Chapter 4

Restricted Diffusion

Anisotropic differential operators are widely used for image enhancement and restoration. However the capability of smoothly extending functions to the whole image domain has been hardly exploited.

As stated in chapter 1, functional extensions are governed by parabolic PDE's, which equal that of heat diffusion processes except for the boundary conditions. We saw that the process has naturally associated a metric, given by the diffusion tensor, that locally describes the way heat extends or distributes. Thus an anisotropic heat diffusion is the analytic way of handling a dilation with non-constant elliptic structural elements. In the context of level sets completion the tensor should degenerate/cancel in the gradient direction, which might not guarantee existence of solutions of the associated PDE. In the present chapter, we perform a study of diffusion tensors from the point of view of differential geometry which provides us with a criterion to decide when such degenerate tensors still produce solvable PDE's. We will define a new family of differential operators that locally restrict diffusion to the tangent spaces of image level sets. A particular instance of such operators results in an Anisotropic Contour Closing (ACC) that reduces the completion problem to the definition of a smooth vector field representing the level sets to be extended.

4.1 Restricted Anisotropic Operators

Any second order partial differential operator, namely L, defines, both, a diffusion process:

$$u_t = Lu \quad \text{with} \quad u(x,0) = u_0(x)$$

and a functional extension:

$$Lu = 0 \quad \text{with} \quad u_{|\gamma} = f$$

for γ a curve in the image domain. In both cases, existence and uniqueness (in the weak sense) of smooth solutions is guaranteed if L is strongly elliptic [23], [72], that is, when it defines a scalar product on some functional space. However, extensions focused on level sets continuation should restrict diffusion to a vector field representing the level sets of the solution. Thus, the hypothesis of strong ellipticity is relaxed and we must tackle with operators that degenerate on some vector fields.

Let us give some geometric requirements over the non null space of L that ensure existence of solutions in the case of an operator given in divergence form. In this case, we have that:

$$Lu = \operatorname{div}(J\nabla u)$$

for J a symmetric (semi) positive defined tensor (quadratic form). Strong ellipticity means that all eigenvalues of J are strictly positive, meanwhile tensors having a null space (kernel) of positive dimension will produce degenerate operators. The fact that scalar products are given by symmetric (semi) positive defined tensors, motivates embedding elliptic operators into the framework of Riemmanian geometry to study the degenerate case. Details regarding results on differential geometry can be found in [67].

Let $(\mathbb{R}^n, g)$ be a Riemmanian manifold with the metric, g, given by a tensor J. Since the matrix J is symmetric, it diagonalizes [44] (considered as linear map) in an orthonormal basis that completely describes the metric. If Q is the coordinate change, then we have that, as bilinear form, $J = Q\Lambda Q^t$, for Λ the eigenvalue matrix. In this context, **isotropic** diffusion corresponds to equal eigenvalues, **anisotropic** to distinct and strictly positive and **restricted** [1] to the case of null eigenvalues. That is, the restricted diffusion is given by a diffusion tensor, $\tilde{J}$, defined by the following eigenvalue matrix:

$$\Lambda = \left(\begin{array}{ccc|ccc} \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_m & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right)$$

Indeed we will only consider the homogeneous case $\lambda_i = 1$, for all i. Let us determine under what conditions a degenerated metric makes sense. Let $\xi_1, \ldots, \xi_k$ be the eigenvectors of positive norm and denote by $\mathcal{D} = \langle \xi_1, \ldots, \xi_k \rangle$ the vector space (distribution) they generate. If such vector space was the tangent space to a sub manifold of $\mathbb{R}^n$ (integral variety of $\mathcal{D}$), then the metric $\tilde{J}$ would be the projection onto its tangent space. Consequently a diffusion process governed by $\tilde{J}$ would not take place in the whole space $\mathbb{R}^n$ but just on the integral manifolds, namely M, of the distribution. We claim that this integrability condition and compactness are the only requirements for a unique solution to:

$$\operatorname{div}(\tilde{J}\nabla u) = 0 \quad \text{with} \quad u_{|\gamma} = f \tag{4.1}$$

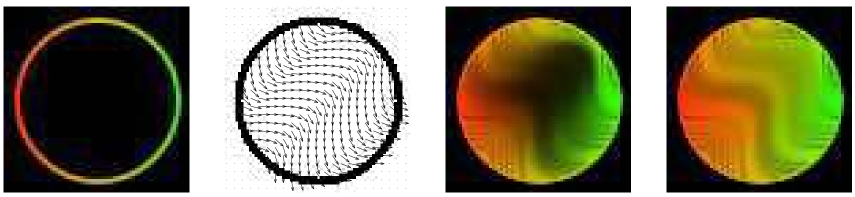
which is as smooth as the boundary function f. We remit the reader to Section 4.3 for rigorous mathematical arguments concerning the above and the next statements. Let us devote the rest of the Section to intuitive reasonings on the precise meaning

[1] The word restricted applies to the the fact that diffusion restricts to the manifolds generated by the vectors of positive eigenvalues

of equation (4.1), those cases which always satisfy the integrability condition and applications to image processing.

Although (4.1) does not coincide with the heat equation for manifolds, the effect of the operator $\mathrm{div}(\tilde{J}\nabla u)$ may be regarded as diffusing on each of the integral manifolds separately. For $u_{|M} = u_M$ not only solves an elliptic second order equation in the manifold M, but also enjoys of the same properties as solutions to the heat equation in $\mathbb{R}^n$:

1. Maximum Principle: the extreme values of the solution u_M are achieved on the boundary $M \cap \partial\Omega$. In the particular case of a single generator, ξ, as M is a curve, the effect of restricting diffusion is that the final extension changes linearly along the level lines of ξ. Figure 4.1 is an example of functional extension in an image. The function to be extended is a color map defined on the ring of fig.4.1(a). The vector, ξ, guiding extension is the sinusoidal of fig.4.1(b) where the extension scope has been restricted to the area enclosed by the ring. The linear rate of change of the final extension (fig.4.1(d)) along the ξ integral curves is better visualized in fig.4.2. The mesh representation of fig.4.2(a) is a cut of the mesh surface of u in the ξ direction, the corresponding plot for u_M is in fig.4.2(b). Because we have not achieved the steady state, the function u_M does not linearly interpolate the values at the boundary as it happens in the plot of fig.4.2(d).

