Inverse problems of the Darboux theory of integrability for planar polynomial differential systems

Chara Pantazi

$X\alpha\rho\alpha \quad \Pi\alpha\nu\tau\alpha\zeta\eta$

Departament de Matemàtiques
Universitat Autònoma de Barcelona

Director: Dr Jaume Llibre

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Problemes inversos de la Teoria de la Integrabilitat de Darboux pel sistemes polinomials diferencials al pla

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Certifico que la present memòria ha estat
realitzada per la Chara Pantazi i constitueix,
la seva Tesi per a aspirar al grau de Doctor en
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Introduction

Differential equations appear in many areas of applied mathematics and physics. For a 2–dimensional system the existence of a first integral determines completely its phase portrait. Of course, the more easiest planar integrable systems are the Hamiltonian ones. The planar integrable systems which are not Hamiltonian can be in general very difficult to detect. Many different methods have been used for studying the existence of first integrals for non–Hamiltonian systems based on: Noether symmetries [7], the Darboux theory of integrability [22], the Lie symmetries [43, 9], the Painlevé analysis [2], the use of Lax pairs [34], the direct method [28, 29], the linear compatibility analysis method [49], the Carlemann embedding procedure [8, 1], the quasimonomial formalism [3], etc. In this work we are interested in the integrability of the planar polynomial differential systems. For such systems there are several notions of integrability, as the previous mentioned, which are not equivalent.

The algebraic theory of integrability is a classical one, which is related with the first part of the Hilbert's 16th problem. This kind of integrability is usually called Darboux integrability, and it provides a link between the integrability of polynomial systems and the number of invariant algebraic curves they have (see Darboux [22] and Poincaré [44]).

Jouanolou [31] extended the planar Darboux theory of integrability to polynomial systems in $\mathbb{R}^n$ or $\mathbb{C}^n$, for extension to other fields see [51]. In [4], [15], [17], [21] and [35], the authors developed the Darboux theory of integrability essentially in $\mathbb{R}^2$ or $\mathbb{C}^2$ considering not only the invariant algebraic curves but also the exponential factors, the independent singular points and the multiplicity of the invariant algebraic curves.

We define a planar polynomial differential system of degree m of the form $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ with $x, y \in \mathbb{C}$ and $t \in \mathbb{C}$ or $\mathbb{R}$. Here we want to study the so called *Darboux integrability* for these planar polynomial systems. A polynomial system is *Darboux integrable* if it has a first integral or a an integrating factor given by a Darboux function. In 1878 Darboux [22] showed that planar polynomial differential systems having an adequate number of invariant algebraic curves have a first integral which can be constructed using such curves.

The Darboux theory of integrability has been improved since its beginning. In Chapter 1 we present a survey on the Darboux theory of integrability for the planar polynomial differential systems. In particular, Darboux showed that for a planar polynomial differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ of degree m with divergence $\text{div}(P, Q)$ which has at least p invariant algebraic curves $f_i = 0$ with cofactors K_i for $i = 1, \ldots, p$, satisfying the relation

$$\sum_{i=1}^{p} \lambda_i K_i + \rho \, \text{div}(P, Q) = 0, \tag{1}$$

for some $\lambda_i \in \mathbb{C}$ not all zero and $\rho \in \{0, 1\}$, there exists a first integral (if $\rho = 0$) or an integrating factor (if $\rho = 1$), which can be constructed using the invariant algebraic curves. Moreover, he showed that relation (1) always occurs with $\rho = 0$ if $p \geq [m(m + 1)/2] + 1$, and with $\rho = 0, 1$ if $p \geq m(m + 1)/2$.

Now we know that if the number of the invariant algebraic curves is at least $p = [m(m + 1)/2] + 2$, then there is a rational first integral, which means that all orbits of the system are contained on algebraic curves, for more details see Section 1.3.

We also present the recent extensions of the Darboux theory of integrability which additionally to the concept of the invariant algebraic curve, incorporate the notions of the exponential factors, independent singular points, the multiplicity of the invariant algebraic curves and the invariants, see Theorem 1.7 or [31, 48, 51, 15, 17, 18, 19, 21, 6, 47] . In particular, if the polynomial differential system has additionally to the p invariant algebraic curves with cofactors K_i, q exponential factors with cofactors L_j, and it satisfies

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j + \rho \, \text{div}(P, Q) + s = 0, \tag{2}$$

with λ_i, $\mu_j \in \mathbb{C}$ not all zero, $\rho \in \{0, 1\}$ and $s \in \mathbb{C}$ then for $s = 0$ we obtain a first integral (if $\rho = 0$) or an integrating factor (if $\rho = 1$), otherwise we have an invariant (i.e. a first integral depending on the time). Moreover, the relation (2) always holds with $\rho = s = 0$ if $p + q \geq m(m+1)/2 + 1$, and with $\rho = 0, 1$ and $s = 0$ if $p + q \geq m(m+1)/2$. For $s \neq 0$ the relation (2) is satisfied with $\rho = 0$ if $p + q \geq m(m+1)/2$, and with $\rho = 0, 1$ if $p + q \geq m(m+1)/2 - 1$. We remark that the first integral describes completely the phase portrait, and an invariant provides information about the asymptotic behavior of the orbits.

In this work we introduce the concept of generalized invariant when $\rho = 1$ and $s \neq 0$ and we present the integrability condition (2), in Section 1.3.

Since the existence of invariant algebraic curves is the key point for the application of the Darboux theory of integrability we also consider the following reciprocal question to the Darboux theory of integrability:

QUESTION 1: *Given a set of algebraic curves which are the planar polynomial differential systems having these curves invariant by the flow?*

We deal with this question in Chapter 2. Firstly, in Theorems 2.1, 2.2, 2.3 and finally in Theorem 2.4 we present a complete answer in a generic case. We note that Theorem 2.4 was announced by Christopher. However, the proof of this theorem was never published. Independently, Żołądek also stated this theorem. However, he never published his complete proof which is based in analytical arguments. Here we present a complete algebraic proof of Theorem 2.4. Additionally, we prove that the generic conditions of the theorem are necessary (see Theorem 2.6). We note that in Theorem 2.4 it appears a strong relation between the degrees of the curve and the degrees of the system. This relation is due to the generic nature of the curves. In particular, when the total degree of the generic curves is the degree of the system increased by one, then the vector field has a very simple form and it has always a Darboux first integral. Hence, this relation between the degrees and the nature of the curves guarantee the Darboux integrability. This result is not true if the curves are not generic and this is proved in Section 2.3. We note that these results have been obtained in collaboration with Christopher, Llibre and Zhang, see [20].

Secondly, in the case where the curves are not generic and their cofactors are known we present Propositions 2.28 and 2.29 inspired in previous results due to Erugguin [24] and Sadovskaia [46].

Finally, a more general answer to this question is given by Theorem 2.34 due to Walcher [50]. Walcher provides the complete expression of the vector fields which have an arbitrary invariant algebraic curve. Of course, in this theorem there is no statement about the bounds of the degrees of the polynomials in contrast with Theorem 2.4. This is due to the fact that the generic conditions are not imposed in Theorem 2.34.

Question 2: *Given a Darboux first integral which are the planar polynomial differential systems having such a first integral?*

In Chapter 3 we present a complete answer to this question through Theorem 3.1.

Additionally, in Corollary 3.2 we show the relation between the total degree of the curves and the degree of the polynomials which appears in the exponential with the degree of the system. In Theorem 3.5 we state that a polynomial system has a Darboux first integral formed by generic curves if and only if the total degree of the curves is the degree of the system increased by one. We note that this theorem improves the conditions for the existence of a first integral in the Darboux theory of integrability using information about the degree and the nature of the invariant algebraic curves. As far as we know, this is the first time that information about the degree of the invariant algebraic curves, instead of the number of these curves, is used for studying the integrability of a polynomial vector field.

In Corollary 3.4 we improve a previous result due to Prelle and Singer [45]. This result will be published in [36].

Prelle and Singer [45], using methods of differential algebra, showed that if a polynomial vector field has an elementary first integral, then it can be computed using Darboux theory of integrability. Singer [48] proved that if a polynomial vector field has a Liouvillian first integral, then it has integrating factors given by Darboux functions. Some related results can be found in [10].

Question 3: *Given a Darboux integrating factor which are the planar polynomial differential systems having such a Darboux integrating factor?*

We study this question in Chapters 4 and 5.

First, in Theorem 4.1 we provide a connection between the degrees of the invariant algebraic curves and the number of them in order to decide about the type of Darboux integrability of the polynomial differential system. This result improves statement (e) of the Darboux Theorem 1.7.

Second, in Theorem 4.2 we present a general family of polynomial differential systems having a Darboux function as an integrating factor. Although, it is a large family of systems, it is not the most general one. We believe that further conditions should be imposed in order to obtain the complete family of such systems.

Third, we characterize polynomial differential systems having a generic Darboux integrating factor, i.e. an integrating factor formed by generic curves.

In Theorem 4.3 we characterize all polynomial systems with an integrating factor formed by one irreducible generic curve $f = 0$, i.e. we characterize all polynomial systems having an integrating factor of the form f^λ with $\lambda \in \mathbb{C}$. We know that such systems in general have a Liouvillian first integral. From Theorem 4.3 and its proof we have that depending on the values of λ it appears an additional invariant algebraic curve or an exponential factor. Moreover, for such systems we can always obtain a Darboux first integral. We note that the additional invariant algebraic curve may not be generic with $f = 0$. An interesting point of this theorem is the relation between the degrees of the curves and the degrees of the polynomials which appear in the exponential factors with the degree of the system. Hence, we have that the sum of all these degrees is the degree of the system increased by one, the curves in general are not generic. Moreover, such systems have a Darboux first integral. This is due to the existence of a generic integrating factor.

Walcher in Theorem 5.1 characterize all polynomial differential systems with an integrating factor of the form f^{-1} where f is a curve without singular points. In Corollary 5.2 we present an easier expression of such vector fields. Additionally, in Theorem 5.3 imposing the generic conditions for the reducible factors of f we obtain an expression of the vector field similar to the one presented in Theorem 5.1. Moreover, in Theorems 5.1 and 5.3 we point out the different use of the generic conditions.

In Theorem 5.3 we characterize polynomial differential systems having a generic integrating factor. Similar to Theorem 4.3 it appears an additional invariant algebraic curve which may not be generic with the other curves. Additionally, the total degree of the curves is the degree of the system increasing by one. In general, a polynomial system having a Darboux integrating factor it has always a Liouvillian first integral and does not always have a Darboux first integral, (see Example 4.11). However, under the assumptions of Theorem 5.3 not only we can guarantee the existence of a Darboux first integral but also we provide an algorithm in order to construct it. We note that there are some values of the parameters which are not covered by Theorem 5.3.

Rudolf Winkel in [52] conjectured: *For a given algebraic curve $f = 0$ of degree $m \geqslant 4$ there is in general no polynomial vector field of degree less than $2m - 1$ leaving invariant $f = 0$ and having exactly the ovals of $f = 0$ as limit cycles.*

In the appendix we show that this conjecture is not true.

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Chapter 1

The Darboux theory of integrability

1.1 Introduction

In 1878 Darboux [22] showed how can be constructed the first integrals of planar polynomial differential systems possessing sufficient invariant algebraic curves. In particular, he proved that if a planar polynomial differential system of degree m has at least $[m(m+1)/2]+1$ invariant algebraic curves, then it has a first integral, which has an easy expression in function of the invariant algebraic curves. The version of the Darboux theory of integrability that we summarize in Theorem 1.7, improves Darboux's original exposition because we also take into account the exponential factors, the independent singular points, the rational first integrals, and the invariants.

Good extensions of the Darboux theory of integrability to polynomial systems in $\mathbb{C}^n$ are due to Jouanolou [31] and Weil [51]. In [15], [17], [18], [19], [21] and [47] the authors developed the Darboux theory of integrability essentially in $\mathbb{C}^2$ considering not only the invariant algebraic curves but also the exponential factors, the independent singular points and the multiplicity of the invariant algebraic curves. Recently, in [40] the Darboux theory of integrability is extended to regular algebraic hypersurfaces (see also [38]). Moreover, Singer in [48] proved that

Darboux theory of integrability allows to compute the Liouvillian first integrals of polynomial differential systems. In [6] were introduced the Darboux invariants, here we introduce the generalized Darboux invariants, see Section 1.3.

The Darboux theory of integrability works for complex polynomial ordinary differential equations (and of course in particular for the real ones). We consider the *polynomial (differential) system* in $\mathbb{C}^2$ defined by

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1.1}$$

where P and Q are polynomials in the variables x and y. The independent variable t can be real or complex, this is not relevant in the Darboux theory of integrability. If A is a polynomial, we denote by δA the *the degree of the polynomial* A. The *degree* m of the polynomial system is defined by $m = \max\{\delta P, \delta Q\}$ and we write $\delta X = m$.

Associated to the polynomial differential system (1.1) in $\mathbb{C}^2$ there is the *polynomial vector field*

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}, \tag{1.2}$$

in $\mathbb{C}^2$. Sometimes, the polynomial vector field X will be denoted simply by (P, Q).

In Section 1.2 we present the basic notions that we will use in this work. The Darboux Theory of integrability is summarized in Theorem 1.7 and is presented in Section 1.3. We note that the version of Theorem 1.7 that we provide here is original on the statements about the Darboux invariants and on the generalized Darboux invariants.

1.2 Basic notions

Algebraic curves is the starting point of the Darboux theory of integrability.

An algebraic curve $f(x, y) = 0$ in $\mathbb{C}^2$ with $f \in \mathbb{C}[x, y]$ is an *invariant algebraic curve* of a polynomial system (1.1) if

$$\dot{f} = Xf = \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q = Kf, \tag{1.3}$$

for some polynomial $K \in \mathbb{C}[x,y]$ called the *cofactor* of the invariant algebraic curve $f = 0$. We note that since the polynomial vector field has degree m, then any cofactor has at most degree $m-1$.

We observe that for the points of the curve $f = 0$ the right hand side of (1.3) is zero. This means that the gradient $(\partial f/\partial x,\ \partial f/\partial x)$ is orthogonal to the vector field (P,Q) at these points. Therefore the vector field (P,Q) is tangent to the curve $f = 0$. Hence, the curve $f = 0$ is formed by trajectories of the vector field (P,Q). This explains why the algebraic curve $f = 0$ is invariant under the flow of the vector field (P,Q).

The following result show that we can reduce the study of the invariant algebraic curves, to study the irreducible invariant algebraic curves in $\mathbb{C}[x,y]$, (for a proof see [13]).

Proposition 1.1. *Suppose that $f \in \mathbb{C}[x,y]$ and let $f = f_1^{n_1} \cdots f_r^{n_r}$ be the factorization of f in irreducible factors over $\mathbb{C}[x,y]$. Then for a polynomial system (1.1), $f = 0$ is an invariant algebraic curve with cofactor K_f if and only if $f_i = 0$ is an invariant algebraic curve for each $i = 1,\ldots,r$ with cofactor K_{f_i}. Moreover, $K_f = n_1 K_{f_1} + \cdots + n_r K_{f_r}$.*

For a given system (1.1) of degree m the calculation of the invariant algebraic curves is a very hard problem (maybe we could also say that in some cases is an unrealistic problem) because in general we don't have any evidence about the degree of a curve. Hence, for a system of a fixed degree m does not always exist a bound for the degree of the invariant curves. However, imposing additionally conditions either for the structure of the system or for the nature of the curves we can have an evidence of a such a bound.

The following Proposition suggest a bound of the degree of the invariant curve.

Proposition 1.2. *Let $f \in \mathbb{C}[x,y]$ irreducible satisfying the generic condition (i). If the curve $f = 0$ is invariant of the vector field (1.1) of degree m then $\delta f \leq m+1$.*

Proof: see Corollary 4 of [13] or Proposition 10 of [14]. ∎

A *first integral* of system (1.1) on an open subset U of $\mathbb{C}^2$ is a nonconstant analytic function $H : U \to \mathbb{C}$ which is constant on every solution curve $(x(t), y(t))$

of (1.1) on U. This means that $H(x(t), y(t)) = c$ with $c \in \mathbb{C}$ for every time t for which the solution $(x(t), y(t))$ is defined on U. If we denote by

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$$

the vector field associated to system (1.1), then H is a first integral in U if and only if $XH \equiv 0$ on U. We say that polynomial system (1.1) is *integrable* on U if there is a first integral on U.

