} \Phi_{l,n}(x) Y_l(X) f_{l,n}(y). \tag{7.19}$$

Here, $Y_l(X)$ are the generalized spherical harmonics in Σ *i.e.* , a complete set of solutions of the Klein-Gordon equation on Σ:

$$P_\Sigma\, Y_l(X) \equiv \frac{1}{R^2}\left[-\Delta_{(\gamma)} + \xi\mathcal{R}_{(\gamma)}\right] Y_l(X) = \lambda_l^2\, Y_l(X)\ , \tag{7.20}$$

with eigenvalues λ_l^2 and degeneracy[3] g_l. If we now require $\Phi_{l,n}(x)$ to satisfy the Klein-Gordon equation on the Minkowskian factor of the space-time M_4 with masses $m_{l,n}^2$,

$$\left[-\Box + m_{l,n}^2\right]\Phi_{l,n}(x) = 0\ , \tag{7.21}$$

we are left with a radial equation for the modes $f_{l,n}(y)$ of the form

$$e^{2\sigma}\left[-e^{-\tau}\partial_y e^{\tau}\partial_y + m^2 - \xi F(y) + \lambda_l^2 e^{-2\rho}\right] f_{l,n} = m_{l,n}^2 f_{l,n}\ . \tag{7.22}$$

This equation is valid for any warp factors σ and ρ, and can be viewed as an eigenvalue problem for the orbifold modes $f_{l,n}$ and the physical masses $m_{l,n}$. Both of them depend in general on the 'internal' index l. In this Chapter we consider the case of two equal warp factors, with

$$\rho(y) = \sigma(y) = -k|y|\ . \tag{7.23}$$

With $D = D_1 + D_2 + 1$, we can specialize Eq. (7.22) to this case as

$$\left[-e^{(3-D)\sigma}\partial_y e^{(D-1)\sigma}\partial_y + m^2 e^{2\sigma} - \xi F(y) e^{2\sigma}\right] f_{l,n} = (m_{l,n}^2 - \lambda_l^2) f_{l,n}. \tag{7.24}$$

We note that the operator in the l.h.s. *does not* depend on the internal index l. Accordingly, in this case neither the modes $f_{l,n}$ nor the combination $q_n^2 \equiv m_{l,n}^2 - \lambda_l^2$ depend on l. In other words, the dependence on l and n of the masses is factorized for this geometry,

$$m_{l,n}^2 = q_n^2 + \lambda_l^2. \tag{7.25}$$

Therefore, from now on we shall drop this index in Z. On the other hand, Eq. (7.24) is similar to the one which arises in the RS model, and the most general solution can still be written in terms of Bessel functions:

$$f_n^\beta(y) = \epsilon_\beta(y)\left[A_n^\beta J_\nu\left(\frac{q_n}{k}e^{-\sigma}\right) + B_n^\beta Y_\nu\left(\frac{q_n}{k}e^{-\sigma}\right)\right] \tag{7.26}$$

where for notational convenience we have defined

$$\epsilon_\beta(y) = e^{-\frac{D-1}{2}\sigma(y)}\begin{cases} y/|y| & \beta = \text{twisted} \\ 1 & \beta = \text{untwisted}, \end{cases} \tag{7.27}$$

and

$$\nu^2 = \frac{m^2}{k^2} - D(D-1)\xi + \frac{(D-1)^2}{4}\ . \tag{7.28}$$

The index β has been introduced in order to discriminate the two possible cases of Φ being untwisted ($f_n(-y) = f_n(y)$) or twisted ($f_n(-y) = -f_n(y)$). Imposing the appropriate boundary

[3] Although we assume P_Σ to be either positive semidefinite or positive definite, the label $l = 0$ always refers to the zero eigenvalue, i.e., $\lambda_0 = 0$, the existence of this eigenvalue being set by g_0 being 0 or 1.

conditions, which can be obtained by integrating equation (7.24) across the orbifold fixed points, we find that the eigenvalues q_n are determined by the transcendental equation:

$$F_\nu^\beta\left(\frac{q_n}{ka}\right) = 0 \ , \tag{7.29}$$

where

$$F_\nu^\beta(z) = \begin{cases} Y_\nu(az)J_\nu(z) - J_\nu(az)Y_\nu(z) & \beta = \text{twisted} \ , \\ y_\nu(az)j_\nu(z) - j_\nu(az)y_\nu(z) & \beta = \text{untwisted} \ . \end{cases} \tag{7.30}$$

As in the RS model, the combinations of Bessel functions relevant to the untwisted case are given by

$$\begin{aligned} j_\nu(z) &= \frac{1}{2}(D-1)(1-4\xi)J_\nu(z) + zJ'_\nu(z) \ , \\ y_\nu(z) &= \frac{1}{2}(D-1)(1-4\xi)Y_\nu(z) + zY'_\nu(z) \ , \end{aligned}$$

This completes the Kaluza-Klein reduction of the bulk scalar field.

In the following we will report only on the case of untwisted fields, although the case of twisted fields can be obtained at ease with simple modifications of our calculation.

7.2 Combining ADD and RS

In this Section we propose a scenario where supersymmetry is broken at a scale $\eta_{\rm SUSY}$ not far below the cutoff scale M, and the hierarchy between the electroweak and the effective Planck scales is generated by a combination of redshift and large volume effects. Also, we discuss the range of possible values for the dynamical (the moduli $R_\pm$) and the fixed scales (the cutoff M and the SUSY breaking scale $\eta_{\rm SUSY}$).

From Eqs. (7.12) and (7.11), we see that the relation between the four dimensional effective Planck mass and the higher dimensional one (in the four dimensional effective theory using the Einstein frame metric $\hat{g}_{\mu\nu}$) is

$$m_P^2 \approx (MR)^{D_2}\frac{M}{k}M^2. \tag{7.31}$$

We shall assume that the masses of particles (located at $y = y_-$) are somewhat below the cutoff M. In the four dimensional theory, these masses are redshifted down to $\sim aM$. Then, the EW/Planck hierarchy is given by

$$h^2 \equiv a^2\frac{M^2}{m_P^2} \sim \frac{a^2}{(RM)^{D_2}}\frac{k}{M} \sim 10^{-32}. \tag{7.32}$$

Thus, the EW/Planck hierarchy h is explained in this model due to a combination of *redshift* [2] and *large volume* [1] effects (even though the branes are of codimension 1). The crucial ingredient in order for the large volume effect to be efficient (aside from having a long orbifold), is that the additional extra space Σ exponentially grows as one moves away from the negative tension brane (see Fig. 7.1). In this way, matter is allowed to propagate along a small Σ, of size R_-, whereas gravity is diluted since it propagates through a much larger Σ, of effective size R_+. Since the gauge interactions must not be diluted by an analogous effect, we have to assume that the compactification scale on the negative tension brane $1/R_-$ is close to the fundamental cutoff M.[4]

Our model solves the hierarchy problem in a fashion very similar to the models considered in [212, 213], with two concentric branes embedded in a non compact bulk. In this references, the hierarchy and the positions of the branes are naturally stabilized by a generalization of the Goldberger and Wise mechanism [5] (see also [214]).

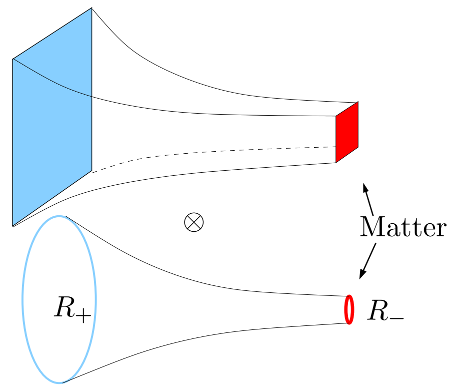


Figure 7.1: Matter can propagate along the additional extra space Σ of size R_-, but gravity samples a much bigger space.

Let us now examine the constraints that we have on the moduli and the physical values they can take. First of all, we are thinking of an inter-brane distance $d = |y_- - y_+|$ somewhat larger than the inverse curvature scale $1/k$ of the bulk, in order to have a substantial redshift factor $a = e^{-kd}$. On the other hand, the smallest physical length scale is given by the size of Σ at the negative tension brane, R_-. This cannot be smaller than the fundamental length of the theory M^{-1} though, as argued in the previous paragraph, it should be close to it. There is a tighter technical restriction which we shall use coming from the result for the potential that we obtain in the next Section, (7.66). This is organized as a power series in $r_\pm = kR_\pm$, and can be trusted only when $1/R_+$ is larger than the curvature scale. The same holds for $1/R_-$, since it is a factor a^{-1} above (recall that the ratio R_-/R_+ coincides with the redshift factor a). Incidentally, we remark that this corresponds to the physical situation where the size of the internal manifold Σ is *everywhere* smaller than the inter-brane distance $\sim 1/k$.[5] So, we must assume a separation between the fundamental cutoff M and the curvature scale k at least of order a. This leads to the following scenario.

Consider a supersymmetric theory where the SUSY breaking scale is given by η_{SUSY}.Then, the bulk cosmological constant $\Lambda \sim k^2 M^{D-2}$ is expected to be proportional to η^D_{SUSY}, which

[4]Keeping only the Σ-zero mode in the action for a $D-1$ dimensional Yang-Mills field F_{AB} at $y = y_-$ with coupling constant $g^2_{*(D-1)} \sim M^{5-D}$, one obtains $\int d^{D-1}x\sqrt{g_{(D-1)-}}\ \frac{1}{g^2_{*(D-1)}} F_{AB}F_{CD}\, g^{AC}_{(D-1)-}\, g^{BD}_{(D-1)-} \simeq \int d^4x\sqrt{\hat{g}}\, R_-^{D_2}\, \frac{1}{g^2_{*(D-1)}}\, F_{\mu\nu}F_{\rho\sigma}\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma}$, where we used that $\tilde{g}_{\mu\nu} \simeq \hat{g}_{\mu\nu}$. Thus, the four dimensional YM coupling is identified as $g^2_{*(4)} = g^2_{*(D-1)}/R_-^{D_2} \sim 1/(MR_-)^{D_2}$.

[5]This means that in a certain range of energy the model is effectively 5 dimensional. In Appendix 6.1, we derive the form of the dimensionally reduced theory down to 5 dimensions.

leads to

$$k \sim \left(\frac{\eta_{\rm SUSY}}{M}\right)^{D/2} M \ll M \tag{7.33}$$

Even if SUSY is broken not far below the cut-off scale, this may lead to a curvature scale k many orders of magnitude below M, due to the large exponent in (7.33). If the moduli $R_\pm$ are stabilized near the values $R_+ \sim 1/k$ and $R_- \sim 1/M$, then $a \sim k/M$ and from (7.32), the hierarchy is given by

$$h \sim \left(\frac{k}{M}\right)^{(D-2)/2} \sim \left(\frac{\eta_{\rm SUSY}}{M}\right)^{(D^2-1)/4}. \tag{7.34}$$

Note that the required hierarchy is obtained with $\eta_{\rm SUSY}$ within one order of magnitude of the cut-off M for $D = 11$, and less than 3 orders of magnitude below M for $D = 6$.

This shows how the problem of the stabilization of a large hierarchy works in this model. Having introduced a small separation between the SUSY breaking and the cutoff scales, we obtain a stable very flat warped space-time, $k \ll M$. If the potential (7.66) can stabilize the moduli $R_\pm$ near the values, $R_+ \sim 1/k$ and $R_- \sim 1/M$, then the effective Planck mass is very large as compared to the EW scale. Whether or not the effective potential (7.66) can do this job is addressed in Section 7.4.

Let us discuss the physical scales in the model in some detail. As illustrated in Fig. 7.1, the branes are of codimension 1, so that matter (residing on the negative tension brane, at $y = y_-$) can propagate through a physical extra dimensional space of size $\sim R_-$. The mass scales on this brane are redshifted by a factor a, thus the mass of the first KK excitations of matter fields is $1/R$. Then, from collider physics, we have to set the compactification scale $1/R \gtrsim$ TeV, at least.

In contrast, gravity propagates through the whole bulk space, and its KK spectrum is analogous to that obtained in Sec. 7.1 for a scalar field. In particular, there are three kinds of modes, excited along the orbifold only, along Σ only or along both, as (7.25) shows. The masses m_Σ of the first graviton KK modes along Σ are of order $1/R$. However, the modes winding along the orbifold only (the Σ zero mode) have masses given by $m_{\rm orb} \sim ak$, as in the RS model (the curvature scale times the redshift factor). In the approximation of everywhere small Σ that we are considering, $kR_\pm \ll 1$, this means that these modes are a factor a lighter than the modes propagating along Σ.

This allows us to assume the SUSY breaking scale $\eta_{\rm SUSY}$ and the cutoff M are such that $k \sim$ TeV, obtaining quite small masses for the graviton orbifold KK modes $m_{\rm orb} \sim a$TeV. Below, we show that such a small value does not conflict with observations, since the coupling of these modes to matter is very suppressed. This is consistent with the assumption made above that in the higher dimensional theory the masses of matter fields are near the cutoff M, since they are redshifted to $aM \sim$ TeV, which compatible with the electroweak scale. Also, since we are considering the limit of everywhere small internal space $kR_\pm \lesssim 1$, setting $k \sim$ TeV implies that the masses of matter KK fields is large enough, $1/R \gtrsim$ TeV.

Thus, from the point of view of the 4 dimensional effective theory, KK modes from the matter fields appear at $1/R \sim$ TeV. Since the curvature scale k of the bulk is close to $1/R$, this coincides with the scale where gravity becomes higher dimensional.

In summary, we are lead to consider distribution of scales illustrated in Fig. 7.2. We set the cutoff M and the SUSY breaking scale $\eta_{\rm SUSY} \lesssim M$ such that the curvature scale of the bulk is

$k \sim$ TeV. We assume that some mechanism can stabilize R_- near the fundamental length $1/M$ and $R_+ \sim 1/k$. As a consequence, the masses of the graviton KK modes along the orbifold are $m_{\rm orb} \sim a$TeV, and for the modes along Σ are $m_\Sigma \sim$ TeV. [6]

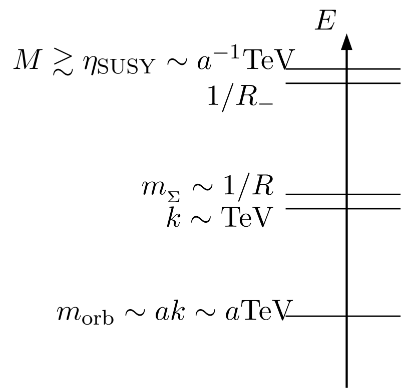


Figure 7.2: The mass $m_{\rm orb}$ of the first KK excitations along the orbifold is much smaller than the mass m_Σ of the modes excited along the internal manifold. There is essentially the same hierarchy between the fundamental cutoff M and the scales that determine the solution, k and R.

The curvature of this background varies from $\sim M$ on the positive tension brane, to $k \lesssim aM$ on the negative tension brane. One might argue that supersymmetry is needed in order to stabilize this other hierarchy of scales, however, supersymmetry is a common ingredient in theories coming from $M-$theory, and the aim of this Chapter is to find out whether or not in such models quantum effects can stabilize the moduli. So, we consider a scenario with supersymmetry in D dimensions with particles that acquire a mass of order the SUSY breaking scale $\sim k$.

It remains to be seen that, indeed, the graviton KK modes along the orbifold are unobservable, in spite of their relatively small masses $m_{\rm orb} \sim ak$. We see from Eqs. (7.26,7.27) that the (unnormalized) wave function of the KK modes grows exponentially as $e^{(D-1)k|y|/2}$, signaling that the more warped the extra dimensions are, the more localized on the negative tension brane these modes are. This implies [210] that the coupling of these graviton KK modes is amplified with respect to that of the zero mode ($\sim 1/m_P$) by a factor $a^{-(D-1)/4} = h^{-(D-1)/2(D-2)}$. Then, they are much more weakly coupled to matter $\sim 1/(10^8{\rm TeV})$ than in the RS model ($\sim 1/{\rm TeV}$). Thus, in spite of their relatively small mass, these KK gravitons cannot be seen individually in accelerators. Moreover, since they are associated with only one off-the-brane dimension, they do not have as large a multiplicity as in the usual large volume mechanism [1], and so they do not significantly cool stars. The total rate of emission of any of such gravitons at a given energy $E <$ TeV can be estimated as the coupling squared times the number of states with masses

[6] Another interesting possibility consists of setting $m_{\rm orb} \sim$ TeV so that $k \sim a^{-1}$TeV and $M \sim a^{-2}$TeV. This could be realized in a scenario with the SUSY breaking scale $\eta_{\rm SUSY} \sim 1/R \sim k$ and the masses of particles of order k, from the D dimensional viewpoint. In this scenario, the EW/Planck hierarchy is given by $h^2 \sim a^2k^2/m_P^2$. If the moduli are stabilized so that $R_+ \lesssim 1/k$ and $R_- \gtrsim 1/M$, then $h \sim (k/M)^D$, thus needing less separation between $\eta_{\rm SUSY}$ and M in order to explain same hierarchy h. Moreover, one can see that the potential (7.66) generates masses for the moduli larger than in the scenario presented so far. However, the bulk cosmological constant Λ would be much larger than $\eta^D_{\rm SUSY}$.

lighter than E [1, 3],

$$\left(\frac{1}{a^{(D-1)/4}m_P}\right)^2 \frac{E}{m_{\rm orb}} \sim h^{(D-5)/(D-2)}\frac{E}{\rm TeV}\frac{1}{\rm TeV^2},$$

which is very small for the energies available inside stars.

Other similar scenarios that use a mixture of ADD and RS mechanisms to generate the hierarchy are considered in [215, 213, 212].

7.3 Effective Potential

The one loop effective action $S_{\it eff}$ can be expressed as the sum over the contributions of each mode, $S^{l,n}_{\it eff}$

$$S_{\it eff} = \sum S^{l,n}_{\it eff} \ .$$

The previous expression can be evaluated in a variety of ways (see for instance [6, 216]). Dimensional regularization of the 4−dimensional Minkowski directions to $4-2\epsilon$ leads to the following expression for the vacuum energy contribution to the effective action

$$S_{\it eff} = -\int d^{4-2\epsilon}x\ V^{\rm reg}(s)\ , \tag{7.35}$$

with

$$V^{\rm reg}(s) = -\frac{1}{2}(4\pi)^s\mu^{2\epsilon}\Gamma(s){\sum_{n,l}}' g_l(q_n^2+\lambda_l^2)^{-s}\ , \tag{7.36}$$

where the prime in the sum assumes that the zero mass mode is excluded (since it does not contribute) and $s=-2+\epsilon$. The renormalization scale μ is introduced for dimensional reasons. It is convenient to separate $V^{\rm reg}$ into three contributions[7]

$$V^{\rm reg}(s) = V_\Sigma(s) + V_{RS}(s) + V_*(s)\ , \tag{7.37}$$

where

$$V_\Sigma(s) \;=\; -\frac{\mu^{2\epsilon}}{2(4\pi)^{-s}}\Gamma(s)\sum_{l=1}^{\infty} g_l\lambda_l^{-2s}\ , \tag{7.38}$$

$$V_{RS}(s) \;=\; -g_0\frac{\mu^4}{2(4\pi)^{-s}}(ka/\mu)^{-2s}\Gamma(s)\sum_{n=1}^{\infty} x_n^{-2s}\ , \tag{7.39}$$

$$V_*(s) \;=\; -\frac{\mu^4}{2(4\pi)^{-s}}(ka/\mu)^{-2s}\Gamma(s)\sum_{n,l=1}^{\infty} g_l\left(x_n^2+y_l^2\right)^{-s}\ , \tag{7.40}$$

with $x_n = q_n/ka$ and $y_l = \lambda_l/ka$. Thus, V_Σ retains the contributions from the orbifold zero mode (present only for an untwisted field), V_{RS} is the contribution from the Σ zero mode (which

[7] Here we define $\lambda_0 = 0$, so that the existence of such a zero eigenvalue or not is controlled by g_0. If $g_0 \neq 0$, the RS contribution comes about explicitly and introduces a divergence which needs to be canceled by a corresponding contribution coming from V_*. This cancellation provides a non-trivial check of our evaluation. The case of a strictly positive definite operator, can be obtained by putting g_0 to zero.

coincides with the potential in the RS model), and V_* includes the contribution from mixed states. It is clear from Eqs. (7.29) and (7.30) that x_n depends on a only. Since λ_l scales like $1/R$, we can factor out the dependence on this modulus, defining dimensionless eigenvalues $\hat{\lambda}_l = R\lambda_l$, that do not depend on R. If Σ is a one-parameter space, then λ_l cannot depend on any other *shape* moduli. However, here we are interested in the dependence on the *breathing* mode R only. So, in general, $y_l = \hat{\lambda}_l/(kaR)$ depends on the moduli described in Sec. 7.1 through R_-.

The first term in (7.37) V_Σ results from the KK excitations along the internal manifold. It can be expressed in terms of the generalized ζ function associated to the Laplacian P_Σ defined on Σ (see Eq. (7.20)),

$$\zeta(s) \equiv \zeta(s|P_\Sigma) = \sum_{l=1}^{\infty} g_l \hat{\lambda}_l^{-2s}. \tag{7.41}$$

using the previous rescaling we can recast V_Σ as

$$V_\Sigma(-2+\epsilon) = -\frac{1}{32\pi^2 R^4}\,(\mu R)^{2\epsilon}\,\Gamma(-2+\epsilon)\zeta(-2+\epsilon)\ , \tag{7.42}$$

where we redefined the renormalization constant μ. The previous expression can be elegantly dealt with by using the Mittag-Leffler representation for the ζ function, which proves to be a very useful tool to handle the pole structure of the ζ function, since the residues at the poles are determined by geometrical quantities of Σ (See for example [190]). As shown in Appendix C,

$$\zeta(s) = \frac{1}{\Gamma(s)}\left\{\sum_{p=0}^{\infty}\frac{\tilde{C}_p}{s-D_2/2+p} + f(s)\right\}\ , \tag{7.43}$$

where $\tilde{C}_p = C_p - g_0\,\delta_{p,D_2/2}$ and C_p are the (integrated) Seeley-DeWitt coefficients of the operator P_Σ on Σ, p runs over the positive half integers and $f(s)$ is an entire function. In fact, the sum (7.43) runs over half integers, but, since Σ has no internal boundaries, the coefficients $C_{i/2}$ are zero. Relation (7.43) can now be used to regulate V_Σ, and a simple calculation gives

$$V_\Sigma(s=-2+\epsilon) = -\frac{1}{32\pi^2 R^4}(4\pi R^2\mu^2)^\epsilon\left[\Omega_{-2} + C_{D_2/2+2}\frac{1}{\epsilon}\right], \tag{7.44}$$

where Ω_{-2} is the constant term in the power series of $\Gamma(s)\zeta(s)$ around $s=-2$ (see Eq. (C.17)).