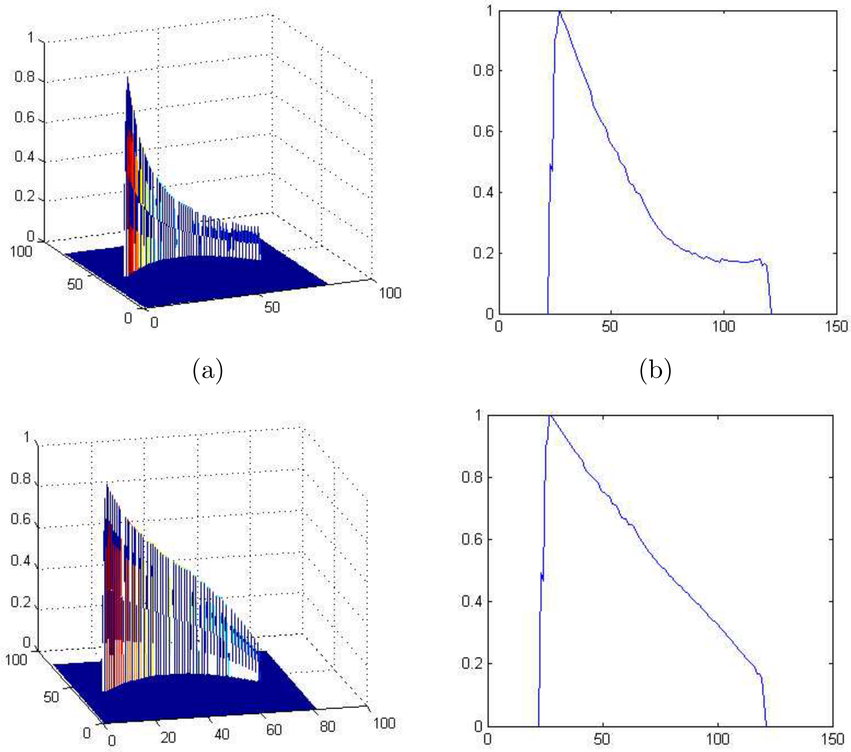


(a) (b) (c) (d)

Figure 4.1: General Extension: Function to extend (a), extension vector (b), intermediate step (c) and final extension (d)

2. Smoothness: the function u_M is smooth, provided $f_{|M}$ is also differentiable. Let us show the reader that the hypothesis that ξ can be integrated (i.e. induces a foliation) on the domain is essential in order to guarantee convergence to smooth functions. The vector shown in fig.4.3(a) has a singular point at the center of the image since all its integral curves meet there. Although the extension process still exists, it converges to the sharp image of fig.4.3(b), which has a jump discontinuity at the center of the image as it shows the angular cut of the mesh of fig.4.3(c).

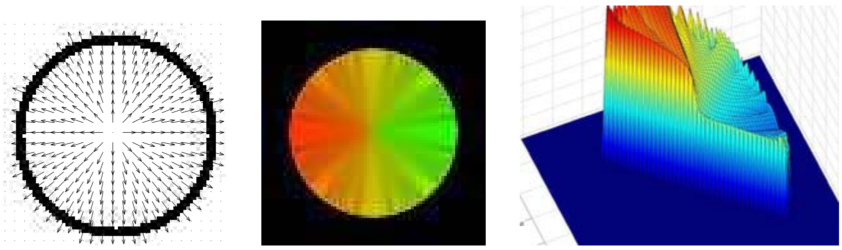
The integrability condition ensuring existence of solutions is a standard result on differential manifolds known as the Frobenius Theorem [67]. The latter states that there exist integral manifolds for a distribution $\mathcal{D}$ provided that the vectors generating $\mathcal{D}$ fulfill an algebraic condition ($\mathcal{D}$ involutive). That is, a local condition



(a) (b)

(c) (d)

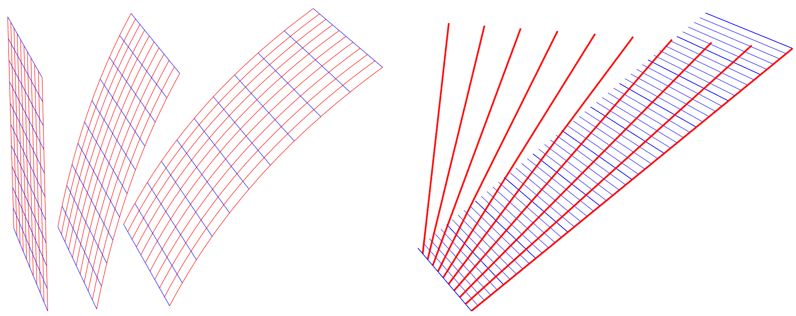
Figure 4.2: Rate of change along integral curves: intermediate step (a) with function plot (b) and final state (c) with function plot (d)



(a) (b) (c)

Figure 4.3: Singular Case: vector field (a), extension (b) and angular cut (c)

on the potential tangent spaces ensures that there will exist manifolds having $\mathcal{D}$ as tangent space. To be precise, the integrability condition is given in terms of the Lie bracket of $\xi_1, \ldots, \xi_k$ and, intuitively, measures whether the integral curves of the fields generated by the distribution can form a mesh or not. We note that for $k \geq 2$ the integral curves of the vectors ξ_k will not, in general, tangle into a web. For instance, $\xi_1 = \partial_y = (0,1,0)$ and $\xi_2 = -y\partial_x + \partial_z = (-y,0,1)$ generate the curves of fig.4.4(b), which as do not knit a mesh will never produce a surface. Meanwhile, the integral curves of $\xi_1 = \partial_y$ and $\xi_2 = x\partial_x + \partial_z$ form mesh surfaces (fig.4.4(a)) which decompose the space in layers (leaves of the foliation). An important remark for the foregoing discussion is that in the case of a single vector ξ the integrability condition is always satisfied.



(a) (b)

Figure 4.4: Frobenius Theorem: integrable (a) and non integrable (b) distributions

The instances of the process 4.1 relevant in image processing are the following:

1. **Image Restoration**

   If M is closed and transversal to the foliation and $\nabla f \subset \mathcal{D}^{\perp}$, then the leaves of the foliation, M, describe the level sets of the extension u, coinciding in the case of an $n-1$ dimensional distribution. This follows from the fact that if $\nabla f \perp \mathcal{D}$, then we have that $\partial\Omega \cap M \subset \{f \equiv const\}$ and, thus, $u_{|M}$ is constant. Besides if $\dim \mathcal{D} = n-1$, as $\nabla f = \nabla u$, our restricted diffusion conforms to the Gestalt principle that requires that level sets continuation should be differentiable at boundary junctions. That is, the solutions suit to the idea of reliable image restoration.

   In this case, the restricted diffusion should be regarded as a simultaneous integration of all manifolds having as tangent space $\mathcal{D}$. This drives (4.1) very close to the system of PDE's used by [4] for image in-painting. The main advantage of our formulation is that it admits a simple implementation using a finite difference Euler scheme for a non-linear heat equation.