An analytic function $R : U \to \mathbb{C}$ which is not identically zero on U is called an *integrating factor* of system (1.1) on U if satisfies

$$XR = -\mathrm{div}(X)R,$$

in the domain of definition of R. As usual the *divergence* of the vector field X is defined by

$$\mathrm{div}(X) = \mathrm{div}(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

Suppose that U is simply connected, then the first integral associated to the integrating factor R is given by

$$H(x, y) = \int R(x, y)P(x, y)dy + f(x), \tag{1.4}$$

satisfying the condition

$$\frac{\partial H}{\partial x} = -RQ. \tag{1.5}$$

From the definition of integrating factor R, we have that $X(R) = -\mathrm{div}(P, Q)R$. This implies that $R = 0$ is an invariant curve (in general non–algebraic) of X with cofactor the polynomial $-\mathrm{div}(P, Q)$. In addition, the existence of two different integrating factors yields directly to a first integral as we note in the following proposition.

Proposition 1.3. *If polynomial system (1.1) has two integrating factors R_1 and R_2 on the open subset U of $\mathbb{C}^2$, then on the open set $U \setminus \{R_2 = 0\}$ the function R_1/R_2 is a first integral.*

Another notion strictly connected with the one of the integrating factor is the notion of the inverse integrating factor.

Let $R : U \to \mathbb{C}$ an integrating factor of system (1.1) and $W = U \setminus \{R = 0\}$. We define $V = 1/R : W \to \mathbb{C}$ as an *inverse integrating factor* of system (1.1). The inverse integrating factor V satisfies the linear partial differential equation

$$X(V) = \mathrm{div}(P, Q)V. \tag{1.6}$$

From the definition of the inverse integrating factor V, we have that $V = 0$ is an invariant curve (in general non–algebraic) of X with cofactor the polynomial $\mathrm{div}(P, Q)$.

The inverse integrating factor of a polynomial system (1.1) contains a lot of useful information. On one hand using relation (1.4) yields into the expression of a first integral in a simple connected open set and on the other hand, the set $\{V = 0\}$ contains all the limit cycles which are in W, see [27]. Moreover, in [10], it has been proved the local existence and the uniqueness of an analytic inverse integrating factor under adequate assumptions. From [10] and [11], it follows, in general, that it is more easy to look for an expression of the inverse integrating factor than one of the integrating factor or of the first integral.

Another useful notion in the Darboux theory of integrability is the notion of an exponential factor and is due to Christoper, [17].

Let $h, g \in \mathbb{C}[x, y]$ be relatively prime in the ring $\mathbb{C}[x, y]$. The function $\exp(g/h)$ is called an *exponential factor* of the polynomial system (1.1) if for some polynomial $K \in \mathbb{C}[x, y]$ of degree at most $m - 1$ it satisfies the equation

$$X\left(\exp\left(\frac{g}{h}\right)\right) = K \exp\left(\frac{g}{h}\right). \tag{1.7}$$

We say that K is the *cofactor* of the exponential factor $\exp(g/h)$.

Proposition 1.4. *If $\exp(g/h)$ is an exponential factor with cofactor K for a polynomial system (1.1) and if h is not a constant, then $h = 0$ is an invariant algebraic curve with cofactor K_h, and g satisfies the equation $Xg = gK_h + hK$.*

We should note that exponential factors of the form $\exp(g/h)$ (respectively $\exp(g)$) appear when the invariant algebraic curve $h = 0$ (respectively the invariant straight line at infinity when we projectivize the vector field X) has geometric multiplicity larger than 1, for more details see [21].

Real polynomial systems are very special because whenever they have a complex invariant algebraic curve or a complex exponential factor they also have as invariant the conjugate ones as we note in the following propositions (for a proof see [19]).

Proposition 1.5. *For a real polynomial system (1.1), $f = 0$ is a complex invariant algebraic curve with cofactor K if and only if $\bar{f} = 0$ is a complex invariant algebraic curve with cofactor $\bar{K}$. Here conjugation of polynomials denotes conjugation of the coefficients of the polynomials.*

Proposition 1.6. *For a real polynomial system (1.1) the complex function* $\exp\left(g/h\right)$ *is an exponential factor with cofactor K if and only if the complex function* $\exp\left(\bar{g}/\bar{h}\right)$ *is an exponential factor with cofactor $\bar{K}$.*

Hence, for real polynomial systems the Darboux integrability maybe forced by the existence of complex invariant algebraic curves or complex exponential factors.

An *invariant* of a real polynomial system (1.1) in the open subset U of $\mathbb{C}^2$ is a non–constant analytic function I in the variables x, y and t such that $I(x(t), y(t), t)$ is constant on all solution curves $(x(t), y(t))$ of system (1.1) contained in U.

For a polynomial differential system the existence of a first integral $H(x, y)$ implies that drawing the curves $H(x, y)$ =constant we can describe completely the phase portrait of such a system. While the existence of an invariant will provide information about the α– or the ω–limit of the orbits of the system, where the time t is real.

The existence of singular points improve the original version of Darboux Theorem 1.7, see [15]. We denote by $\mathbb{C}_{m-1}[x, y]$ the space of all complex polynomials of degree $m - 1$ and and we note that $\dim_{\mathbb{C}} \mathbb{C}_{m-1}[x, y] = m(m + 1)/2$. Let $K(x, y) = \displaystyle\sum_{i+j=0}^{m-1} a_{ij} x^i y^j \in \mathbb{C}_{m-1}[x, y]$. We consider the isomorphism

$$K \longrightarrow (a_{00},\ a_{10},\ a_{01}, \ldots,\ a_{m-1,0},\ a_{m-2,1}, \ldots,\ a_{0,m-1}),$$

i.e. we identify the linear vector space $\mathbb{C}_{m-1}[x, y]$ with $\mathbb{C}^{m(m+1)/2}$.

We say that r singular points $(x_k, y_k) \in \mathbb{C}^2$, for $k = 1, \ldots, r$, of a real polynomial system (1.1), are *independent* with respect to $\mathbb{C}_{m-1}[x, y]$ if the intersection of the r hyperplanes

$$\sum_{i+j=0}^{m-1} a_{ij} x_k^i y_k^j = 0, \qquad k = 1, \dots, r,$$

in $\mathbb{C}^{m(m+1)/2}$, is a linear subspace of dimension $[m(m+1)/2] - r$.

We note that Bezout Theorem [25] guarantee that the maximum number of complex isolated singular points of a polynomial system (1.1) is m^2, and the maximum number of complex independent isolated singular points of the system is $m(m+1)/2 < m^2$ for $m \geq 2$.

A singular point (x_0, y_0) of a polynomial system (1.1) is *weak* if $\operatorname{div}(P,Q)(x_0, y_0) = 0$.

1.3 The method of Darboux

The presentation of the Darboux theory of integrability can be summarized in Theorem 1.7 and as far as we know is the most new version. This version is partially original on his statements about the Darboux invariants and original on the generalized Darboux invariants. The other statements are well known and we will not prove them, for a proof see, for instance, [19]. In Theorem 4.1 we obtain further improvements of Theorem 1.7 but for a better presentation of this result we prefer to present it in chapter 4.

The following theorem it will be mentioned as Darboux theorem.

Theorem 1.7. *Suppose that a polynomial system (1.1) of degree m admits p irreducible invariant algebraic curves $f_i = 0$ with cofactors K_i for $i = 1, \dots, p$; q exponential factors $F_j = \exp(g_j/h_j)$ with cofactors L_j for $j = 1, \dots, q$; and r independent singular points $(x_k, y_k) \in \mathbb{C}^2$ such that $f_i(x_k, y_k) \neq 0$ for $i = 1, \dots, p$ and $k = 1, \dots r$. Of course, every h_j factorizes in product of the factors $f_1, \cdots f_q$, except if it is equal to 1. Let V be a C^1 solution of equation (1.6) defined in an open subset W of $\mathbb{C}^2$ (i.e. V is an inverse integrating factor). Then the following statements hold.*

(a) *There exist* $\lambda_i, \mu_i \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{i=1}^{q} \mu_j L_j = 0, \qquad (D_{fi})$$

if and only if the (multi–valued) function

$$H(x, y) = f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q}, \tag{1.8}$$

is a first integral of system (1.1). Moreover, for real systems the function (1.8) is real.

(b) *If* $p + q + r = [m(m + 1)/2] + 1$*, then there exist* $\lambda_i, \mu_j \in \mathbb{C}$ *not all zero satisfying condition* (D_{fi})*.*

(c) *If* $p+q+r \geq [m(m+1)/2]+2$*, then system (1.1) has a rational first integral, and consequently all orbits of the system are contained in invariant algebraic curves.*

(d) *There exist* $\lambda_i, \mu_j \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j + div(P, Q) = 0, \qquad (D_{if}),$$

if and only if the function (1.8) is an integrating factor of system (1.1). Moreover, for real systems the function (1.8) is real.

(e) *If* $p + q + r = m(m + 1)/2$ *and the* r *independent singular points are weak, then there exist* $\lambda_i, \mu_j \in \mathbb{C}$ *not all zero satisfying at least one of the conditions* (D_{fi}) *or* (D_{if})*.*

(f) *There exist* $\lambda_i, \mu_i \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j + s = 0, \qquad (D_{in})$$

with $s \in \mathbb{C} \setminus \{0\}$*, if and only if the (multi–valued) function*

$$I(x, y, t) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} \exp(st) \tag{1.9}$$

is an invariant of system (1.1). Moreover, for real systems this function is real.

(g) *There exist $\lambda_i, \mu_i \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j + div(P,Q) + s = 0, \qquad (D_{gin})$$

with $s \in \mathbb{C} \setminus \{0\}$, if and only if the (multi–valued) function

$$G(x,y,t) = V f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} \exp(st), \tag{1.10}$$

is an invariant of system (1.1). Moreover, for real systems this function is real.

(h) *If $p+q = [m(m+1)/2] - 1$, then there exist $\lambda_i, \mu_i \in \mathbb{C}$ not all zero satisfying at least one of the conditions (D_{fi}), (D_{if}), (D_{in}) or (D_{gin}).*

Proof: For a proof of statements (a)–(e) see [19]. Here we will prove statements (f) and (h). Statement (g) can be proved in a similar way.

Clearly the function $I(x,y,t)$ is an invariant of system (1.1) if and only if $X(I) = 0$. Then, from the equalities

$$\begin{aligned} X\left(f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} \exp(st)\right) &= \\ X\left(f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} \exp(st)\right)\left(\sum_{i=1}^{p} \lambda_i \frac{X f_i}{f_i} + \sum_{j=1}^{q} \mu_j \frac{X F_j}{F_j} + s\right) &= \\ \left(f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} \exp(st)\right)\left(\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j + s\right), & \end{aligned}$$

the first part of statement (f) follows.

Supose now that X is a real vector field. If among the invariant algebraic curves of X a complex conjugate pair $f = 0$ and $\overline{f} = 0$ occurs, then the invariant (1.9) has a real factor of the form $f^{\lambda} \bar{f}^{\bar{\lambda}}$, which is the multi–valued real function

$$\left[(\mathrm{Re}\, f)^2 + (\mathrm{Im}\, f)^2\right]^{\mathrm{Re}\,\lambda} \exp\left(-2\,\mathrm{Im}\,\lambda \arctan\left(\frac{\mathrm{Im}\, f}{\mathrm{Re}\, f}\right)\right), \tag{1.11}$$

if $\mathrm{Im}\,\lambda\,\mathrm{Im}\, f \neq 0$. We note that if in (D_{fi}) the coefficient of a cofactor K or L is λ, then the coefficient of the cofactor $\bar{K}$ or $\bar{L}$ is $\bar{\lambda}$, because conjugating such equality K goes over to $\bar{K}$ and L to $\bar{L}$.

If among the exponential factors of X a complex conjugate pair $F = \exp(h/g)$ and $\overline{F} = \exp(\overline{h}/\overline{g})$ occurs, the invariant (1.9) has a real factor of the form

$$\left(\exp\left(\frac{h}{g}\right)\right)^{\mu}\left(\exp\left(\frac{\overline{h}}{\overline{g}}\right)\right)^{\overline{\mu}} = \exp\left(2\,\mathrm{Re}\left(\mu\,\frac{h}{g}\right)\right). \tag{1.12}$$

In short, the function (1.9) is real, and the proof of statement (f) is completed.

Let K be the divergence of system (1.1). All polynomials K_i, L_j, K belong to the vector space $\mathbb{C}_{m-1}[x, y]$ of dimension $m(m + 1)/2$. The number $s \neq 0$ is identified with the corresponding polynomial of degree 0 of $\mathbb{C}_{m-1}[x, y]$. Therefore, we have $p + q + 2$ polynomials K_i, L_j, K and s in $\mathbb{C}_{m-1}[x, y]$. Since from the assumptions $p + q + 2 = m(m + 1)/2 - 1$, either K is a linear combination of the polynomials K_i,L_j and s, or a linear combination of those polynomials is zero. In the first case if s does not appear in the linear combination, then we obtain the equality (D_{if}), and if s appears, then we obtain the equality (D_{gin}) perhaps with a constant times s instead of s. In the second case if s does not appear in the linear combination, then we obtain the equality (D_{fi}), and if s appears, then we obtain the equality (D_{in}) perhaps with a constant times s instead of s. Hence, statement (h) is proved. ∎

The functions (1.9) and (1.10) are called *Darboux invariants* and *generalized Darboux invariants*, respectively.

A function of the form

$$f_1^{\lambda_1}\cdots f_p^{\lambda_p}\exp\left(\frac{g_1}{h_1}\right)^{\mu_1}\cdots\exp\left(\frac{g_q}{h_q}\right)^{\mu_q}, \tag{1.13}$$

is called a *Darboux function.* If system (1.1) has a first integral or an integrating factor of the form (1.13) where $f_i = 0$ and $\exp(g_i/h_i)$ are invariant algebraic curve and exponential factors of system (1.1) respectively and $\lambda_i, \mu_j \in \mathbb{C}$, then system (1.1) is called *Darboux integrable.*

From Proposition 1.4 we have that the irreducible factors of the polynomials h_j are some f_i's and we can write

$$\left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1}\cdots\left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q} = \exp\left(\frac{\mu_1 g_1}{h_1}+\cdots+\frac{\mu_q g_q}{h_q}\right) = \exp\left(\frac{g}{f_1^{n_1}\cdots f_p^{n_p}}\right),$$

where $\mu_1, \cdots, \mu_q \in \mathbb{C}$, $n_1, \cdots, n_p \in \mathbb{N}\bigcup\{0\}$ and the polynomial g of $\mathbb{C}[x,y]$ is coprime with f_i if $n_i \neq 0$. We denote by $l = \max\left\{\sum_{i=1}^{r} n_i \delta f_i, \delta g\right\}$.

Hence, a Darboux function can be written into the form

$$R(x,y) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp\left(\frac{g}{f_1^{n_1} \cdots f_p^{n_p}}\right), \tag{1.14}$$

where $f_1, \cdots, f_p$ are irreducible polynomials in $\mathbb{C}[x,y]$, $\lambda_1, \cdots, \lambda_p \in \mathbb{C}$, $n_1, \cdots, n_p \in \mathbb{N}\bigcup\{0\}$ (i.e. the n_i are non–negative integers) and the polynomial g of $\mathbb{C}[x,y]$ is coprime with f_i if $n_i \neq 0$.

The associated first integral to a Darboux integrating factor is called a *Liouvillian first integral.*

We observe that the relation

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j + \rho \operatorname{div}(P,Q) + s = 0, \tag{D}$$

with $\lambda_i, \mu_j \in \mathbb{C}$ and $\rho, s \in \mathbb{C}$, contains all the information about the Darboux theory of integrability for polynomial differential systems in $\mathbb{C}^2$. Thus,

(1) If $s = \rho = 0$ we obtain condition (D_{fi}), and we have a Darboux first integral.