The term proportional to the RS contribution has been computed in [7, 11, 12, 13]. Without going into details, we write such term as follows:

$$V_{RS} = -g_0\frac{k^4}{32\pi^2}(k/\mu)^{-2\epsilon}\left\{-d_4\frac{1}{\epsilon}\left(1+a^{4-2\epsilon}\right) + c_1 + a^4 c_2 - 2a^4\mathcal{V}(a)\right\}, \tag{7.45}$$

where we have introduced (see Eqs. (5.3),6.59)

$$\mathcal{V}(a) = \int_0^\infty dz z^3 \ln\left(1 - \frac{k_\nu(z)}{k_\nu(az)}\frac{i_\nu(az)}{i_\nu(z)}\right) \tag{7.46}$$

and the coefficients c_1 and c_2 do not depend on a. Here, the coefficient d_4 depends on the mass and non-mininal coupling of Φ, and is defined through Eqs. (7.56,7.52,7.50).

Before entering into the discussion of the higher dimensional contribution due to the mixed KK states V_*, we can foresee now some of the details of the computation. As mentioned above, the case when Σ is a torus corresponds to a toroidal compactification of a slice of higher dimensional AdS space. Since it is a maximally symmetric space, all the geometric invariants are constant, and so proportional to the brane tensions. Thus, the only possible divergence that can appear is of the form $\int d^4x(1+a^{(D-1)})$. However, regardless of the dimension of Σ, the contribution from V_{RS} contains a divergence of the form $\int d^4x(1+a^4)$. Of course, what happens is that aside from the higher dimensional divergence, V_* also contains another divergence that cancels the RS one. This feature occurs not only when Σ is a torus. Rather, it is completely general. As we show next and in Appendix C, the divergence of $V_* + V_{RS}$ proportional to $\int d^4x(1+a^4)$ is *always* controlled by a geometric invariant related to Σ (which trivially vanishes for a torus). This ensures that if this divergence persists, it is because one can build some operator that behaves like it in this background.

Let us now turn to the evaluation of V_*. First of all, let us concentrate on the sum

$$\Gamma(s) \sum_{n,l=1}^{\infty} g_l \left(x_n^2 + y_l^2\right)^{-s} \ . \tag{7.47}$$

This is not straightforward to compute, however the method developed in [217, 198, 197] allows us to perform such a calculation. Since, in our case, the evaluation does not present any particular difficulty, we will be brief and address the reader to the original references for an introduction to the details of the method.

The residue theorem permits us to express the sum (7.47) as a contour integral and an appropriate choice of the contour of integration leaves us with

$$\Gamma(s)\frac{\sin(\pi s)}{\pi} \sum_l g_l \int_{y_l}^{\infty} (x^2 - y_l^2)^{-s} \frac{d}{dx} \ln\left[F_\nu(ix)\right] dx \ , \tag{7.48}$$

which, by changing variable and by using some known properties of the Bessel functions can be recast as

$$\frac{1}{\Gamma(1-s)} \sum_l g_l\, y_l^{-2s} \int_1^{\infty} (z^2-1)^{-s} \frac{d}{dz} \ln\left[P_\nu(y_l z)\right] dz, \tag{7.49}$$

where

$$P_\nu(z) = F_\nu(iz) = \frac{2}{\pi}\left[k_\nu(z) i_\nu(az) - k_\nu(az) i_\nu(z)\right], \tag{7.50}$$

and

$$\begin{aligned} i_\nu(z) &= zI'_\nu(z) + \frac{1}{2}(D-1)(1-4\xi)I_\nu(z) \\ k_\nu(z) &= zK'_\nu(z) + \frac{1}{2}(D-1)(1-4\xi)K_\nu(z). \end{aligned}$$

We can regulate relation (7.49) using the asymptotic expansions for the Bessel functions. The large z behaviour of i_ν and k_ν can be written as

$$\begin{aligned} i_\nu(z) &= \sqrt{\frac{z}{2\pi}} e^{z} \Theta^{(i)}(z) \ , \\ k_\nu(z) &= \sqrt{\frac{\pi z}{2}} e^{-z} \Theta^{(k)}(z) \ , \end{aligned} \tag{7.51}$$

where $\Theta^{(k)}(z) = \Theta^{(i)}(-z)$ is a power series in $1/z$ beginning with 1. Thus, we can recast the integrand of Eq. (7.49) in the form

$$P_\nu(y_l z) = -\frac{\sqrt{a}}{\pi} y_l z e^{(1-a)y_l z} \Theta^{(i)}(y_l z)\Theta^{(k)}(a y_l z) \left[1 - \frac{k_\nu(y_l z) i_\nu(a y_l z)}{i_\nu(y_l z) k_\nu(a y_l z)}\right] . \tag{7.52}$$

Up to a constant term, $\ln P_\nu$ can be split as

$$\ln\left[P_\nu(y_l z)\right] = \mathcal{H}_l^{(1)}(z) + \mathcal{H}_l^{(2)}(z) + \mathcal{H}_l^{(3)}(z)$$

with

$$\begin{aligned} \mathcal{H}_l^{(1)}(z) &= \ln z + (1-a) y_l z \ , \\ \mathcal{H}_l^{(2)}(z) &= \ln\left[\Theta^{(i)}(y_l z)\Theta^{(k)}(a y_l z)\right] \ , \\ \mathcal{H}_l^{(3)}(z) &= \ln\left[1 - \frac{k_\nu(y_l z) i_\nu(a y_l z)}{i_\nu(y_l z) k_\nu(a y_l z)}\right] \ . \end{aligned} \tag{7.53}$$

and correspondingly,

$$V_*(s) = V_*^{(1)}(s) + V_*^{(2)}(s) + V_*^{(3)}(s),$$

with

$$V_*^{(\alpha)} = -\frac{\mu^4}{2(4\pi)^{-s}} (ka/\mu)^{-2s} \frac{1}{\Gamma(1-s)} \sum_{l=1}^{\infty} g_l y_l^{-2s} \int_1^\infty (z^2-1)^{-s} \frac{d}{dz} \ln\left[\mathcal{H}_l^{(\alpha)}(z)\right] dz \qquad (\alpha = 1,2,3). \tag{7.54}$$

The evaluation of $V_*^{(1)}(s)$ is analogous to the one for $V_\Sigma(s)$ and, once more, the Mittag-Leffler expansion allows to express the result in terms of the heat-kernel coefficients of the operator P_Σ on Σ. We find

$$\begin{aligned} V_*^{(1)}(s = -2+\epsilon) = -\frac{1}{32\pi^2 R^4} (\mu R)^{2\epsilon} \Bigg\{ & \left[\frac{1}{2} C_{2+D_2/2} + \frac{1}{2\sqrt{\pi}} C_{5/2+D_2/2} \frac{1-a}{kaR}\right] \frac{1}{\epsilon} \\ + & \ \frac{1}{2}\Omega_{-2} + \frac{1}{2\sqrt{\pi}} \Omega_{-5/2} \frac{1-a}{kaR} \Bigg\} \end{aligned} \tag{7.55}$$

The second term $V_*^{(2)}(s)$ can be evaluated[8] using the explicit form of $\Theta^{(i)}$ and $\Theta^{(k)}$:

$$\ln\left(\Theta^{(i)}(z)\Theta^{(k)}(az)\right) \simeq \sum_{j=1}^{\infty} \left(1 + \frac{(-1)^j}{a^j}\right) d_j \ z^{-j} \qquad \text{for} \quad z \gg 1, \tag{7.56}$$

the coefficients d_j can be obtained by simply Taylor expanding the logarithm. Using (7.56) and treating the sum over the eigenvalues y_l as in the case of V_Σ (see App. (C)), we can write $V_*^{(2)}$ as

$$\begin{aligned} V_*^{(2)}(s = -2+\epsilon) &= \frac{1}{32\pi^2 R^4} (\mu R)^{2\epsilon} \\ & \sum_{j=1}^{\infty} \frac{d_j}{\Gamma(j/2)} \left\{ \left[C_{2+D_2/2-j/2} - g_0 \delta_{4,j}\right] \frac{1}{\epsilon} + \Omega_{-2+j/2} \right\} \left((kaR)^j + (-kR)^j\right) . \end{aligned} \tag{7.57}$$

[8] Strictly speaking we are using an asymptotic expansion and therefore the equality sign is not exact. However, the approximation we are making is reasonable because the integration range vary from 1 to ∞ and the argument of $\Theta^{(i)}$ and $\Theta^{(k)}$ is large in the region $R \ll 1$ and $aR \ll 1$.

The third term in $V_*^{(3)}(s)$ is finite by construction, and we can put safely $s=-2$,

$$V_*^{(3)}(s=-2) = -\frac{1}{64\pi^2 R^4}\sum_{l=1}^{\infty} g_l \hat{\lambda}_l^4 \int_1^{\infty} (z^2-1)^2 \frac{d}{dz}\mathcal{H}_l^{(3)}(z)dz \ . \tag{7.58}$$

Combining the previous results, we obtain the unrenormalized Casimir energy:

$$\begin{aligned} V^{\rm reg} = -\frac{1}{32\pi^2 R^4} & \left[\sum_{j=-1}^{\infty} \left[(kR_-)^j + (-kR_+)^j\right] \left\{\gamma_j + (\beta_j - g_0 d_4 \delta_{4,j})\frac{1}{\epsilon}\,(\mu R)^{2\epsilon}\right\} \right. \\ & + \ g_0 (kR)^4 \left\{ c_1 + a^4 c_2 - 2a^4 \mathcal{V}(a) - \frac{1}{\epsilon}\left(1+a^{4-2\epsilon}\right)(k/\mu)^{-2\epsilon} \right\} \\ & + \left. 2\sum_{l=1}^{\infty} g_l \hat{\lambda}_l^4 \mathcal{V}_l(a,R_-) \right] \end{aligned} \tag{7.59}$$

where

$$\beta_j = \begin{cases} (1/2\sqrt{\pi})C_{5/2+D_2/2} & \text{for} \quad j=-1 \\ (3/2)\ C_{2+D_2/2} & \text{for} \quad j=0 \\ -(d_j/\Gamma(j/2))\,C_{2-j/2+D_2/2} & \text{otherwise,} \end{cases} \tag{7.60}$$

and we understand that the Seeley-DeWitt coefficients C_i are zero if $i<0$,

$$\gamma_j = \begin{cases} (1/2\sqrt{\pi})\Omega_{-5/2} & \text{for} \quad j=-1 \\ (3/2)\Omega_{-2} & \text{for} \quad j=0 \\ -(d_j/\Gamma(j/2))\Omega_{j/2-2} & \text{otherwise,} \end{cases} \tag{7.61}$$

and

$$\mathcal{V}_l(a,R_-) = \int_1^{\infty} dz\ z(z^2-1)\ln\left(1 - \frac{k_\nu(y_l z)}{k_\nu(y_l a z)}\frac{i_\nu(y_l a z)}{i_\nu(y_l z)}\right). \tag{7.62}$$

Equation (7.59) shows that as we advanced above, the lower dimensional divergence coming from the RS contribution V_{RS} is always canceled, independently of the structure of the internal manifold Σ. On the other hand, the contribution from the KK modes along Σ only (lower dimensional, as well) may give a divergence corresponding to $j=0$. This is controlled by the Seeley-DeWitt coefficient $C_{D_2/2+2}$, and gives $1/2$ of the resulting $3/2$ factor in β_0, the rest coming from the mixed states in V_*. In particular, if D_2 is odd, then there is no such divergence (if Σ is boundaryless), in accordance with the absence of any operator that behaves as $\int d^4x\ 1/R^4$ in the background, in this case.

To conclude this Section, we briefly comment on the differences appearing when we consider a twisted bulk field. First, since there is no orbifold zero mode, its contribution V_Σ is not present. One can show that the asymptotic behaviour of the function P_ν differs in two powers of the argument, originating a change of sign in the contribution to β_0 and γ_0 from $V_*^{(1)}$. Of course, the d_j coefficients also change, and can be read from [7, 12]. In brief, one needs to change the d_j by the corresponding one, and the $3/2$ factor in β_0 and γ_0 by $-1/2$. Also, we haven't included any brane mass terms or kinetic terms, relevant for the untwisted case only (aside from the ones arising from the coupling to curvature). In principle, these can be different on each brane. This changes our result in that we would have different coefficients, $d_j^{\pm}$, for the $r_\pm$ series.

Until now, we have computed the unrenormalized Casimir energy (7.59) using dimensional regularization. This allows us to isolate the divergent terms, of the form

$$\Gamma^{\rm div} = \frac{1}{\epsilon}\frac{1}{32\pi^2}\frac{1}{R^4}\int d^4x \sum_{j=-1}^{D_2+4} \beta_j (a^j + (-1)^j)(kR)^j, \tag{7.63}$$

with β_j given by (7.60). A finite number of divergences appear because we have computed the one loop contribution to the effective potential.

It is well known (see *e.g.* [197]) that the divergences present in the effective action are given by the Seeley-DeWitt coefficient $C_{D/2}$ related to the operator in (7.16) on our D dimensional background space-time. Since this has boundaries, nonzero boundary terms are present for any dimension. Moreover, since the extrinsic curvature is constant in the space-time we are considering, several powers of the intrinsic curvature of the boundaries are present. Finally, it is easily shown that once any possible bulk term is evaluated on the background solution, it can be recast as boundary term for this specific solution.[9]

So, we shall consider boundary term of the form

$$\sum_{\pm}\int d^{(D-1)}x\sqrt{g_{(D-1)\,\pm}}\,\mathcal{R}^N_{(D-1)\,\pm}, \tag{7.64}$$

where $N = 0, 1, 2, \ldots$ and $\mathcal{R}_{(D-1)\,\pm}$ denotes the (intrinsic) curvature computed from the induced metrics on the branes $g_{(D-1)\pm}$. Using the explicit expression for the metric tensor (7.1), a simple calculation shows the previous term generates a contribution proportional to R^{D_2-2N} coming from the brane at y_+, and a contribution of the form $R^{D_2-2N}a^{4+D_2-2\epsilon-2N}$ from the other brane. Then, it is clear that all the divergences in (7.63) can be dealt with operators of the form (7.64). Specifically, we can take the following expression as the counter-term needed to renormalize the effective action:

$$\begin{aligned} S_j^{CT} &= \frac{1}{32\pi^2\epsilon}\int d^{D_2}X d^{4-2\epsilon}x \left\{\sqrt{g_{(D-1)+}}\kappa_j^+\mathcal{R}_+^{(D_2+4-j)/2} + \sqrt{g_{(D-1)-}}\kappa_j^-\mathcal{R}_-^{(D_2+4-j)/2}\right\} \\ &= \frac{1}{32\pi^2\epsilon}\int d^{4-2\epsilon}x\frac{R^j}{R^4}\left\{\kappa_j^+ + \kappa_j^- a^{j-2\epsilon}\right\} \end{aligned} \tag{7.65}$$

The index j here runs over the integers comprised between -1 and $4+D_2$, and $\kappa_j^{\pm}$ are renormalization constants. We recall that, from (7.63) and (7.60), the divergences occur for j even only if D_2 is even, and for j odd when D_2 odd.

In the process of subtracting the counter-terms, finite contributions to the vacuum energy with a logarithmic dependence on the moduli are generated. The renormalized expression can be written as

$$\begin{aligned} V(R_\pm) = -\frac{1}{32\pi^2R^4}\quad\Bigg[\quad & \sum_{j=-1}^{\infty}\Bigg\{(\beta_j - g_0\,d_4\,\delta_{4,j})\left[(kR_-)^j\ln(kR_-)^2 + (-kR_+)^j\ln(kR_+)^2\right] \\ + \quad & \left(\gamma_j - \beta_j\ln{(k/\mu)^2}\right)\left[(kR_-)^j + (-kR_+)^j\right]\Bigg\} \\ + \quad & g_0(kR)^4\left\{c_1 + a^4c_2 - 2a^4\mathcal{V}(a)\right\} + 2\sum_{l=1}^{\infty} g_l\hat{\lambda}_l^4\mathcal{V}_l(a,R_-)\Bigg] \end{aligned} \tag{7.66}$$

[9] For instance, $\int d^Dx\,\sqrt{g_{(D)}}\Lambda = \sum_{\pm}\int d^{(D-1)}x\,\sqrt{g_{(D-1)\pm}}\,\sigma_\pm$, with $\sigma_\pm = \mp 2\Lambda/(D-1)k$.

A few remarks are now in order. First of all, note that we recast the result in order to isolate the μ dependent terms. Such terms are not computable from our effective theory, rather they have to be fixed by imposing a set of renormalization conditions. Secondly, notice that the result is valid for D_2 even as well as for D_2 odd, and the heat-kernel coefficients automatically take this into account.

An important remark concerns the divergence proportional to $R_+^4 + R_-^4$. This is the divergence present in the RS contribution [7, 12]. For D_2 odd, it is not reproduced by any of the counter-terms in (7.64). However, this is not a problem because such divergence is canceled by the corresponding one coming from (7.58) for $j = 4$.

7.4 Stabilization

The result we have obtained so far (7.66) is a potential $V(R_\pm)$ for the two moduli describing the background. Using $r_\pm \equiv kR_\pm$, it can be cast as

$$V(r_\pm) = -\frac{1}{32\pi^2 R^4}\left[V_+(r_+) + V_-(r_-) + v(r_+, r_-)\right], \tag{7.67}$$

where $v(R, r)$ contains the 'non-local' part, and

$$V_\pm(r_\pm) = \sum_{j=-1}^{\infty} (\mp 1)^j \left\{\gamma_j r_\pm^j + (\beta_j - g_0 d_4 \delta_{j,4})\, r_\pm^j \ln r_\pm^2 - \alpha_j^\pm r_\pm^j\right\}, \tag{7.68}$$

Here, the coefficients $\alpha_j^\pm$ are understood to be finite renormalization constants, and are nonzero when the corresponding logarithmic term is nonzero. This is dictated by β_j being zero or not (*i.e.*, whether or not such a term is divergent), with the sole exception of $j = 4$. If $d_4 = 0$ and the Laplacian P_Σ (see Eq. (7.20)) has one zero eigenvalue, $g_0 = 1$, the logarithmic terms corresponding to $j = 4$ are not associated to any divergence of the effective action, and $\alpha_4^\pm = 0$. This situation arises, for example, when Σ is a torus.

Note that the sum goes from -1 to ∞ and we recall that from Eq. (7.60), all the β_j with $j > 4 + D_2$ vanish identically. Thus, the term $\beta_j r_-^j$ appears with j running from -1 to $D_2 + 4$, and the same holds for the terms with $\alpha_j^\pm$ (there are a finite number of divergent terms).

One interesting feature of the effective potential (7.67) in both cases with D_2 even and odd is that the two leading terms in the small $r_\pm$ limit (corresponding to $j = -1, 0$) do not depend on the mass m nor the non-minimal coupling constant ξ. This means that if we consider equal number of fermionic and bosonic degrees of freedom, these terms cancel identically even with non supersymmetric masses. From now on, we will focus on this case, one motivation being that the models considered here arise mainly in string theories, and the field content of the effective theories indeed contain equal number of bosonic and fermionic degrees of freedom. The only change is that the sum in Eq. (7.68) will begin at $j = 1$ instead of $j = -1$. As mentioned above, the effective potential contains a finite number of renormalization parameters $\alpha_j^\pm$. Their values are not computable from our effective theory. Rather, we shall fix them by requiring some renormalization conditions, which determine the values for the moduli as well. Since the moduli must be stabilized, we demand

$$\partial_{r_+} V(r_\pm) = \partial_{r_-} V(r_\pm) = 0\ , \tag{7.69}$$

and in order to match the observed value of the effective four dimensional cosmological constant, we shall impose

$$V(r_\pm)|_{\min} \simeq 10^{-122} m_P^4. \tag{7.70}$$

We are interested in the limit when the size of Σ is everywhere smaller than the orbifold size, $r_+ \lesssim 1$ and $r_- \ll 1$. One can show[10] that in this limit the non-local term $v(r_\pm)$ is exponentially suppressed, and we can approximate the potential by the 'local' terms $V_\pm(r_\pm)$. Moreover, since we consider only the positive powers of $r_\pm$ in $V_\pm$, the potential at the minimum is dominated by r_+. Then, conditions (7.69) and (7.70) reduce to

$$V'_+(r_+) = V'_-(r_-) = 0, \qquad \text{and} \qquad R^{-4} V_+(r_+)|_{\min} \simeq 10^{-122} m_P^4. \tag{7.72}$$

To investigate whether this potential can stabilize the moduli, we consider separately the cases with flat and curved Σ.

Flat Σ

This case corresponds to a toroidal compactification of a $4 + D_2 + 1$ dimensional RS model (with two codimension one branes). In this case, all the divergences have the same form, because all geometric invariants are constant and thus proportional to the brane tensions. Thus, there will appear a logarithmic term in the $(4 + D_2)$-th power of $r_\pm$. As can also be derived from Eqns. (7.66), (7.60) and (7.61), setting $C_j = 0$ for all $j \neq 0$, and $g_0 = 1$, there is another logarithmic term corresponding to $j = 4$.