2. **Image Enhancement and Filtering**

   The time-dependent parabolic equation associated to the restricted diffusion:

   $$u_t = \mathrm{div}_{\mathbb{R}^n}(\tilde{J}\nabla u)$$

   with initial condition, $u(x,0) = u_0(x)$ an arbitrary image, is an image filtering operator, provided that the vector ξ is a smooth approximation of the image level sets. Because diffusion is performed along image level sets, image features are enhanced, in the sense that they are of uniform gray-level in the final image. Although this version of (4.1) resembles the diffusion of [9], there are some relevant differences. First of all, the degenerate differential operator:

$$u_t = \frac{1}{\sqrt{1+|\nabla u|^2}} u_{\xi\xi} \tag{4.2}$$

used in [9] can not be interpreted as a diffusion restricted to the integral curves of ξ. This follows because, given a function $u = u(x_1, \ldots, x_n)$ on $\mathbb{R}^n$, its divergence on a manifold embedded in $\mathbb{R}^n$ is given by:

$$\begin{aligned}\mathrm{div}_M(J\nabla u) &= \sum_m \langle \nabla u, \xi_m\rangle_M \cdot \mathrm{div}_M(\xi_m) + \sum_m \xi_m(\langle \nabla u, \xi_m\rangle_M) = \\ &= \sum_m u_{\xi_m} \cdot \mathrm{div}_M(\xi_m) + \sum_m u_{\xi_m\xi_m} + \sum_m \langle \nabla u, \xi_m(\xi_m)\rangle\end{aligned}$$

for $u_{\xi_m\xi_m}$ the second derivative in the direction ξ_m. It follows that even in the case of $\mathrm{div}(\xi_m) = 0$, the equation (4.2) can either be written in divergence form nor interpreted as diffusing the function just on the integral curves of ξ. Meanwhile the development of (4.1):

$$\mathrm{div}_{\mathbb{R}^n}(\tilde{J}\nabla u) = \sum_m \langle \nabla u, \xi_m\rangle_M \cdot \mathrm{div}_{\mathbb{R}^n}(\xi_m) + \sum_m \xi_m(\langle \nabla u, \xi_m\rangle_M)$$

is a second order operator on the integral manifolds, which coincides with a heat kernel in the case of null divergence.

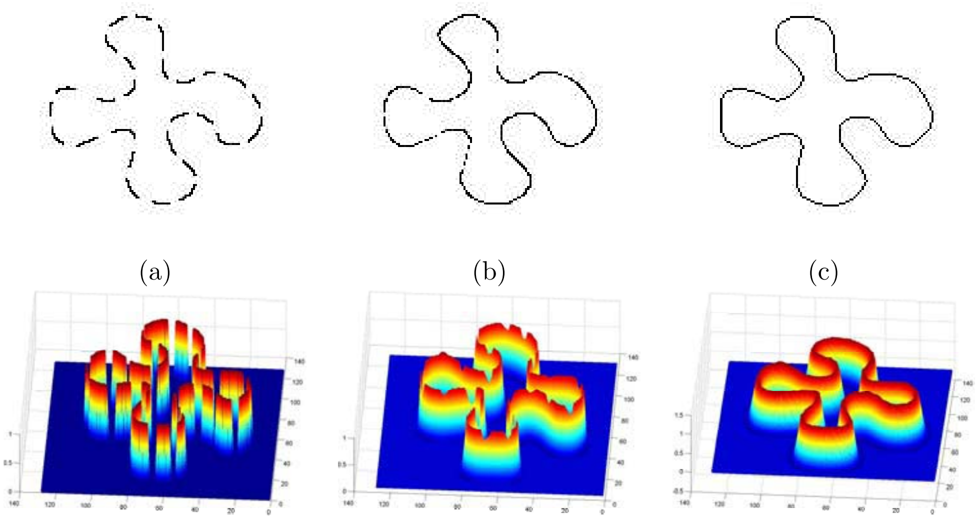
Besides lacking of a geometric interpretation, it is not guaranteed that steady states of (4.2) are non trivial. This fact forces adding the usual close-to-data constraint [62] or relying on a given number of iterations to ensure preservation of the image most relevant features. Its geometric nature makes our restricted diffusion evolution equation converge to a non trivial image that preserves the original image main features as curves of uniform gray level.

3. **Contour Closing**

   If the manifold M is included in one of the leaves of the foliation associated to $\mathcal{D}$ and $f_{|M} = const$, the restricted diffusion corresponds to curve continuation and surface gap filling. This is the case we will focus on.

4.2 Anisotropic Contour Closing

We model the contour completion process as follows. Denote by γ the set of points to connect. Let us assume that ξ is a unitary vector field defined on a band around the curve in the image domain that smoothly extends the curve unit tangent. If we consider a metric $\tilde{J}$ with eigenvectors $\eta = \xi^{\perp}$ and ξ and eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$, then equation (1.6) tells us that closed contours of the initial image are preserved during the evolution. Meanwhile for incomplete level curves, the effect of



(a) (b) (c)

(d) (e) (f)

Figure 4.5: Gap filling: clover (a), ridges of its mask extension (b), (c), image graph of uncomplete clover (d), intermediate step (e) and closing (f).

distributing heat only in the tangent direction, makes these curves evolve towards a closed contour of uniform gray level.

Therefore if we use this restricted anisotropic operator to extend a binary map of a unconnected curve (i.e. its characteristic function), the final state will be a binary map of a closed model of the uncomplete initial contour. This process is the Anisotropic Completion of Contours we suggest:

$$\begin{aligned} u_t &= \operatorname{div}(\tilde{J}\nabla u) \\ u_{|\gamma} &= u_0 \end{aligned} \tag{4.3}$$

with u_0 the characteristic function of the opened contours, χ_γ, and the diffusion tensor $\tilde{J}$ as described in the previous paragraph. Figure 4.5 illustrates the different stages in the process of gap filling for an incomplete clover (fig.4.8(a)). Ridges of the final characteristic function (fig.4.8(f)) correspond to the reconstructed complete contour (fig.4.8(c)).

From the above considerations, computation of an extension conforming to the image reduces to giving a smooth vector field representing its level curves.

4.2.1 Coherence Vector Fields

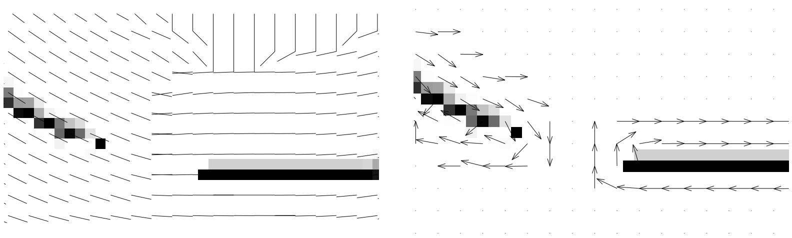
This section is devoted to the computation of a extension, ξ, of the unit tangent of an unconnected curve γ_0 smooth in a band surrounding the curve. Following the ideas presented in [26], we will use the Structure Tensor, St, upon a suitable function to compute the vector field ξ. Notice that since orientations do not play any role in the diffusion process, the eigenvectors of St serve to design diffusion tensors [72].