(2) If $s = 0$ and $\rho \neq 0$ then condition (D_{if}) holds, and there exists a Darboux integrating factor and a Liouvillian first integral.

(3) If $s \neq 0$ and $\rho = 0$ we obtain relation (D_{in}), and we have a Darboux invariant.

(4) If $s \neq 0$ and $\rho \neq 0$ we have the relation (D_{gin}), and there exists a generalized Darboux invariant.

Chapter 2

On polynomial systems having invariant algebraic curves

2.1 Introduction

The concept of the invariant algebraic curves is the key point in the Darboux theory of integrability. For a given system Darboux note that the existence of the invariant algebraic curves provides an important information about the behavior of the system. In that chapter we deal with the following question: Find the polynomial systems having a given set of invariant algebraic curves?

We say that the algebraic curves $f_1 = 0, \cdots, f_p = 0$ are *generic* if satisfy the following *generic conditions*:

(i) There are no points at which f_i and its first derivatives are all vanish.

(ii) The highest order terms of f_i have no repeated factors.

(iii) If two curves intersect at a point in the finite plane, they are transversal at this point.

(iv) There are no more than two curves $f_i = 0$ meeting at any point in the finite plane.

(v) There are no two curves having a common factor in the highest order terms.

First we characterize all the polynomial vector fields having one invariant generic curve.

Theorem 2.1. *Assume that the vector field $X = (P, Q)$ of degree m has an invariant algebraic curve $C = 0$ of degree c, and that C satisfies the generic condition (i).*

(a) If $(C_x, C_y) = 1$, then X has the following normal form:

$$\dot{x} = AC - DC_y, \qquad \dot{y} = BC + DC_x, \tag{2.1}$$

where A, B and D are suitable polynomials.

(b) If C satisfies the generic condition (ii), then X has the normal form (2.1) with $a, b \leq m - c$ and $d \leq m - c + 1$. Moreover, if the highest order term C^c of C does not have the factors x and y, then $a \leq p - c$, $b \leq q - c$ and $d \leq min\{p, q\} - c + 1$.

We note that the first statement of Theorem 2.1 is referring to polynomial systems having one curve $C = 0$ invariant such that $(C_x, C_y) = 1$. If the curve $C = 0$ satisfies the generic condition (i), then such systems are given by the expression (2.1). If additionally the algebraic curve satisfies the generic condition (ii), then we can provide bounds about the degrees of the polynomials that appears in systems (2.1). We also note that statement (a) of Theorem 2.1 is very similar to the one of Theorem 2.34(b) due to Walcher presented in Section 2.5. What actually happens is that in Theorem 2.34(b) do not appear explicitly the condition $(C_x, C_y) = 1$. However, in that expression of the vector field is used the Hamiltonian removing the common factors and is denoted as $(C_x, C_y)^*$. Hence, both Theorems 2.1(a) and 2.34(b) use exactly the same conditions and they state the same expression for the vector field.

Now we consider polynomial differential systems having two invariant algebraic curves satisfying the generic conditions (i) and (iii). In Theorem 2.2(a) we present the expression (2.2) for such systems. If in addition the curves satisfy the generic conditions (ii) and (v) then we can obtain bounds for the degree of the polynomials that appear in system (2.2).

Theorem 2.2. *Assume that $C = 0$ and $D = 0$ are different irreducible invariant algebraic curves of the vector field $X = (P, Q)$ of degree m, and that they satisfy the generic conditions (i) and (iii).*

(a) If $(C_x, C_y) = 1$ and $(D_x, D_y) = 1$, then X has the normal form

$$\dot{x} = ACD - EC_yD - FCD_y, \quad \dot{y} = BCD + EC_xD + FCD_x, \tag{2.2}$$

(b) If C and D satisfy conditions (ii) and (v), then X has the normal form (2.2) with $a, b \leq m - c - d$ and $e, f \leq m - c - d + 1$.

The next theorem founds the expressions of the polynomial vector fields having some invariant algebraic curves. Similar to Theorems 2.1 and 2.2 we need the generic conditions (i) and (iii). A possible third line needs the condition (iv) in order to obtain an expression of the vector field. We note that the bounds for the degrees of the polynomials appearing in expression (2.3) are due to the generic conditions (ii) and (v).

Theorem 2.3. *Let $C_i = 0$ for $i = 1, \cdots, p$ be different irreducible invariant algebraic curves of the vector field $X = (P, Q)$ with $\delta C_i = c_i$. Assume that C_i satisfy the generic conditions (i), (iii) and (iv). Then*

(a) If $(C_{ix}, C_{iy}) = 1$ for $i = 1, \cdots, p$, then the vector field $X = (P, Q)$ has the normal form

$$\dot{x} = \left(B - \sum_{i=1}^{p} \frac{A_i C_{iy}}{C_i} \right) \prod_{i=1}^{p} C_i, \qquad \dot{y} = \left(D + \sum_{i=1}^{p} \frac{A_i C_{ix}}{C_i} \right) \prod_{i=1}^{p} C_i, \tag{2.3}$$

where B, D and A_i are suitable polynomials.

(b) If C_i satisfy the generic conditions (ii) and (v), then X has the normal form (2.3) with $b, d \leq m - \sum_{i=1}^{p} c_i$ and $a_i \leq m - \sum_{i=1}^{p} c_i + 1$.

Theorem 2.4 was stated by Christopher and use strongly the generic nature of the curves. He does not only affirms that the only vector fields having degree less than the total degree of the invariant algebraic curves minus one is the zero vector field, but also provides the simplest form of such systems. Christopher stated this theorem in several papers without proof like [17] and [33], and it was used in other papers as [4] and [37]. The proof of Theorem 2.4 that we present here circulated as the preprint [16] but was never published. Żołądek in [53] (see also Theorem 3 of [54]) stated a similar result to Theorem 2.4, but as far as we know the paper [53] has not been published. In any case Żołądek's approach to Theorem 2.4 is analytic, while the approach that we present here is completely algebraic [20].

Theorem 2.4. *Let $f_i = 0$ for $i = 1, \cdots, p$, be irreducible invariant algebraic curves in $\mathbb{C}^2$, and set $r = \sum_{i=1}^{p} \delta f_i$. We assume that all f_i satisfy the generic conditions (i)–(v). Then any polynomial vector field $X = (P, Q)$ of degree m tangent to all $f_i = 0$ satisfies one of the following statements.*

(a) *If $r < m + 1$ then*

$$X = Y\left(\prod_{i=1}^{p} f_i\right) + \sum_{i=1}^{p} h_i \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j\right) X_{f_i}, \tag{2.4}$$

where $X_{f_i} = (-f_{iy}, f_{ix})$ is a Hamiltonian vector field, the h_i are polynomials of degree no more than $m - r + 1$, and the Y is a polynomial vector field of degree no more than $m - r$.

(b) *If $r = m + 1$ then*

$$X = \sum_{i=1}^{p} \alpha_i \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j\right) X_{f_i}, \tag{2.5}$$

with $\alpha_i \in \mathbb{C}$.

(c) *If $r > m + 1$ then $\mathbb{X} = 0$.*

Theorems 2.1, 2.2 and 2.3 and 2.4 will be proved in Section 2.2.

Statement (b) of Theorem 2.4 yields to a corollary due to Christopher and Kooij [33]. They showed that system (2.5) has the integrating factor $(f_1 \cdots f_p)^{-1}$, and consequently the system is Darboux integrable.

Proposition 2.5. *Under the assumptions of Theorem 2.4(b) a polynomial system (2.5) has an integrating factor of the form* $(f_1 \cdots f_p)^{-1}$ *and a first integral of the form* $f_1^{\lambda_1} \cdots f_p^{\lambda_p}$.

The following result shows that the generic conditions are necessary in order that the statements of Theorem 2.4 hold [20].

Theorem 2.6. *If one of the conditions (i)–(v) of Theorem 2.4 is not satisfied, then the statements of Theorem 2.4 do not hold.*

We prove Theorem 2.6 in Section 2.2.

From Theorem 2.4(b) and from Proposition 2.5 we have that the vector field X satisfying the generic conditions (i)–(v) and $r = m + 1$ is Darboux integrable.

In Section 2.3 we provide two examples of polynomial systems satisfying all assumptions of Theorem 2.4 with $r = m + 1$ except either (ii) or (iii) and which are not Darboux integrable. Until now there are very few proofs of polynomial systems which are not Darboux integrable, see for instance Jouanolou [31] and Moulin Ollagnier et al [42].

Theorem 2.7. *There are values of the parameters a and b for which system*

$$\begin{aligned} \dot{x} &= y(ax - by + b) + x^2 + y^2 - 1, \\ \dot{y} &= bx(y - 1) + a(y^2 - 1), \end{aligned} \tag{2.6}$$

is not Darboux integrable.

In Section 2.4 we deal with systems having some arbitrary invariant algebraic curves when their cofactors are known, see Propositions 2.28 and 2.29. These results are inspired in previous results due to Erugguin [24] and Sadovskaia [46].

In general, if the curves are arbitrary, we cannot guarantee any relation between the degree of the system and the degree of the curves. Additionally, due to Proposition 2.6 the form of the polynomial systems is not controlled by Theorem 2.4.

We dedicate the last section of this chapter to the presentation of a result due to Walcher [50]. Thus, Theorem 2.34 provides the complete expression of the vector fields which have an arbitrary invariant algebraic curve. Of course, in this theorem there is no statement about the bounds of the degrees of the polynomials since the generic conditions are not imposed.

2.2 Proofs of Theorems

In what follows if we have a polynomial A we will denote its degree by a. If we do not say anything we denote by C^c the homogeneous part of degree c for the polynomial C.

Also we should use intensively the Hilbert's Nullstellensatz property (see for instance, [25]):

Set $A, B_i \in \mathbb{C}[x,y]$ for $i = 1, \cdots, r$. If A vanishes in $\mathbb{C}^2$ whenever the polynomials B_i vanish simultaneously, then there exist polynomials $M_i \in \mathbb{C}[x,y]$ and a nonnegative integer n such that $A^n = \sum_{i=1}^{r} M_i B_i$. In particular, if all B_i have no common zero, then there exist polynomial M_i such that $\sum_{i=1}^{r} M_i B_i = 1$.

In order to proof Theorem 2.1(b) we should use the following Lemma.

Lemma 2.8. *If C^c has no repeated factors, then $(C_x, C_y) = 1$.*

Proof: Suppose that $(C_x, C_y) \neq 1$. Then there exists a polynomial A nonconstant such that $A|C_x$ and $A|C_y$. Here $A|C_x$ means that the polynomial A divides the polynomial C_x. Therefore, $A^a|(C^c)_x$ and $A^a|(C^c)_y$. By the Euler theorem for homogeneous polynomials we have that $x(C^c)_x + y(C^c)_y = cC^c$. So $A^a|C^c$. Since A^a, $(C^c)_x$, $(C^c)_y$ and C^c are homogeneous polynomials of $\mathbb{C}[x,y]$ and A^a divides $(C^c)_x$, $(C^c)_y$ and C^c , the linear factors of A^a having multiplicity m, must be linear factors of C^c having multiplicity $m+1$. This last statement follows easily identifying the linear factors of the homogeneous polynomial $C^c(x,y)$ in two variables with the roots of the polynomial $C^c(1,z)$ in the variable z. Hence, A^a is a repeated factor of C^c. It is in contradiction with the assumption. ∎

Proof of Theorem 2.1: (a) Since there are no points at which C, C_x and C_y vanish simultaneously, from Hilbert's Nullstellensatz we obtain that there exist polynomials E, F and G such that

$$EC_x + FC_y + GC = 1. \tag{2.7}$$

As C satisfies equation (1.2), we get from (1.2) and (2.7) that

$$K = (KE + GP)C_x + (KF + GQ)C_y.$$

Substituting K into (1.2), we get

$$[P - (KE + GP)C]C_x = -[Q - (KF + GQ)C]C_y.$$

Since $(C_x, C_y) = 1$, there exists a polynomial D such that

$$P - (KE + GP)C = -DC_y, \qquad Q - (KF + GQ)C = DC_x.$$

This proves that X takes the form (2.1) with $A = KE + GP$ and $B = KF + GQ$.

(b) From (a) and Lemma 2.8 we get that the vector field X has the normal form (2.1). Without loss of generality we can assume that $p \le q$.

We first consider the case that C^c has neither factor x nor y. So we have $(C^c, (C^c)_x) = 1$ and $(C^c, (C^c)_y) = 1$, where $(C^c)_x$ denotes the derivative of C^c with respect to x. In (2.1) we assume that $a > p - c$, otherwise the statement follows. Then $d = a + 1$. Moreover, from the highest order terms of (2.1) we get

$$A^a C^c = D^{a+1} C_y^{c-1},$$

where C_y^{c-1} denotes the homogeneous part with degree $c-1$ of C_y. Since $(C^c, C_y^{c-1}) = 1$, there exists a polynomial F such that

$$A^a = FC_y^{c-1}, \qquad D^{a+1} = FC^c.$$

In (2.1) we replace A by $A - FC_y$ and D by $D - FC$, so the degrees of polynomials under consideration reduce by one. We continue this process and do the same for $\dot{y}$ until we reach a system of the form

$$\dot{x} = AC - DC_y, \qquad \dot{y} = BC + EC_x, \tag{2.8}$$

with $a \le p - c$, $d \le p - c + 1$, $b \le q - c$ and $e \le q - c + 1$. Since $C = 0$ is an invariant algebraic curve of (2.8), from (1.2) we get

$$C(AC_x + BC_y) + C_x C_y (E - D) = KC.$$

This implies that there exists a polynomial R such that $E - D = RC$, because C with C_x and C_y are coprime.

If $e \geq d$, then $r = e - c$. We write $BC + EC_x = (B + RC_x)C + DC_x$ and denote $B + RC_x$ again by B, then system (2.8) has the form (2.1) where A, B and D have the required degrees.

If $e < d$, then $r = d - c$. We write $AC - DC_y = (A + RC_y)C - EC_y$ and denote $A + RC_y$ again by A, then system (2.8) has the form (2.1) where A, B and E instead of D have the required degrees. This proves the second part of (b).

Now we prove the first part of (b). We note that even though C^c has no repeated factor, C^c with C_x^{c-1} or C_y^{c-1} may have a common factor in x or y (for example $C^3 = x(x^2 + y^2)$, $C^3 = y(x^2 + y^2)$ or $C^4 = xy(x^2 + y^2)$). In order to avoid this difficulty we rotate the initial system slightly such that C^c has no factors in x and y. Then, applying the above method to the new system we get that the new system has a normal form (2.1) with the degrees of A, B and D as those of the second part of (b).

We claim that under affine changes system (2.1) preserves its form and the upper bound of the polynomials, i.e. $a, b \leq m - c$ and $d \leq m - c + 1$. Indeed, using the affine change of variables $u = a_1x + b_1y + c_1$ and $v = a_2x + b_2y + c_2$ with $a_1b_2 - a_2b_1 \neq 0$, system (2.1) becomes

$$\dot{u} = (a_1A + b_1B)C - (a_1b_2 - a_2b_1)DC_v, \qquad \dot{v} = (a_2A + b_2B)C + (a_1b_2 - a_2b_1)DC_u.$$

Hence, the claim follows. This completes the proof of (b), and consequently we have the proof of the theorem. ∎

Proof of Theorem 2.2: Since $(C, D) = 1$, the curves C and D have finitely many intersection points. By assumption (i) at each of such points there is at least one non–zero first derivative of both C and D. In a similar way to the proof of the claim inside the proof of Theorem 2.1, we can prove that under an affine change of the variables, system (2.2) preserves its form and the bound for the degrees of A, B, E and F. So, we rotate the initial system slightly such that all first derivatives of C and D are not equal to zero at the intersection points.