Thus, the expression for the potential reduces to

$$\begin{aligned} V_\pm(r_\pm) \;\approx\; & \Big\{ \mp \gamma_1 r_\pm + \gamma_2 r_\pm^2 \mp \gamma_3 r_\pm^3 + \left(\gamma_4 - d_4 \ln r_\pm^2\right) r_\pm^4 + \ldots \\ & + \; (\mp 1)^{4+D_2} \beta_{4+D_2} r_\pm^{4+D_2} \ln r_\pm^2 - \alpha^\pm_{4+D_2} r_\pm^{4+D_2} + \ldots \Big\}. \end{aligned} \tag{7.73}$$

To illustrate better how the stabilization mechanism works in these cases, we shall discuss in more detail the six dimensional example with $\Sigma = S^1$.

[10] For instance, consider a the six dimensional example, with $\Sigma = S^1$. As described in more detail in Subsect. 7.4, the generalized zeta function is related to the Riemann zeta function. In this case the we can easily work out the asymptotic behaviour of the nonlocal contribution due to the mixed KK states $\mathcal{V}_l(a, R_-)$ defined in (7.62). If we keep the first term in the asymptotic expansion of the Bessel functions (7.51)

$$\begin{aligned} \mathcal{V}_l(a, R_-) \;&\sim\; \int_1^\infty dz (z^3 - z) \ln\left(1 - e^{-2(1-a) y_l z}\right) \\ &= -\frac{1}{8} \frac{1}{y_l^4 (1-a)^4} \left\{ 4\,(1-a)^2 y_l^2 \mathrm{Li}_3\left(e^{-2(1-a)y_l}\right) + 6\,(1-a) y_l\, \mathrm{Li}_4\left(e^{-2(1-a)y_l}\right) + 3\, \mathrm{Li}_5\left(e^{-2(1-a)y_l}\right) \right\}. \end{aligned}$$

Taking only the first term in the series of the poly-logarithms for small arguments $\mathrm{Li}_n(z) \approx z$, recalling that $y_l = \hat\lambda_l / r_-$ with $\hat\lambda_l = l$, $g_l = 2$ and summing over $l = 1, 2, \ldots$, we find to leading order

$$\sum_{l=1}^\infty g_l \hat\lambda_l^4 \mathcal{V}_l(a, R_-) \sim -\frac{1}{(1-a)^2} r_-^2 e^{-2(1-a)/r_-}. \tag{7.71}$$

Thus, this contribution is safely negligible in the limit of small internal space size $r_- \ll 1$.

The Laplacian (7.20) on this flat manifold is $P_\Sigma = \partial_X^2/R^2$, and its generalized zeta function (7.41,C.2) is related to the Riemann zeta function through

$$\zeta(s|\partial_\theta^2) = 2\zeta_R(2s).$$

The pole structure of $\zeta_R(2s)$ is easily found and one immediately identifies

$$\beta_j = \begin{cases} -4d_5/3 & \text{for } j=5, \\ 0 & \text{otherwise,} \end{cases} \tag{7.74}$$

and

$$\gamma_j = \begin{cases} -2/945 & \text{for } j=-1, \\ 3\zeta_R'(-4) & \text{for } j=0, \\ 4\zeta_R'(-2)d_2 & \text{for } j=2, \\ -d_4 & \text{for } j=4, \\ -\dfrac{8d_j\zeta_R(j-4)}{(j-4)(j-2)} & \text{otherwise.} \end{cases} \tag{7.75}$$

From (7.73), the potential is of the form

$$V_\pm(r_\pm) \approx \left\{ \mp\gamma_1 r_\pm + \gamma_2 r_\pm^2 \mp \gamma_3 r_\pm^3 + \left(\gamma_4 - d_4 \ln r_\pm^2\right) r_\pm^4 \mp \beta_5 r_\pm^5 \ln r_\pm^2 - \alpha_5^\pm r_\pm^5 + \dots \right\}. \tag{7.76}$$

As we mentioned above, the renormalization constants $\alpha_5^\pm$ arise from a finite renormalization $\delta\tau_\pm$ of the brane tensions,

$$\delta\tau_\pm \int d^5x \sqrt{g_{(5)\pm}} = \frac{2\pi}{R^4} \int d^4x\, \delta\tau_\pm R_\pm^5,$$

so that $\alpha_5^\pm = 2\pi\delta\tau_\pm/k^5$. The size of $\delta\tau_\pm$ is expected to be set by the SUSY breaking scale $\eta_{\rm SUSY}$ so that $\alpha_5^\pm$ are large in principle. Then, the main contributions to this potential arise from the fifth and the first powers. The extremum condition for the r_- modulus can be well approximated by

$$\delta\tau_- \simeq \frac{\gamma_1}{10\pi}\frac{1}{r_-^4}k^5.$$

Setting the natural value $\delta\tau_- \sim \eta_{\rm SUSY}^5$, we obtain

$$r_- \sim \left(\frac{M}{\eta_{\rm SUSY}}\right)^{1/2} \frac{k}{M},$$

so that indeed R_- is stabilized just above the fundamental scale $1/M$ without fine tuning.

As for r_+, we have two conditions for just one variable, $\delta\tau_+$. The idea is to use the renormalization constant $\delta\tau_+$ in order to satisfy $V_+|_{\rm min} \simeq 0$, and then using this value in $V_+' = 0$, the r_+ is determined. In order to be consistent, we should obtain $r_+ \lesssim 1$. In such a case, we can foresee from Eq. (7.76) that if α_5^+ has to compensate for the potential at the minimum, it has to be of order one. But this means that $\delta\tau_+$ is fine tuned to a value $\sim k^5$ instead of $\eta_{\rm SUSY}^5$.

Imposing explicitly these conditions, we obtain

$$\delta\tau_+ \sim \frac{\gamma_1}{6\pi}\frac{1}{r_+^4}k^5 \sim k^5,$$

and

$$r_+ \simeq \frac{4\gamma_1}{3\gamma_2} \sim 1.$$

We can easily check that for a twisted field this ratio is $\simeq 0.6$, in agreement with the assumption we made above. For the untwisted case, this ratio depends on the boundary and bulk masses, so it can be made small generically. In conclusion, besides the fine tuning needed in order to match the four dimensional cosmological constant, no tuning is needed for the Planck/EW hierarchy in this case.

A simple computation gives the mass that this potential induces for the canonical moduli $\widehat{\psi}$ and $\widehat{\varphi}$ of Section 7.1

$$m_\psi^2 \simeq -\frac{\gamma_1}{24\pi^2}\frac{k}{R^3 m_P^2} \sim -\gamma_1 (hk)^2 \sim (1/mm)^2 \tag{7.77}$$

$$m_\varphi^2 \simeq -\frac{\gamma_1}{192\pi^2}\, a^{-2}\, \frac{k}{R^3 m_P^2} \sim -\gamma_1 (a\text{TeV})^2, \tag{7.78}$$

where we used $1/R \sim k \sim$ TeV and Eq. (7.34). The mass of $\widehat{\psi}$ is of the order of the inverse millimeter, which is large enough in order not to cause deviations from Newton's law at short distances. Since, as shown in Section 7.1the coupling of $\widehat{\psi}$ to matter is suppressed by a Planckian factor, its effects in accelerators are negligible as well. On the other hand, the mass for the modulus φ is of 10 KeV size. From (7.15), its coupling to matter is suppressed as $1/(10^4\text{TeV})$.

Let us briefly discuss the stabilization when we consider a higher dimensional flat Σ. As mentioned above, with more flat dimensions, the renormalization constants related to the brane tensions $\alpha^\pm_{D-1}$ appear with higher powers of $r_\pm$. The only change with respect to the case above is that the condition $V'_- = 0$ now reads

$$\delta\tau_- \sim \frac{1}{r_-^{D-2}} k^{D-1},$$

and assuming a natural value for $\delta\tau_-$ given by η_{SUSY}^{D-1}, we obtain again $r_- \sim (k/M)(M/\eta_{\text{SUSY}})^{1/2}$. Thus, for any dimension D the modulus R_- is stabilized without fine tuning near $1/M$.

As for the modulus R_+, we expect the potential (7.73) to stabilize it near k once the fine tuning of $\delta\tau_+ \sim k^{D-1}$ needed for the cosmological constant is performed.

We can compute the masses for the moduli for an arbitrary number of flat internal dimensions. We find that the mass for the ψ is always millimetric, whereas $m_\varphi \sim a$TeV increases with D_2, ranging from 10 KeV for $D_2 = 1$ to 100MeV for $D_2 = 6$. The coupling of φ to matter, of strength (see Eq. (7.15))

$$1/\left(h^{-1/(D-2)}\ \text{TeV}\right)\ ,$$

is comprised between $\sim 1/(10^4\text{TeV})$ for $D_2 = 1$ and $\sim 1/(100\text{TeV})$ for large D_2. This guarantees that it hasn't been produced at colliders, or has any effect in star cooling.

Curved Σ

When Σ is not flat, besides the divergences proportional to brane tensions terms (giving rise to the power $r_\pm^{D-1}$ in $V_\pm$), the potential has more divergences. For instance, there can appear divergences proportional to curvature terms, which give rise to the powers $r_\pm^{D-3}$. Accordingly, terms with fewer powers of $r_\pm$ are due to higher powers of the curvature, and in general the effective potential takes the form

$$V_\pm(r_\pm) = \sum_{j=1}^{\infty} (\mp 1)^j \left\{ \gamma_j r_\pm^j + \left(\beta_j r_\pm^j - g_0 d_4 \delta_{j,4} \right) \ln r_\pm^2 \right\} - \sum_{j=1}^{D-1} \alpha_j^\pm r_\pm^j. \tag{7.79}$$

As in the previous case for the brane tensions, the size of the renormalization constants in front of these operators are expected to be of order the cutoff scale M (or $\eta_{\rm SUSY}$). Finite renormalization terms of boundary operators behave as,

$$M^j \int d^{D-1}x \sqrt{g_{(D-1)\pm}}\, \mathcal{R}^{(D-1-j)/2} = \frac{1}{R^4} \int d^4x\, (MR_\pm)^j = \frac{1}{R^4} \int d^4x\, \alpha_j^\pm r_\pm^j,$$

and we conclude that the dimensionless renormalization constants in (7.79) are large, $\alpha_j^\pm \sim (M/k)^j \gg 1$. Thus, these terms are a series in $MR_\pm > 1$ rather than in $kR_\pm < 1$, the dominant terms being with the highest powers, that is, the brane tension and the curvature terms. As a first approximation, we can neglect the remaining terms, and minimum condition for R_- is reached naturally for $R_- \sim 1/M$, which is what we need (see Fig. 7.2).

However, we see that in order to obtain $R_+ \sim 1/k$, we need to tune the ratio of α_{D-1} and α_{D-3}. Besides, the tuning corresponding to the cosmological constant is still needed.

In principle, we could consider the case when the heat kernel coefficient $C_1(P_\Sigma)$ is zero, which can happen for some value of the non-minimal coupling ξ. We see from (7.60) that in this case there is no divergence in the potential corresponding to the curvature terms.[11] Then, assuming that the next nonzero coefficient is C_2, the two powers that dominate the potential are $(MR_\pm)^{D-1}$ and $(MR_\pm)^{D-5}$. However, in order to stabilize R_+ near $1/k$, again we have to do one fine tuning. We can say that in general, the presence of any other divergence, besides the brane tension, spoils the efficiency of the potential in stabilizing the moduli at well separated scales.

We conclude that, for curved Σ the potential can naturally stabilize the moduli but without a large hierarchy.

7.5 Dimensional reduction and the 5D scalar-tensor model

In Section 7.2, we have argued that there exists a range of energies where the theory is effectively 5 dimensional, as illustrated in Fig. 7.2. In this Section we show the dimensional reduction procedure from $D = 5 + D_2$ dimensions down to 5 dimensions, which allows contact with the language of the last Chapter [15]. The reduction from the higher dimensional theory (7.2) to 5 dimensions is performed by the compactification on the internal manifold Σ. This amounts to keeping only the Σ-zero modes of the fields defined in D dimensions.

Recall that we denote collectively the four dimensional Minkowski coordinates x^μ and the orbifold x^5 by x^α. For the sake of simplicity, we shall consider only the breathing mode of Σ in the internal components of the metric. As well, we shall freeze the $\{\alpha, i\}$ components (the graviphotons) to zero. Thus, the ansatz for the metric that we shall adopt depends on the

[11] The same thing cannot happen for the brane tension terms, since the corresponding coefficient is $C_0(P_\Sigma) = 1$ always.

internal coordinates X^i only through the background geometry on Σ, and on x^α through the five dimensional graviton $g^{(5)}_{\alpha\beta}$, and a dilaton σ,

$$ds^2 = g^{(5)}_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta + R^2 e^{2\sigma(x^\gamma)}\gamma_{ij}dX^i dX^j \ . \tag{7.80}$$

As for the sigma model scalars, we shall also freeze them to their value in the background, $\phi^a = \phi^a(X^i)$.

The action (7.2) corresponding to this ansatz is

$$S_5 = -v_\Sigma R^{D_2} \Bigg[\int d^5x\sqrt{g_{(5)}}e^{D_2\sigma}\left\{M^{D-2}\left(\mathcal{R}_{(5)} - D_2(D_2-1)(\partial\sigma)^2_{(5)}\right) + \Lambda\right\} + \int d^4x\sqrt{g_{(5)+}}e^{D_2\sigma}\,\tau_+ + \int d^4x\sqrt{g_{(5)-}}e^{D_2\sigma}\,\tau_-\Bigg], \tag{7.81}$$

where $g^{(5)\pm}_{\mu\nu}$ denote the metrics on the branes induced by $g^{(5)}_{\alpha\beta}$ and we have performed the X integration. We can rewrite this action in the (5 dimensional) Einstein frame, given by $g^E_{\alpha\beta} = e^{2D_2\sigma/3}g^{(5)}_{\alpha\beta}$,

$$S_5 = -\ M_5^3\int d^5x\sqrt{g_E}\left\{\mathcal{R}_E + \frac{1}{2}(\partial\phi)^2_E + \Lambda_5 e^{c\phi}\right\} \tag{7.82}$$

$$-\ \int d^4x\sqrt{g_{E+}}\ \tau_{5+}\ e^{c\phi/2} - \int d^4x\sqrt{g_{E-}}\ \tau_{5-}\ e^{c\phi/2} \tag{7.83}$$

where

$$c^2 = \frac{2}{3}\frac{D_2}{D_2+3},$$

the canonical scalar field is $\phi = -(2D_2/3c)\sigma$, $g^{E\pm}_{\mu\nu}$ are the metrics on the branes induced by $g^E_{\alpha\beta}$, the 5 dimensional Planck mass is given by $M_5^3 = v_\Sigma R^{D_2}M^{D-2}$, $\Lambda_5 = M^{2-D}\Lambda$ and $\tau_{5\pm} = v_\Sigma R^{D_2}\tau_\pm$.

The action (7.82) coincides with the 5 dimensional scalar-tensor model considered in Chapter 6 [15]. It was found there that this model has a solution with a power-law warp factor of the form

$$\begin{aligned} ds^2_E &= a^2_E(z)\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right), \\ \phi_0(z) &= -\sqrt{6\beta(\beta+1)}\,\ln(z/z_0) \qquad \text{with} \qquad \mathrm{a_E(z) = (z/z_0)^\beta} \end{aligned} \tag{7.84}$$

with $\beta = 2/(3c^2-2) = -(D_2+3)/3$.[12]

The brane operators induced by quantum effects on this background are given by positive powers of the extrinsic curvature scale (see *e.g.* [15]) $\mathcal{K}_{E\pm} = \beta/z_\pm a_{E\pm} = \beta z_\pm^{-(\beta+1)}$,

$$\int d^4x\sqrt{g_{E\pm}}\mathcal{K}^n_{E\pm} = \int d^4x\left(\frac{z_\pm}{z_0}\right)^{(4-n)\beta}\frac{1}{z^n_\pm} \propto \int d^4x\ r_\pm^{4+(4/3)D_2-(n/3)D_2}, \tag{7.85}$$

[12] In terms of the proper coordinate (in the 5 dimensional Einstein frame) $y_E \propto z^{\beta+1}$, $a_E(y_E) = (y_E/y_0)^q$ with $q = 2/3c^2 = (D_2+3)/D_2$.

where $n = 1, 2, 3\ldots$, we used that the conformal coordinate $z = e^{ky}$ is the same in 5 and in D dimensions, and $a_\pm = 1/kz_\pm$. On the other hand, we have seen in Section 7.3 that the operators generated by the effective potential due to bulk fields in the model (7.2) are of the form

$$\int d^4x\sqrt{g_\pm}\mathcal{R}_\pm^N \propto \int d^4x r_\pm^{4+D_2-2N} \tag{7.86}$$

where $\mathcal{R}_\pm$ is the intrinsic curvature computed with the induced metrics on each brane, $g^\pm_{\mu\nu}$. Here $N = 0, 1, \ldots, [D/2]$, and [ ] denotes the integer part. Now we can identify that these operators correspond to a number of powers of the extrinsic curvature operator (7.85) given by

$$n = \frac{6}{D_2}\, N + 1.$$

We note that all the induced operators can be cast as powers of the extrinsic curvature for $D_2 = 1, 2, 3$ and 6 only, having in the $D_2 = 6$ case a one-to-one correspondence. For any other value of D_2, there exist higher dimensional local operators that are not simply powers of $\mathcal{K}_{E}\pm$, but of some power of e^ϕ in the 5 dimensional effective theory (7.82).

As well, last Chapter [15] raised the question that the path integral measure of a bulk scalar field in the effective five dimensional theory (7.82) quantized on the warped vacuum configuration (7.84) is ambiguously defined. The nontrivial profile of the scalar ϕ permits to define many different conformal frames, all of them equivalent at the classical level. However, the path integral measure can be defined covariantly with respect to any of them. It turns out that the term proportional to

$$\frac{\ln z_+}{z_+^4} + \frac{\ln z_-}{z_-^4}$$

in the potential depends on this choice. Several arguments can be given in favor of possible 'preferred' frames. For instance, with a measure covariant with respect to the 5 dimensional Einstein frame metric $g^{E}_{\alpha\beta}$, this term is present. But if one chooses covariance with respect to $g^{(5)}_{\alpha\beta}$, there is no such term. However, in the model presented here, there is no ambiguity in the choice of the measure since in the D dimensional theory there is no scalar with nontrivial profile along the orbifold. In the computation presented here, the choice of the measure shows up (see Chapter 6 [15]) when we subtract the divergences Eq. (7.65), covariant precisely with respect to the higher dimensional Einstein frame metric $g^{(D)}_{MN}$. As a result, when we take into account both the 5 dimensional modes (the Σ KK zero mode) together with the D dimensional ones (the KK modes excited along the Σ as well), we have found that there is a remaining contribution of this form, see Eqns. (7.66,7.60,7.61). Anyhow, it should be noted that these Coleman-Weinberg-like terms do not play a very relevant role in stabilizing of the moduli.

7.6 Discussion

In this Chapter, we have investigated the role of quantum effects arising from bulk fields in higher dimensional brane models. Specifically, we have considered a class of warped brane models whose topology is $M_4 \times \Sigma \times S^1/Z_2$, where Σ is a D_2 dimensional one-parameter compact manifold, M_4 is the four dimensional Minkowski space and both M_4 and Σ directions are warped as in the Randall Sundrum model, with two branes of codimension one sitting at the orbifold fixed points. Aside from the usual negative cosmological constant, a bulk sigma model scalar field theory is used as the source of gravity in the cases of a curved internal manifold Σ. We have identified the relevant moduli fields characterizing the background, and found the classical action in the moduli approximation (as well as the coupling of the moduli to matter fields sitting on the branes).

We have computed the contribution to the one-loop effective action from generic bulk scalar fields at lowest order (i.e. the Casimir energy). The computation, similar to the one for the RS model, is technically more complex since there are KK modes propagating along Σ, resulting in a dependence of the Kaluza-Klein masses on the eigenvalues of the Klein-Gordon operator on Σ. However, for the specific choice of space-time we made, where the warp factors for Σ and for the Minkowski factor M_4 are the same, the physical KK masses split as in the usual factorisable geometries. Using the Mittag-Leffler expansion for the generalized $\zeta-$function we were able to express the Casimir energy in terms of heat-kernel coefficients of the internal space Σ, so that the presence of each divergence is dictated by a certain heat-kernel coefficient. This simplifies the renormalization of the result. An interesting nontrivial check of our result is the fact that the RS divergence (which is lower dimensional in this model and which appears as the contribution of the Σ zero mode) cancels out in the final result, as it should, once all contributions are added. We renormalized the effective potential by subtracting suitable counter-terms proportional to a number of boundary or bulk local operators. Since we work in dimensional regularization, the subtraction is performed in the regularized space, with $(4-2\epsilon)+D_2+1$ dimensions. As a result, there is a mismatch in the powers of the moduli appearing in the divergent terms, and a number of logarithmic terms (in the moduli) appear in the renormalized expression for the effective potential.

As an application, we proposed a scenario where SUSY is broken at a scale just below the fundamental cutoff M. This makes the curvature scale of the background to be a few orders of magnitude below M. As a result, a large hierarchy is generated by a combination of redshift [2] and a large volume effects [1]. The key point for the latter to be efficient (in spite of having codimension one branes) is that the size of the internal manifold Σ (present in the bulk and on the brane) grows as one moves away from the TeV brane, where matter lives. Therefore, this behaves effectively as a brane with a *small* Σ extra space, attached to which there is a *large* Σ space where only gravity propagates.

As for the stabilization, we find that, generically, the potential induced by bulk fields can generate sizeable masses for the moduli compatible with a large hierarchy with no need of fine tuning if Σ is flat. If it is curved, the effective potential can naturally stabilize the moduli but without a large hierarchy.