The Structure Tensor is usually employed to determine the direction of maximum contrast change of an image u in a robust way [72]. Given an integration scale, ρ, it is defined as the mean of the projection matrices, $P(\nabla u_\sigma)$, onto a regularized image gradient:

$$St_\rho = G_\rho * P(\nabla u_\sigma) = G_\rho * (\nabla u_\sigma \otimes \nabla u_\sigma) = G_\rho * \left(\nabla u_\sigma \nabla u_\sigma^T\right)$$

where G_ρ denotes a centered gaussian of variance ρ and $\nabla u_\sigma = G_\sigma * \nabla u$. The eigenvector of minimum eigenvalue, ξ, (coherence direction) corresponds to the level sets unit tangent direction.

Since by convolving with a gaussian we obtain solutions to the heat equation, we have that the Structure Tensor benefits from the regularizing and extension properties of diffusion processes. The matrix St_ρ is the solution to the heat equation with initial condition the projection matrix onto the image gradient vector space. This implies that the coherence direction, ξ, is a infinitely differentiable field [24] that regularizes and extends the level curves unit tangent space. This property is the key point for the definition of the different Coherence Vector Fields. We will note by γ_0 the contour to be closed and by t_0 its tangent space.

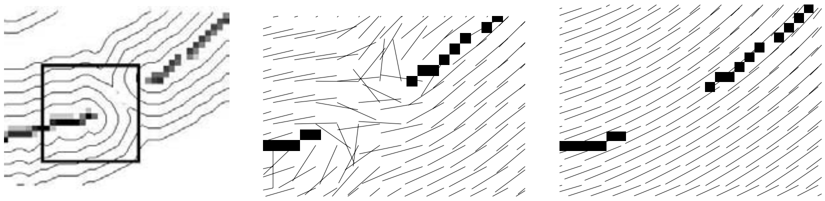


(a) (b)

Figure 4.6: LVF extension (a) of tangent vector at a gap (b)

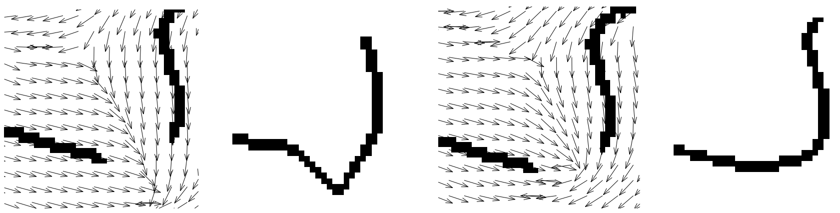
1. **Linear Vector Fields (LVF).**

   They are the coherence direction of the Structure Tensor compute over the characteristic function χ_{γ_0}. By the above properties, ξ is a smooth vector field defined (i.e. non zero) in a neighborhood of γ_0 that correctly matches t_0 at gap boundaries and coincides with it at other points. Figure 4.6 shows the gradient of χ_{γ_0} (fig.4.6 (b)) used to compute the vector ξ of fig.4.6 (a). Since ξ scope may not be large enough as to fill all gaps, it must be updated during the extension process. Intuitively, this dynamic process yields closed shapes that resemble the one we would get if we drew the tangent at the boundary points of the original curve and intersected the lines. This might lead to undesirable wrong models (fig.4.8 (b)) when the angle between the unit tangent of two consecutive pieces is too acute as the vector field becomes singular (fig.4.8 (a)). Besides, this pathology of the vector guiding the extension difficulties stopping the process.



(a) (b) (c)

Figure 4.7: DVF extension: distance map (a), tangent spaces (b), DVF (c)



(a) (b) (c) (d)

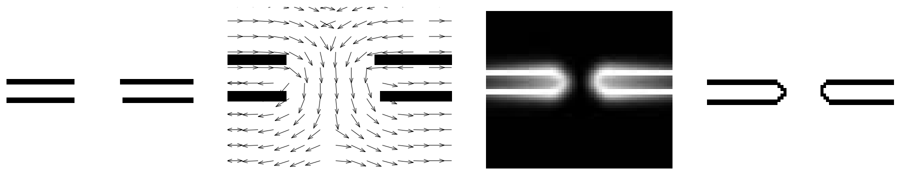
Figure 4.8: Corners Extension: LVF (a), LVF closure (b), DVF (c) and closure (d)

Note that LVF is nothing but the coherence direction of the convolution $G_\sigma * \chi_{\gamma_0}$, which level sets might be regarded as a propagation of the original curve. The fact that geometric flows are the natural way of propagating and deforming shapes leads to:

2. **Distance Vector Fields (DVF).**

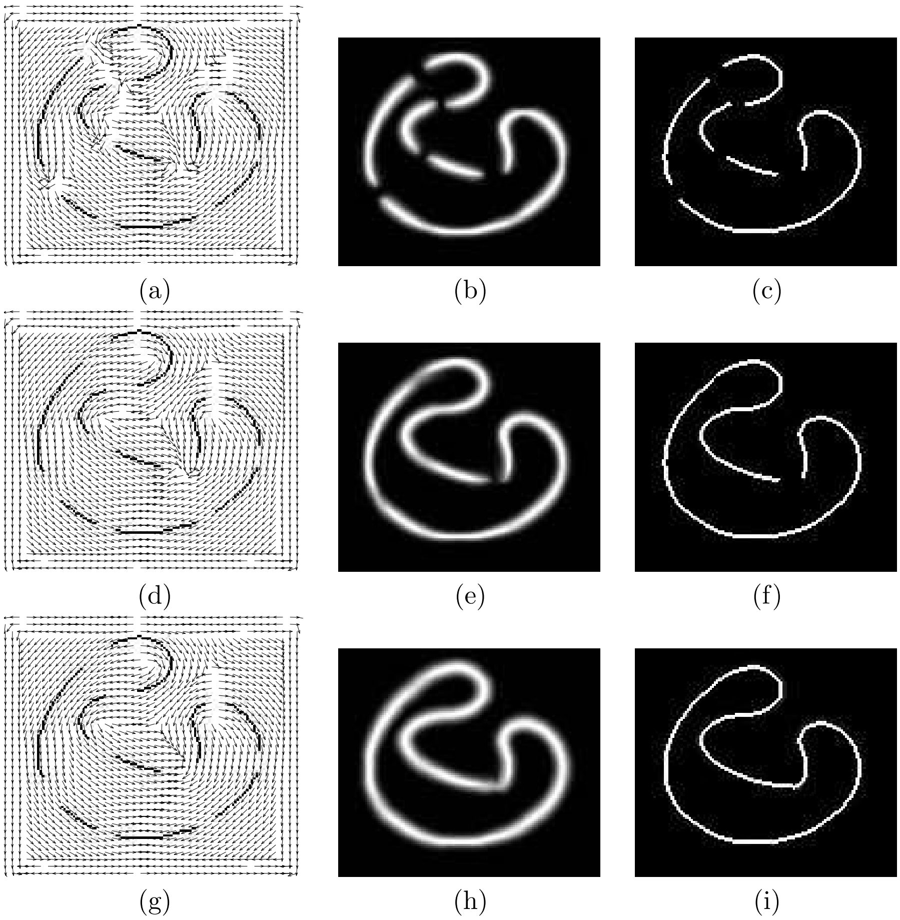
   Let u be a function representing the evolution of γ_0 under a monotonous geometric flow, $\gamma_t = \beta \overrightarrow{n}$, with $\beta > 0$. That is, the level sets, $u \equiv c$, coincide with the evolution of γ_0 at time $t = c$. Notice that at a suitable time/scale, ρ, (gap dependent) the level curves of this distance map have only two connected components that model a closure of γ_0 (fig.4.7(a)). It follows that the coherence direction, ξ, models a closure of γ_0 (fig.4.7(c)) that smoothly interpolates t_0 at gaps and is not singular at corners (fig.4.8(c)). The shapes obtained are smooth models (fig.4.8(d)) of the original shape based on the principle of minimum distance for joining boundary points. Although, this is a desirable property, in the particular case of a distance between contours smaller than the gap size, LCV is preferable, as DVF closed models do not conform to the shape yielded by our visual system. For instance, DVF reconstruction of the broken lines of fig. 4.9(a) are not two straight lines, as expected, but the curves of fig.4.9 (b). We remit the reader to the experiments in chapter 5.2 for illustrative examples on the choice of the coherence vector field in practical applications