From the Hilbert's Nullstellensatz, there exist polynomials M_i, N_i and R_i,

$i = 1, 2$ such that

$$M_1C + N_1D + R_1D_y = 1, \quad M_2C + N_2D + R_2C_y = 1. \tag{2.9}$$

By Theorem 2.1 we get that

$$P = A_1C - E_1C_y = G_1D - F_1D_y, \tag{2.10}$$

for some polynomials A_1, E_1, G_1 and F_1. Moreover, using the first equation of (2.9) we have $F_1 = SC+TD+UC_y$ for some polynomials S, T and U. Substituting F_1 into (2.10) we obtain that

$$(A_1 + SD_y)\,C + (-G_1 + TD_y)\,D + (-E_1 + UD_y)\,C_y = 0. \tag{2.11}$$

Using the second equation of (2.9) and (2.11) to eliminate C_y we get

$$-E_1 + UD_y = VC + WD, \tag{2.12}$$

for some polynomials V and W. Substituting (2.12) into (2.11), we have

$$(A_1 + SD_y + VC_y)\,C = (G_1 - TD_y - WC_y)\,D.$$

Since $(C, D) = 1$, there exists a polynomial K such that

$$A_1 + SD_y + VC_y = KD, \qquad G_1 - TD_y - WC_y = KC. \tag{2.13}$$

Substituting E_1 of (2.12) and A_1 of (2.13) into (2.10), then we have

$$P = KCD - SCD_y + WC_yD - UC_yD_y. \tag{2.14}$$

Similarly, we can prove that there exist some polynomials K', S', W' and U' such that

$$Q = K'CD + S'CD_x - W'C_xD + U'C_xD_x. \tag{2.15}$$

Since C is an invariant algebraic curve of $X = (P, Q)$, we have that $PC_x + QC_y = K_CC$ for some polynomial K_C. Using (2.14) and (2.15) we get

$$\begin{aligned} K_CC \;\; &= \;\; C\left[D\left(KC_x + K'C_y\right) - SC_xD_y + S'C_yD_x\right] \\ &\quad\; +C_xC_y\left[D(W - W') - UD_y + U'D_x\right]. \end{aligned}$$

As C, C_x and C_y are coprime, there exists a polynomial Z such that

$$D(W-W')-UD_y+U'D_x=ZC. \tag{2.16}$$

Substituting the expression $DW-UD_y$ into (2.14), we get

$$P=KCD-SCD_y+W'C_yD-U'C_yD_x+ZCC_y. \tag{2.17}$$

Since $D=0$ is an invariant algebraic curve of X, we have $PD_x+QD_y=K_DD$ for some polynomial K_D. Using (2.15) and (2.17) we get

$$\begin{aligned} K_DD &= D\left[C\left(KD_x+K'D_y\right)+W'\left(C_yD_x-C_xD_y\right)\right] \\ &\quad +D_x\left[CD_y(-S+S')+U'\left(C_xD_y-C_yD_x\right)+ZCC_y\right]. \end{aligned}$$

As D and D_x are coprime, there exists a polynomial M such that

$$CD_y(-S+S')+U'\left(C_xD_y-C_yD_x\right)+ZCC_y=MD. \tag{2.18}$$

The curves C and D are transversal implies that C, D and $C_xD_y-C_yD_x$ have no common zeros. From Hilbert's Nullstellensatz, there exist some polynomials M_3, N_3 and R_3 such that

$$M_3C+N_3D+R_3\left(C_xD_y-C_yD_x\right)=1. \tag{2.19}$$

Eliminating the term $C_xD_y-C_yD_x$ from (2.18) and (2.19), we obtain that $U'=IC+JD$ for some polynomials I and J. Hence, equation (2.18) becomes

$$\begin{gathered} C\left[I\left(C_xD_y-C_yD_x\right)+D_y(-S+S')+ZC_y\right] \\ +D\left[J\left(C_xD_y-C_yD_x\right)-M\right]=0. \end{gathered}$$

Since $(C,D)=1$, there exists a polynomial G such that

$$\begin{aligned} &M=J\left(C_xD_y-C_yD_x\right)+GC, \\ &I\left(C_xD_y-C_yD_x\right)+D_y(-S+S')+ZC_y=GD. \end{aligned}$$

Substituting ZC_y-SD_y and U' into (2.17) we obtain that

$$P=(K+G)CD-\left(IC_x+S'\right)CD_y+\left(W'-JD_x\right)DC_y.$$

This means that P can be expressed in the form (2.14) with $U = 0$.

Working in a similar way, we can express Q in the form (2.15) with $U' = 0$. Thus, (2.16) is reduced to $D(W - W') = ZC$. Hence, we have $W = W' + HC$ for some polynomial H. Consequently, $Z = HD$. Therefore, from (2.18) we obtain that $CD_y(-S + S') = D(M - HCC_y)$. Since $(C, D) = 1$ and $(D, D_y) = 1$, we have $S = S' + LD$ for some polynomial L. Substituting W and S into (2.14) we obtain that P and Q have the form (2.2). This proves statement (a).

As in the proof of Theorem 2.1 we can prove that under suitable affine change of variables the form of system (2.2) and the bound of the degrees of the polynomials A, B, E and F are invariant. So, without loss of generality we can assume that the highest order terms of C and D are neither divisible by x nor y.

By the assumptions, the conditions of statement (a) hold, so we get that X has the form (2.2). If the bounds of the degrees of A, B, E and F are not satisfied, we have by (2.2) that

$$\begin{aligned} &A^a C^c D^d - E^e C_y^{c-1} D^d - F^f C^c D_y^{d-1} = 0, \\ &B^b C^c D^d + E^e C_x^{c-1} D^d + F^f C^c D_x^{d-1} = 0. \end{aligned} \tag{2.20}$$

We remark that if one of the numbers $a+c+d$, $e+c-1+d$ and $f+c+d-1$ is less than the other two, then its corresponding term in the first equation of (2.20) is equal to zero. The same remark is applied to the second equation of (2.20). From the hypotheses it follows that C^c and C_y^{c-1} are coprime, and also D^d and D_y^{d-1}, and C^c and D^d, respectively. Hence, from these last two equations we obtain that there exist polynomials K and L such that $E^e = KC^c$, $F^f = LD^d$, and

$$A^a = KC_y^{c-1} + LD_y^{d-1}, \quad B^b = -KC_x^{c-1} - LD_x^{d-1}.$$

We rewrite equation (2.2) as

$$\begin{aligned} \dot{x} &= (A - KC_y - LD_y)\, CD - (E - KC)C_y D - (F - LD)CD_y, \\ \dot{y} &= (B + KC_x + LD_x)\, CD + (E - KC)C_x D + (F - LD)CD_x. \end{aligned}$$

Thus, we reduce the degrees of A, B, E and F in (2.2) by one. We can continue this process until the bounds are reached. This completes the proof of statement (b). ∎

Proof of Theorem 2.3: We use induction to prove this Theorem. By Theorems 2.1 and 2.2 we assume that for any l with $2 \le l < p$ we have

$$P = \sum_{i=1}^{l} \left(B_i - \frac{A_i C_{iy}}{C_i} \right) \prod_{i=1}^{l} C_i, \qquad Q = \sum_{i=1}^{l} \left(D_i + \frac{A_i C_{ix}}{C_i} \right) \prod_{i=1}^{l} C_i,$$

where $\sum_{i=1}^{l} B_i = B$ and $\sum_{i=1}^{l} D_i = D$. Since $C_{l+1} = 0$ is an invariant algebraic curve, from Theorem 2.1 we get that there exist some polynomials E, G and H such that

$$\begin{aligned} P &= \sum_{i=1}^{l} \left(B_i - \frac{A_i C_{iy}}{C_i} \right) \prod_{i=1}^{l} C_i = E C_{l+1} - G C_{l+1,y}, \\ Q &= \sum_{i=1}^{l} \left(D_i + \frac{A_i C_{ix}}{C_i} \right) \prod_{i=1}^{l} C_i = H C_{l+1} + G C_{l+1,x}. \end{aligned} \tag{2.21}$$

Now we consider the curves

$$K_j = \prod_{\substack{i=1 \\ i \neq j}}^{l} C_i = 0, \qquad j = 1, \cdots, l.$$

From the assumptions we obtain that there is no points at which all the curves $K_i = 0$ and $C_{l+1} = 0$ intersect. Otherwise, at least three of the curves $C_i = 0$ for $i = 1, \cdots, l+1$ intersect at some point. Hence, there exist polynomials U and V_i for $i = 1, \cdots, l$ such that

$$U C_{l+1} + \sum_{i=1}^{l} V_i K_i = 1. \tag{2.22}$$

Using this equality, we can rearrange (2.21) as

$$\begin{aligned} (E - GUC_{l+1,y})\, C_{l+1} &= \sum_{i=1}^{l} (B_i C_i - A_i C_{iy} + G V_i C_{l+1,y})\, K_i, \\ (H + GUC_{l+1,x})\, C_{l+1} &= \sum_{i=1}^{l} (D_i C_i + A_i C_{ix} - G V_i C_{l+1,x})\, K_i. \end{aligned} \tag{2.23}$$

Using (2.22) and (2.23) to eliminate C_{l+1} we obtain that

$$E - GUC_{l+1,y} = \sum_{i=1}^{l} I_i K_i, \quad H + GUC_{l+1,x} = \sum_{i=1}^{l} J_i K_i,$$

for some polynomials I_i and J_i. Substituting these last equalities into (2.23), we have

$$\begin{aligned}\sum_{i=1}^{l}\left(B_iC_i - A_iC_{iy} + GV_iC_{l+1,y} - I_iC_{l+1}\right)K_i &= 0,\\ \sum_{i=1}^{l}\left(D_iC_i + A_iC_{ix} - GV_iC_{l+1,x} - J_iC_{l+1}\right)K_i &= 0.\end{aligned} \tag{2.24}$$

It is easy to check that the expression multiplying K_i in the two summations of (2.24) are divisible by C_i. Hence, there exist polynomials L_i and F_i for $i = 1, \cdots, l$ such that

$$\begin{aligned}B_iC_i - A_iC_{iy} + GV_iC_{l+1,y} - I_iC_{l+1} &= L_iC_i,\\ D_iC_i + A_iC_{ix} - GV_iC_{l+1,x} - J_iC_{l+1} &= F_iC_i.\end{aligned} \tag{2.25}$$

So, from (2.24) we get that $\sum_{i=1}^{l} L_i = 0$ and $\sum_{i=1}^{l} F_i = 0$. This implies that (2.21) can be rewritten as

$$P = \sum_{i=1}^{l}\left((B_i - L_i)C_i - A_iC_{iy}\right)K_i, \quad Q = \sum_{i=1}^{l}\left((C_i - F_i)C_i + A_iC_{ix}\right)K_i. \tag{2.26}$$

Moreover, we write (2.25) in the form

$$\begin{aligned}(B_i - L_i)C_i - A_iC_{iy} &= I_iC_{l+1} - GV_iC_{l+1,y} &= P_i,\\ (D_i - F_i)C_i + A_iC_{ix} &= J_iC_{l+1} + GV_iC_{l+1,x} &= Q_i.\end{aligned} \tag{2.27}$$

It is easy to see that C_i and C_{l+1} are invariant algebraic curves of the system $\dot{x} = P_i, \dot{y} = Q_i$. So, from statement (a) of theorem 2.2 we can obtain that

$$\begin{aligned}P_i &= (B_i - L_i)C_i - A_iC_{iy} &= X_iC_iC_{l+1} - Y_iC_{iy}C_{l+1} - N_iC_iC_{l+1,y},\\ Q_i &= (D_i - F_i)C_i + A_iC_{ix} &= Z_iC_iC_{l+1} + Y_iC_{ix}C_{l+1} + N_iC_iC_{l+1,x}.\end{aligned}$$

Substituting these last two equations into (2.26), we obtain that X takes the form (2.3) with the $l + 1$ invariant algebraic curves $C_1, \cdots, C_{l+1}$. From induction we have finished the proof of statement (a).

The proof of statement (b) is almost identical with those of Theorem 2.2(b), so we shall omit it here. Hence, this ends the proof of the Theorem. ∎

Proof of Theorem 2.4: From Theorem 2.3 it follows statement (a) of Theorem 2.4.

By checking the degrees of polynomials A_i, B and D in statement (b) of Theorem 2.3 we obtain statement (b) of Theorem 2.4.

From statement (a) of Theorem 2.3, we can rearrange the initial system such that it has the form (2.3). But from statement (b) of Theorem 2.3 we must have $B = 0$, $D = 0$ and $A_i = 0$. This proves statement (c) of Theorem 2.4. ∎

Proof of Theorem 2.6: The proof is formed by the following examples. First, we consider the case $r < m + 1$. That is, the sum of degrees of the given invariant algebraic curves is less than the degree of the system plus one.

Example 2.9.

The algebraic curve $f = y^3 + x^3 - x^2 = 0$ satisfies all conditions of Theorem 2.4 excepting (i). The cubic system

$$\dot{x} = 2x - 2x^3 - 3xy^2 + y^3, \quad \dot{y} = \frac{4}{3}y + x^2 - 3x^2y - 3y^3, \tag{2.28}$$

has $f = 0$ as an invariant algebraic curve. We claim that system (2.28) does dot have the form (2.4). Otherwise, it can be written in the form

$$\dot{x} = A(y^3 + x^3 - x^2) + D(-3y^2),$$
$$\dot{y} = B(y^3 + x^3 - x^2) + D(3x^2 - 2x),$$

where A, B and D are polynomials. It is in contradiction with (2.28), because in the first equation of (2.28) there is a linear term.

Example 2.10.

The algebraic curve $f = y - x^2 = 0$ satisfies all conditions of Theorem 2.4 excepting (ii). The polynomial system of degree m with $m \geq 2$:

$$\dot{x} = D(y) + xE(y) + A(x,y)(y - x^2), \quad \dot{y} = 2xD(y) + 2yE(y) + B(x,y)(y - x^2), \tag{2.29}$$

has $f = 0$ as an invariant algebraic curve, where $\delta D, \delta E = m - 1$, and $\delta A, \delta B \leq m - 2$. We can write system (2.29) in the form (2.4), i.e.

$$\dot{x} = A(x,y)(y - x^2) + D(y) + xE(y),$$
$$\dot{y} = (B(x,y) + 2E(y))(y - x^2) + 2x(D(y) + xE(y)).$$

But then $\delta(B(x,y) + 2E(y)) = m - 1 > m - \delta C$.

Example 2.11.

The algebraic curves $f_1 = x^2 + y^2 - 1 = 0$ and $f_2 = y - 1 = 0$ satisfy all conditions of Theorem 2.4 excepting (iii). The cubic system

$$\dot{x} = -1 - y + x^2 + xy + y^2 + x^2y + y^3 = P, \quad \dot{y} = (y + x^2 + y^2)(y - 1) = Q, \tag{2.30}$$

has $f_1 = 0$ and $f_2 = 0$ as invariant algebraic curves. We claim that system (2.30) cannot be written in the form (2.4). Otherwise, Q can be written as

$$Q = B(x^2 + y^2 - 1)(y - 1) + D2x(y - 1),$$

where B and D are polynomials. However, there not exist polynomials B and D such that

$$B(x^2 + y^2 - 1) + 2xD = y + x^2 + y^2. \tag{2.31}$$

Because if the equality holds, then B must contain the monomial $-y$. Let ay^t be the monomial of B with the highest degree $t \geq 1$ and without the variable x. Then the left hand side of (2.31) contains the monomial ay^{t+2}. It is in contradiction with the right hand side of (2.31).

Example 2.12.

The algebraic curves $f_1 = x = 0$, $f_2 = y = 0$ and $f_3 = x + y = 0$ satisfy all conditions of Theorem 2.4 excepting (iv). The cubic system

$$\dot{x} = (1 + x + y + x^2 + xy)x = P, \quad \dot{y} = (1 + x^2 + 2xy + y^2)y = Q, \tag{2.32}$$

has these three curves as invariant algebraic curves. We claim that system (2.32) cannot be written in the form (2.4). Otherwise, the polynomial Q can be written as

$$Q = Bxy(x + y) + Dy(x + y) + Exy,$$

where B, D and E are polynomials. But it is in contradiction with (2.32).

Example 2.13.

The algebraic curves $f_1 = xy + 1 = 0$ and $f_2 = y = 0$ satisfy all conditions of Theorem 2.4 excepting (v). The cubic system

$$\dot{x} = 1 + x + y - x^2 + x^3 + 2xy^2 = P, \quad \dot{y} = (x + y - x^2 + xy - y^2)y = Q, \tag{2.33}$$

has $f_1 = 0$ and $f_2 = 0$ as invariant algebraic curves. If we write this system in the form (2.4), then we have

$$Q = B(xy + 1)y + Dy^2,$$

where B and D are polynomials. Comparing it with (2.33), we get that B cannot be a constant. So, $\delta B > 0 = m - r$, which is in contradiction with statement (a) of Theorem 2.4.