In the model we have considered, the size of the internal space Σ is everywhere small compared with the size of the orbifold. Therefore, there is a range of energy scales where the model is effectively five dimensional (this feature is common to the Hořava-Witten model [9, 10]). From the five-dimensional point of view, the model contains a dilaton field in the bulk, which causes a power-law warp factor in the Einstein frame, $a(y) \propto y^q$, where y is the proper distance along the extra dimension. The power q is related to the number of additional dimensions through $q = (D_2+3)/D_2$ [15], which leads to $1 < q \leq 4$. Five dimensional models with power-law warp factors were investigated in Chapter 6 [15], where it was argued that the counterterms at the orbifold fixed points can naturally stabilize the moduli corresponding to the positions of the branes. However, a large hierarchy isnot expected unless the power is substantially large, $q \gtrsim 10$. This conclusion is consistent with the results of the present Chapter, which correspond to relatively small q. In this case, the large hierarchy can only be stabilized naturally if the internal space is flat. This case is special because the only possible counterterms are renormalizations of the higher dimensional brane tensions.

The above arguments suggest that a large hierarchy may be obtained by considering more general warped models, where a larger power exponent q is obtained after reducing to five dimensions. In such cases, the stabilization of a hierarchy without fine tuning is expected even if the internal manifold Σ is curved and all sorts of counterterms are present.

Chapter 8

Quantum self-consistency of $\mathrm{AdS} \times \Sigma$ brane models

Many generalizations of the Randall-Sundrum model fall in the quite general class of higher dimensional warped solutions studied in [209], where a $D-$dimensional system of gravity plus Yang-Mills is considered. The base spacetime is described by the following line element:

$$ds^2 = e^{2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2\rho(y)}R^2\gamma_{ij}dX^i dX^j + dy^2, \tag{8.1}$$

where the coordinates x^μ parametrize D_1 dimensional Minkowski space M_4, the coordinates X^i cover a D_2-dimensional compact internal manifold Σ of radius R and the coordinate $y \in [-\pi r, \pi r]$ parametrizes the orbifold. We define $D = D_1 + D_2 + 1$ and take $D_1 = 4$.

Depending on the geometry of the internal space, Einstein equations lead to different types of warp factors $\sigma(y)$ and $\rho(y)$: when the internal space is a Ricci flat manifold and the Yang-Mills flux is switched off, the general result for the warp factors is given by a combination of exponentials. Simpler solutions with

$$\sigma(y) = \rho(y) = -k|y|, \tag{8.2}$$

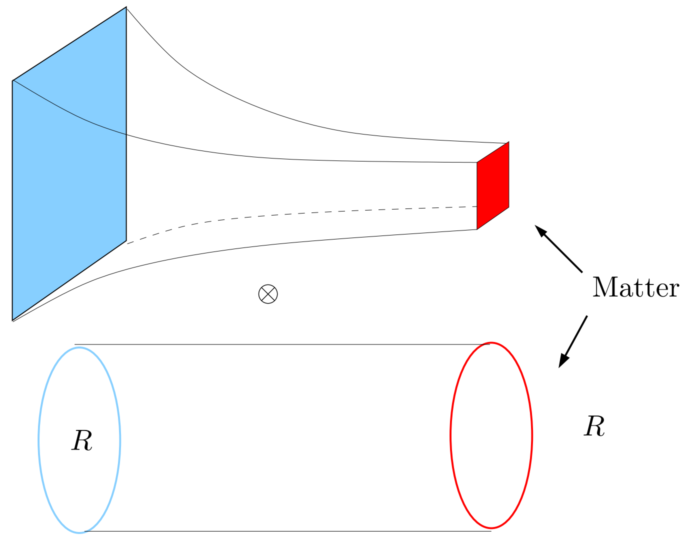
are found when the bulk cosmological constant is taken to be negative. Additionally, the condition of Ricci flatness of Σ can be relaxed at the price of introducing some extra bulk matter, like, for instance, a scalar field with hedgehog configuration [207, 208]. In the above cited papers, the set-up allows for the presence of one brane only; two brane models can be constructed by gluing two slices of the previous spacetime and imposing the Z_2-identification.

Solutions of the type $AdS_{D_1+1} \times \Sigma$ are, instead, found when the internal space is non Ricci flat without adding any extra bulk matter. In such case the warp factor along Σ is constant and we can take

$$\rho(y) = 0, \sigma(y) = -k|y|. \tag{8.3}$$

We note that in such case the requirement of a negative higher dimensional cosmological constant can be relaxed.

In Chapter 7 [16], we have considered scenarios of the type (6.10), (8.2) and evaluated the effective potential arising from bulk fields. Here, we extend the previous results to the second type of spacetimes (6.10), (8.3), illustrated in Fig. 8.



In contrast with the previous Chapter, there is only one modulus here, since the radius of Σ is fixed by Einstein's equations. From Eq. (8.1), it is clear that this line element corresponds to the *direct* product space $\mathrm{AdS}_5 \times \Sigma$. This model represents a straightforward generalization of the RS model, where we add some compact space Σ at each point of the RS construction. It might be expected that upon compactification on Σ, only the the zero mode in the KK decomposition on Σ is relevant. Thus, for small R, the RS model is recovered. We shall see that this the case for the effective potential. The contribution from the Σ KK modes is negligible and the potential has the same form as for in RS model (5.6). Furthermore, the contribution from the Σ zero mode of the bulk YM gauge fields is exactly the same as in the RS model, hence it can stabilize a large hierarchy without fine tuning, giving the radion a sizable mass.

8.1 The Model

In this Section we describe the background solution and and discuss the relevant scales of the problem. As mentioned above, the line element (8.1) with $\sigma(y) = -k|y|$ corresponds to a bulk spacetime of the form of the direct product of 5D Anti de Sitter space AdS_5 and a compact manifold Σ, which we will assume to be one-parameter.

Randjbar-Daemi and Shaposhnikov have considered this type of solutions and showed that they arise from a system of gravity plus Yang-Mills fields [207, 209], with bulk action given by

$$S_{BG} = \int d^D x \sqrt{g} \left\{ M^{D-2}\mathcal{R} - \Lambda - \frac{1}{4g_*^2}\mathcal{F}_{IJ}\mathcal{F}^{IJ} \right\} . \tag{8.4}$$

The equations of motion can be obtained in the standard way, and once the ansatz for the metric tensor (8.1), (8.3) is used, the following independent equations are obtained:

$$k^2 = -\frac{M^{2-D}\Lambda}{D_1(D-2)} + \frac{M^{2-D}}{D_1(D-2)}\frac{F^2}{4g_*^2 R^4} , \tag{8.5}$$

$$\frac{\Omega}{R^2} = \frac{M^{2-D}}{D-2}\left\{\Lambda + \frac{2D-D_2-4}{D_2}\frac{F^2}{4g_*^2R^4}\right\} , \tag{8.6}$$

where we have expressed the curvature $\mathcal{R}_\Sigma$ of the internal manifold in terms of its radius R (Ω is a constant):

$$\mathcal{R}_\Sigma = \frac{D_2\Omega}{R^2} ,$$

and

$$\frac{F^2}{R^4} = g^{IM}g^{JN}\mathcal{F}_{IJ}\mathcal{F}_{MN} .$$

The previous equations (8.5), (8.6) allow us to determine the radius of the internal manifold and the Yang-Mills flux in terms of Λ, M and k:

$$R^2 = \Omega D_2\mathcal{P}^2 , \tag{8.7}$$

$$\frac{F^2}{4g_*^2} = D_2^2\Omega^2\mathcal{P}^4(\Lambda + D_1(D-2)k^2M^{D-2}) , \tag{8.8}$$

where, for notational convenience, we have defined

$$\mathcal{P}^{-2} = 2M^{2-D}\Lambda + D_1(2D-D_2-4)(D-2)k^2.$$

One immediately notice that the radius R of Σ is fixed at classical level at the price of tuning the Yang-Mills flux according to (8.8). Furthermore, it can be easily shown that R is stable around this value (see *e.g.* [218]). In this sense we point out some analogy with the recent works [219, 220], concerning the direct product of Minkowski space and a 2-sphere. In their case the radius of the 2-sphere is stabilized by the flux and a relaxing the tuning of such flux would induce a de Sitter or anti-de Sitter geometry rather than Minkowski. The same is also true in our case with the additional modification of the warp factor.

Since we are considering two branes embedded in such a spacetime, we have to add to the action appropriate brane tension terms. It is easy to see that there are no solutions of the type considered here, if the brane action contains only an isotropic tension term, and the requirement of conservation of the higher dimensional energy-momentum tensor along with the junction conditions forces us to introduce such anisotropy[1] [222, 223, 221].

The brane energy-momentum tensor is then given by:

$$\begin{aligned} T_\mu^\nu &= \delta(y)\ \mathrm{diag}\left(\tau_-^{\mathcal{M}}\delta_\mu^\nu\ ,\ \ \tau_-^\Sigma\delta_i^j\right) + \\ &+ \ \delta(y-\pi r)\ \mathrm{diag}\left(\tau_+^{\mathcal{M}}\delta_\mu^\nu\ ,\ \ \tau_+^\Sigma\delta_i^j\right) . \end{aligned} \tag{8.9}$$

The spacetime we are considering can then be constructed by gluing two copies of a slice of the bulk space and imposing the Z_2-identification. The Israel junction conditions fix the brane tensions to be

$$\tau_\pm^{\mathcal{M}} = \frac{D_1-1}{D_1}\tau_\pm^\Sigma = \frac{D_1-1}{D_1}\left(\mp 4D_1kM^{D-2}\right) . \tag{8.10}$$

We can now look at the physical scales to see whether such class of models suggests anything about the gauge hierarchy problem.

[1] The source for such anisotropy can be due to different contributions to the vacuum energy or also due to a background three-form field [221].

By integrating out the extra dimensions we can write a relation between the four- and higher-dimensional Planck scales

$$m_P^2 = \frac{v_\Sigma}{D_1-2}(MR)^{D_2}\frac{M}{k}M^{D_1-2}\ . \tag{8.11}$$

Then, the EW/Planck hierarchy can then be written, for $D_1 = 4$, as

$$h^2 \equiv a^2\frac{M^2}{m_P^2} \sim \frac{a^2}{(RM)^{D_2}}\frac{k}{M} \sim 10^{-32}\ . \tag{8.12}$$

We see that, analogously to [16], the hierarchy h is expressed in terms of a and R, however in the present case it is not possible to use both the redshift and large volume effects as in our previous work [16]. To see this, we remind that in the case of equal warpings the crucial ingredient was that the internal manifold was growing exponentially away from the negative tension brane located at $y = y_-$ and this was diluting gravity as in models with large extra dimension. On the other hand, gauge interactions, confined on the negative tension brane, were not diluted because the size of Σ at $y = y_-$ was of order of the fundamental cut-off.

Here the situation is different as we are considering the direct product AdS $\times\,\Sigma$. In such case, the size, R, of the internal manifold has to be everywhere small, if we require that the extra $\Sigma-$dimensions remain invisible to ordinary matter, confined on the wall. Since R is determined at classical level, Einstein equations leave us with a first 'consistency' check on such class of models if we were going to construct any (pseudo-)realistic scenario.

If we express the cosmological constant by factoring out two powers of the mass,

$$\Lambda \sim \lambda^2 M^{D-2}\ , \tag{8.13}$$

relations (8.7), (8.8) can be recast in the following form:

$$\frac{F^2}{4g_*^2} \sim \frac{M^{D-2}}{(k^2+\lambda^2)} \tag{8.14}$$

$$R^2 \sim \frac{1}{k^2+\lambda^2}\ . \tag{8.15}$$

Now, a natural assumption is that the bulk cosmological constant is of the same order as the higher dimensional Planck scale, $\lambda \sim M$, and k smaller than M, implying

$$R \sim M^{-1}\ , \tag{8.16}$$

meaning that the size of the internal manifold is of order of the cut-off and thus satisfying the requirement of small R. The previous relation also implies that

$$(kR)^2 \sim \frac{k^2}{\lambda^2+k^2} << 1\ . \tag{8.17}$$

This last condition will be tacitly used in the subsequent computation of the effective potential.

From the gauge hierarchy point of view, this class of models does not suggest any improvement with respect to the RS model. As one can see from (8.12) and (8.16), the hierarchy is resolved only through redshift effects. Obviously, one could relax the condition $\lambda \sim M$, but this in turn would interchange the gauge hierarchy problem with the need for an 'ad hoc' tuning of the bulk cosmological constant, as we would have to justify a value of λ different from its natural value M.

8.2 KK reduction

As we pointed out in our previous Chapter, quantum effects in scenarios with more than one extra dimension can be qualitatively different from models without internal spaces and can, in principle, provide new ways of addressing the hierarchy. It then seems reasonable to ask the same question in relation to the class of models described previously.

Therefore we devote this Section to the computation of the one-loop effective potential arising from a massive bulk scalar field $\Phi(x, X, y)$ coupled non-minimally to the higher dimensional curvature. We also point out that, as noted in [8, 105], it is possible to relate the effective potential from a bulk scalar with the one arising from a gauge field, the computation being virtually the same. It is possible to do so by appropriately fixing the non-minimal coupling and the bulk mass of the scalar field to (we take $D_1 = 4$)

$$\xi = 1/8 \,, \tag{8.18}$$

$$m^2 = -k^2/2 \,. \tag{8.19}$$

The field equation for $\Phi(x, X, y)$ is given by the Klein-Gordon equation

$$\left[-\Box_D + m^2 + \xi\mathcal{R}\right]\Phi = 0 \,, \tag{8.20}$$

where $\mathcal{R}$ is the higher dimensional curvature and $\Box_D$ the D'Alembertian, both computed from the metric (8.1), (8.3).

Standard Kaluza-Klein theory tells us that such a higher dimensional field can be expressed in terms of a complete set of modes, which describe a tower of fields with masses quantized according to some eigenvalue problem. Such a decomposition is, of course, arbitrary, however a convenient choice is

$$\Phi(x, X, y) = \sum_{l,n} Y_l(X)\Phi_{l,n}(x) f_{l,n}(y) \,, \tag{8.21}$$

where again $Y_l(X)$ are the Σ spherical harmonics (*cf.* Eq. (7.20))

$$\begin{aligned} P_\Sigma Y_l(X) &= \frac{1}{R^2}\Big[-\Delta_{(\gamma)} + \xi\mathcal{R}_{(\gamma)}\Big]\, Y_l(X) \\ &= \frac{1}{R^2}\,\hat\lambda_l^2\, Y_l(X) \,, \end{aligned} \tag{8.22}$$

with eigenvalues $\hat\lambda_l^2$ (independent of R) and degeneracy g_l. If we now require $\Phi_{l,n}(x)$ to satisfy the Klein-Gordon equation in Minkowski spacetime, M_4, with masses $m_{l,n}^2$,

$$\left[-\Box + m_{l,n}^2\right]\Phi_{l,n}(x) = 0 \,, \tag{8.23}$$

equation (8.20) leaves us with a radial equation for the modes $f_{l,n}(y)$

$$\mathcal{D}_y f_{l,n} = m_{l,n}^2 f_{l,n} \,, \tag{8.24}$$

where the differential operator $\mathcal{D}_y$ is given by

$$\mathcal{D}_y = e^{2\sigma}\left[-e^{-D_1\sigma}\partial_y e^{D_1\sigma}\partial_y + \mu_l^2 - 2D_1\xi\sigma''\right] \,, \tag{8.25}$$

and

$$\mu_l^2 = m^2 + \frac{1}{R^2}\hat{\lambda}_l^2 - D_1(D_1+1)k^2\xi \ . \tag{8.26}$$

The most general solution to (8.24) can be written in terms of Bessel functions and by imposing the appropriate boundary conditions, we find that the eigenvalues m_n are determined by the transcendental equation:

$$F^{\beta}_{\nu_l}\left(\frac{m_{n,l}}{ka}\right) = 0 \ . \tag{8.27}$$

The function $F^{\beta}_{\nu_l}(z)$ is given by

$$F^{\beta}_{\nu_l}(z) = Y^{\beta}_{\nu_l}(az)J^{\beta}_{\nu_l}(z) - J^{\beta}_{\nu_l}(az)Y^{\beta}_{\nu_l}(z) \ , \tag{8.28}$$

where

$$\nu_l^2 = \frac{\mu_l^2}{k^2} + \frac{D_1^2}{4} \ , \tag{8.29}$$

and

$$J^{\beta}_{\nu_l}(z) = J_{\nu_l}(z) \tag{8.30}$$

for twisted field configurations ($f_{n,l}(-y) = -f_{n,l}(y)$) or

$$J^{\beta}_{\nu_l}(z) = j_{\nu_l}(z) = \frac{1}{2}D_1(1-4\xi)J_{\nu_l}(z) + zJ'_{\nu_l}(z) \ , \tag{8.31}$$

for untwisted ones ($f_{n,l}(-y) = f_{n,l}(y)$). Analogous expressions are valid also for $Y^{\beta}_{\nu_l}(z)$. In the following we focus on the case of untwisted fields only.

The one loop effective action S_{eff} can be expressed as the sum over the contributions of each mode [216]:

$$S_{\mathit{eff}} = -\int d^{4-2\epsilon}x \ V(s) \ , \tag{8.32}$$

with

$$V(s) = -\frac{\mu^{2\epsilon}}{2(4\pi)^2}\Gamma(s){\sum_{n,l}}' g_l m_{n,l}^{-2s} \ , \tag{8.33}$$

where the prime in the sum assumes that the zero mass mode is excluded and $s = -2+\epsilon$. We are using dimensional regularization and continuing along Minkowski spacetime ($4 \to 4-2\epsilon$) and μ is a renormalization scale introduced for dimensional reasons.

In order to evaluate the sum in (8.33), we find convenient to separate the $\hat{\lambda}_0-$mode from the rest of the tower[2]:

$$V(s) = V_{RS}(s) + V_*(s) \ . \tag{8.34}$$

The first term corresponds to the usual Randall-Sundrum contribution:

$$V_{RS}(s) = -\frac{(ka)^4}{2(4\pi)^2}(ka/\mu)^{-2\epsilon}\Gamma(s){\sum_{n}}' g_0 x_{n,0}^{-2s} \ , \tag{8.35}$$

[2] This procedure is not essential, however, by performing such spitting, the RS contribution comes about explicitly. Moreover, the RS divergence has to cancel when the two contributions are summed and this provides a non-trivial check of the calculation

with $x_{n,l} = m_{n,l}/ka$. This term, present only when the eigenvalue $\hat{\lambda}_0 = 0$, has been evaluated in [7, 12] and we report the result without further comments:

$$\begin{aligned} V_{RS} &= -g_0 \frac{k^4}{32\pi^2} (k/\mu)^{-2\epsilon} \Big\{ -d_4 \frac{1}{\epsilon} \left(1 + a^{4-2\epsilon}\right) \\ &\quad + c_1 + a^4 c_2 - 2a^4 \mathcal{V}(a) \Big\}, \end{aligned} \tag{8.36}$$

where (see Eqs. (5.3),(6.59))

$$\mathcal{V}(a) = \int_0^\infty dz z^3 \ln\left(1 - \frac{k_\nu(z)}{k_\nu(az)} \frac{i_\nu(az)}{i_\nu(z)}\right) \tag{8.37}$$

and the coefficients c_1 and c_2 do not depend on a. The remaining term in (8.34) is given by

$$V_*(s) = -\frac{(ka)^4}{2(4\pi)^2} (ka/\mu)^{-2\epsilon} \Gamma(s) \sum_{n,l=1}^{\infty} g_l x_{n,l}^{-2s} , \tag{8.38}$$

and can be handled in the usual manner by transforming it into a contour integral and by deforming the contour appropriately, according to a general technique developed in [217, 198] (See [197] for a comprehensive review). Standard manipulations lead to

$$\begin{aligned} V_*(s) &= -\frac{(ka)^4}{2(4\pi)^2} \frac{(ka/\mu)^{-2\epsilon}}{\Gamma(1-s)} \\ &\quad \sum_{l=1}^{\infty} g_l \int_0^\infty dz z^{-2s} \frac{d}{dz} \ln P_{\nu_l}(z) \end{aligned} \tag{8.39}$$

where

$$P_{\nu_l}(z) = F_{\nu_l}(iz) = i_{\nu_l}(az) k_{\nu_l}(z) - i_{\nu_l}(z) k_{\nu_l}(az) , \tag{8.40}$$

and

$$\begin{aligned} i_{\nu_l}(z) &= z I'_{\nu_l}(z) + \frac{1}{2} D_1 (1 - 4\xi) I_{\nu_l}(z) , \\ k_{\nu_l}(z) &= z K'_{\nu_l}(z) + \frac{1}{2} D_1 (1 - 4\xi) K_{\nu_l}(z) . \end{aligned}$$

Now we have to analytically continue the previous expression (8.39) to the left of $\Re(s) < 1/2$. A possible way of achieving this is to employ the uniform asymptotic expansion (UAE). This is because the order of the Bessel function depends explicitly on the eigenvalues $\hat{\lambda}_l$. In order to apply the UAE, we rescale the integral (8.39), $z \to \nu_l z$:

$$\begin{aligned} V_*(s) &= -\frac{(ka)^4}{2(4\pi)^2} \frac{(ka/\mu)^{-2\epsilon}}{\Gamma(1-s)} \\ &\quad \sum_{l=1}^{\infty} g_l \int_0^\infty d(\nu_l z) (\nu_l z)^{-2s} \frac{d}{d(\nu_l z)} \ln P_{\nu_l}(\nu_l z) , \end{aligned} \tag{8.41}$$

and to isolate the divergent part, we express the integrand as its large ν_l portion plus terms leading to finite contributions.