We have chosen a non convex contour in order to illustrate the dynamic process of contour completion in fig.4.10. If gaps are not too large DVF can be computed



(a) (b) (c) (d)

Figure 4.9: DVF pathology: lines (a), DVF (b), final extension (c), DVF closure (d)



(a) (b) (c) (d) (e) (f) (g) (h) (i)

Figure 4.10: Dynamic Closing of Contours

once at the beginning of the process. However, in the general case, in order to get the maximum accuracy as possible, we recommend updating DVF over the ridges of the current image evolution every k iterations. The initial DVF (fig.4.10(a)) closes the smallest gaps and reduces the length of the gaps in the ridges (fig.4.10(c)) of the evolved image (fig.4.10(b)). DVF over the distance map of image ridges yields a vector field (fig.4.10(d)) that closes all gaps with the exception of the largest one (fig.4.10(e), (f)), which is the most delicate since we must interpolate a piece of high curvature. The DVF that finally closed this gap is the vector field of (fig.4.10(g)). Ridges of the final extension (fig.4.10(h)) are the reconstructed shape of fig.4.10(i) and represent a smooth and accurate completion of the contour regardless of the magnitude of the curvature.

4.3 Mathematical Issues

This section deals with the mathematical arguments that yield existence of solutions to the restricted diffusion. In mathematical formal terms, the problem is stated as follows. Let Ω be a compact subset of $\mathbb{R}^n$ with a smooth boundary, $\partial\Omega$, and let $f = f(x_1, \dots, x_n)$ be a smooth function defined on $\partial\Omega$. We seek for solutions to the following extension problem:

$$\operatorname{div}(\tilde{J}\nabla u) = 0 \quad \text{with} \quad u_{|\partial\Omega} = f \tag{4.4}$$

for $\tilde{J}$ the projection matrix onto a distribution $\mathcal{D} = \langle \xi_1, \dots, \xi_k \rangle$ transversal to $\partial\Omega$ almost everywhere:

$$\tilde{J} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_k \end{pmatrix} \left(\begin{array}{c|c} Id_k & 0 \\ \hline 0 & 0 \end{array} \right) (\xi_1, \dots, \xi_k)$$

Because the diffusion tensor cancels on some curves, standard arguments on elliptic PDE do not apply. One way of proving existence and uniqueness to (4.4) could be through viscosity solutions [23], [20]. We, instead, will approach the problem from a differential geometry point of view and show that (4.4) yields a heat equation on the integral varieties of the distribution $\mathcal{D}$ (if it is involutive), which coincides with the laplacian operator [21] for manifolds provided that the divergence of the fields ξ_i is zero. Existence of solutions on a generic leave of the folliation suffices to guarantee existence and uniqueness of functions $u = u(x_1, \dots, u_n)$ solving the restricted diffusion problem (4.4) in $\mathbb{R}^n$.

4.3.1 Solutions to the Problem on Manifolds

In order to show that (4.4) is solvable on a generic leave, M, we will first compute its expression in local coordinates and, then, construct a solution.

Relation between Restricted Diffusion and the Heat Equation for Manifolds

We start with a brief introduction to notations and definitions. For a given Riemanian manifold, N, its scalar product will be noted by $\langle \cdot, \cdot \rangle_N$, the divergence of a vector field by $\mathrm{div}_N(\cdot)$, Lie derivatives by L and the Lie bracket by $[\cdot,\cdot]$. We recall that if $(x_1,\ldots,x_N)$ are local coordinates, then the partial derivatives ∂_{x_i} are a basis of the tangent spaces T_pN and its dual, dx_i, are a basis of the cotangent (linear forms) T_p^*N.

It follows that, in local coordinates, a vector field ξ equals $\sum \xi^k \partial_{x_k}$, the derivative, $\xi(f)$, of any smooth function $f : N \to \mathbb{R}$ is given by:

$$\xi(f) = \sum \xi^k \partial_{x_k} f$$

and the metric by a (symmetric) matrix $(g_{ij})_{ij}$. Furthermore, if $g = \det(g_{ij})$ is its determinant, then the volume form equals $\omega = g^{1/2} dx_1 \ldots dx_N$. The divergence of a vector field ξ can be defined as the Lie derivative of ω along ξ:

$$L_\xi \omega = \mathrm{div}_N(\xi)\omega$$

The quantity measures the change of a unit of volume along the integral curves of ξ (fig. ....). The fact [67] that:

$$\xi(\omega(\partial_{x_1},\ldots,\partial_{x_1})) = (L_\xi\omega)(\partial_{x_1},\ldots,\partial_{x_1}) - \sum_i \omega(\partial_{x_1},\ldots,[\xi,\partial_{x_i}],\ldots,\partial_{x_1})$$

yields that the expression of $\mathrm{div}_N(\xi)$ in local coordinates is:

$$\mathrm{div}_N(\xi) = g^{-1/2}\xi(g^{1/2}) + \sum_i \partial_{x_i}(\xi^i) \tag{4.5}$$