Next, we consider the case $r = m + 1$. That is, the sum of the degrees of the given invariant algebraic curves is equal to the degree of the system plus one.

Example 2.14.

The curve $f = x^2 + x^3 + y^3 = 0$ satisfies all conditions of Theorem 2.4 excepting (i). The quadratic systems with $f = 0$ as an invariant algebraic curve can be written as

$$\dot{x} = \frac{3}{2}ax + \frac{3}{2}ax^2 - by^2, \qquad \dot{y} = \frac{2}{3}bx + ay + bx^2 + \frac{3}{2}axy,$$

where a and b are arbitrary complex numbers. Obviously, if $a \neq 0$ this system cannot have the form (2.5).

Example 2.15.

The curve $f = y - x^3 = 0$ satisfies all conditions of Theorem 2.4 excepting (ii). It is an invariant algebraic curve of the system

$$\dot{x} = 1 + x - x^2 + xy, \quad \dot{y} = 3y + 3x^2 - 3xy + 3y^2.$$

This system cannot be written in the form (2.5).

Example 2.16.

The curves $f_1 = x^2 + y^2 - 1 = 0$ and $f_2 = y - 1 = 0$ satisfy all conditions of Theorem 2.4 excepting (iii). Moreover, $f_1 = 0$ and $f_2 = 0$ are invariant algebraic curves of system (2.6). However, system (2.6) does not have the form (2.5) if $a \neq 0$.

Example 2.17.

The curves $f_1 = x + iy = 0$, $f_2 = x - iy = 0$ and $f_3 = x = 0$ satisfy all conditions of Theorem 2.4 excepting (iv). The quadratic system

$$\dot{x} = -b(x^2 + y^2) + x + y(ax + by), \quad \dot{y} = k(x^2 + y^2) + y - x(ax + by),$$

has $f_1 = 0$, $f_2 = 0$ and $f_3 = 0$ as invariant algebraic curves, but this system cannot take the form (2.5).

Example 2.18.

The curves $f_1 = xy - 1 = 0$ and $f_2 = x = 0$ satisfy all conditions of Theorem 2.4 excepting (v). They are invariant algebraic curves of the system

$$\dot{x} = (1 - 2x + y)x, \quad \dot{y} = 1 - y + xy - y^2.$$

Obviously, this system does not have the form (2.5).

Last we give the counterexamples for the case $r > m + 1$. That is, the sum of the degrees of the invariant algebraic curves is larger than the degree of the system plus one.

Example 2.19.

The algebraic curve $f = x^4 + x^3 + y^4 = 0$ satisfy all conditions of Theorem 2.4 excepting (i). The quadratic systems having $f = 0$ as an invariant algebraic curve are

$$\dot{x} = ax + ax^2, \quad \dot{y} = \frac{3}{4}ay + axy.$$

So, statement (c) of Theorem 2.4 is not satisfied.

Example 2.20.

The algebraic curve $f = y - x^4 = 0$ satisfy all conditions of Theorem 2.4 excepting (ii). The quadratic systems having $f = 0$ as an invariant algebraic curve are

$$\dot{x} = ax + bx^2 + cxy, \quad \dot{y} = 4ay + 4bxy + 4cy^2.$$

They are not zero unless $a = b = c = 0$.

Example 2.21.

The algebraic curves $f_1 = x^2 + y^2 - 1 = 0$, $f_2 = y - 1 = 0$ and $f_3 = 4x + 3y + 5 = 0$ satisfy all conditions of Theorem 2.4 excepting (iii). However, the quadratic system $\dot{x} = y(2x - y + 1) + x^2 + y^2 - 1$, $\dot{y} = x(y - 1) + 2y^2 - 2$ has these three curves as invariant algebraic curves.

Example 2.22.

The algebraic curves $f_1 = x = 0$, $f_2 = y = 0$ and $f_3 = x + y = 0$ satisfy all conditions of Theorem 2.4 excepting (iv). The linear systems having these three curves as invariant algebraic curves are $\dot{x} = ax$, $\dot{y} = ay$. They are not zero unless $a = 0$.

Example 2.23.

The algebraic curves $f_1 = xy - 1$, $f_2 = y$ and $f_3 = y + 1$ satisfy all conditions of Theorem 2.4 excepting (v). The quadratic system with f_1, f_2 and f_3 as invariant algebraic curves are

$$\dot{x} = a - bx - (a + b)xy, \quad \dot{y} = by(y + 1).$$

They are not zero unless $a = b = 0$.

From these fifteen examples it follows the proof of Theorem 2.6. ∎

2.3 On the non-existence of Darboux first integrals

We note that all quadratic systems having an ellipse and a straight line tangent into the ellipse can be written into the form (2.6). System (2.6) has the invariant circle $f_1 = x^2 + y^2 - 1 = 0$ with cofactor $K_1 = 2(x + ay)$ and the invariant straight line $f_2 = y - 1 = 0$ with cofactor $K_2 = bx + ay + a$. We also note that f_1 and f_2 are tangent at the point $(0, 1)$.

Proof of Theorem 2.7: The proof is separated into three parts. The first part shows that there exists a set Ω_1 of values of the parameters a and b such that system (2.6) has only the given two invariant algebraic curves. The second part give a proof that there exists a set Ω_2 of values of a and b such that systems (2.6) have no exponential factors. Moreover, $\Omega_1 \cap \Omega_2 \neq \emptyset$. The last step contributes to prove that system (2.6) is not Darboux integrable for $a, b \in \Omega_1 \cap \Omega_2$.

We should use the following result (for a proof, see [17]).

Lemma 2.24. *Assume that system $\dot{x} = P,\ \dot{y} = Q$ with degree m has an invariant algebraic curve C of degree n. Let C_n, P_m and Q_m be the homogeneous parts of C with degree n, P and Q with degree m. Then the irreducible factor of C_n divides $yP_m - xQ_m$.*

The first part is formed by the following proposition, which is related to the existence of invariant algebraic curves of system (2.6).

Proposition 2.25. *For each $b \neq 1 \pm \dfrac{1}{p}$ with $p \in \mathbb{N}$ there exists a numerable set Υ such that if $a \in \mathbb{R} \backslash (\Upsilon \cup \{0\})$, then system (2.6) has no irreducible invariant algebraic curves different from $f_1 = 0$ and $f_2 = 0$.*

Proof: Assume that $C = \sum\limits_{i=0}^{n} C_i(x,y) = 0$ be an invariant algebraic curve of system (2.6) with cofactor $K = K_1 + K_0$, where C_i and K_i are homogeneous polynomials of degree i. From the definition of invariant algebraic curve, i.e. (1.2) we have

$$\begin{aligned}&\left[x^2 + axy + (1-b)y^2 + by - 1\right] \sum_{i=1}^{n} C_{ix} \\ &\qquad + \left[bxy + ay^2 - bx - a\right] \sum_{i=1}^{n} C_{iy} = (K_1 + K_0) \sum_{i=0}^{n} C_i.\end{aligned}$$

Equating the terms with the same degree we obtain

$$\begin{aligned} L[C_{n-i}] &= K_1 C_{n-i} + K_0 C_{n-i+1} - by C_{n-i+1,x} + bx C_{n-i+1,y} \\ &\qquad + C_{n-i+2,x} + a C_{n-i+2,y}, \quad i = 0, 1, \cdots, n+2, \end{aligned} \tag{2.34}$$

where $C_i = 0$ for $i < 0$ and $i > n$, and L is the partial differential operator

$$L = \left[x^2 + axy + (1-b)y^2\right] \frac{\partial}{\partial x} + \left[bxy + ay^2\right] \frac{\partial}{\partial y}.$$

For system (2.6) we have $yP_2 - xQ_2 = (1-b)y(x^2+y^2)$. So, from Lemma 2.24 we can assume that

$$C_n = (x^2+y^2)^l y^m, \qquad n = 2l+m.$$

Substituting C_n into (2.34) with $i=0$ and doing some computations we get

$$K_1 = (2l+mb)x + a(2l+m)y.$$

Set $C_{n-1} = \sum_{i=0}^{n-1} c_{n-1-i}x^{n-1-i}y^i$. Substituting C_{n-1}, C_n and K_1 into (2.34) with $i=1$ and doing some calculations, we obtain

$$\begin{aligned}
&\sum_{i=0}^{n-1}(m-1-i+ib-mb)c_{n-1-i}x^{2l+m-i}y^i - \sum_{i=0}^{n-1} ac_{n-1-i}x^{2l+m-1-i}y^{i+1}\\
&+\sum_{i=0}^{n-1}(2l+m-1-i)(1-b)c_{n-1-i}x^{2l+m-2-i}y^{i+2}\\
=\ & K_0(x^2+y^2)^l y^m + mbx(x^2+y^2)^l y^{m-1}\\
=\ & \sum_{i=0}^{l} K_0\binom{l}{i}x^{2l-2i}y^{m+2i} + \sum_{i=0}^{l} mb\binom{l}{i}x^{2l+1-2i}y^{m+2i-1}.
\end{aligned}$$

This equation can be written as

$$\begin{aligned}
&\sum_{i=0}^{n}[(m-1-i+ib-mb)c_{n-1-i} - ac_{n-i}\\
&\qquad +(2l+m+1-i)(1-b)c_{n+1-i}]x^{2l+m-i}y^i\\
=\ & \sum_{i=0}^{l} K_0\binom{l}{i}x^{2l-2i}y^{m+2i} + \sum_{i=0}^{l} mb\binom{l}{i}x^{2l+1-2i}y^{m+2i-1},
\end{aligned}$$

where $c_i = 0$ for $i<0$ and $i>n-1$. Equating the coefficients of x^iy^j in the above equation, we get

$$\begin{aligned}
&[m-i-1+(i-m)b]c_{2l+m-1-i} - ac_{2l+m-i}\\
&\qquad +(2l+m+1-i)(1-b)c_{2l+m+1-i} = 0, \quad i=0,1\cdots,m-2, \quad (2.35)\\
&[(2i-1)b-2i]c_{2l-2i} - ac_{2l+1-2i},\\
&\qquad -(2l+2-2i)(b-1)c_{2l+2-2i} = mb\binom{l}{i}, \quad i=0,1,\cdots,l, \quad (2.36)\\
&[2i(b-1)-1]c_{2l-2i-1} - ac_{2l-2i}\\
&\qquad -(2l+1-2i)(b-1)c_{2l+1-2i} = K_0\binom{l}{i}, \quad i=0,1,\cdots,l. \quad (2.37)
\end{aligned}$$

From the assumptions and (2.35) we can prove easily that $c_{2l+j} = 0$ for $j = 1, \cdots, m-1$. Equations (2.36) and (2.37) can be written as

$$\begin{aligned} c_{2l-2i} &= \frac{1}{2i(b-1)-b}\left[ac_{2l+1-2i} + (2l+2-2i)(b-1)c_{2l+2-2i} + mb\binom{l}{i}\right], \\ c_{2l-2i-1} &= \frac{1}{2i(b-1)-1}\left[ac_{2l-2i} + (2l+1-2i)(b-1)c_{2l+1-2i} + K_0\binom{l}{i}\right], \end{aligned} \tag{2.38}$$

with $i = 0, 1, \cdots, l$. It is easy to check that

$$c_{2l} = -m, \qquad c_{2l-1} = am - K_0.$$

From (2.38) with $i = 1$ we get that

$$\begin{aligned} c_{2l-2} &= \frac{a}{b-2}(am-K_0) - ml = \mathcal{B}_1(a,b,l)(am-K_0) - m\binom{l}{1}, \\ c_{2l-3} &= \left[\frac{a^2}{(2b-3)(b-2)} + l - \frac{b-1}{2b-3}\right](am-K_0) = \mathcal{B}_2(a,b,l)(am-K_0). \end{aligned}$$

In what follows we use the induction to find the coefficients c_{2l-i} for $i = 4, \cdots, 2l$. Assume that for $i = h$ we have

$$c_{2l-2h} = \mathcal{B}_{2h-1}(a,b,l)(am-K_0) - m\binom{l}{h}, \quad c_{2l-1-2h} = \mathcal{B}_{2h}(a,b,l)(am-K_0),$$

where $\mathcal{B}_{j-1}(a,b,l)$ for $j = 2h, 2h+1$, are polynomials in a where coefficients are function of b and l and the highest order terms of the form

$$a^{j-1}/\prod_{i=2}^{j}[(i-1)b-i]. \tag{2.39}$$

Then from (2.38) with $i = h+1$ we get

$$\begin{aligned} c_{2l-2h-2} &= \frac{1}{2(h+1)(b-1)-b}\{a\mathcal{B}_{2h}(a,b,l)(am-K_0) \\ &\quad +(2l-2h)(b-1)\left[\mathcal{B}_{2h-1}(a,b,l)(am-K_0) - m\binom{l}{h}\right] + mb\binom{l}{h+1}\right\} \\ &= \mathcal{B}_{2h+1}(a,b,l)(am-K_0) - m\binom{l}{h+1}, \\ c_{2l-2h-3} &= \frac{1}{2(h+1)(b-1)-1}\left\{a\left[\mathcal{B}_{2h+1}(a,b,l)(am-K_0) - m\binom{l}{h+1}\right]\right. \\ &\quad \left. +(2l-2h-1)(b-1)\mathcal{B}_{2h}(a,b,l)(am-K_0) + K_0\binom{l}{h+1}\right\} \\ &= \mathcal{B}_{2h+2}(a,b,l)(am-K_0). \end{aligned}$$

where

$$\begin{aligned}
\mathcal{B}_{2h+1} &= \frac{1}{2(h+1)(b-1)-b}[a\mathcal{B}_{2h}+(2l-2h)(b-1)\mathcal{B}_{2h-1}],\\
\mathcal{B}_{2h+2} &= \frac{1}{2(h+1)(b-1)-1}\left[a\mathcal{B}_{2h+1}-\binom{l}{h+1}+(2l-2h-1)(b-1)\mathcal{B}_{2h}\right],
\end{aligned}$$

are polynomials in a of degree $2h+1$ and $2h+2$ respectively, in which the highest order terms are the form (2.39) for $j=2h+2$ and $j=2h+3$, respectively. Hence, from (2.38) and using induction we obtain that for $h=0,1,\cdots,2l$

$$c_{2l-h}=\mathcal{B}_{h-1}(a,b,l)(am-K_0)+\frac{(-1)^{h+1}-1}{2}m\binom{l}{h/2}.$$

Moreover, from the first equation of (2.38) with $i=l$, i.e. $ac_0+(b-1)c_1+K_0=0$ we get

$$a[\mathcal{B}_{2l-1}(am-K_0)-m]+(b-1)\mathcal{B}_{2l-2}(am-K_0)=K_0.$$

This means that

$$[a\mathcal{B}_{2l-1}+(b-1)\mathcal{B}_{2l-2}-1](am-K_0)=0.$$

Since $a\mathcal{B}_{2l-1}+(b-1)\mathcal{B}_{2l-2}-1$ is a polynomial of degree $2l$ in the variable a, it has at most $2l$ real roots, denoted by S_l the set of the roots. Then, for $a\in\mathbb{R}\backslash S_l$ we must have $K_0=am$.

Obviously, $\Upsilon=\cup_{l=1}^{\infty}S_l$ is a numerable set. Moreover, for each $a\in\mathbb{R}\backslash\Upsilon$ and $l\in\mathbb{N}$ we have $K_0=am$. So, if C is an invariant algebraic curve of the above form, it has the cofactor $K=K_1+K_0=(2l+mb)x+a(2l+m)y+am=2(x+ay)l+(bx+ay+a)m$.