By using (D.3), (D.4), we can recast (8.41) as the sum of three terms:

$$V_*(s) = V_1 + V_2 + V_3 , \tag{8.42}$$

with

$$V_j = -\frac{(ka)^4}{2(4\pi)^2}\frac{(ka/\mu)^{-2\epsilon}}{\Gamma(1-s)}\sum_{l=1}^{\infty} g_l \nu_l^{2s} \int_0^\infty dz z^{-2s}\frac{d}{dz}\ln H_j(z) \ , \tag{8.43}$$

and

$$H_1(z) = (1+a^2z^2)^{1/4}e^{-\nu_l\eta(az)}(1+z^2)^{1/4}e^{\nu_l\eta(z)} \ ,$$

$$H_2(z) = \Sigma^{(I)}_{\nu_l}(z)\Sigma^{(K)}_{\nu_l}(az) \ ,$$

$$H_3(z) = 1-e^{2\nu_l(\eta(az)-\eta(z))}\frac{\Sigma^{(I)}_{\nu_l}(az)\Sigma^{(K)}_{\nu_l}(z)}{\Sigma^{(I)}_{\nu_l}(z)\Sigma^{(K)}_{\nu_l}(az)} \ ,$$

where $\eta(z)$ is defined in Appendix D. The first term is straightforward to evaluate and gives

$$V_1 = -\frac{(ka)^4}{8(4\pi)^2}(ka/\mu)^{-2\epsilon}\Big\{\Gamma(s)\zeta_\nu(s)(1+a^{2s}) - -\frac{1}{2\sqrt{\pi}}\Gamma(s-1/2)\zeta_\nu(s-1/2)(1-a^{2s})\Big\} \ . \tag{8.44}$$

The second one is slightly more involved to evaluate. The uniform asymptotic expansion (D.10), (D.11) allows us to write

$$V_2 = \frac{(ka)^4}{2(4\pi)^2}(ka/\mu)^{-2\epsilon}\left\{\sum_{n=1}^{\infty}\sum_{k=0}^{n}(1+(-1)^n a^{2s}) \sigma_{n,k}\frac{\Gamma(s+n/2+k)}{\Gamma(k+n/2)}\zeta_\nu(s+n/2)\right\} \ . \tag{8.45}$$

In order to deal with the sum over the eigenvalues ν_l, we have defined the following base $\zeta-$function:

$$\zeta_\nu(s) = \sum_{l=1}^{\infty} g_l\nu_l^{-2s} = \sum_{l=1}^{\infty} g_l\left(\frac{\hat\lambda_l^2}{(kR)^2}+\nu^2\right)^{-s} \ , \tag{8.46}$$

where

$$\nu^2 = \frac{m^2}{k^2} - D_1(1+D_1)\xi + \frac{D_1^2}{4} \ .$$

The last term in (8.42) is the usual non local contribution and, since it is finite by construction, we can safely put $\epsilon = 0$:

$$V_3 = \frac{(ka)^4}{(4\pi)^2}\sum_{l=1}^{\infty} g_l\mathcal{V}_l(a) \tag{8.47}$$

where

$$\mathcal{V}_l(a) = \int_0^\infty dz z^3 \ln\left\{1-\frac{i_{\nu_l}(az)k_{\nu_l}(z)}{i_{\nu_l}(z)k_{\nu_l}(az)}\right\} \ . \tag{8.48}$$

In order to make the $R-$dependence in (8.44) and (8.45) explicit, it is convenient to rescale the above defined $\zeta-$function by expanding the binomial. A simple calculation gives:

$$\zeta_\nu(s) = \frac{(kR)^{2s}}{\Gamma(s)} \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} (kR\nu)^{2q} \Gamma(s+q) \zeta(s+q) \tag{8.49}$$

where

$$\zeta(s) = \sum_{l=1}^{\infty} g_l \hat{\lambda}_l^{-2s} \tag{8.50}$$

does not depend on R. The use of (8.49) allows us to express the result in terms of the generalized $\zeta-$function (8.50) and the additional (Mittag-Leffler) representation for the $\zeta-$function can then be used to deal with the pole structure of (8.50) and express the residues at the poles in terms of geometrical quantities [190]. The Mittag-Leffler representation for the $\zeta-$function associated with the operator P_Σ (see, for example, [16]) is

$$\zeta(s) = \frac{1}{\Gamma(s)} \left\{ \sum_{p=0}^{\infty} \frac{\tilde{C}_p}{s - D_2/2 + p} + f(s) \right\} , \tag{8.51}$$

where $\tilde{C}_p = C_p - g_0 \delta_{p,D_2/2}$ and the C_p are the heat-kernel coefficients of the operator P_Σ, p runs over the positive half integers and $f(s)$ is an entire function. As in the case of [16], since the internal space Σ is boundaryless the heat-kernel coefficients of semi-integer order are zero. Relation (8.51) can now be used to regulate the effective potential and some calculations lead to

$$\begin{aligned} V(s) \quad = \quad & \frac{(ka)^4}{2(4\pi)^2} (ka/\mu)^{-2\epsilon} (kR)^{2s} \\ & \sum_{n=-1}^{\infty} \sum_{q=0}^{\infty} (1 + (-1)^n a^{2s}) \left(\frac{1}{\epsilon} \mathbf{a}_{n,q} + \mathbf{b}_{n,q} \right) (kR)^{2q+n} \\ & - \frac{g_0 k^4}{2(4\pi)^2} (k/\mu)^{-2\epsilon} \left\{ -\frac{d_4}{\epsilon} \left(1 + a^{4-2\epsilon}\right) + c_1 + a^4 c_2 \right\} \\ & + \frac{(ka)^4}{(4\pi)^2} \left\{ g_0 \mathcal{V}(a) + \sum_{l=1}^{\infty} g_l \mathcal{V}_l(a) \right\} \end{aligned} \tag{8.52}$$

where the coefficients of the previous expression can be written as

$$\mathbf{a}_{n,q} = \frac{(-1)^q}{q!} \nu^{2q} \tilde{C}_{2+D_2/2-n/2-q} \mathbf{A}_n \tag{8.53}$$

where

$$\begin{aligned} \mathbf{A}_{-1} &= \frac{1}{8\sqrt{\pi}} , \\ \mathbf{A}_0 &= -\frac{1}{4} , \\ \mathbf{A}_n &= \sum_{k=0}^{n} S_{n,k} , \quad \text{for } n > 1 \\ S_{n,k} &= \frac{\Gamma(k+n/2+s)}{\Gamma(k+n/2)\Gamma(n/2+s)} \sigma_{n,k} \end{aligned} \tag{8.54}$$

The coefficients $\mathbf{b}_{n,q}$ are related to the $\mathbf{a}_{n,q}$ via the following correspondence

$$\mathbf{b}_{n,q} = \mathbf{a}_{n,q}(\tilde{C}_p \to \Omega_{-p}) \ ,$$

where the Ω_p represent the finite part in the power series of $\Gamma(s)\zeta(s)$.

A check on the previous result is provided by the cancellation of the (lower dimensional) RS divergence, given by

$$g_0 \frac{k^4(1+a^4)}{32\pi^2\epsilon}(\Delta_0 + \Delta_2\nu^2 + \Delta_4\nu^4) \ , \tag{8.55}$$

where

$$\begin{aligned} \Delta_0 &= -\frac{27}{128} + \frac{3}{8}\Delta - \frac{1}{2}\Delta^2 + \frac{1}{2}\Delta^3 - \frac{1}{4}\Delta^4 \\ \Delta_2 &= \frac{13}{16} - \Delta + \frac{1}{2}\Delta^2 \\ \Delta_4 &= -\frac{1}{8} \\ \Delta &= \frac{1}{2}D_1(1-4\xi) \ . \end{aligned} \tag{8.56}$$

A simple inspection of (8.52) shows that the relevant terms for such a cancellation are the ones corresponding to the couples $(n,q) = (0,2)\ ,(2,1)\ ,(4,0)$. Such terms can be easily extracted from (8.52) and the use of the coefficients $\sigma_{n,k}$ (the relevant ones are reported in Appendix D, (D.12)) shows that the RS divergence is indeed canceled.

The result for the vacuum energy (8.52) is divergent and needs to be renormalized. The counterterm action can be constructed analogously to the case of two equal warpings [16]:

$$\begin{aligned} S_{n,q} &= \frac{1}{32\pi^2\epsilon}\sum_{\pm}\int d^Dx\sqrt{g_\pm}\,\kappa_\pm^{(n,q)}\ \mathcal{R}_\pm^{(4+D_2-n-2q)/2} = \\ &= \frac{1}{32\pi^2\epsilon}\int d^{4-2\epsilon}x\ \frac{(kR)^{2q+n}}{R^4}\left\{a^{4-2\epsilon}+(-1)^n\right\}\kappa^{(n,q)} \end{aligned}$$

where we have defined (the factor proportional to v_Σ has been reabsorbed in the coefficients $\kappa^{(n,q)}$)

$$\kappa_-^{(n,q)} = (-1)^n\kappa_+^{(n,q)} = k^{2q+n}\kappa^{(n,q)} \ , \tag{8.57}$$

and it is easy to see that all the divergences can be reabsorbed in counterterms of the previous type. Once we subtract the counter-terms, we arrive at the following expression for the renormalized effective potential

$$\begin{aligned} V(a) &= \frac{(1/R)^4}{2(4\pi)^2}\sum_{n=-1}^{\infty}\sum_{q=0}^{\infty}\left[\mathbf{a}_{n,q}\ln(\mu R)^2+\mathbf{b}_{n,q}\right]\ (kR)^{2q+n}\left(a^4+(-1)^n\right) \\ &\quad -\frac{g_0k^4}{2(4\pi)^2}\left[c_1+a^4c_2+(1+a^4)d_4\ln(k/\mu)^2\right] \\ &\quad +\frac{(ka)^4}{(4\pi)^2}\left[g_0\mathcal{V}(a)+\sum_{l=1}^{\infty}g_l\mathcal{V}_l(a)\right] \ . \end{aligned} \tag{8.58}$$

8.3 Radion stabilization and quantum self-consistency

In the previous Section we have computed and renormalized the Casimir energy arising from a massive bulk scalar field non-minimally coupled to the curvature and from a massless bulk gauge field. So we are now in the position to see whether or not quantum effects provide a reasonable stabilization mechanism for the class of models of the type AdS $\times\,\Sigma$. To this aim, let us consider the full action S, where we include the contribution S_{eff} arising from a quantized field:

$$S = S_{BG} + S_{eff}, \tag{8.59}$$

where we generically write the quantum contribution as

$$S_{eff} = -\int d^4x\sqrt{\tilde{g}}V(a). \tag{8.60}$$

S_{BG} is the classical background action obtained by using the ansatz for the metric (8.1) (with $\eta_{\mu\nu} \to \tilde{g}_{\mu\nu}(x)$) in (8.4) and by integrating out the extra $D_2 + 1$ dimensions. Now, varying the full action S with respect to $\tilde{g}_{\mu\nu}(x)$

$$\frac{\delta S}{\delta \tilde{g}_{\mu\nu}} = 0\ , \tag{8.61}$$

and requiring that the minimum is at $\tilde{g}_{\mu\nu}(x) = \eta_{\mu\nu}$ will tell us whether or not the classical solution is spoiled by quantum effects. On the other side, varying S with respect to the radion a

$$\frac{\delta S}{\delta a} = 0\ , \tag{8.62}$$

at $\tilde{g}_{\mu\nu}(x) = \eta_{\mu\nu}$, will tell us whether we can obtain an exponentially small hierarchy, $a = e^{-\pi kr}$ (with $kr \sim 12$), in which case such solution also solves the hierarchy problem. We want to stress that one can have solutions that satisfy (8.61) but not (8.62) and therefore are self-consistent but do not solve the hierarchy problem. A simple calculation shows that (8.61) and (8.62) are equivalent to

$$V'(a) = V(a) = 0\ , \tag{8.63}$$

where the prime denotes derivative with respect to a. Equation (8.63) is exactly the same as the *renormalization conditions* that we imposed in the RS model.

Different situations arise depending on the bulk field content. One possibility is to consider a the gauge field in the bulk only. The gauge fields split into a classical plus a quantum part,

$$A_\mu = A_\mu^C + A_\mu^Q\ , \tag{8.64}$$

and the quantum contribution comes from A_μ^Q. We shall consider the AdS$_5$ components only, which have a zero vev and do not couple to the Yang-Mills flux configuration. Alternatively, we can quantize a bulk scalar field on the classical background (8.4).

We recast the result for the effective potential as follows:

$$V(a) = \frac{k^4}{32\pi^2}\Big\{\gamma_+ + a^4\gamma_- + V_{NL}(a)\Big\} \tag{8.65}$$

where γ_+ and γ_- do not depend on a. The non-local contribution,

$$V_{NL}(a) = a^4\mathcal{V}(a) + a^4\sum_{l=1}^{\infty} g_l\mathcal{V}_l(a)\ , \tag{8.66}$$

is slightly more involved to inspect, however, in our case, it is sufficient to see that the contribution coming from the massive Kaluza-Klein modes (involving the sum over l) is highly suppressed with respect to the (RS) zero-mode term, proportional to $\mathcal{V}(a)$. This can be shown by noticing that the dominant contribution to the integral in $\mathcal{V}_l(a)$ comes from the region $z \lesssim 1$. Expanding the integrand in such region allows one to see that $\mathcal{V}_l(a)$ goes like $a^{2\nu_l}$ and a simple inspection of the sum tells us that the non-local contribution coming from the massive KK states is proportional to powers of $a^{1/(kR)}$. The non local contribution can then be approximated as

$$\Gamma_{NL}(a) \simeq a^4 \mathcal{V}(a)\ . \tag{8.67}$$

Fixing the field content of the theory (or the bulk parameters) will uniquely determine the function $\mathcal{V}(a)$. (Such term has been evaluated for any of ν in [8]). By expanding the integrand for small a, one finds that for bulk gauge fields

$$\mathcal{V}(a) = \frac{\beta}{\ln a}\ , \tag{8.68}$$

with β being $a-$independent. Instead, for a bulk scalar, one has to distinguish three possibilities: when the order of the Bessel functions is $\nu = 0$, this corresponds to taking

$$\xi = \frac{4m^2 + D_1^2 k^2}{4D_1(D_1+1)k^2}, \tag{8.69}$$

when $\nu = 1$ and this corresponds to fixing the values of ξ and m according to (8.18) and (8.19), and finally, when ν is different from the two previous values[3]. In the first case, we find

$$\mathcal{V}(a) = \frac{\beta}{\alpha + \ln a}\ , \tag{8.70}$$

where α and β do not depend on a. The second case, obviously, gives back relation (8.68), whereas in the third case $\mathcal{V}(a)$ is proportional to a^N with $N \geq 4$.

The previous relations along with the self-consistency condition (8.63) allow us to see in which cases we obtain a solution to the hierarchy problem with the bonus for the solution to be self-consistent.

By using the expression for the effective potential (8.65) and (8.68), we find that the solution to (8.63) for the gauge field in the limit of $a \ll 1$, is

$$a \sim e^{-\beta/\gamma_-} \tag{8.71}$$

which shows that there is no need of any fine tuning in order to get an exponentially small a.

For bulk scalar fields, one can easily check that fixing the parameters ξ and m according to (8.19) or (8.69) provides also a small hierarchy, whereas in the other cases no solution to (8.63) is found for small values of a.

[3]Fixing the bulk matter content can also be understood as a sort of tuning, which can be removed only by a more fundamental theory that leads to the specified field content. Moreover, special values of the mass of bulk scalars are unstable under quantum corrections unless supersymmetry is present

8.4 Discussion

In this Chapter, we have investigated the role of quantum effects arising from bulk fields in higher dimensional brane models. Specifically, we have considered a class of warped brane models whose topology is $AdS_5 \times \Sigma$, where Σ is a D_2 dimensional one-parameter compact manifold and two branes of codimension one are placed at the orbifold fixed points.

We have seen that such a set-up can be obtained from Einstein-Yang-Mills theory. Contrarily to the case studied in the previous Chapter [16], where both the radion a and the radius of Σ are undetermined classically, here the radius of the internal space Σ is stabilized at a size comparable with the higher dimensional cut-off once the Yang-Mills flux is tuned according to (8.8). This guarantees that, when matter is placed on the wall, the extra dimensions in the $\Sigma-$direction remain invisible, as it must be. On the other hand, the fact that the size of the internal manifold is of order $1/M$, does not suggest any new way of addressing the hierarchy, which is resolved only through a redshift effect coming from the AdS direction.

We evaluated the renormalized one-loop effective action at lowest order, namely the Casimir energy for generic bulk fields. The resulting scalar effective potential can be related to the one arising from quantized gauge fields with suitable values of the mass and nonmininal coupling constant.

The computation is similar to the one carried out in Chapter 7[16], with some technical differences due to the explicit presence of the eigenvalues of the scalar operator on the manifold Σ in the order of the Bessel functions. This can be effectively dealt with by using the uniform asymptotic expansion of the modes, which turned out slightly more involved than the corresponding computation in the case of Chapter 7 [16]. On the other hand, the Mittag-Leffler expansion for the generalized $\zeta-$function allowed us to express the Casimir energy in terms of heat-kernel coefficients of the internal space Σ as in the case previously considered in Chapter 7 [16]. The same non trivial check of the cancellation of the RS divergence works. Also the renormalization is carried out analogously to Chapter 7 [16] by subtracting suitable counter-terms proportional to a number of boundary or bulk local operators.

Our analysis indicates that the Casimir forces can stabilize the radion without fine tuning thanks to any KK modes whose index of the Bessel functions is 0 or 1. The latter is obtained with the zero mode of a bulk gauge field, in analogy with the Randall Sundrum case [8].

For scalar fields, this leads to two extreme situations occur. One possibility is that the splitting between the modes is small, *i.e.* $kR \gg 1$. If the lowest lying mode ($l = 0, 1$) has an index $\nu < 1$, then some KK modes will have ν_l very close to 1. In this case, irrespective of the properties of Σ, these modes will generate a large contribution to the potential and stabilise a large hierarchy. However, the mass of these modes is unstable to quantum effects and supersymmetry has to be invoked.

The other situation corresponds to well separated KK modes, *i.e.* $kR \ll 1$. In this case, the index ν_l of the KK modes is larger than one. Thus, no KK mode will induce a sizable contribution to the potential, so that only the zero mode can contribute significantly. If Σ is curved and the scalar field is nonminimally coupled, the contribution from the zero mode is also small. This can be seen from (8.22), (8.26) and (8.29), which imply that $\nu > 1$ in this case.

This does not happen for the gauge boson, whose zero mode has an index $\nu = 1$, irrespective of Σ being flat or curved. The difference is that the scalar field couples to the Ricci scalar, whereas the gauge boson couples to the Ricci tensor, as in (3.28). Since the space under consideration is factorisable, the Ricci tensor is box-diagonal and the non-compact components depend on k but not on R. Thus, the curvature of Σ does not enter the equation of motion of the zero mode of A_μ. In contrast with the scalar field, the gauge boson zero mode in the AdS $\times$ Σ model is identical to the gauge boson in the RS model. In particular, the Casimir energy that it induces can stabilize the hierarchy.

Chapter 9

Conclusions

The hierarchy problem of particle physics can be solved in the context of the the Brane World (BW) scenario in a variety of ways. They all have in common that the ratio of the electroweak to the Planck scales in the four dimensional effective theory is a function of the size of the extra dimensions. This is parametrized by the radion field and is massless at tree level. Thus, a complete solution to the hierarchy problem must include some stabilization mechanism that fixes the radion to the suitable value and gives it a large enough mass in order to avoid unwanted scalar interactions.

In this thesis, we explore the possibility that quantum effects provide such a mechanism. This is particularly appealing, since no other ingredient would be needed to solve the problem. In particular, we concentrate on the Casimir forces among the branes arising from gravity or eventually other fields propagating in the bulk.

In ADD scenarios, with flat and large bulk [1], it does not seem easy that the Casimir energy can stabilize a large Planck/EW hierarchy [4]. For dimensional reasons, one has a balance of different powers of the radius. Since this has to be much larger than the fundamental scale $1/M$, large numbers must be introduced at some point and this mechanism is no longer natural.

However, in models with warped extra dimensions this may not be the case. The scaling of different terms competing in the effective potential strongly depend on the bulk geometry. For instance, in the Randall Sundrum model [2] a bulk cosmological constant term depends exponentially on the radius. This opens the possibility that the Casimir energy combined with such terms generates naturally the hierarchy. This is indeed the case for a bulk gauge field [8], though not for the graviton or generic scalar fields [7].

We have presented three classes of brane models where the Casimir forces (or the quantum effects) are responsible for the large Planck/EW hierarchy. The first is presented in Chapter 6 [15] and constitutes a generalization of the RS model where a scalar field is included in the bulk. Several supergavity models arising as effective theories of string theories include bulk scalars with exponential potentials. These models admit vacuum configurations where the scale factor is a power of the proper distance $a(y) \propto y^q$. For any q, two moduli are found, corresponding to the brane positions, $y_\pm$.

In these models, the hierarchy problem is resolved by a redshift effect, as in RS. Since the warp factor is not so steep, the bulk size has to be large as well. We have obtained the bound

$q \geq 5/4$ in order for the model to generate the hierarchy. The couplings of the moduli to matter reveal that one of them is very strongly coupled for $q \lesssim 10$. This means that the stabilization mechanism must generate an almost Planckian mass for this modulus.

We have computed the effective potential due to generic fields in the bulk. This is of the Coleman Weinberg form for $y_\pm$ and can stabilize naturally the moduli at exponentially large/small values. In turn, this means that the Planck/EW hierarchy is naturally generated as well. For $q \lesssim 10$, the induced moduli masses are not large enough. Thus, the Casimir force induced by bulk fields (of any spin) is an efficient stabilization mechanism that leads to a full solution of the hierarchy problem in these models as long as the warp factor is steep enough, $q \gtrsim 10$.

The second example is discussed in Chapter 7 [16]. It consists in an extension of the RS model with a compact internal space that shares the same exponential warp factor with the non compact dimensions. As in the previous model there are two moduli, corresponding to the sizes of the internal space at the brane locations, $R_\pm$. None of them is coupled to matter with a coupling stronger than electroweak.