Given a smooth function, f, it has an associated form, df, defined as:

$$df(\xi) = \xi(f) = \sum \xi^k \partial_{x_k}(f)$$

The gradient, ∇f, of such function it is defined [21] as $j^{-1}(df)$, for:

$$j : T_pN \to T_p^*N$$

the isomorphism given by the metric $j(\xi)(\eta) := \langle \xi, \eta \rangle_N$. It follows that:

$$\langle \nabla f, \xi \rangle_N = j(j^{-1}(df))(\xi) = \xi(f) \tag{4.6}$$

The above formulae (4.5), (4.6) are the only expressions we need to compute (4.4) in the local coordinates of a generic leave M. If M is a manifold embedded in $\mathbb{R}^n$, then given a coordinate chart:

$$\begin{array}{cccc} \phi : & U \subset \mathbb{R}^k & \longrightarrow & M \subset \mathbb{R}^n \\ & s = (s_1,\ldots,s_k) & \mapsto & (\phi_1(s),\ldots,\phi^n(s)) = x \end{array}$$

The vector fields ∂_{s_i} are given by $\partial_{s_i}(\phi) = (\phi^1_{s_i},\ldots,\phi^n_{s_i}) = \phi^1_{s_i}\partial_{x_1} + \cdots + \phi^n_{s_i}\partial_{x_n}$ and the scalar product by $(g_{ij})_{ij} = (\langle \partial_{x_i}, \partial_{x_j} \rangle_{\mathbb{R}^n})_{ij}$. In the case that M is an integral manifold of a distribution $\mathcal{D}$, its generators $\xi_1, \ldots, \xi_k$ can be expressed in, both, s and x coordinates with the relation:

$$\xi_m = \xi_m(x_1, \ldots, x_n) = \sum_j \tilde{\xi}_m^j \partial_{x_j} = \xi_m(s_1, \ldots, s_k) = \sum_k \xi_m^k \partial_{s_k} =$$
$$= \sum_k \xi_m^k (\sum_i \phi_{s_k}^i \partial_{x_i}) = \sum_j (\sum_k \xi_m^k \phi_{s_k}^i) \partial_{x_j} \tag{4.7}$$

Equipped with the above tools and notations formula (4.4) writes:

$$\begin{aligned} \mathrm{div}_{\mathbb{R}^n}(\tilde{J}\nabla u) &= \mathrm{div}_{\mathbb{R}^n}(\langle \nabla u, \xi_m \rangle_{\mathbb{R}^n} \cdot \xi_m) = \\ &= \sum_m \langle \nabla u, \xi_m \rangle_{\mathbb{R}^n} \cdot \mathrm{div}_{\mathbb{R}^n}(\xi_m) + \sum_m \xi_m(\langle \nabla u, \xi_m \rangle_{\mathbb{R}^n}) = \\ &= \sum_m \langle \nabla u, \xi_m \rangle_M \cdot \mathrm{div}_{\mathbb{R}^n}(\xi_m) + \sum_m \xi_m(\langle \nabla u, \xi_m \rangle_M) \end{aligned} \tag{4.8}$$

where the last equality follows from the relations (4.6) and (4.7) in the case that $\mathrm{div}_{\mathbb{R}^n}(\tilde{J}\nabla u)$ is restricted to M. Before deducing from the above expression the equation in local coordinates that yields existence and uniqueness of solutions to (4.4) on M, let us show the relation between (4.8) and heat equations on manifolds.

As in the case of $\mathbb{R}^n$, heat equations on manifolds are given by divergence operators $\mathrm{div}_M(J\nabla u)$, for J a symmetric positive defined 2-form (i.e. a metric). It follows that if $\xi_1, \ldots, \xi_k$ are the unitary vector fields, then:

$$\mathrm{div}_M(J\nabla u) = \sum_m \langle \nabla u, \xi_m \rangle_M \cdot \mathrm{div}_M(\xi_m) + \sum_m \xi_m(\langle \nabla u, \xi_m \rangle_M) \tag{4.9}$$

The Laplacian corresponds to the case $J = Id$.

If the vector fields have null divergence in $\mathbb{R}^n$, then the two expressions (4.8) and (4.9) are equal to:

$$\sum_m \xi_m(\langle \nabla u, \xi_m \rangle_M) = \sum_m \xi_m(\langle \nabla u, \xi_m \rangle_{\mathbb{R}^n}) = \sum_m \xi_m(u_{\xi_m})$$

where u_{ξ_m} is the partial derivative (in $\mathbb{R}^n$) in the direction $\xi_m(x_1, \ldots, x_n)$. This coincidence of formulations leads to:

Proposition 4.3.1 *If $\mathrm{div}_{\mathbb{R}^n}(\xi_i) = 0$, for all i, then there exists a unique solution to the problem (4.4) on each integral manifold of $\mathcal{D}$, provided that they are complete (compact).*

Proof. The proof is straightforward by existence of heat kernels in manifolds [21]. □

Remark: We note that, in the 2 dimensional case, it suffices that $\xi = \nabla f^{\perp}$.

In the general case, we need the expression (4.8) in local coordinates for a suitable open covering (so that boundary conditions are taken into account) to construct a solution. Although this constitutes an extra theoretic effort, from the computational point of view the restricted diffusion has several advantages over the heat equation for manifolds. Non existence of an explicit formulation for the heat kernel, makes that, in any case, both equations (4.8), (4.9) are solved numerically. The best way of handling equations over manifolds (curves) is via an implicit formulation [65]. A main advantage is that (4.4) provides with a global expression for (4.8) in $\mathbb{R}^n$ coordinates which is easily integrated using an explicit Euler scheme for non-linear heat equations. Besides because the operator (4.8) is of second order it enjoys from the same smoothing and regularizing properties as (4.9).

Existence of Solutions and Properties

Let us first give the expression in local coordinates. Using the same notations as in the previous Section, we have that, in a generic local chart $(s_1, \dots, s_k)$, the operator is given by:

$$\begin{aligned}\mathrm{div}_{\mathbb{R}^n}(\tilde{J}\nabla u)_{|M} &= \sum_m \langle \nabla u, \xi_m\rangle_M \cdot \mathrm{div}_{\mathbb{R}^n}(\xi_m) + \sum_m \xi_m(\langle \nabla u, \xi_m\rangle_M) = \sum_m f_m(s)\xi_m(u) + \\ &+ \sum_m \xi_m(\xi_m(u)) = \sum_m u_{\xi_m\xi_m} + \sum_m f_m(s)\xi_m(u) + \langle \nabla u, \xi_m(\xi_m)\rangle_M\end{aligned}$$

where $\xi_m(\xi_m)$ stands for the vector filed $\sum_k \xi_m(\xi_m^k)\partial_{s_k}$. The second order term $\sum_m u_{\xi_m\xi_m}$ is of elliptic type. In fact, since the vectors ξ_k are orthonormal, we have the following matrix equalities:

$$(\xi_m^k)^{-1}_{km}(u_{s_is_j})_{ij}(\xi_m^k)_{km} = (\xi_m^k)^t_{km}(u_{s_is_j})_{ij}(\xi_m^k)_{km} = (u_{\xi_i\xi_j})_{ij}$$