Moreover, we can check that $C^*=(x^2+y^2-1)^l(y-1)^m=0$ is an invariant algebraic curve with cofactor K. If $D=C-C^*\neq 0$, then $D=0$ is also an invariant algebraic curve with the cofactor K. But D has degree $d\leq 2l+m-2$. Again using Lemma 2.24 we can assume that the highest order homogeneous term of D is of the form $D_d=(x^2+y^2)^{l'}y^{m'}$ with $d=2l'+m'$. Then, from the above proof we should have the linear part of K is $K_1=(2l'+m'b)x+a(2l'+m')y$. It is in contradiction with the last paragraph. Hence, we must have $C=C^*$. This proves that for $b\neq 1\pm\frac{1}{p}$ with $p\in\mathbb{N}$ and $a\in\mathbb{R}\backslash(\Upsilon\cup\{0\})$ system (2.6) has only the irreducible invariant algebraic curves $x^2 + y^2 = 1$ and $y = 1$. This proves the proposition. ∎

Now we should prove that for some values of parameters system (2.6) has no exponential factors.

Proposition 2.26. *For each $b \notin \mathbb{Q}$ there exists a numerable set $\Upsilon^* \supset \Upsilon$ such that if $a \in \mathbb{R}\backslash(\Upsilon^* \cup \{0\})$, then system (2.6) has no exponential factors.*

Proof: From Proposition 2.25 system (2.6) has only the invariant algebraic curves $f_1 = x^2 + y^2 - 1 = 0$ and $f_2 = y - 1 = 0$. If system (2.6) has an exponential factor, we can assume that it has the form $F = \exp\left(\dfrac{G}{f_1^{l_1} f_2^{l_2}}\right)$ with a cofactor L, where l_1 and l_2 are non–negative integers. Since the invariant algebraic curve $f_1^{l_1} f_2^{l_2} = 0$ has the cofactor $K = l_1K_1 + l_2K_2 = 2l_1(x + ay) + l_2(bx + ay + a)$, from (**??**) we get that G satisfies the following equation

$$\begin{aligned}
&[x^2 + axy + (1-b)y^2 + by - 1]G_x + (bxy + ay^2 - bx - a)G_y \\
&= [2l_1(x + ay) + l_2(bx + ay + a)]G + L(x^2 + y^2 - 1)^{l_1}(y - 1)^{l_2}. \quad (2.40)
\end{aligned}$$

Set $\delta G = n$. Since $\delta L \leq 1$, we can assume that $L = L_1 + L_0$, where L_i are homogeneous polynomials of degree i.

Case 1: $n + 1 < 2l_1 + l_2$. By equating the homogeneous terms of highest degree in (2.40) we obtain first that $L_1 = 0$, and after that $L_0 = 0$, and so $L = 0$. Thus G is an invariant algebraic curve. Moreover, from the assumption of this proposition we obtain that $G = cf_1^{l_1} f_2^{l_2}$, where c is a constant. Then, $F =$ constant, and it cannot be an exponential factor.

Case 2: $n + 1 = 2l_1 + l_2$. Then we have $L_1 = 0$. Set $G = \sum_{i=0}^{n} G_i(x, y)$ with G_i homogeneous polynomials of degree i and $G_n = \sum_{i=0}^{n} a_i x^{n-i} y^i$, where a_i are constants. Then, equating the terms of (2.40) with degree $n + 1$ we get that

$$\begin{aligned}
&\left[x^2 + axy + (1-b)y^2\right] \sum_{i=0}^{n} (n-i)a_i x^{n-i-1} y^i + \left(bxy + ay^2\right) \sum_{i=0}^{n} i a_i x^{n-i} y^{i-1} \\
= \; &[2l_1(x + ay) + l_2(bx + ay)] \sum_{i=0}^{n} a_i x^{n-i} y^i + L_0 \left(x^2 + y^2\right)^{l_1} y^{l_2}.
\end{aligned}$$

Using the relation $n+1=2l_1+l_2$ we can write this last equation as

$$\sum_{i=0}^{2l_1+l_2+1}\{[(b-1)(i-l_2)-1]a_i+aa_{i-1}+(1-b)(2l_1+l_2+1-i)a_{i-2}\}x^{2l_1+l_2-i}y^i$$
$$=L_0\sum_{i=0}^{l_1}\left(\begin{array}{c} l_1 \\ i \end{array}\right)x^{2l_1-2i}y^{2i+l_2},$$

where $a_i=0$ for $i<0$ and $i>n$. The last equation is equivalent to

$$[(b-1)(i-l_2)-1]a_i+aa_{i-1}+(1-b)(2l_1+l_2+1-i)a_{i-2}=0, \qquad i=0,1,\cdots,l_2-1 \tag{2.41}$$

$$\sum_{j=0}^{2l_1+1}[(b-1)j-1]a_{j+l_2}+aa_{j+l_2-1}-(1-b)(2l_1+1-j)a_{j+l_2-2}\}x^{2l_1-j}y^{j+l_2}$$
$$=L_0\sum_{i=0}^{l_1}\left(\begin{array}{c} l_1 \\ i \end{array}\right)x^{2l_1-2i}y^{2i+l_2}. \tag{2.42}$$

Since $b\neq 1\pm\frac{1}{k}$ for $k\in\mathbb{N}$, from (2.41) we get that $a_i=0$ for $i=0,1,\cdots,l_2-1$. From (2.42) we obtain that for $i=0,1,\cdots,l_1$

$$\begin{aligned}&[(b-1)(2i+1)-1]\,a_{2i+1+l_2}+aa_{2i+l_2}-(1-b)(2l_1-2i)a_{2i+l_2-1}=0,\\ &[(b-1)2i-1]\,a_{2i+l_2}+aa_{2i+l_2-1}-(1-b)(2l_1+1-2i)a_{2i+l_2-2}=L_0\left(\begin{array}{c} l_1 \\ i \end{array}\right).\end{aligned} \tag{2.43}$$

Solving (2.43) for $i=0,1,\cdots,l_1-1$ and its second equation with $i=l_1$ we get that

$$a_{l_2+h}=\overline{\mathcal{B}}_h(a)L_0, \quad k=0,1,\cdots,2l_1,$$

where $\overline{\mathcal{B}}_h(a)$ is a polynomial of degree h in a whose coefficients are rational functions in b and l_1. The highest order term of $\overline{\mathcal{B}}_h(a)$ in a is $-1/\prod_{j=0}^{h}[(1-b)j+1]$. So the first equation of (2.43) with $i=l_1$ is $a\overline{\mathcal{B}}_{2l_1}L_0=0$. Since the coefficient of L_0 is a polynomial of degree $m+1$, there exists at most $m+1$ values of a such that it is equal to zero. We denote by $\overline{\gamma}_{l_1}$ the set of such a. Hence, if $a\notin\overline{\gamma}_{l_1}$ we must have $L_0=0$. This means that $L=0$. So, system (2.6) has no exponential factors for $a\notin\overline{\gamma}_{l_1}$.

Case 3: $n = 2l_1 + l_2$. Let $L_1 = L_{10}x + L_{01}y$. Using the notations for G and G_n introduced in the study of Case 2, equating the terms of (2.40) with degree $n+1$ and doing some computations we get that

$$\begin{aligned}&\sum_{i=0}^{n+2}[(b-1)(i-l_2)]a_i + (1-b)(n+2-i)a_{i-2}]x^{n+1-i}y^i \\ &= L_{10}\sum_{i=0}^{l_1}\begin{pmatrix} l_1 \\ i \end{pmatrix} x^{2l_1-2i+1}y^{2i+l_2} + L_{01}\sum_{i=0}^{l_1}\begin{pmatrix} l_1 \\ i \end{pmatrix} x^{2l_1-2i}y^{2i+l_2+1},\end{aligned}$$

where $a_i = 0$ for $i < 0$ and $i > n$. These equations are equivalent to

$$\begin{aligned}&(i-l_2)a_i - (n+2-i)a_{i-2} = 0, \quad i = 0, 1, \cdots, l_2 - 1, \\ &(b-1)2ja_{2j+l_2} + (1-b)(2l_1+2-2j)a_{2j+l_2-2} = L_{10}\begin{pmatrix} l_1 \\ j \end{pmatrix}, \\ &(b-1)(2j+1)a_{2j+l_2+1} + (1-b)(2l_1+1-2j)a_{2j+l_2-1} = L_{01}\begin{pmatrix} l_1 \\ j \end{pmatrix},\end{aligned} \tag{2.44}$$

where $j = 0, 1, \cdots, l_1$.

From the first equation of (2.44) we obtain that $a_i = 0$ for $i = 0, 1, \cdots, l_2 - 1$. Hence, the first equation of (2.44) with $j = 0$ induces to $L_{10} = 0$. Thus, we have

$$a_{2j+l_2} = \frac{2l_1+2-2j}{2j}a_{2j+l_2-2}, \quad j = 1, \cdots l_1.$$

i.e. $a_{l_2+2j} = \begin{pmatrix} l_1 \\ j \end{pmatrix} a_{l_2}$, $j = 1, \cdots l_1$. From the second equation of (2.44) with $j = 0, 1, \cdots, l_1 - 1$ and using induction, we can prove that

$$a_{2j+1+l_2} = \frac{\mu_j}{b-1}L_{01}, \qquad j = 0, 1, \cdots, l_1 - 1,$$

with $\mu_j > 0$. Now the second equation of (2.44) with $j = l_1$ can be written as $(1 + \mu_{l_1})L_{01} = 0$. This implies that $L_{01} = 0$. Moreover, we have $a_{2j+1+l_2} = 0$ for $j = 0, 1, \cdots, l_1 - 1$.

From the above calculations we get that $L = L_0$ and

$$G_n = \sum_{i=0}^{n} a_i x^{n-i}y^i = a_{l_2}(x^2+y^2)^{l_1}y^{l_2}.$$

Since $a_{l_2} \neq 0$, without loss of generality we assume that $a_{l_2} = 1$.

Equating the terms of (2.40) with degree n we get that

$$[x^2 + axy + (1-b)y^2]G_{n-1,x} + (bxy + ay^2)G_{n-1,y} =$$
$$[2l_1(x+ay) + l_2(bx+ay)]G_{n-1} - byG_{nx} + bxG_{ny} + l_2 aG_n + L_0(x^2+y^2)^{l_1} y^{l_2}.$$

Let $G_{n-1} = \sum_{i=0}^{n-1} b_i x^{n-1-i} y^i$. Substituting G_{n-1} into the above equation and doing some computations, we can obtain that

$$\sum_{i=0}^{n} \{[l_2 - 1 - i + b(i - l_2)]b_i - ab_{i-1} + (1-b)(n+1-i)b_{i-2}\} x^{n-i} y^i$$
$$= bl_2 \sum_{i=0}^{l_1} \begin{pmatrix} l_1 \\ i \end{pmatrix} x^{2l_1+1-2i} y^{l_2+2i-1} + (l_2 a + L_0) \sum_{i=0}^{l_1} \begin{pmatrix} l_1 \\ i \end{pmatrix} x^{2l_1-2i} y^{l_2+2i},$$

where $b_i = 0$ for $i < 0$ and $i > n-1$. From this equation we obtain that

$$\begin{aligned}
&[l_2 - 1 - j + b(j - l_2)]b_j - ab_{j-1} + (1-b)(n+1-j)b_{j-2} = 0, \\
&\qquad\qquad j = 0, 1, \cdots, l_2 - 2, \\
&[-2i + b(2i-1)]b_{2i+l_2-1} - ab_{2i+l_2-2} + (1-b)(2l_1 + 2 - 2i)b_{2i+l_2-3} \\
&\qquad\qquad = bl_2 \begin{pmatrix} l_1 \\ i \end{pmatrix}, \\
&[-2i - 1 + 2bi]b_{2i+l_2} - ab_{2i+l_2-1} + (1-b)(2l_1 + 1 - 2i)b_{2i+l_2-2} \\
&\qquad\qquad = (l_2 a + L_0) \begin{pmatrix} l_1 \\ i \end{pmatrix},
\end{aligned} \tag{2.45}$$

with $i = 0, 1, \cdots, l_1$.

From the first equation of (2.45) we can prove that $b_j = 0$ for $j = 0, 1, \cdots, l_2 - 2$. From (2.45) with $i = 0, 1, \cdots, l_1 - 1$ and its first equation with $i = l_1$, working in a similar way to the proof of Proposition 2.25 we can prove that

$$b_{l_2+2i-1} = \widetilde{\mathcal{B}}_{2i-1}(a)L_0 - l_2 \begin{pmatrix} l_1 \\ i \end{pmatrix}, \quad b_{l_2+2i} = \widetilde{\mathcal{B}}_{2i}(a)L_0, \quad i = 0, 1, \cdots, l_1,$$

where $\widetilde{\mathcal{B}}_k(a)$ is a polynomial of degree k in a whose coefficients are rational functions in b and l_1. Using the last equation of (2.45) with $i = l_1$ we get

$$[a\widetilde{\mathcal{B}}_{2l_1-1} + (b-1)\widetilde{\mathcal{B}}_{2l_1-2} + 1]L_0 = 0.$$

For every given b and l_1 there exist at most $2l_1$ values of a for which $a\widetilde{\mathcal{B}}_{2l_1-1}+(b-1)\widetilde{\mathcal{B}}_{2l_1-2}+1$ is equal to zero. We denote by $\widetilde{\gamma}_{l_1}$ the set of such a. Then if $a \notin \widetilde{\gamma}_{l_1}$, we have $L_0 = 0$. So for every b satisfying the assumption of the proposition, if $a \notin \cup\widetilde{\gamma}_{l_1}$ system (2.6) has no exponential factors.

Case 4: $n > 2l_1 + l_2$. Using the notations of Case 2 for G and G_n, from (2.40) we get that

$$[x^2 + axy + (1-b)y^2]G_{nx} + (bxy + ay^2)G_{ny} = [2l_1(x + ay) + l_2(bx + ay)]G_n.$$

Working in similar way to the previous case we can prove that the coefficients a_i in G_n satisfy the following equations

$$[n - i - 2l_1 + b(i - l_2)]a_i + a(n - 2l_1 - l_2)a_{i-1} + (1 - b)(n + 2 - i)a_{i-2} = 0,$$

with $i = 0, 1, \cdots, n+1$. Since $b \notin \mathbb{Q}$, from these equations we obtain that $a_i = 0$. So, $G_n = 0$. This implies that system (2.6) has no exponential factors.

Summing up these four cases the proof of the proposition follows. ∎

In this last step we prove that for each $b \notin \mathbb{Q}$, if $a \in \mathbb{R}\backslash(\Upsilon^* \cup \{0\})$ system (2.6) is not Darboux integrable.

Suppose that the assumptions of Proposition 2.26 are satisfied. Then, by Propositions 2.25 and 2.26 we get that system (2.6) has only the invariant algebraic curves $x^2 + y^2 = 1$ with cofactor $K_1 = 2(x + ay)$ and $y = 1$ with cofactor $K_2 = bx + ay + a$, and has no exponential factors. We can check easily that under these assumptions do not exist $\lambda_1, \lambda_2 \in \mathbb{C}$ not all zero such that $\lambda_1K_1 + \lambda_2K_2 = 0$ or $\lambda_1K_1+\lambda_2K_2 = -\mathrm{div}(P, Q) = -(2+b)x-3ay$, where $P = y(ax-by+b)+x^2+y^2-1$ and $Q = bx(y - 1) + a(y^2 - 1)$. Hence, from the Darboux theory of integrability (see for instance [15] or [13]) it follows that system (2.6) is not Darboux integrable. We have finished the proof of Theorem 2.7.

As a corollary of Theorem 2.7 is the following result which shows that there are polynomial systems with an invariant algebraic curve whose highest order term have repeated factors such that they are not Darboux integrable.

Corollary 2.27. *There exist values of the parameters a and b for which system*

$$\begin{aligned}\dot{x} &= (1 - b)(x^2 + 2y - 1) - (ax - b)(y - 1) = P(x, y),\\ \dot{y} &= -(bx + 2ay - a)(y - 1) = Q(x, y),\end{aligned} \tag{2.46}$$

is not Darboux integrable.

Proof of corollary 2.27: System (2.46) has the invariant algebraic curves $f_1 = x^2+2y-1=0$ with cofactor $K_1 = 2[(1-b)x-ay+a]$ and $f_2 = y-1=0$ with cofactor $K_2 = -(bx+2ay-a)$. We note that the highest order term of f_1 has a repeated factor x.