In these models, the hierarchy is generated by a combination of redshift [2] and large volume [1] effects. In a scenario with supersymmetry broken very close to the cutoff $\eta_{SUSY} \lesssim M$, the bulk curvature scale is rather below the cutoff $k \ll M$. If the moduli R_- and R_+ are stabilized close to M and k, then the Planck/EW hierarchy is given by a large power of the ratio η_{SUSY}/M, depending on the spacetime dimension. The 16 orders of magnitude separating the Planck and the EW scales are obtained from η_{SUSY} and M separated by less than one order of magnitude for $D = 11$ and less than 3 for $D = 6$.

The Casimir energy induced by generic bulk fields can be computed for a generic compact space. We have found that it naturally stabilizes the moduli $R_\pm$ to the required values (k and M resp.) for a flat internal space. A natural stabilization of the hierarchy with curved internal space seems to require more general combination of the warp factors for the compact an non compact spaces.

The last example is considered in Chapter 8 [17], and consist of a higher dimensional generalization of the RS model where the additional internal space is not warped. These models arise as solutions to the Einstein's equations with Yang Mills fields in the bulk. These solutions fix the size of the internal space, so they have only one destabilized modulus at tree level, the radion. For the model to be phenomenologically acceptable, the volume of the internal space must be small. Thus, the hierarchy is generated by a redshift effect, as in the RS model.

We have computed the effective potential induced by generic bulk fields for an arbitrary internal manifold. For the gauge field, the KK modes along the internal space produce a negligible contribution to the effective potential and the contribution from the zero mode is the same as in the RS model. Thus, the quantum effects arising from the YM gauge fields present in the model naturally stabilize the hierarchy.

In conclusion, quite generically, the Casimir effect may stabilize a large hierarchy naturally in warped brane models. Thus, the hierarchy problem may be solved without the need to introduce other ingredients in these scenarios.

Models with extra dimensions present attractive ways to solve long standing problems in particle physics and cosmology. The implications of these models can be tested in the near future. In particular, the radion is the first detectable signature in a large number of models with extra dimensions. We believe that further research in this topic will be interesting to reveal the viability of these theories.

Appendix A

The Heat Kernel

In Chapter 4 we have seen that the effective potential induced by a bulk field with an action of the form $S^{(\Phi)} = (1/2)\int \Phi\, P\, \Phi$ can be expressed in terms of the zeta function associated to the operator P (see Eqns. (4.39) and (4.38)), as

$$V^D \equiv \frac{\mu^\epsilon}{2\mathcal{A}} \mathrm{Tr}\ln\left(\frac{P(D)}{\mu^2}\right) = -\frac{\mu^\epsilon}{2\mathcal{A}} \lim_{s\to 0} \partial_s \zeta(s, D), \tag{A.1}$$

where

$$\zeta(s, D) = \mathrm{Tr}\left[\left(\frac{P(D)}{\mu^2}\right)^{-s}\right]. \tag{A.2}$$

Using a Mellin transform, the zeta function can be related to the so called *heat kernel* operator $\mathrm{Tr}\left[e^{-\xi^2 P(D)}\right]$ (where the Tr is the L_2 trace) according to

$$\zeta(s, D) = \frac{2\mu^{2s}}{\Gamma(s)} \int_0^\infty \frac{d\xi}{\xi} \xi^{2s}\ \mathrm{Tr}\left[e^{-\xi^2 P(D)}\right]. \tag{A.3}$$

The heat kernel is a well known object, and for small values of ξ, it admits an asymptotic expansion

$$\mathrm{Tr}\left[f\ e^{-\xi^2 P(D)}\right] \sim \sum_{n=0}^{\infty} \xi^{n-D} a^D_{n/2}(f, P), \tag{A.4}$$

with coefficients $a_{n/2}$ depending on geometric quantities related to the manifold and the operator P. These are the so-called Seeley-De Witt coefficients. For odd n, they have boundary contributions only. Here, we give the explicit form of the first of these for a Dirichlet field with a bulk operator $P = -(\Box + E)$.

$$a^D_{1/2}(f, P) = \frac{-(4\pi)^{\frac{(1-D)}{2}}}{4} \sum_{i=\pm} \int_{y_i} \sqrt{g_i} f(x)\, d^{D-1}x, \tag{A.5}$$

$$a^D_{3/2}(f, P) = \frac{-(4\pi)^{\frac{(1-D)}{2}}}{384} \sum_{i=\pm} \int_{y_i} \sqrt{g_i}\, d^{D-1}x \Big\{ f \quad \Big(\quad 96E + 16\mathcal{R} - 8\mathcal{R}_{yy} + 7\mathcal{K}^2 - 10\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}\Big) \\ + \quad O(f_{;y}\,,\ f_{;yy}) \Big\}. \tag{A.6}$$

The most relevant for our purposes will be $a^D_{5/2}$:

$$
\begin{aligned}
a^D_{5/2}(f,P) = \frac{-(4\pi)^{\frac{(1-D)}{2}}}{5760} \sum_{i=\pm} \int_{y_i} \sqrt{g_i}\, d^{D-1}x \Big\{ f \quad \Big(& \quad 720E^2 - 450\mathcal{K}\, E_{;y} + 360 E_{;yy} \\
+ & \quad 15\big(7\mathcal{K}^2 - 10\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} - 8\mathcal{R}_{yy} + 16\mathcal{R}\big)\, E \\
+ & \quad 20\mathcal{R}^2 - 48\,\Box\mathcal{R} - 17\mathcal{R}^2_{yy} - 8\mathcal{R}_{ab}\mathcal{R}^{ab} + 8\mathcal{R}_{abcd}\mathcal{R}^{abcd} \\
- & \quad 20\mathcal{R}_{yy}\,\mathcal{R} + 16\mathcal{R}_{yy}\mathcal{R} - 10\mathcal{R}_{yy}\mathcal{R}_{yy} + 12\mathcal{R}_{;yy} + 15\mathcal{R}_{yy;yy} \\
+ & \quad 16\mathcal{K}_{\mu\nu}\mathcal{K}^{\nu\rho}\mathcal{R}^{\mu}_{\rho} + 32\mathcal{K}^{\mu\nu}\mathcal{K}^{\rho\sigma}\mathcal{R}_{\mu\rho\nu\sigma} - \frac{215}{8}\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}\,\mathcal{R}_{yy} \\
- & \quad 25\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}\,\mathcal{R} - \frac{47}{2}\mathcal{K}_{\mu\nu}\mathcal{K}^{\nu}_{\rho}\mathcal{R}^{\rho}{}_{y\mu y} - \frac{215}{16}\mathcal{R}_{yy}\,\mathcal{K}^2 \\
+ & \quad \frac{35}{2}\mathcal{R}\,\mathcal{K}^2 + 14\mathcal{K}_{\mu\nu}\mathcal{R}^{\mu\nu}\,\mathcal{K} + \frac{49}{4}\mathcal{K}^{\mu\nu}\mathcal{R}_{\mu y\nu y}\,\mathcal{K} - 42\mathcal{R}_{;y}\,\mathcal{K} \\
- & \quad \frac{65}{128}\mathcal{K}^4 - \frac{141}{32}\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}\,\mathcal{K}^2 + \frac{17}{2}\mathcal{K}_{\mu\nu}\mathcal{K}^{\nu\rho}\mathcal{K}^{\mu}_{\rho}\,\mathcal{K} \\
+ & \quad \frac{777}{32}(\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu})^2 - \frac{327}{8}\mathcal{K}_{\mu\nu}\mathcal{K}^{\nu\rho}\mathcal{K}_{\rho\sigma}\mathcal{K}^{\mu\sigma}\Big) \\
+ & \quad O(f_{;y}, ..., f_{;yyyy})\ \Big\}. \qquad \text{(A.7)}
\end{aligned}
$$

Here the notation is as follows. E is a general scalar function, $\mathcal{R}^a{}_{bcd} = -\Gamma^a{}_{bc,d} + \cdots$ is the Riemann tensor, $\mathcal{R}_{bc} = \mathcal{R}^a{}_{bac}$ is the Ricci tensor and $\mathcal{R} = \mathcal{R}_{ab}g^{ab}$ is the curvature scalar. The extrinsic curvature is given by $\mathcal{K}_{\mu\nu} \equiv (1/2)\partial_y g_{\mu\nu}$, where $g_{\mu\nu}(y)$ is the induced metric on y-constant hypersurfaces, and $\mathcal{K} = \mathcal{K}_{\mu\nu}\, g^{\mu\nu}$. The vector normal to the boundary is ∂_y so the normal components are simply the y components. The a, $b, \cdots$ indices run over the extra coordinate, and over the directions tangential to the branes, μ, $\nu, \cdots$. The omitted terms, represented by $O(f_{;y}, ...)$, are linear combinations of the derivatives of f with coefficients which depend on $\mathcal{K}_{\mu\nu}$, E and its derivatives.

Appendix B

Dimensional regularization *vs.* zeta function regularization

Zeta function regularization

Let us rederive the main result of Section 6.2 using zeta function regularization. For simplicity, we shall restrict attention to the case $\theta = 1$. In this case, we have explicitly calculated the integral (4.56) along the conformal path using both methods described in Section 4.4. This exercise leads to the result

$$V = V_0 - \frac{1}{(4\pi)^2}\left[-d_4\left(\frac{1}{z_-^4}\ln(z_-/z_0)^\beta + \frac{1}{z_+^4}\ln(z_+/z_0)^\beta\right) + \hat{\alpha}\left(\frac{1}{z_-^4} + \frac{1}{z_+^4}\right)\right], \tag{B.1}$$

where $\hat{\alpha} = (\beta/3072)\{144 + \beta\big(784 - 1692\beta + 335\beta^2 - 192(3\beta - 14)(3\beta - 2)\xi\big)\}$. The zeta function associated with the operator P_0 is given by

$$\zeta_0(s) = \mathcal{A}\int \frac{d^4k}{(2\pi)^4}\sum_i \left(\frac{k^2 + m_n^2}{\mu^2}\right)^{-s}. \tag{B.2}$$

Performing the momentum integrals in (B.2), we have

$$\zeta_0(s) = \mathcal{A}\frac{\mu^{2s} z_-^{2s-4}\Gamma(s-2)}{(4\pi)^2\Gamma(s)}\tilde{\zeta}(2s-4), \tag{B.3}$$

Substituting (B.3) into (4.52), we have

$$V_0(z_+, z_-) = -\frac{1}{2(4\pi)^2 z_-^4}\left[\left\{\ln(\mu z_-) + \frac{3}{4}\right\}\tilde{\zeta}(-4) + \tilde{\zeta}'(-4)\right], \tag{B.4}$$

and from (6.57) we obtain

$$V_0(z_+, z_-) = \frac{1}{(4\pi)^2}\left[\frac{\mathcal{I}_K}{z_<^4} + \frac{\mathcal{I}_I}{z_>^4} + \frac{\mathcal{V}(\tau)}{z_>^4}\right] + \frac{d_4}{(4\pi)^2}\left[\frac{1}{z_+^4}\ln(\mu z_+) + \frac{1}{z_-^4}\ln(\mu z_-)\right]. \tag{B.5}$$

Substituting in (B.1) we recover Eq. (6.64) up to finite renormalization of μ.

V_0 in dimensional regularization

We shall now reproduce the result for V_0 by using dimensional regularization. Again, this is a redundant exercise: the calculation of an effective potential (be it V or V_0) will give the same answer whether it is done in dimensional or in zeta function regularization. Nevertheless, it is interesting to do it explicitly since this calculation is closest in spirit to the standard flat space calculations in four dimensions.

Adding up the dimensionally regularized effective potential per comoving 4-volume due to all KK modes, we have

$$V_0^{\rm reg} = \frac{1}{2}\mu^\epsilon \sum_n \int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \ln\left(\frac{k^2+m_n^2}{\mu^2}\right). \tag{B.6}$$

Performing the momentum integration for each mode, we obtain

$$V_0^{\rm reg} = -\frac{1}{2(4\pi)^2}(4\pi\mu^2)^{\epsilon/2}\frac{1}{z_-^{4-\epsilon}}\Gamma(-2+\epsilon/2)\sum_n \tilde{m}_n^{4-\epsilon}, \tag{B.7}$$

where we used $\tilde{m}_n$ defined in (6.52). This regularized expression for the effective potential is finite when the real part of ϵ is sufficiently large. Performing analytical continuation in ϵ, the summation over KK modes $\sum_n \tilde{m}_n^{4-\epsilon}$ can be identified with the zeta function $\tilde{\zeta}(-4+\epsilon)$. The pole part proportional to $1/\epsilon$ is identified with

$$V_0^{\rm div} = -\frac{1}{\epsilon}\frac{1}{2(4\pi)^2 z_-^4}\tilde{\zeta}(-4). \tag{B.8}$$

Subtracting this divergent part, we get the renormalized expression for the effective potential as

$$V_0 = V^{\rm reg} - V^{\rm div} = -\lim_{\epsilon\to 0}\frac{1}{2(4\pi)^2}\left[\frac{(4\pi\mu^2)^{\epsilon/2}}{z_-^{4-\epsilon}}\left(\frac{1}{\epsilon}+\frac{3}{4}-\frac{\gamma}{2}\right)\tilde{\zeta}(-4+\epsilon) - \frac{1}{\epsilon}\tilde{\zeta}(-4)\right]. \tag{B.9}$$

Consequently, we find that the dimensional regularization method reproduces the previous result (B.5).

Here, one remark is in order. In a usual 4-dimensional problem, the divergent part is given by the Seeley-De Witt coefficient a_2. In the present case, since the background is 4-dimensional flat space, a_2 consists of only one term proportional to m_n^4 for each KK mode. Subtraction of this counter term for each mode leads to the expression $\sum_n m_n^4 \ln m_n$, which is still divergent. This is not a surprising fact because the problem is essentially 5-dimensional as it is indicated by the existence of the infinite tower of the KK modes. Usually, this point is bypassed in the literature by evaluating the divergent sum with the help of the generalized zeta function.

Alternative regularization

In Section 6.2 and 4.4, we were interested in the class of conformally related operators P_θ, and therefore it was important to do a dimensional extension of the spacetime such that all geometries labeled by θ would be conformally flat. In other words, we added dimensions "parallel" to the brane, whose "size" was also affected by the warp factor. It should be stressed that this was done for computational convenience, since V^D is independent of the parameter θ only in this regularization. Putting aside computational considerations, nothing prevents us to extend the spacetime in any way we please, and the results should still be the same. To illustrate this point, let us consider an alternative dimensional extension of our 5-dimensional curved space $\mathcal{M}$ to a simple direct product space given by $\mathbb{R}^{-\epsilon} \times \mathcal{M}$. The dimensional regularization is done by an analytic continuation of the number of added dimensions ϵ, while the manifold $\mathcal{M}$ is kept unchanged.

Since this is a direct product space, the eigenvalues of the $(5-\epsilon)$-dimensional D'Alembertian are given by a simple summation of those in each space, $\mathcal{M}$ and $\mathbb{R}^{-\epsilon}$,

$$\lambda_{(5-\epsilon)} = k_\omega^2 + \lambda.$$

Then, the dimensionally regularized effective potential per unit comoving volume is given by

$$V^{\rm reg} = \frac{1}{2}\mu^\epsilon \sum_\lambda \int \frac{d^{-\epsilon}k_\omega}{(2\pi)^{-\epsilon}} \ln\left(\frac{k_\omega^2+\lambda}{\mu^2}\right). \tag{B.10}$$

This quantity is evaluated by introducing the function

$$\Upsilon(s) = \sum_\lambda \int \frac{d^{-\epsilon}k_\omega}{(2\pi)^{-\epsilon}} \left(\frac{k_\omega^2+\lambda}{\mu^2}\right)^{-s} = (1/4\pi)^{-\epsilon/2}\frac{\Gamma(s+\epsilon/2)}{\Gamma(s)}\zeta_1(s+\epsilon/2),$$

where ζ_1 is the zeta function defined in (4.39) with $\theta = 1$. Then, using (B.11), we can rewrite (B.10) as

$$V^{\rm reg} = -\frac{1}{2\mathcal{A}}\Upsilon'(0) = -\frac{1}{2\mathcal{A}}(1/4\pi)^{-\epsilon/2}\Gamma(\epsilon/2)\zeta_1(\epsilon/2). \tag{B.11}$$

The above expression contains the object ζ_1, which we have encountered in zeta function regularization. However, it should be noted that now the regularization parameter is not s, the argument of the function Υ, but the dimension of the product space ϵ. Therefore, even after we take the limit $s \to 0$, $V^{\rm reg}$ still diverges as $1/\epsilon$.

The divergent piece in the dimensional regularization in the D dimensional problem is the Seeley-De Witt coefficient $a_{D/2}$. From (4.39) and (4.40), it is easy to relate this divergent piece with the value of $\zeta_1(0)$ as[1]

$$V^{\rm div} = \frac{-a_{5/2}(P_1)}{\epsilon\mathcal{A}} = \frac{-1}{\epsilon\mathcal{A}}\zeta_1(0). \tag{B.12}$$

Subtracting this divergent piece from (B.11), we obtain the renormalized value

$$V = V^{\rm reg} - V^{\rm div} = -\frac{1}{2\mathcal{A}}\Big(\zeta_1'(0) - \{\ln\left(1/4\pi\right) + \gamma\}\zeta_1(0)\Big), \tag{B.13}$$

where γ is Euler's gamma. This coincides with Eq. (4.52) for $\theta = 1$ up to a redefinition of μ (note that with our conventions, $\zeta_1'(0)$ depends on μ.)

[1]Strictly speaking $a_{5/2}^D$ has to be evaluated in the regularized $(5+\epsilon)$-dimensional space. However, since the added ϵ dimensions are trivial, it is identical to $a_{5/2}(P_1)$.

Appendix C

The Mittag Leffler expansion

In the present Appendix we prove Eq. (7.43). It is well known that for a strictly positive definite Laplacian P with eigenvalues λ_P, the associated zeta function $\zeta(s|P) = \sum \lambda_P^{-s}$ admits a *Mittag-Leffler* expansion, of the form

$$\zeta(s|P) = \frac{1}{\Gamma(s)} \left\{ \sum_{p=0}^{\infty} \frac{C_p(P)}{s - D_2/2 + p} + f(s|P) \right\} \tag{C.1}$$

where $C_p(P)$ are the Seeley-DeWitt coefficients related to P, with $p \in \mathbb{N}/2$ and the function $f(s|P)$ is analytic for all finite s. This is a very useful relation since it neatly shows the pole structure of the zeta function, in terms of geometrical invariants.

A number of comments are in order before we proceed. The representation (C.1) may appear to differ from the one introduced in Eq. (4.42). First of all, in this Appendix, we deal with dimensionless eigenvalues, such as $\hat{\lambda}_l$ appearing in (7.41) or in (8.22). Thus, there is no need to introduce an arbitrary renormalization scale μ, as in (4.42). On the other hand, Eq. (4.42) contains an arbitrary (length) scale Λ. In Eq. (C.1), this is a dimensionless parameter that has been set to one. The sum in (4.42) runs over non-negative integers, whereas in (C.1) it runs over non-negative half integers. Both the $a_{n/2}$ and the C_p are Seeley-DeWitt coefficients. We denote them differently in order to distinguish that they are associated to the full or the internal space respectively.

We need to generalize equation (C.1) to operators with one zero eigenvalue, $g_0 = 1$. Consider a positive semidefinite differential operator P_Σ with eigenvalues λ_l^2 and assume that there is one zero eigenvalue. As usual in these cases, one defines the generalized ζ function excluding this eigenvalue (see Eq. (7.41),(8.50)),

$$\zeta(s) = \sum_{l=1}^{\infty} \hat{\lambda}_l^{-2s}. \tag{C.2}$$

Our main task is to express $\zeta(s)$ in terms of geometrical quantities in the form

$$\zeta(s) = \frac{1}{\Gamma(s)} \left\{ \sum_{p=0}^{\infty} \frac{\tilde{C}_p}{s - D_2/2 + p} + f(s) \right\}, \tag{C.3}$$

with $p \in \mathbb{N}/2$, for some $\tilde{C}_p$ related to the Seeley-DeWitt coefficients of P_Σ. First of all, let us introduce the regulated zeta function associated to the operator[1] $P_\Sigma^\mu \equiv P_\Sigma + \mu^2/R^2$,

$$\zeta_\mu(s) = \sum_{l=0}^{\infty} (\hat{\lambda}_l^2 + \mu^2)^{-s}. \tag{C.4}$$

Now it is trivial to express the function $\zeta(s)$ in terms of $\zeta_\mu(s)$,

$$\zeta(s) = \lim_{\mu\to 0} \left(\zeta_\mu(s) - \mu^{-2s}\right), \tag{C.5}$$

and it is obvious that, understood as this limit, $\zeta(s)$ is infrared finite for any s, even though $\zeta_\mu(s)$ is only IR finite for $\mathrm{Re}(s) \leq 0$. By construction, P_Σ^μ is strictly positive definite, so $\zeta_\mu(s)$ admits the expansion

$$\zeta_\mu(s) = \frac{1}{\Gamma(s)} \left\{ \sum_{p=0}^{\infty} \frac{C_p(\mu)}{s - D_2/2 + p} + g(\mu, s) \right\}, \tag{C.6}$$

where the function $g(\mu, s)$ is analytic and the Seeley-DeWitt coefficients $C_p(\mu)$ depend polynomially on μ.