Invariance of traces under linear coordinate changes, yields that $\sum_m u_{s_ms_m} = \sum_m u_{\xi_m\xi_m}$. It follows that, in local coordinates (4.8) equals:

$$\underbrace{\sum_m u_{s_ms_m}}_{\text{2nd order}} + \underbrace{\sum_m f_m(s)\xi_m(u) + \langle \nabla u, \xi_m(\xi_m)\rangle_M}_{\text{1st order}} \tag{4.10}$$

which is an elliptic second order operator. General arguments on elliptic PDE yield the following result:

Theorem 4.3.1 *Let M be a compact Riemmannian manifold embedded in $\mathbb{R}^n$ with boundary ∂M. If ξ_m are orthonormal vectors fields and f is a smooth function, then the following boundary problems have a unique smooth solution:*

1. $\mathrm{div}_{\mathbb{R}^n}(\tilde{J}\nabla u)_{|M} = f$, *with* $u_{|\partial M} = 0$
2. $\mathrm{div}_{\mathbb{R}^n}(\tilde{J}\nabla u)_{|M} = 0$, *with* $u_{|\partial M} = f$
3. $u_t = \mathrm{div}_{\mathbb{R}^n}(\tilde{J}\nabla u)_{|M}$, *with* $u_{|\partial M} = f$ *and initial condition a smooth function* u_0.

for $\tilde{J}$ the projection matrix onto $\langle \xi_1, \dots, \xi_k\rangle$.

Proof. The operator $\mathrm{div}_{\mathbb{R}^n}(\tilde{J}\nabla u)_{|M}$ will be noted by Lu for short.

1. $\mathrm{div}_{\mathbb{R}^n}(\tilde{J}\nabla u)_{|M} = f$, with $u_{|\partial M} = 0$

   Since M is compact, whatever coordinate covering, it exists a finite sub covering, U_i. Let φ_i be a partition of unity subordinated to this covering and note $f_{U_i} = \varphi_i f$. For any such coordinate set, the equation converts into the elliptic PDE given by (4.10). It follows that the local problem:

   $$Lu^i = f_{U_i} \qquad u_{|\partial U_i} = 0$$

   has a unique smooth solution defined on U_i. We claim that the function, u, defined as:

   $$u(p) = \sum_i u^i(p)$$

   is a solution. First note that it is well defined as the summation is finite. Second, by linearity of L, we have that:

   $$Lu = \sum_i Lu^i = \sum_i f_{U_i} = (\sum \varphi_i)f = f$$

   as $\sum_i \varphi_i(p) = 1$. It only remains to check that on the boundary ∂M the function cancels. If V_k is the subset of U_i intersecting the boundary, then $\partial V_k \cap \partial M$ is a covering. It follows that, for any $p \in \partial M$, $u(p) = \sum_k u^k(p) = 0$.

2. $\mathrm{div}_{\mathbb{R}^n}(\tilde{J}\nabla u)_{|M} = 0$, with $u_{|\partial M} = f$

   If $\tilde{u}$ solves:

   $$\mathrm{div}_{\mathbb{R}^n}(\tilde{J}\nabla \tilde{u})_{|M} = \tilde{f}, \qquad u_{|\partial M} = 0$$

   for $\tilde{f} = Lf$. Then $u = f - \tilde{u}$ solves 2.

3. $u_t = \mathrm{div}_{\mathbb{R}^n}(\tilde{J}\nabla u)_{|M}$, with $u_{|\partial M} = f$ and initial condition u_0

   It follows by existence of solutions to 2, by general arguments on PDE's.

□

Solutions enjoy from identical properties than solutions to the problem in Euclidean space:

Proposition 4.3.2 *The solution to the extension problems 2 and 3 of Theorem 4.3.1 is infinitely smooth and fulfills the maximum principle:*

$$\max_M u = \max_{\partial M} u = \max_{\partial M} f, \ \textit{for the case 2}$$

$$\max_t u(t) = \max(u_0, f), \ \textit{for the case 3}$$

Proof. Straightforward since both properties are local and the expressions in local coordinates given by (4.10) satisfy them. □

Remark: If the boundary function f is not smooth, the solution is $\mathcal{C}^\infty$ only on the interior of the gap.

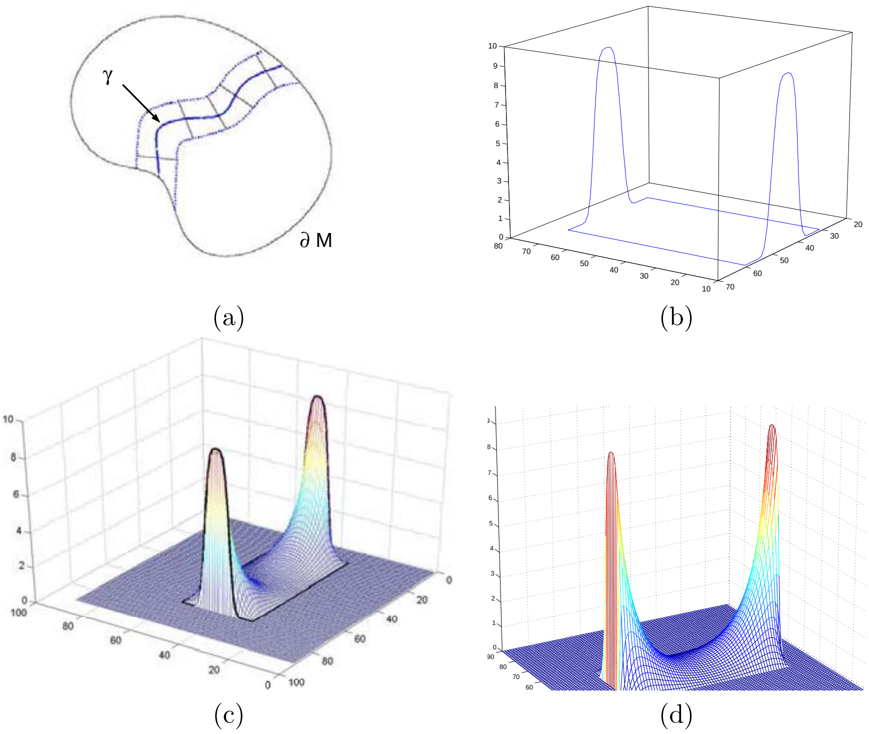


Figure 4.11: Extension on coordinate chart: tubular chart (a), function on the boundary (b), extension (c) and maximum principle (d)

Figure 4.11 illustrates the behavior of a solution to the parabolic problem 3 in a strip-like coordinate chart joining two pieces of the boundary ∂M. We note that we can always assume this kind of covering. This follows because, as M is geodesically complete (it is compact), for any two points on the boundary there exists a geodesic, γ, achieving the distance between them. Now by the embedding $M \hookrightarrow \mathbb{R}^n$, this geodesic is also a curve in $\mathbb{R}^n$, where it has a trivial normal bundle. Projecting the latter onto the distribution, we have that the geodesic normal bundle in M is also trivial. Any tubular neighborhood associated to a basis, $\eta_1, \ldots, \eta_{k-1}$, of the bundle yields a strip-shaped coordinate chart:

$$(s, r_1, \ldots, r_{k-1}) \mapsto \gamma(s) + \sum r_j \eta_j$$

joining to pieces of ∂M. The drawing in fig.1.11(a) outlines the construction of such strip coordinate chart and the plot of fig.4.11(b) represents the function φf in the case of a constant boundary function. On such a band around γ, the function interpolating the boundary values (fig.4.11(c)) satisfies the maximum principle and is of hyperbolic type (fig.4.11(d)) in the two dimensional case. The sum of contributions for all open charts would yield the constant function on M.