Since $y-1=0$ is invariant by system (2.46), after the change of variables

$$\overline{x} = \frac{x}{y-1}, \qquad \overline{y} = \frac{y}{y-1}, \qquad t = \frac{\tau}{y-1},$$

system (2.46) becomes into the form of system (2.6), i.e.

$$\begin{array}{rclcl} \frac{d\overline{x}}{d\tau} & = & \overline{y}(a\overline{x}-b\overline{y}+b)+\overline{x}^2+\overline{y}^2-1 & = & \overline{P}(x,y), \\ \frac{d\overline{y}}{d\tau} & = & b\overline{x}(\overline{y}-1)+a(\overline{y}^2-1) & = & \overline{Q}(x,y). \end{array} \tag{2.47}$$

Let $C(x,y)$ be a polynomial of degree n, and set $\overline{C}(\overline{x},\overline{y}) = (\overline{y}-1)^n C\left(\frac{\overline{x}}{\overline{y}-1}, \frac{\overline{y}}{\overline{y}-1}\right)$. We claim that if $C(x,y)=0$ is an invariant algebraic curve of system (2.46) with cofactor $K(x,y)$ and $\overline{C} \not\equiv$ constant, then $\overline{C}(\overline{x},\overline{y}) = 0$ is an invariant algebraic curve of system (2.47) with cofactor

$$\overline{K} = (\overline{y}-1)K\left(\frac{\overline{x}}{\overline{y}-1}, \frac{\overline{y}}{\overline{y}-1}\right) + n\frac{\overline{Q}}{\overline{y}-1}.$$

Indeed, straightforward calculations show that

$$\begin{array}{rcl} \overline{P}\,\overline{C}_{\overline{x}} + \overline{Q}\,\overline{C}_{\overline{y}} & = & (\overline{y}-1)^n\left[\frac{\overline{P}(\overline{y}-1)-\overline{Q}\overline{x}}{(\overline{y}-1)^2}C_x - \frac{\overline{Q}}{(\overline{y}-1)^2}C_y + n\frac{\overline{Q}}{\overline{y}-1}C\right] \\ & = & (\overline{y}-1)^n\left[(\overline{y}-1)PC_x + (\overline{y}-1)QC_y + n\frac{\overline{Q}}{\overline{y}-1}C\right] \\ & = & (\overline{y}-1)^{n+1}KC + n(\overline{y}-1)^n\frac{\overline{Q}}{\overline{y}-1}C = \overline{K}\,\overline{C}. \end{array}$$

This proves the claim.

Now we claim that if $F(x,y) = \exp\left(\frac{G(x,y)}{H(x,y)}\right)$ is an exponential factor of system (2.46) with cofactor $L(x,y)$, then

$$\overline{F}(\overline{x},\overline{y}) = \exp\left(G\left(\frac{\overline{x}}{\overline{y}-1}, \frac{\overline{y}}{\overline{y}-1}\right) / H\left(\frac{\overline{x}}{\overline{y}-1}, \frac{\overline{y}}{\overline{y}-1}\right)\right),$$

is an exponential factor of system (2.47) with cofactor $\overline{L}(\overline{x},\overline{y}) = (\overline{y}-1)L\left(\frac{\overline{x}}{\overline{y}-1},\frac{\overline{y}}{\overline{y}-1}\right)$. In fact, we have

$$\begin{aligned}
\overline{P}\,\overline{F}_{\overline{x}} + \overline{Q}\,\overline{F}_{\overline{y}} &= \exp\left(\frac{G}{H}\right)H^{-2}\left[\left(\frac{\overline{P}(\overline{y}-1)-\overline{Q}\overline{x}}{(\overline{y}-1)^2}G_x - \frac{\overline{Q}}{(\overline{y}-1)^2}G_y\right)H\right.\\
&\qquad\left. - \left(\frac{\overline{P}(\overline{y}-1)-\overline{Q}\overline{x}}{(\overline{y}-1)^2}H_x - \frac{\overline{Q}}{(\overline{y}-1)^2}H_y\right)G\right]\\
&= (\overline{y}-1)\exp\left(\frac{G}{H}\right)H^{-2}\left[\left(PG_x+QG_y\right)H - \left(PH_x+QH_y\right)G\right]\\
&= (\overline{y}-1)\left(PF_x+QF_y\right)\\
&= (\overline{y}-1)LF = (\overline{y}-1)L\left(\frac{\overline{x}}{\overline{y}-1},\frac{\overline{y}}{\overline{y}-1}\right)\overline{F}.
\end{aligned}$$

This proves the claim.

From these two claims and the proof of Theorem 2.7 we obtain that there exist values of a and b for which systems (2.46) and (2.6) have only two irreducible invariant algebraic curves and no exponential factors. Hence, for such values of $a \neq 0$ and b system (2.46) is not Darboux integrable. Otherwise, system (2.6) would have a Darboux integral, in contradiction with Theorem 2.7. Hence, the proof of Corollary 2.27 is completed. ∎

2.4 Polynomial systems with arbitrary set of invariant algebraic curves

In this section we are interesting to construct a polynomial vector field having an arbitrary set of invariant algebraic curves when their cofactors are known. First we consider the case of two algebraic curves.

Proposition 2.28. *Let $f_1, f_2 \in \mathbb{C}[x,y]$ be two irreducible polynomials such that $J_{12} = f_{1x}f_{2y} - f_{1y}f_{2x} \neq 0$. Let $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ be a polynomial differential system having $f_1 = 0$ and $f_2 = 0$ as invariant with cofactors K_1 and K_2*

respectively, then we have

$$
\begin{aligned}
P(x,y) &= \frac{1}{J_{12}}\,(K_1 f_1 f_{2y} - K_2 f_2 f_{1y}),\\
Q(x,y) &= \frac{1}{J_{12}}\,(-K_{f_1} f_1 f_{2x} + K_{f_2} f_2 f_{1x}).
\end{aligned}
\tag{2.48}
$$

Proof: Suppose that $\dot{x} = P(x,y), \dot{y} = Q(x,y)$ is a polynomial system having $f_1 = 0$ and $f_2 = 0$ as invariant algebraic curves with cofactors K_1 and K_2 respectively, we have

$$
\begin{aligned}
Pf_{1x} + Qf_{1y} &= K_1 f_1,\\
Pf_{2x} + Qf_{2y} &= K_2 f_2.
\end{aligned}
\tag{2.49}
$$

Multiplying the first equation of (2.49) by f_{2y} and the second one of (2.49) by f_{1y} and abstracting both relations we get

$$P(f_{1x}f_{2y} - f_{1y}f_{2x}) = K_1 f_1 f_{2y} - K_2 f_2 f_{1y}.$$

In a similar way, we get

$$Q(f_{1y}f_{2x} - f_{1x}f_{2y}) = K_1 f_1 f_{2x} - K_2 f_2 f_{1x}.$$

Hence, we have

$$
\begin{aligned}
PJ_{12} &= K_1 f_1 f_{2y} - K_2 f_2 f_{1y},\\
-QJ_{12} &= K_1 f_1 f_{2x} - K_2 f_2 f_{1x},
\end{aligned}
$$

and so we get system (2.48). ∎

We note that we are interested into construct all polynomial systems of degree m having $f_1 = 0$ and $f_2 = 0$ as invariant algebraic curves then, since K_1 and K_2 are cofactors, their degrees must be at most $m-1$.

Proposition 2.29. *Let $f_1, f_2 \in \mathbb{C}[x,y]$ such that $J_{12} \in \mathbb{C} \setminus \{0\}$. Then, all polynomial systems $\dot{x} = P,\ \dot{y} = Q$ having invariant the curves $f_1 = 0$ and $f_2 = 0$ can be written into the form*

$$
\begin{aligned}
\dot{x} &= \mu_1 f_1 f_{2y} - \mu_2 f_2 f_{1y},\\
\dot{y} &= -\mu_1 f_1 f_{2x} + \mu_2 f_2 f_{1x},
\end{aligned}
\tag{2.50}
$$

where μ_1, μ_2 are arbitrary polynomials.

Proof: The proof follows directly from the arguments of the proof of Proposition 2.28 and setting $\mu_1 = K_1/J_{12}$ and $\mu_2 = K_2/J_{12}$ we get system (2.50). ∎

We note that straight lines always satisfy the conditions of Proposition 2.29.

The expression of system (2.50) is not actually a new one. It appears in [46] however, there do not appear the condition that the curves should satisfy condition $J_{12} \in \mathbb{C} \setminus \{0\}$. Here, we present an example which proves that in general the form of system (2.50) does not hold in the case where the curves do not satisfy this condition.

Example 2.30.

We consider the system

$$\begin{array}{rcl} \dot{x} & = & -1 - y + x^2 + xy + y^2 + x^2y + y^3, \\ \dot{y} & = & (x^2 + y^2 + y)(y - 1), \end{array} \tag{2.51}$$

which has the two invariant curves $f_1 = x^2 + y^2 - 1$ and $f_2 = y - 1$ with cofactors $K_1 = 2x + 2xy + 2y^2$ and $K_2 = y + x^2 + y^2$. We note that $J_{12} = f_{1x}f_{2y} - f_{1y}f_{2x} = 2x \neq 0$. We observe that system (2.51) is of the form (2.48). However, since $K_{f_1}/J_{12} \notin \mathbb{C}[x, y]$ and $K_{f_2}/J_{12} \notin \mathbb{C}[x, y]$ system (2.51) cannot be written into the form (2.50). So, Proposition 3.2 of [46] is not true.

If in addition polynomial system $\dot{x} = P,\ \dot{y} = Q$ has a third invariant curve with cofactor K_3 then an interesting property holds.

Proposition 2.31. *System (2.48) admits an additional irreducible invariant algebraic curve $f_3 = 0$ with $f_3 \in \mathbb{C}[x, y]$ with cofactor K_3 if and only if*

$$\begin{vmatrix} f_{1x} & f_{1y} & K_1f_1 \\ f_{2x} & f_{2y} & K_2f_2 \\ f_{3x} & f_{2y} & K_3f_3 \end{vmatrix} = 0, \tag{2.52}$$

or equivalently,

$$K_1f_1J_{23} - K_2f_2J_{13} + K_3f_3J_{12} = 0. \tag{2.53}$$

Proof: Working in a similar way as in proof of Proposition 2.28 we get that

$$\begin{array}{rcl} PJ_{12} & = & -K_2f_2f_{1y} + K_1f_1f_{2y}, \\ PJ_{13} & = & -K_3f_3f_{1y} + K_1f_1f_{3y}, \\ PJ_{23} & = & -K_3f_3f_{2y} + K_2f_2f_{3y}. \end{array}$$

So, we have

$$
\begin{aligned}
&PJ_{12}f_{3x} + PJ_{23}f_{1x} - PJ_{13}f_{2x} \\
&= -K_2f_2f_{1y}f_{3x} + K_1f_1f_{2y}f_{3x} - K_3f_3f_{2y}f_{1x} + K_2f_2f_{3y}f_{1x} + K_3f_3f_{1y}f_{2x} - K_1f_1f_{3y}f_{2x} \\
&= K_2f_2(f_{1x}f_{3y} - f_{1y}f_{3x}) - K_1f_1(f_{2x}f_{3y} - f_{2y}f_{3x}) - K_3f_3(f_{1x}f_{2y} - f_{1y}f_{2x}) \\
&= K_2f_2J_{13} - K_1f_1J_{23} - K_3f_3J_{12},
\end{aligned}
$$

and therefore we get

$$
J_{12}(Pf_{3x} + K_3f_3) - J_{13}(Pf_{2x} + K_2f_2) + J_{23}(Pf_{1x} + K_1f_1) \;=\; 0,
$$

or

$$
\begin{vmatrix} f_{1x} & f_{1y} & Pf_{1x} + K_1f_1 \\ f_{2x} & f_{2y} & Pf_{2x} + K_2f_2 \\ f_{3x} & f_{2y} & Pf_{3x} + K_3f_3 \end{vmatrix} = 0,
$$

and equivalently we get the relation (2.52). ∎

Eruguin, in the paper [24], found the forms of differential systems having the invariant curves $\omega_1(x,y) = 0$ and $\omega_2(x,y) = 0$ where ω_1, ω_2 are C^r functions. We denote with

$$
J = \frac{\partial \omega_1}{\partial x}\frac{\partial \omega_2}{\partial y} - \frac{\partial \omega_1}{\partial y}\frac{\partial \omega_2}{\partial x}
$$

the Jacobian of these two functions. Hence, according to his paper, all differential systems having the invariant curves $\omega_1(x,y) = 0$ and $\omega_2(x,y) = 0$ and $J \neq 0$ can be written into the form

$$
\begin{aligned}
\dot{x} &= \frac{1}{J}\left(\frac{\partial \omega_2}{\partial y}F_1(\omega_1,x,y) - \frac{\partial \omega_1}{\partial y}F_2(\omega_2,x,y)\right), \\
\dot{y} &= -\frac{1}{J}\left(\frac{\partial \omega_2}{\partial x}F_1(\omega_1,x,y) + \frac{\partial \omega_1}{\partial x}F_2(\omega_2,x,y)\right),
\end{aligned}
\tag{2.54}
$$

where F_1, F_2 are functions such that

$$
\begin{aligned}
F_1(\omega_1,x,y)\big|_{\omega_1=0} &\equiv 0, \\
F_1(\omega_2,x,y)\big|_{\omega_2=0} &\equiv 0.
\end{aligned}
$$

We note that for polynomial differential systems with irreducible invariant algebraic curves, Eruguin's result coincides with Proposition 2.28. This is due to the fact that the polynomial F_1 vanishes whenever vanishes polynomial ω_1. Then, from

Hilbert's Nullstellansatz relation we have that there is $M_1 \in \mathbb{C}[x,y]$ and a positive integer n such that $F_1^n = M_1\omega_1$. Since ω_1 is an irreducible algebraic curve it must divides the polynomial F_1. Hence, there is a polynomial $K_1 \in \mathbb{C}[x,y]$ such that $F_1 = K_1\omega_1$. By similar arguments we get that $F_2 = K_2\omega_2$ for some $K_2 \in \mathbb{C}[x,y]$. However, we should note that according Eruguin's result, for polynomial differential systems, the polynomials F_1 and F_2 are of arbitrary degree, and therefore the polynomials K_1 and K_2 should be polynomials of arbitrary degree. So, for polynomial systems, Proposition 2.28 can be obtained from Eruguin's results.

Lemma 2.32. *Let $F, P, Q \in \mathbb{C}[x,y]$, P and Q are coprime, and $D = gcd(F_x, F_y)$. Suppose $X = (P,Q)$ has F as a first integral. Then there exits a polynomial $G \in \mathbb{C}[x,y]$ such that*

$$X = G\left(-\frac{F_y}{D}, \frac{F_x}{D}\right). \tag{2.55}$$

Proof: Since $X(F) = 0$ we have that $PF_x + QF_y = 0$ which yields to

$$P\frac{F_x}{D} + Q\frac{F_y}{D} = 0. \tag{2.56}$$

Since P and Q are coprime and F_x/D and F_y/D are also coprime, from (2.56) we get that $(F_x/D)/Q$ and $(F_y/D)/P$. Therefore there exists a polynomial $G \in \mathbb{C}[x,y]$ such that $P = -GF_y/D$, and from (2.56) we have that $Q = HF_x/D$. This completes the proof. ∎

The next proposition is essentially due to A. Gasull.

Proposition 2.33. *Let $f_j \in \mathbb{C}[x,y]$ for $j = 1, \ldots, q$, and let $F = \prod_{j=1}^{q} f_j$. Suppose that $F = 0$ is an invariant algebraic curve for the polynomial vector fields X_i with cofactor K_i for $i = 1, 2$, and that $K_2 \neq 0$. If $D = gcd(F_x, F_y)$, then*

$$X_1 = \frac{1}{K_2}\left(K_1X_2 - G\left(-\frac{F_y}{D}, \frac{F_x}{D}\right)\right), \tag{2.57}$$

where G is any polynomial which allows that X_1 be a polynomial vector field.