From Eqns.(C.5,C.6) it follows that

$$\zeta(s) = \frac{1}{\Gamma(s)} \lim_{\mu\to 0} \left\{ \sum_{p=0}^{\infty} \frac{C_p(\mu)}{s - D_2/2 + p} + g(\mu, s) - \Gamma(s)\mu^{-2s} \right\}. \tag{C.7}$$

The next step is to isolate the poles from the last term in the previous formula. We do this expanding $\Gamma(s)$ as

$$\Gamma(s) = \Gamma(1, s) + \sum_{p=0}^{\infty} \frac{b_{2p}}{(s + 2p)}, \tag{C.8}$$

where $\Gamma(z, s)$ is the incomplete gamma function, and the coefficients of the expansion are given by $b_{2p} = (-1)^{2p}/p!$. Using (C.8) we have:

$$\Gamma(s)\mu^{-2s} = \sum_{p=0}^{\infty} b_{2p} \frac{\mu^{4p}}{s + 2p} + \sum_{p=0}^{\infty} h_p(\mu, s) + \Gamma(1, s)\mu^{-2s}, \tag{C.9}$$

where we have defined

$$h_p(\mu, s) = b_{2p} \frac{\mu^{-2s} - \mu^{4p}}{s + 2p}. \tag{C.10}$$

Eq. (C.9) allows us to write

$$\zeta(s) = \frac{1}{\Gamma(s)} \lim_{\mu\to 0} \left\{ \sum_{p=0}^{\infty} \frac{\tilde{C}_p(\mu)}{s - D_2/2 + p} + f(\mu, s) \right\}, \tag{C.11}$$

where

$$f(\mu, s) = g(\mu, s) - \sum_{p=0}^{\infty} h_p(\mu, s) - \Gamma(1, s)\mu^{-2s}, \tag{C.12}$$

[1]Note that this mass is fictitious and has nothing to neither with the physical bulk mass m (8.20) nor with the renormalization scale.

and the *modified* coefficients $\tilde{C}_p(\mu)$ are then given by

$$\tilde{C}_{D_2/2+2p}(\mu) = C_{D_2/2+2p}(\mu) - b_{2p}\mu^{4p}. \tag{C.13}$$

We note that the only coefficients which are modified are $C_{D_2/2}, C_{D_2/2+1}, C_{D_2/2+2}, \ldots$. Taking the limit $\mu \to 0$, we obtain the main result of this Appendix

$$\zeta(s) = \frac{1}{\Gamma(s)} \left\{ \sum_{p=0}^{\infty} \frac{\tilde{C}_p}{s - D_2/2 + p} + f(s) \right\}, \tag{C.14}$$

with

$$\tilde{C}_p \equiv \lim_{\mu \to 0} \tilde{C}_p(\mu) = C_p(0) - \delta_{2p,D_2}, \tag{C.15}$$

where $\delta_{p,p'}$ is the Kronecker delta, the limit of $C_p(\mu)$ can be taken because they are polynomials in μ, and the function $f(s) = f(0,s)$ is analytic and finite by construction, although it can be explicitly checked from (C.12).

In conclusion, the result (C.14,C.15) implies that for a Laplacian with one zero eigenvalue, there also exists a Mittag-Leffler-like expansion for the 'primed' zeta function (C.2,7.41,8.50), changing only the Seeley-DeWitt coefficient $C_{D_2/2}$ by $C_{D_2/2} - 1$. That is why we can consider Eq. (C.6) valid in general (with either $g_0 = 1$ or 0), replacing $C_{D_2/2}$ by $C_{D_2/2} - g_0$.

It is now easy to expand around $s = p$ and a simple calculation gives

$$\Gamma(s)\zeta(s)\,|_{s=p} = \frac{\tilde{C}_{D_2/2-p}}{s-p} + \Omega_p + \mathcal{O}((s-p)^2), \tag{C.16}$$

with

$$\Omega_p \equiv \sum_{p' \neq D_2/2-p} \frac{\tilde{C}_{p'}}{p + p' - D_2/2} + f(p). \tag{C.17}$$

Appendix D

Bessel function asymptotics

Asymptotic expansions for large arguments

The asymptotic expansion for large argument z of the Bessel functions I_ν and K_ν can be written as

$$\begin{aligned} I_\nu(z) &= \sqrt{\frac{z}{2\pi}} e^{z} \Theta^{(I)}(z) \ , \\ K_\nu(z) &= \sqrt{\frac{\pi z}{2}} e^{-z} \Theta^{(K)}(z) \ , \end{aligned} \tag{D.1}$$

with

$$\begin{aligned} \Theta^{(I)}(z) &\simeq \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\nu + j + 1/2)}{2^j j! \Gamma(\nu - j + 1/2)} z^{-j} \ , \\ \Theta^{(I)}(z) &\simeq \Theta^{(K)}(-z) \ . \end{aligned} \tag{D.2}$$

Uniform Asymptotic Expansion and computation of the coefficients $\sigma_{n,k}$

In the present Appendix, we report the relevant formulas concerning the uniform asymptotic expansion (UAE) for the altered Bessel functions, $i_{\nu_l}(z)$ and $k_{\nu_l}(z)$, used in the computation of V_2. By using the results reported in [197, 224], we find

$$i_{\nu_l}(\nu_l z) = \frac{\nu_l e^{\nu_l \eta}}{\sqrt{2\pi\nu_l}} (1+z^2)^{1/4} \Sigma^{(I)}_{\nu_l}(z) \ , \tag{D.3}$$

$$k_{\nu_l}(\nu_l z) = -\sqrt{\frac{\pi\nu_l}{2}} e^{\nu_l \eta} (1+z^2)^{1/4} \Sigma^{(K)}_{\nu_l}(z) \ , \tag{D.4}$$

with

$$\Sigma^{(I)}_{\nu_l}(z) = \frac{1}{2\nu_l\sqrt{1+z^2}} D_1(1-4\xi)\Sigma_1 + \Sigma_2 \ , \tag{D.5}$$

and

$$\Sigma^{(K)}_{\nu_l}(z) = \frac{1}{2\nu_l\sqrt{1+z^2}} D_1(1-4\xi)\Sigma_3 - \Sigma_4 \ , \tag{D.6}$$

where

$$t = \frac{1}{\sqrt{1+z^2}} ,$$
$$\eta(z) = \sqrt{1+z^2} + \ln\left(\frac{z}{1+\sqrt{1+z^2}}\right) .$$

The functions Σ_I are given by

$$\Sigma_1 = \sum_{k=0}^{\infty} \frac{u_k}{\nu_l^k} ,$$
$$\Sigma_2 = \sum_{k=0}^{\infty} \frac{v_k}{\nu_l^k} ,$$
$$\Sigma_3 = \sum_{k=0}^{\infty} (-1)^k \frac{u_k}{\nu_l^k} ,$$
$$\Sigma_4 = \sum_{k=0}^{\infty} (-1)^k \frac{v_k}{\nu_l^k} .$$

with the coefficients of the previous expansions expressed by the following recursion relations:

$$u_{k+1}(t) = \frac{1}{2}t^2(1-t^2)u_k'(t) + \frac{1}{8}\int_0^t (1-5x^2)u_k(x)dx$$
$$v_{k+1}(t) = u_{k+1}(t) - \frac{1}{2}t(1-t^2)u_k(t) - t^2(1-t^2)u_k'(t)$$

with $u_0(t) = 1$. It is possible to expand the previous functions in powers of ν_l:

$$\Sigma_{\nu_l}^{(I)}(z) = 1 + \sum_{j=1}^{\infty} \frac{p_j(t)}{\nu_l^j} , \qquad \text{(D.7)}$$

$$\Sigma_{\nu_l}^{(K)}(z) = 1 + \sum_{j=1}^{\infty} (-1)^j \frac{p_j(t)}{\nu_l^j} , \qquad \text{(D.8)}$$

where

$$p_j(t) = \frac{D_1(1-4\xi)}{2} t u_{j-1} + v_j . \qquad \text{(D.9)}$$

It is now easy to see that, in order to obtain the coefficients $\sigma_{n,k}$, we only need to expand the logarithm of $\Sigma_{\nu_l}^{(I,K)}(z)$:

$$\ln\left(1 + \sum_{j=1}^{\infty} \frac{p_j(t)}{\nu_l^j}\right) = \sum_{n=1}^{\infty}\sum_{k=0}^{n} \sigma_{n,k} \frac{t^{n+2k}}{\nu_l^n} . \qquad \text{(D.10)}$$

$$\ln\left(1 + \sum_{j=1}^{\infty} (-1)^j \frac{p_j(t)}{\nu^j}\right) = \sum_{n=1}^{\infty}\sum_{k=0}^{n} (-1)^n \sigma_{n,k} \frac{t^{n+2k}}{\nu_l^n} . \qquad \text{(D.11)}$$

The coefficients $\sigma_{n,k}$ can be obtained by using any symbolic manipulation program. We report here only the ones needed to cancel the RS divergence:

$$\begin{aligned}
\sigma_{4,0} &= -\frac{27}{128}+\frac{3}{8}\Delta-\frac{1}{2}\Delta^2+\frac{1}{2}\Delta^3-\frac{1}{4}\Delta^4 \ , \\
\sigma_{2,1} &= \frac{5}{8}-\frac{1}{2}\Delta \ , \\
\sigma_{2,0} &= -\frac{3}{16}+\frac{1}{2}\Delta-\frac{1}{2}\Delta^2 \ ,
\end{aligned} \tag{D.12}$$

with Δ given by (8.56).

Asymptotic form of $\mathcal{V}(a)$

The behaviour of $\mathcal{V}(a)$ defined in (5.3,6.59) for $a \ll 1$ is given by

$$\mathcal{V}(a) = -\frac{2}{\nu\Gamma(\nu)^2}\left(\frac{a}{2}\right)^{2\nu}\int_0^\infty dt\, t^{2\nu+3}\frac{K_\nu(t)}{I_\nu(t)} + \mathcal{O}\left(a^4\ln a\right). \tag{D.13}$$

This corresponds to a large separation between branes. In this limit the integral is generically negligible compared with the logarithmic terms in (6.64). The special case when $\nu = 1$ or 0 does not follow this pattern. The precise form of the expansion for small a is explicitly given in [8].

The limit of small separation between branes corresponds to $1-a \ll 1$. In this limit, the integral can be approximated by taking the arguments of the Bessel functions to be large. Using the asymptotic expansion

$$\frac{I_\nu(a\rho)K_\nu(\rho)}{K_\nu(a\rho)I_\nu(\rho)} \sim e^{-2(1-a)\rho}, \tag{D.14}$$

we have

$$\begin{aligned}
\mathcal{V} &\approx \int_0^\infty d\rho\rho^3 \ln\left(1-e^{-2(1-a)\rho}\right) \\
&= -\frac{1}{2^6(1-a)^4}\int_0^\infty dx\frac{x^4}{e^x-1} = -\frac{3\zeta(5)}{8(1-a)^4}.
\end{aligned} \tag{D.15}$$

In the second equality, we performed integration by parts and a change of variable. Here, ζ is the usual Riemann's zeta function $\zeta(5) = 1.03693\cdots$. Using the relation

$$\zeta'(-4) = \frac{3}{4\pi^4}\zeta(5), \tag{D.16}$$

and substituting in (6.64), we find that in the limit of small brane separation, the effective potential reduces to the one we had found in the massless conformally coupled case, given in equation (5.7):

$$V(z_+, z_-) \sim -\frac{A}{|z_+ - z_-|^4}. \tag{D.17}$$

Equation (D.16) is a particular case of a more general formula

$$\zeta'(2n) = \frac{(2n)!}{2^{2n+1}\pi^{2n}}\zeta(2n+1),$$

valid for positive integer n. This can be derived from the perhaps better known relation [225]

$$\zeta(1-z) = 2^{1-z}\pi^{-z}\zeta(z)\Gamma(z)\cos(\pi z/2),$$

by setting $z = 2n+1$ after differentiation of both sides of the equation with respect to z.

Bibliography

[1] N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. **B429** 263;
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G. Dvali, *ibid.* **B436** 257.

[2] L. Randall, R. Sundrum, Phys. Rev. Lett. **83** (1999) 3370.

[3] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Rev. **D59** (1999) 086004 .

[4] N. Arkani-Hamed, S. Dimopoulos and J. March-Russell, Phys. Rev. D **63**, 064020 (2001) [arXiv:hep-th/9809124].

[5] W. D. Goldberger and M. B. Wise, Phys. Rev. Lett. **83**, 4922 (1999).

[6] P. Candelas, S. Weinberg, Nucl. Phys. **B237** (1984) 397.

[7] J. Garriga, O. Pujolas and T. Tanaka, "Radion effective potential in the brane-world," Nucl. Phys. B **605**, 192 (2001) [arXiv:hep-th/0004109].

[8] J. Garriga and A. Pomarol, Phys. Lett. B **560**, 91 (2003) [arXiv:hep-th/0212227].

[9] P. Horava, E. Witten, Nucl. Phys. **B460** (1996) 506;
P. Horava, E. Witten, Nucl. Phys. **B475** (1996) 94.

[10] A. Lukas, B.A. Ovrut, K.S. Stelle, D. Waldram, Phys. Rev. **D59** (1999) 086001.

[11] D.J. Toms, Phys. Lett. **B484** (2000) 149.

[12] A. Flachi, D.J. Toms, Nucl. Phys. **B610** (2001) 144.

[13] W.D. Goldberger, I.Z. Rothstein, Phys. Lett. **B491** (2000) 339.

[14] O. Pujolàs, Master Thesis, UAB june 2000;
Int. J. Theor. Phys. **40**, 2131 (2001) [arXiv:hep-th/0103193].

[15] J. Garriga, O. Pujolàs., T. Tanaka, "Moduli effective potential in warped-brane world compactifications," Nucl. Phys. **B655** (2003) 127, [arXiv:hep-th/0111277].

[16] A. Flachi, J. Garriga, O. Pujolas and T. Tanaka, "Moduli stabilization in higher dimensional brane models," [arXiv:hep-th/0302017].

[17] A. Flachi and O. Pujolas, "Quantum self-consistency of AdS$\times\Sigma$ brane models," [arXiv:hep-th/0304040], to appear in *Phys. Rev. D.*

[18] I. Antoniadis, Phys. Lett. **B246**, 377 (1990).

[19] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B **125** (1983) 136.

[20] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B **125**, 139 (1983); M. Visser, Phys. Lett. B **159**, 22 (1985) [arXiv:hep-th/9910093]; J. D. Lykken, Phys. Rev. D **54**, 3693 (1996) [arXiv:hep-th/9603133].

[21] R. Jackiw and C. Rebbi, Phys. Rev. D **13** (1976) 3398.

[22] R. Jackiw and P. Rossi, Nucl. Phys. B **190** (1981) 681.

[23] G. 't Hooft, Phys. Rev. D **14** (1976) 3432.

[24] K. Akama, Lect. Notes Phys. **176** (1982) 267.

[25] V. A. Rubakov, Phys. Usp. **44**, 871 (2001) [Usp. Fiz. Nauk **171**, 913 (2001)] [arXiv:hep-ph/0104152].

[26] S. Weinberg, Phys. Rev. D **19** (1979) 1277.

[27] L. Susskind, Phys. Rev. D **20**, 2619 (1979).

[28] K. D. Lane, arXiv:hep-ph/0007304.

[29] Y. Hosotani, Phys. Lett. B **126**, 309 (1983).

[30] A. Higuchi, L. Parker, Phys. Rev. **D37** (1988) 2853.

[31] Y. Hosotani, Ann. Phys. **190** (1989) 233; Phys. Lett. **B126** (1983) 309.

[32] D. J. Toms, Phys. Lett. **B126** (1983) 445.

[33] L. H. Ford, Phys. Rev. **D21** (1980) 933.

[34] J. S. Dowker, S. Jadhav, Phys. Rev. **D39** (1989) 1196; ibid. 1196.

[35] S. Mignemi, I. G. Moss, Phys. Rev. **D48** (1993) 3725.

[36] M. E. Shaposhnikov and P. Tinyakov, Phys. Lett. B **515**, 442 (2001) [arXiv:hep-th/0102161].

[37] J. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231;
E. Witten, Adv. Theor. Math. Phys. **2** (1998) 253;
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428** (1998) 105;
M. Henningson and K. Sfetsos, Phys. Lett. B **431** (1998) 63;
W. Muck and K. S. Viswanathan, Phys. Rev. D **58** (1998) 106006;
S. S. Gubser, Phys. Rev. D **63** (2001) 084017;
M. Perez-Victoria, JHEP **0105** (2001) 064.

[38] R. Rattazzi and A. Zaffaroni, JHEP **0104** (2001) 021;

[39] N. Arkani-Hamed, M. Porrati and L. Randall, JHEP **0108** (2001) 017;

[40] G. Nordström, Phys. Zeitsch. **15** (1914) 504.

[41] T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin Math. Phys. Kl. **33** (1921) 966.

[42] O. Klein, Z. Phys. **37** (1926) 895;
O. Klein, Nature (London) **118** (1926) 516.

[43] B. S. De Witt in *Relativity, Groups and Topology*, eds C. and B. S. De Witt (Gordan and Breach, New York, 1964).

[44] R. Kerner, Ann. Inst. Poincare, Sect. **A9** (1968) 29.

[45] E. Witten, "Fermion Quantum Numbers In Kaluza–Klein Theory", The Proc. of Second Shelter Island Meeting (1983) 227.

[46] J. Polchinski, "String Theory. Vol. 1: An Introduction To The Bosonic String," *Cambridge, UK: Univ. Pr. (1998) 402 p.* "String Theory. Vol. 2: Superstring Theory And Beyond," *Cambridge, UK: Univ. Pr. (1998) 531 p.*

[47] T. Appelquist, H. C. Cheng and B. A. Dobrescu, Phys. Rev. D **64**, 035002 (2001) [arXiv:hep-ph/0012100].

[48] M. J. Duff, B. E. Nilsson and C. N. Pope, Phys. Rept. **130** (1986) 1; M. J. Duff, arXiv:hep-th/9410046.

[49] "Modern Kaluza-Klein Theories", (Edited by T. Appelquist, A. Chodos, and P. Freund), Frontiers In Physics v. 65, Addison-Wesley Pub. Com. (1987).

[50] J. M. Overduin and P. S. Wesson, Phys. Rept. **283**, 303 (1997) [arXiv:gr-qc/9805018].

[51] H. C. Lee, "An Introduction To Kaluza-Klein Theories. Proceedings, Workshop, Chalk River, Canada, August 11-16, 1983,"

[52] O. Klein, Nature **118** (1926) 516.

[53] J. Rayski, Acta. Phys. Polon. **27** (1965) 947 and **28** (1965) 87.

[54] A. Trautmann, Rep. Math. Phys. **1** (1970) 29.

[55] Y. M. Cho, J. Math. Phys. **16** (1975) 2029.

[56] Y. M. Cho and P. G. O. Freund, Phys. Rev. **D12** (1975) 1711.

[57] Y. M. Cho and P. S. Yang, Phys. Rev. **D12** (1975) 3789.

[58] S. Randjbar-Daemi, A. Salam and J. Strathdee, Nucl. Phys. B **214** (1983) 491.

[59] S. Randjbar-Daemi, A. Salam and J. Strathdee, Phys. Lett. B **132** (1983) 56.

[60] J. C. Long, H. W. Chan and J. C. Price, Nucl. Phys. B **539** (1999) 23.

[61] J. C. Long, “Laboratory Seach for Extra-Dimensional Effects in the Sub-millimeter Regime”; A. Kapitulnik, “Experimental Tests of Gravity Below 1 mm”, *Talks presented at the International Coference on Physics Beyind Four Dimensions*, ICTP, Triest, Italy (2000).

[62] G. F. Giudice, R. Rattazzi and J. D. Wells, Nucl. Phys. B **544** (1999) 3.

[63] J. L. Hewett, Phys. Rev. Lett. **82**, 4765 (1999) [arXiv:hep-ph/9811356].

[64] M. Bando, T. Kugo, T. Noguchi and K. Yoshioka, Phys. Rev. Lett. **83**, 3601 (1999) [arXiv:hep-ph/9906549].

[65] J. Hewett and J. March-Russell, “Extra Dimensions (New),” in K. Hagiwara *et al.* [Particle Data Group Collaboration], Phys. Rev. D **66**, 010001 (2002).

[66] Y. A. Kubyshin, arXiv:hep-ph/0111027. Lectures given at 11th International School on Particles and Cosmology, Karbardino-Balkaria, Russia, 18-24 Apr 2001.

[67] Y. Uehara, Mod. Phys. Lett. A **17**, 1551 (2002) [arXiv:hep-ph/0203244].

[68] M. Besancon, “Experimental introduction to extra dimensions,” arXiv:hep-ph/0106165.

[69] J. Hewett and M. Spiropulu, Ann. Rev. Nucl. Part. Sci. **52**, 397 (2002) [arXiv:hep-ph/0205106].

[70] A. V. Kisselev, arXiv:hep-ph/0303090.

[71] S. Cullen, M. Perelstein and M. E. Peskin, Phys. Rev. D **62** (2000) 055012.

[72] L. J. Hall and D. R. Smith, Phys. Rev. D **60** (1999) 085008.

[73] A. Perez-Lorenzana, AIP Conf. Proc. **562**, 53 (2001), hep-ph/0008333.

[74] S. Hannestad and G. Raffelt, Phys. Rev. Lett. **87** (2001) 051301.

[75] T. Han, J. D. Lykken and R. Zhang, Phys. Rev. D **59** (1999) 105006.

[76] I. Antoniadis and K. Benakli, hep-ph/0108174.

[77] V. Barger, T. Han, C. Kao and R. J. Zhang, Phys. Lett. B **461** (1999) 34.

[78] C. Adloff *et al.* [H1 Collaboration], Phys. Lett. B **479** (2000) 358.

[79] S. Sigurdsson, Int. J. Mod. Phys. D **10**, 781 (2001) astro-ph/0107169.

[80] S. Cullen and M. Perelstein, Phys. Rev. Lett. **83**, 268 (1999) [arXiv:hep-ph/9903422];
V. D. Barger, T. Han, C. Kao and R. J. Zhang, Phys. Lett. B **461**, 34 (1999) [arXiv:hep-ph/9905474];
C. Hanhart, J. A. Pons, D. R. Phillips and S. Reddy, Phys. Lett. B **509**, 1 (2001) [arXiv:astro-ph/0102063];
S. Hannestad and G. Raffelt, Phys. Rev. Lett. **87**, 051301 (2001) [arXiv:hep-ph/0103201].