This Theorem is the basis for the main result of the chapter:

4.3.2 Solutions to the General Problem

The goal of this section is to prove that (4.4) has a unique solution which is as differentiable as the boundary function f, provided that $\mathcal{D}$ is involutive and that for all integral manifolds, M, the function f is smooth on $\partial\Omega \cap M$. We will first prove existence and uniqueness and, then, approach differentiability of solutions.

Theorem 4.3.2 *Let Ω be a compact subset of $\mathbb{R}^n$ with a smooth boundary, $\partial\Omega$ and $\mathcal{D} = \langle \xi_1, \ldots, \xi_k \rangle$ an involutive distribution. If f is a function defined on $\partial\Omega$ such that restricted to the integral manifolds od $\mathcal{D}$ is smooth, then there is a unique solution to the restricted diffusion problem given by:*

$$\mathrm{div}(\tilde{J}\nabla u) = 0 \quad \textit{with} \quad u_{|\partial\Omega} = f$$

where $\tilde{J}$ is the projection matrix onto $\mathcal{D}$.

Proof. As shown in the previous Sections, the differential operator restricts to a elliptic PDE on each of the integral manifolds, M, of the distribution $\mathcal{D}$, which, by Theorem 4.3.1, has a unique smooth solution, namely u_M, that extends $f_{|\partial\Omega\cap M}$ to the whole leave. Now, by Frobenius Theorem these manifolds foliate the whole space, so that $\forall p \in \mathbb{R}^n$, there exists a unique leave Mp through the point. The function in $\mathbb{R}^n$ defined as $u(p) = u_{Mp}(p)$ solves the general equation. □

Theorem 4.3.3 *Solutions given by Theorem 4.3.2 are as smooth as the boundary function f is on $\partial\Omega$.*

Proof. Because it suffices to check the statement locally, we can assume that we are in a generic coordinate chart given by the Frobenius theorem, where the leaves of the distribution $\mathcal{D}$ correspond to the hyper-planes $\{x_1 = c_1, \ldots, x_M = c_M\}$, for $M = n - k$. We will show that for any such neighborhood cutting $\partial\Omega$, the solution $u(x_1, \ldots, x_n)$ is as differentiable as f. The statement follows by, iteratively, repeating the argument to the function u restricted to the boundary of the closure of an interior neighborhood.

Let us assume that f is $\mathcal{C}^n$ and that the coordinate system is the one given by Frobenius. We recall that by compactness of Ω, the boundary function f is, indeed, uniformly smooth and that by Proposition 4.3.2, we already have that u is uniformly $\mathcal{C}^n$ with respect to the coordinates, $(x_M, \ldots, x_n)$, that define the distribution. In order to prove differentiability with respect to the other coordinates, we note that because the distribution is transversal to $\partial\Omega$ almost everywhere there are only two possibilities:

1. All leaves $\{x_1 = c_1, \ldots, x_M = c_M\}$ are transversal to the boundary.

   In this case (fig.4.12(a)), modulo a permutation of the last coordinates $(x_M, \ldots, x_n)$, the boundary is given by a graph $x_n = \phi(x_1, \ldots, x_{n-1})$. It follows that in coordinates $(Y, X) = (y_1, \ldots, y_M, \tilde{x}_{M+1}, \ldots, \tilde{x}_n) = (x_1, \ldots, x_M, x_{M+1}, \ldots, x_{n-1}, x_n -$

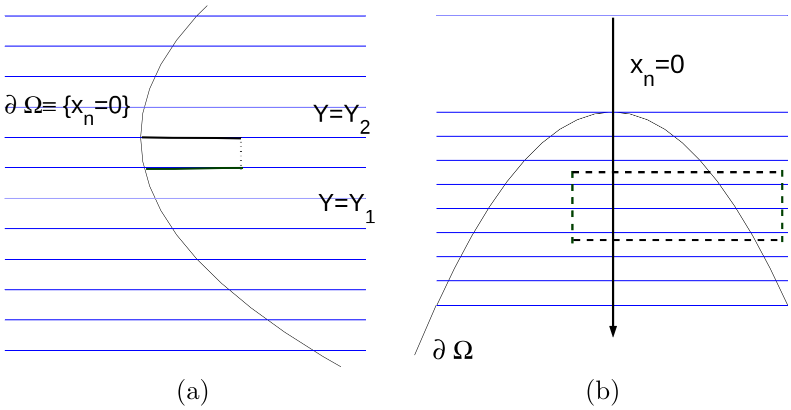


Figure 4.12: Foliation transverse (a) and tangent (b) to the boundary

$\phi(x_1, \ldots, x_{n-1})$), the integral manifolds are given by $\{Y = const\}$ and the boundary by $\partial\Omega = \{x_n = 0\} \supseteq \{X = 0\}$. Using this coordinate system, we have that:

$$\begin{aligned}
&|u(Y_1, X_0) - u(Y_2, X_0)| \leq |u(Y_1, 0) - u(Y_1, X_0)| + |u(Y_1, 0) - u(Y_2, 0)| + \\
&+ |u(Y_2, 0) - u(Y_2, X_0)| \leq |u_{|M}(0) - u_{|M}(X_0)| + |f(Y_1, 0) - f(Y_2, 0)| + \\
&+ |u_{|M}(0) - u_{|M}(X_0)| \leq C_M ||X_0|| + C_M ||X_1|| + C_f ||Y_1 - Y_2||
\end{aligned}$$

The last inequality by uniform continuity of both $u_{|M}$ and f and, holding, for any partial derivative of u of order less or equal to n. We conclude that the function u is as smooth as f.

2. There is a (single) leave tangent to the boundary as in fig.4.12(b).

   As the boundary is not diffeomorphic to any hyper plane defined by the last $n - M$ coordinates, we can not directly apply the above argument. We claim that the solution restricted to the axis $\{x_n = 0\} \supseteq \{X = 0\}$ is as smooth as f, which reduces this case to the previous one. Intuitively, such degree of differentiability follows from the fact that the extension process may be understood as propagating the values of f along the leaves. The formal proof is a straight consequence of the argument given in the first case applied to a neighborhood not containing the tangent leave (like the dotted square of fig.4.12(b)).

□

. . . .