[81] S. Hannestad and G. G. Raffelt, arXiv:hep-ph/0304029.

[82] S. Hannestad and G. G. Raffelt, Phys. Rev. Lett. **88**, 071301 (2002) [arXiv:hep-ph/0110067].

[83] A. Vilenkin and E.P.S. Shellard, "Cosmic Strings And Other Topological Defects," *Cambridge U. Press* (1994).

[84] R. Sundrum, Phys. Rev. **D59**, 085009 (1999) [arXiv:hep-ph/9805471].

[85] G. R. Dvali, I. I. Kogan and M. A. Shifman, Phys. Rev. D **62**, 106001 (2000) [arXiv:hep-th/0006213].

[86] A. Dobado and A.L. Maroto Nucl. Phys. B **592**, 203 (2001) [arXiv:hep-ph/0007100]

[87] H. Murayama and J. D. Wells, Phys. Rev. D **65**, 056011 (2002) [arXiv:hep-ph/0109004].

[88] J. Alcaraz, J.A.R. Cembranos, A. Dobado and A.L. Maroto [arXiv:hep-ph/0212269]

[89] O. DeWolfe, D. Z. Freedman, S. S. Gubser and A. Karch, Phys. Rev. D **62**, 046008 (2000) [arXiv:hep-th/9909134].

[90] J. A. Cembranos, A. Dobado and A. L. Maroto, arXiv:hep-ph/0302041.

[91] R. Sundrum, Phys. Rev. **D59** 085010 (1999) [arXiv:hep-ph/9807348].

[92] Y. Okada, Phys. Lett. B **150**, 103 (1985).

[93] G. Dvali, Phys. Lett. **B459**, 489 (1999) [arXiv:hep-ph/9905204].

[94] N. Arkani-Hamed, L. J. Hall, D. R. Smith and N. Weiner, Phys. Rev. D **62**, 105002 (2000) [arXiv:hep-ph/9912453].

[95] S. Deser, R. Jackiw, and G. 't Hooft, *Ann. Phys.* **152**, 220 (1984).

[96] A. G. Cohen and D.B. Kaplan, hep-th/9910132.

[97] N. Kaloper, J. March-Russell, G. D. Starkman and M. Trodden, Phys. Rev. Lett. **85**, 928 (2000) [arXiv:hep-ph/0002001].

[98] S. Nasri, P. J. Silva, G. D. Starkman and M. Trodden, Phys. Rev. D **66**, 045029 (2002) [arXiv:hep-th/0201063].

[99] A. Albrecht, C. P. Burgess, F. Ravndal and C. Skordis, Phys. Rev. D **65**, 123506 (2002) [arXiv:hep-th/0105261].

[100] A. Albrecht, C. P. Burgess, F. Ravndal and C. Skordis, Phys. Rev. D **65**, 123507 (2002) [arXiv:astro-ph/0107573].

[101] A. Flachi, I.G. Moss, D.J. Toms, Phys. Lett. **B518** (2001) 153.

[102] A. Flachi, I.G. Moss, D.J. Toms, Phys. Rev. **D64** (2001) 105029.

[103] W.D. Goldberger and M.B. Wise Phys. Rev. **D60**, 107505 (1999) [hep-ph/9907218];

[104] A. Pomarol, Phys. Lett. B **486**, 153 (2000) [arXiv:hep-ph/9911294].

[105] T. Gherghetta and A. Pomarol, Nucl. Phys. B **586**, 141 (2000) [arXiv:hep-ph/0003129].

[106] H. Davoudiasl, J. L. Hewett and T. G. Rizzo, Phys. Rev. Lett. **84**, 2080 (2000) [arXiv:hep-ph/9909255].

[107] N. D. Birrell and P. C. Davies, "Quantum Fields In Curved Space," *Cambridge, Uk: Univ. Pr.* (1982) 340p.

[108] C. Csaki, M. Graesser, L. Randall and J. Terning, Phys. Rev. D **62**, 045015 (2000) [arXiv:hep-ph/9911406].

[109] J. Garriga and T. Tanaka, Phys. Rev. Lett. **84**, 2778 (1999) [hep-th/9911055].

[110] C. Charmousis, R. Gregory and V. A. Rubakov, Phys. Rev. D **62**, 067505 (2000) [arXiv:hep-th/9912160].

[111] T. Chiba, Phys. Rev. D **62**, 021502 (2000) [gr-qc/0001029].

[112] W. D. Goldberger and M. B. Wise, Phys. Lett. B **475**, 275 (2000) [arXiv:hep-ph/9911457].

[113] G. F. Giudice, R. Rattazzi and J. D. Wells, Nucl. Phys. B **595**, 250 (2001) [arXiv:hep-ph/0002178].

[114] K. m. Cheung, Phys. Rev. D **63**, 056007 (2001) [arXiv:hep-ph/0009232].

[115] D. Dominici, B. Grzadkowski, J. F. Gunion and M. Toharia, arXiv:hep-ph/0206192.

[116] G. Azuelos *et al.*, Part XVI, arXiv:hep-ph/0204031.

[117] J. L. Hewett and T. G. Rizzo, arXiv:hep-ph/0202155.

[118] M. Battaglia, S. De Curtis, A. De Roeck, D. Dominici and J. F. Gunion, [arXiv:hep-ph/0304245].

[119] S. Bae, P. Ko, H. S. Lee and J. Lee, Phys. Lett. B **487**, 299 (2000) [arXiv:hep-ph/0002224].

[120] G. D. Kribs, in *Proc. of the APS/DPF/DPB Summer Study on the Future of Particle Physics (Snowmass 2001)* ed. N. Graf, eConf **C010630**, P317 (2001) [arXiv:hep-ph/0110242].

[121] J. L. Hewett and T. G. Rizzo, in *Proc. of the APS/DPF/DPB Summer Study on the Future of Particle Physics (Snowmass 2001)* ed. N. Graf, eConf **C010630**, P338 (2001) [arXiv:hep-ph/0112343].

[122] A. Das and A. Mitov, Phys. Rev. D **66**, 045030 (2002) [arXiv:hep-th/0203205].

[123] H. Davoudiasl, J. L. Hewett and T. G. Rizzo, Phys. Lett. B **493**, 135 (2000) [arXiv:hep-ph/0006097].

[124] U. Mahanta and A. Datta, Phys. Lett. B **483**, 196 (2000) [arXiv:hep-ph/0002183].

[125] T. Han, G. D. Kribs and B. McElrath, Phys. Rev. D **64**, 076003 (2001) [arXiv:hep-ph/0104074].

[126] M. Chaichian, A. Datta, K. Huitu and Z. h. Yu, Phys. Lett. B **524**, 161 (2002) [arXiv:hep-ph/0110035].

[127] M. Chaichian, K. Huitu, A. Kobakhidze and Z. H. Yu, Phys. Lett. B **515**, 65 (2001) [arXiv:hep-ph/0106077].

[128] S. Bae and H. S. Lee, Phys. Lett. B **506**, 147 (2001) [arXiv:hep-ph/0011275].

[129] S. C. Park, H. S. Song and J. Song, Phys. Rev. D **65**, 075008 (2002) [arXiv:hep-ph/0103308].

[130] S. R. Choudhury, A. S. Cornell and G. C. Joshi, arXiv:hep-ph/0012043.

[131] K. Cheung, arXiv:hep-ph/0305003.

[132] A. Lukas, B.A. Ovrut and D. Waldram, Phys. Rev. **D60** (1999) 086001; Phys. Rev. **D61** (2000) 023506.

[133] N. Kaloper and A. Linde, Phys. Rev. **D59** (1999) 101303; N. Kaloper, Phys. Rev. **D60** (1999) 123506.

[134] P. Binétruy, C. Deffayet and D. Langlois, Nucl. Phys. **B565** (2000) 269.

[135] P. Kanti, I. Kogan, K.A. Olive and M. Pospelov, Phys. Lett. **B468** (1999) 31.

[136] P. Kanti, I. I. Kogan, K. A. Olive and M. Pospelov, Phys. Rev. D **61**, 106004 (2000) [arXiv:hep-ph/9912266].

[137] P. Kanti, K. A. Olive and M. Pospelov, Phys. Lett. B **538**, 146 (2002) [arXiv:hep-ph/0204202].

[138] H. Kim, Phys. Lett. **B478** (2000) 285.

[139] R. Hofmann, P. Kanti and M. Pospelov, Phys. Rev. D **63**, 124020 (2001) [arXiv:hep-ph/0012213].

[140] M. Gell-Mann and B. Zwiebach, Phys. Lett. **B141** (1984) 333; Nucl. Phys. **B260** (1985) 569.

[141] T. Tanaka and X. Montes, Nucl. Phys. B **582**, 259 (2000) [arXiv:hep-th/0001092].

[142] C. Csaki, M. L. Graesser and G. D. Kribs, Phys. Rev. D **63**, 065002 (2001) [arXiv:hep-th/0008151].

[143] S. Kalyana Rama, Phys. Lett. B **495**, 176 (2000) [arXiv:hep-th/0010121].

[144] O. DeWolfe, D.Z. Freedman, S.S. Gubser and A. Karch, Phys. Rev. **D62** (2000) 046008;
D. Youm, Phys. Rev. **D62** (2000) 084007;
Z. Chacko and A.E. Nelson, Phys. Rev. **D62** (2000) 085006;
C. Csaki, J. Erlich, T. Hollowood and Y. Shirman, Nucl. Phys. **B581** (2000) 309;
U. Ellwanger, hep-th/0001126;
C. Csaki, J. Erlich, C. Grojean and T. Hollowood, Nucl. Phys. **B584** (2000) 359;
K. Maeda and D. Wands, Phys. Rev. **D62** (2000) 124009;
J. Cline and H. Firouzjani, hep-ph/0005235; Phys. Lett. **B495** (2000) 271;
C. Barcelo and M. Visser, Phys. Rev. **D63** (2001) 024004;
A. Mennim and R.A. Battye, hep-th/0008192;
N. Arkani-Hamed, S. Dimopoulos, N. Kaloper and R. Sundrum, Phys. Lett. **B480** (2000) 193;
S. Kachru, M. Schulz and E. Silverstein, Phys. Rev. **D62** (2000) 045021;
S. Forste, Z. Lalak, S. Lavignac and H.P. Nilles, Phys. Lett. **B481** (2000) 360; JHEP **0009** (2000) 034;
B. Grinstein, D.R. Nolte and W. Skiba, Phys. Rev. **D62** (2000) 086006;
P. Binetruy, J.M. Cline and C. Grojean, Phys. Lett. **B489** (2000) 403;
A. Kehagias and K. Tamvakis, Phys. Lett. B **504**, 38 (2001) [arXiv:hep-th/0010112]; Mod. Phys. Lett. A **17**, 1767 (2002) [arXiv:hep-th/0011006].

[145] V. Barger, T. Han, T. Li, J.D. Lykken and D. Marfatia, Phys. Lett. **B488** (2000) 97.

[146] M. A. Luty and R. Sundrum, Phys. Rev. D **62**, 035008 (2000) [arXiv:hep-th/9910202].

[147] M. A. Luty and R. Sundrum, Phys. Rev. D **64**, 065012 (2001) [arXiv:hep-th/0012158].

[148] B. Grinstein, D. R. Nolte and W. Skiba, Phys. Rev. D **63**, 105016 (2001) [arXiv:hep-th/0012202].

[149] P. Brax, C. van de Bruck, A. C. Davis and C. S. Rhodes, Phys. Rev. D **67**, 023512 (2003) [arXiv:hep-th/0209158].

[150] T. Damour and K. Nordtvedt, Phys.Rev.Lett. **70**, 2217 (1993)

[151] T. Damour and K. Nordtvedt, Phys.Rev.D **48**, 3436 (1993)

[152] D.I. Santiago, D. Kalligas and R. Wagoner, Phys.Rev.D **58**, 124005 (1998)

[153] J. E. Kim and B. Kyae, Phys. Lett. B **486**, 165 (2000) [arXiv:hep-th/0005139].

[154] P. Kanti, K. A. Olive and M. Pospelov, Phys. Rev. D **62**, 126004 (2000) [arXiv:hep-ph/0005146].

[155] H. B. Kim, Phys. Lett. B **478**, 285 (2000) [arXiv:hep-th/0001209].

[156] P. Kanti, S. c. Lee and K. A. Olive, Phys. Rev. D **67**, 024037 (2003) [arXiv:hep-ph/0209036].

[157] S. Kanno and J. Soda, Phys. Rev. D **66**, 083506 (2002) [arXiv:hep-th/0207029].

[158] T. Shiromizu and K. Koyama, Phys. Rev. D **67**, 084022 (2003) arXiv:hep-th/0210066.

[159] T. Shiromizu, K. Koyama and K. Takahashi, "Effective theory for close limit of two branes," arXiv:hep-th/0212331.

[160] S. Kanno and J. Soda, Phys. Rev. D **66**, 043526 (2002) [arXiv:hep-th/0205188].

[161] S. Kanno, M. Sasaki and J. Soda, Prog. Theor. Phys. **109**, 357 (2003) arXiv:hep-th/0210250.

[162] G. Huey, P. J. Steinhardt, B. A. Ovrut and D. Waldram, Phys. Lett. B **476**, 379 (2000) [arXiv:hep-th/0001112].

[163] M. Dine, arXiv:hep-th/0001157.

[164] S. D. Unwin, Phys. Lett. B **103** (1981) 18.

[165] T. Appelquist and A. Chodos, Phys. Rev. Lett. **50**, 141 (1983); Phys. Rev. D **28**, 772 (1983).

[166] A. Delgado, A. Pomarol and M. Quiros, Phys. Rev. D **60**, 095008 (1999) [arXiv:hep-ph/9812489].

[167] A. Delgado, A. Pomarol and M. Quiros, JHEP **0001**, 030 (2000) [arXiv:hep-ph/9911252].

[168] E. Ponton and E. Poppitz, JHEP **0106**, 019 (2001) [arXiv:hep-ph/0105021].

[169] S. Mukohyama, Phys. Rev. **D63** (2001) 044008.

[170] P. Kanti, K. A. Olive and M. Pospelov, Phys. Lett. B **481**, 386 (2000) [arXiv:hep-ph/0002229].

[171] N. Sago, Y. Himemoto, M. Sasaki, Phys. Rev. **D65** (2001) 024014;
S. Kobayashi, K. Koyama, J. Soda, Phys.Lett. **B501** (2001) 157.

[172] W. Naylor, M. Sasaki, Phys. Lett. **B542** (2002) 289.

[173] I. G. Moss, W. Naylor, W. Santiago-German and M. Sasaki, "Bulk quantum effects for de Sitter branes in AdS(5)", [arXiv:hep-th/0302143].

[174] I. G. Moss, W. Santiago-German, W. Naylor and M. Sasaki, arXiv:hep-th/0303003.

[175] S. Nojiri, S. Odintsov, S. Zerbini, Class. Quantum Grav. **17** (2000) 4855.

[176] E. Elizalde, S. Nojiri, S. Odintsov, S. Ogushi, Phys. Rev. D **67**, 063515 (2003) [arXiv:hep-th/0209242].

[177] A.O. Barvinsky, "Braneworld effective action and the origin of inflation", hep-th/0107244.

[178] Y. Himemoto, M. Sasaki, Phys. Rev. **D63** (2001) 04015.

[179] I. G. Moss and W. Naylor, "Quantum corrections to the kinetic term in the Randall-Sundrum model", [arXiv:hep-th/0106228].

[180] A. A. Saharian and M. R. Setare, Phys. Lett. **B552** (2003) 119

[181] I. Brevik, K. A. Milton, S. Nojiri and S. D. Odintsov, Nucl. Phys. **B599** (2001) 305.

[182] H. B. Casimir, Kon. Ned. Akad. Wetensch. Proc. **51** (1948) 793.

[183] G. Plunien, B. Muller and W. Greiner, Phys. Rept. **134** (1986) 87.

[184] M. Bordag, U. Mohideen and V. M. Mostepanenko, Phys. Rept. **353**, 1 (2001) [arXiv:quant-ph/0106045].

[185] S. K. Lamoreaux, Am. J. Phys. **67**, 850 (1999).

[186] M. J. Sparnaay, Physica **24**, 751 (1958).

[187] S.K. Lamoreaux, Phys. Rev. Lett. **78** (1997) 5 ;
U. Mohideen, A. Roy, Phys. Rev. Lett. **81** (1998) 4549 ;
A. Roy, U. Mohideen, Phys. Rev. Lett. **82** (1999) 4380 ;
A. Roy, C.-Y. Lin, U. Mohideen, Phys. Rev. **D60** (1999) 111101(R);
B.W. Harris, F. Chen, U. Mohideen, Phys. Rev. **A62** (2000) 052109 .

[188] H. B. Casmir and D. Polder, Phys. Rev. **73**, 360 (1948).

[189] E. M. Lifshitz, Sov. Phys. JETP **2**, (1956) 73.

[190] S. Blau, M. Visser and A. Wipf, Nucl. Phys. B **310**, 163 (1988).

[191] P. Ramond, "Field Theory: A Modern Primer," *(Frontiers in Physics, 74), Redwood City, USA: Addison-Wesley* (1989) 329p

[192] J. S. Dowker and J. S. Apps, Class. Quant. Grav. **12**, 1363 (1995) [arXiv:hep-th/9502015].

[193] S.W. Hawking, Commun. Math. Phys. **56**, 133 (1977); See also E. Elizalde, S. Odintsov, A. Romeo, A. Bytsenko and S. Zerbini "Zeta Regularization Techniques with Applications", *World Scientific*, Singapore, 1994; E. Elizalde, "Ten physical applications of spectral zeta functions," *Berlin, Germany: Springer (1995) 224 p. (Lecture notes in physics, M35)*; I Buchbinder, S.Odintsov and I. Shapiro, "Effective Action in Quantum Gravity", IOP Publishing, 1992.

[194] B.S. De Witt, "Quantum Field Theory in Curved Space-time" Phys. Rep. **19**, 295 (1975).

[195] T.P. Branson, and P.B. Gilkey, Commun. Partial Diff. Eq. **15**, 245 (1990).

[196] H.P. McKean Jr. and I.M. Singer, J. Diff. Geom. **1**, 43 (1967).

[197] K. Kirsten, *Spectral Functions in Mathematics and Physics*, Chapman & Hall/CRC, Boca Raton, FL (2001).

[198] M. Bordag, E. Elizalde, K. Kirsten, J. Math. Phys. **37** (1996) 895;
M. Bordag, E. Elizalde, K. Kirsten and S. Leseduarte, Phys. Rev. D **56**, 4896 (1997);
M. Bordag, J. S. Dowker, K. Kirsten, Comm. Math. Phys. **182** (1996) 371.

[199] I. Moss and J.S. Dowker, Phys. Lett. B **229**, 261 (1989).

[200] D.V. Vassilevich, "Vector fields on a disk with mixed boundary conditions," J. Math. Phys. **36**, 3174 (1995).

[201] J.S. Dowker and G. Kennedy, J. Phys. A 11, (1978); T. Branson and P.B. Gilkey, Trans. Am. Math. Soc. 344, (1994).

[202] K. Kirsten, Class. Quant. Grav. **15**, L5 (1998) [hep-th/9708081].

[203] *e.g.* D. Youm, Nucl. Phys. B **576**, 106 (2000) [hep-th/9911218];Nucl. Phys. B **596**, 289 (2001) [hep-th/0007252].

[204] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, Phys.Rev.**D64** (2001) 123522. [hep-th/0103239].

[205] R. Gregory, Phys. Rev. Lett. **84** (2000) 2564.

[206] T. Gherghetta, M. Shaposhnikov, Phys. Rev. Lett. **85** (2000) 240.

[207] T. Gherghetta, E. Roessl and M. E. Shaposhnikov, Phys. Lett. **B491** (2000) 353.

[208] I. Oda, Phys. Rev. **D64** (2001) 026002.

[209] S. Randjbar.Daemi, M. Shaposhnikov, Phys. Lett. **B491** (2000) 329.

[210] H. Davoudiasl, J. L. Hewett, T. G. Rizzo, JHEP **0304**, 001 (2003) [arXiv:hep-th/0211377].

[211] S. Randjbar-Daemi, M. Shaposhnikov, Phys. Lett. **B492** (2000) 361.

[212] Z. Chacko, P. J. Fox, A. E. Nelson and N. Weiner, JHEP **0203**, 001 (2002).

[213] Z. Chacko and A. E. Nelson, Phys. Rev. **D62**, 085006 (2000).

[214] T. Multamaki and I. Vilja, Phys. Lett. **B545**, 389 (2002).

[215] C. P. Burgess, J. M. Cline, N. R. Constable and H. Firouzjahi, JHEP **0201**, 014 (2002).

[216] D.J. Toms, Phys. Lett. **B129** (1983) 31.

[217] E. Elizalde, S. Leseduarte, A. Romeo, J. Phys. **A26** (1993) 2409;
S. Leseduarte, A. Romeo, J. Phys. **A27** (1994) 2483.

[218] S. M. Carroll, J. Geddes, M. B. Hoffman and R. M. Wald, Phys. Rev. D **66**, 024036 (2002).

[219] S. M. Carroll and M. M. Guica, "Sidestepping the cosmological constant with football-shaped extra dimensions," [arXiv:hep-th/0302067].

[220] I. Navarro, "Codimension two compactifications and the cosmological constant problem," arXiv:hep-th/0302129.

[221] J. W. Chen, M. A. Luty and E. Ponton, JHEP **0009** (2000) 012

[222] P. Kanti, R. Madden and K. A. Olive, Phys. Rev. **D64** (2001) 044021

[223] I. I. Kogan, S. Mouslopoulos, A. Papazoglou and G. G. Ross, Phys. Rev. **D64** (2001) 124014.

[224] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions*, Dover Pubications, Inc., New York, (1970)

[225] E. T. Whittaker, and G. N. Watson “A Course of Modern Analysis”, *Cambridge, UK: Univ. Pr. (1969) 608 p.*