Proof: Let $X_i = (P_i, Q_i)$ for $i = 1, 2$. Since $F = 0$ is an invariant algebraic curve of the vector field X_1 we have that $X_1(F) = P_1F_x + Q_1F_y = K_1F$. Then

$$K_2X_1(F) - K_1X_2(F) = K_2K_1F - K_1K_2F = 0.$$

Let $\tilde{D} = \gcd(K_2P_1 - K_1P_2, K_2Q_1 - K_1Q_2)$ and $\tilde{X} = (K_2X_1 - K_1X_2)/\tilde{D}$. Since $\tilde{X}$ satisfy the assumptions of Lemma 2.32 we get that

$$\tilde{X} = G_1\left(-\frac{F_y}{D}, \frac{F_x}{D}\right),$$

for some polynomial G_1. Therefore

$$X_1 = \frac{1}{K_2}\left(K_1X_2 + \tilde{D}G_1\left(-\frac{F_y}{D}, \frac{F_x}{D}\right)\right),$$

and taking $G = \tilde{D}G_1$ the proposition follows. ∎

We remark that Proposition 2.33 can be used to find all polynomials vector fields X_1 having the invariant algebraic curves $f_i = 0$ for $i = 1, \ldots, q$, because X_2 can be obtained from Lemma 2.32. We also note that the construction of the vector field X_1 is for an arbitrary given set of algebraic invariant curves. Hence, it can be applied whenever the conditions of Theorem 2.4 are not satisfied. In any case, Proposition 2.33 is difficult to apply because in general it is not easy to determine the polynomial G in such a way that X becomes a polynomial.

2.5 On a Theorem of Walcher

Until now we show that in order to define completely the polynomial vector fields having some invariant algebraic curves we need to impose some conditions. In particular, the condition that we definitely need in order to construct the polynomial vector field, even if we consider just one invariant algebraic curve, is the generic condition (i). Hence, in order to determine a polynomial vector field having an invariant algebraic curve invariant we asked that this curve has no multiple points.

In this section we present Theorem 2.34 due to Walcher [50] and not only covers the case where one curve satisfies the generic condition (i) but also revels the complete structure of the vector fields when this condition does not hold. He based his proof in some results of the classical commutative algebra. In this theorem we denote by $< f_x, f_y >$ the ideal generating by f_x and f_y and by $< f_x, f_y >:< f >$ the quotient ideal.

Theorem 2.34. *(a) The dimension d of*

$$(< f_x, f_y >:< f >) \, / < f_x, f_y >$$

is finite.

(b) The curve f satisfies the generic condition (i) if and only if $d = 0$. In that case the polynomial vector field X has $f = 0$ invariant if and only if

$$X = hf + \rho X_f{}^*, \tag{2.58}$$

with an arbitrary vector field h and an arbitrary polynomial ρ. The vector field $X_f{}^*$ corresponds to the irreducible of the Hamiltonian $X_f = (-f_y, f_x)$, i.e. $X_f{}^*$ is X_f having its components divided by their common divisors.

(c) In case that $d \geq 1$, let $\mu_1, \cdots \mu_d \in \mathbb{C}[x, y]$ such that

$$\mu_1 + < f_x, f_y >, \cdots, \mu_d + < f_x, f_y >$$

form a basis of the above vector space, and let X_i be the vector fields having $f = 0$ invariant with cofactors μ_i $(1 \leq i \leq d)$. Then the vector field X has $f = 0$ invariant if and only if

$$X = hf + \rho X_f{}^* + \sum_{i=1}^{d} \alpha_i X_i, \tag{2.59}$$

with $\alpha_1, \cdots, \alpha_d \in \mathbb{C}$, and $\rho, h, X_f{}^*$ as above.

We note that Theorem 2.34(b) is the same as Theorem 2.1(a) because in the expression (2.58) is used the irreducible vector $X_f{}^*$. Walcher in statement (c) of Theorem 2.34 proved that there is a finite number of terms which contribute to the expression of the vector field having the invariant algebraic curve $f = 0$ when this does not satisfy the generic condition (i). More precisely, if the invariant algebraic curve has multiple points, then the complete expression of the vector field having such curve as invariant algebraic curve is given by system (2.59).

We note that in Theorem 2.34 does not appear bounds of the degrees of the polynomials in the expressions (2.58) and (2.59). As we proved in Theorem 2.6 extra conditions are necessary in order to obtain these bounds.

We also note that the expressions of the vector fields (2.58) and (2.59) correspond to the existence of just one invariant algebraic curve, but this curve can be reducible. The presence of a second curve, especially when there are some tangencies complicates the expression of the vector fields. Theorem (2.2)(a) avoids this complication because it uses the generic conditions (i) and (iii).

Chapter 3

Polynomial systems and Darboux first integrals

3.1 Introduction

The Darboux theory of integrability allows to determine when a polynomial differential system in $\mathbb{C}^2$ has a first integral of the kind $f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp(g/h)$ where f_i, g and h are polynomials in $\mathbb{C}[x, y]$, and $\lambda_i \in \mathbb{C}$ for $i = 1, \ldots, p$. In this chapter we solve the inverse problem, i.e. we characterize the polynomial vector fields in $\mathbb{C}^2$ having the following function

$$H(x, y) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp(g/(f_1^{n_1} \cdots f_p^{n_p})) \tag{3.1}$$

as a *Darboux first integral.*

In the *degree* P/Q is defined as $\delta(P/Q) = \max\{\delta P, \delta Q\}$.

Theorem 3.1. *Let $H(x, y) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp(g/(f_1^{n_1} \cdots f_p^{n_p}))$ be a Darboux function with $f_1, \cdots, f_p$ irreducible polynomials in $\mathbb{C}[x, y]$, $\lambda_1, \cdots, \lambda_p \in \mathbb{C}$, $n_1, \cdots, n_p \in \mathbb{N} \bigcup \{0\}$ and the polynomial g of $\mathbb{C}[x, y]$ is coprime with f_i if $n_i \neq 0$. We denote by l the degree of the rational function $g/(f_1^{n_1} \cdots f_p^{n_p})$. Then, H is a first integral for the polynomial vector field $X = (P,\ Q)$ of degree m with P and Q coprimes if and only if* *(a)* $l + \sum_{i=1}^{p} \delta f_i = m + 1$ *and*

$$X = \left(\prod_{l=1}^{p} f_l^{n_l}\right) \sum_{i=1}^{p} \lambda_i \left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) X_{f_i} - g \sum_{i=1}^{p} n_i \left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) X_{f_i} + \left(\prod_{j=1}^{p} f_j\right) X_g, \tag{3.2}$$

where X_{f_i} *is the Hamiltonian vector field* $(-f_{iy}, f_{ix})$.

Moreover, the vector field given by (3.2) has the integrating factor

$$R_1 = (f_1 \cdots f_p f_1^{n_1} \cdots f_p^{n_p})^{-1}.$$

(b) $l + \sum_{i=1}^{p} \delta f_i > m + 1$ *and* X *is as in (3.2) dividing its components by their greatest common divisor* A*. Moreover,* AR_1 *is a rational integrating factor of* X*.*

In Section 3.2, we will prove Theorem 3.1.

We note that the second part of statement (a) of Theorem 3.1 is in some sense the equivalent to Proposition 2.5 for our inverse problem.

In Remark 3.6 we shall show that the second part of statement (a) cannot be extended to the integrating factors of the form (3.1) with $g \neq 0$. In Section 3.3 we provide examples of all statements of Theorem 3.1.

Corollary 3.2. *Under the assumptions of Theorem 3.1 if (3.1) is a first integral for the polynomial vector field* $X = (P,\ Q)$ *of degree* m *with* P *and* Q *coprimes, then* $l + \sum_{i=1}^{p} \delta f_i \geq m + 1$.

Corollary 3.2 follows directly from Theorem 3.1. Note that Corollary 3.2 says that the degree of a polynomial vector field having the first integral (3.1) is not independent of the degrees of the polynomials appearing in (3.1).

As far as we know, Theorem 3.1 and consequently Corollary 3.2 uses by first time information about the degree of the invariant algebraic curves for studying the integrability of a polynomial vector field, because until now the Darboux theory of integrability only used the number of the invariant algebraic curves of a polynomial vector field for studying its integrability through, either a first integral, or an integrating factor, see Theorem 1.7 (Darboux Theorem).

Prelle and Singer in [45] proved the following result.

Theorem 3.3. *If a polynomial vector field X has a first integral of the form $H(x,y) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp(g/(f_1^{n_1} \cdots f_p^{n_p}))$ where $f_1, \cdots, f_p$ are irreducible polynomials in $\mathbb{C}[x,y]$, $\lambda_1, \cdots, \lambda_p \in \mathbb{C}$, $n_1, \cdots, n_p \in \mathbb{N} \cup \{0\}$ and the polynomial g of $\mathbb{C}[x,y]$ is coprime with f_i if $n_i \neq 0$, then the vector field has an integrating factor of the form*

$$\left(\frac{A(x,y)}{B(x,y)}\right)^{\frac{1}{N}}$$

with $A, B \in \mathbb{C}[x,y]$ and N an integer.

The following corollary not only reproduce the result of Prelle and Singer, Theorem 3.3, but also improve it as follows.

Corollary 3.4. *We assume that the polynomial vector field X has a first integral of the form $H(x,y) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp(g/(f_1^{n_1} \cdots f_p^{n_p}))$ where $f_1, \cdots, f_p$ are irreducible polynomials in $\mathbb{C}[x,y]$, $\lambda_1, \cdots, \lambda_p \in \mathbb{C}$, $n_1, \cdots, n_p \in \mathbb{N} \cup \{0\}$ and the polynomial g of $\mathbb{C}[x,y]$ is coprime with f_i if $n_i \neq 0$. We denote by $l = \delta(g/(f_1^{n_1} \cdots f_p^{n_p}))$.*

(a) If $l + \sum_{i=1}^{p} \delta f_i = m + 1$, then the inverse of the polynomial $f_1 \cdots f_p f_1^{n_1} \cdots f_p^{n_p}$ is an integrating factor.

(b) Otherwise, a function of the form $A(x,y)/(f_1 \cdots f_p f_1^{n_1} \cdots f_p^{n_p})$ with $A \in \mathbb{C}[x,y]$ is an integrating factor.

The results of Corollary 3.4 are strongly related with Proposition 3.2 and Corollary 3.3 of Walcher [50].

Theorem 3.5. *Let $X = (P,Q)$ be a polynomial vector field with P and Q coprime having $f_1 = 0, \cdots, f_p = 0$ as irreducible invariant algebraic curves satisfying the* *generic conditions (i)–(v). Then, X has the first integral $f_1^{\lambda_1} \cdots f_p^{\lambda_p}$ with $\lambda_i \in \mathbb{C}$ if and only if $\sum_{i=1}^{p} \delta f_i = m+1$. Moreover,*

$$X = \sum_{i=1}^{p} \lambda_i \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right) X_{f_i}. \tag{3.3}$$

This theorem improves the conditions for the existence of a first integral in the Darboux theory of integrability using information about the degree and the nature of the invariant algebraic curves, specifically it improves statement (e) of Theorem 1.7. As far as we know, this is the first time that information about the degree of the invariant algebraic curves, instead of the number of these curves, is used for studying the integrability of a polynomial vector field.

Reader could find examples of the previous theorems in Section 3.3.

3.2 Darboux first integrals

We note that if the polynomials P and Q are not coprime, let $A(x,y) \in \mathbb{C}[x,y]$ be the greatest common divisor of P and Q. Then, the change in the independent variable t given by $ds = A(x,y)dt$ transforms the polynomial vector field (1.2) into the polynomial vector field $(P/A, Q/A)$ with P/A and Q/A coprime. Since if $(P/A, Q/A)$ has a first integral, we also have a first integral for (P,Q), in what follows we shall work with polynomial vector fields (P,Q) with P and Q coprime.

Proof of Theorem 3.1: By a direct calculation we prove that system (3.2) in statements (a) and (b) of Theorem 3.1 has (3.1) as a first integral. So, the "only if" part of Theorem 3.1 is proved. Now, we shall prove the "if" part.

We assume that $H = f_1^{\lambda_1} \cdots f_p^{\lambda_p} F$ with $F = \exp\left(g/\left(f_1^{n_1} \cdots f_p^{n_p}\right)\right)$ is a first integral of the polynomial vector field $X = (P, Q)$ of degree m. So, we have

$$
\begin{aligned}
0 &= PH_x + QH_y \\
&= PF\left(\sum_{i=1}^{p} \lambda_i f_i^{\lambda_i - 1} f_{ix} \prod_{\substack{j=1 \\ j \neq i}}^{p} f_j^{\lambda_j} + g_x \left(\prod_{r=1}^{p} f_r^{-n_r}\right)\left(\prod_{j=1}^{p} f_j^{\lambda_j}\right)\right. \\
&\qquad \left. - g\left(\sum_{i=1}^{p} n_i f_i^{n_i - 1} f_{ix} \prod_{\substack{j=1 \\ j \neq i}}^{p} f_j^{n_j}\right)\left(\prod_{r=1}^{p} f_r^{-2n_r}\right)\left(\prod_{j=1}^{p} f_j^{\lambda_j}\right)\right) \\
&\quad + QF\left(\sum_{i=1}^{p} \lambda_i f_i^{\lambda_i - 1} f_{iy} \prod_{\substack{j=1 \\ j \neq i}}^{p} f_j^{\lambda_j} + g_y \left(\prod_{r=1}^{p} f_r^{-n_r}\right)\left(\prod_{j=1}^{p} f_j^{\lambda_j}\right)\right. \\
&\qquad \left. - g\left(\sum_{i=1}^{p} n_i f_i^{n_i - 1} f_{iy} \prod_{\substack{j=1 \\ j \neq i}}^{p} f_j^{n_j}\right)\left(\prod_{r=1}^{p} f_r^{-2n_r}\right)\left(\prod_{j=1}^{p} f_j^{\lambda_j}\right)\right) \\
&= \left[P\left(\sum_{i=1}^{p} \lambda_i f_{ix} \prod_{\substack{j=1 \\ j \neq i}}^{p} f_j + g_x \left(\prod_{r=1}^{p} f_r^{-n_r}\right)\left(\prod_{j=1}^{p} f_j\right)\right.\right. \\
&\qquad \left. - g\sum_{i=1}^{p} n_i f_{ix} \left(\prod_{r=1}^{p} f_r^{-n_r}\right)\left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j\right)\right)
\end{aligned}
$$

$$+Q\left(\sum_{i=1}^{p}\lambda_i f_{iy}\prod_{\substack{j=1\\ j\neq i}}^{p} f_j + g_y\left(\prod_{r=1}^{p} f_r^{-n_r}\right)\left(\prod_{j=1}^{p} f_j\right)\right.$$

$$\left.\left. - g\sum_{i=1}^{p} n_i f_{iy}\left(\prod_{r=1}^{p} f_r^{-n_r}\right)\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right)\right)\right] F\prod_{j=1}^{p} f_j^{\lambda_j-1}.$$

Since the last expression is equal to zero, we can cancel the non–zero product $F\prod_{j=1}^{p} f_j^{\lambda_j-1}$ and we can replace it with the non–zero product $\prod_{r=1}^{p} f_r^{n_r}$. So we get

$$0 = PG_1 + QG_2, \tag{3.4}$$

with

$$\begin{aligned} G_1 &= \left(\sum_{i=1}^{p}\lambda_i f_{ix}\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right)\prod_{r=1}^{p} f_r^{n_r} + g_x\prod_{j=1}^{p} f_j - g\sum_{i=1}^{p} n_i f_{ix}\prod_{\substack{j=1\\ j\neq i}}^{p} f_j, \\ G_2 &= \left(\sum_{i=1}^{p}\lambda_i f_{iy}\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right)\prod_{r=1}^{p} f_r^{n_r} + g_y\prod_{j=1}^{p} f_j - g\sum_{i=1}^{p} n_i f_{iy}\prod_{\substack{j=1\\ j\neq i}}^{p} f_j. \end{aligned}$$

We remark that, since P and Q are coprime, from $PH_x + QH_y = 0$ it follows that H_x and H_y cannot be zero. Consequently, G_1 and G_2 are not zero.

Since P and Q are coprime, from (3.4) we have that P must divide the polynomial G_2, and Q must divide the polynomial G_1, which is impossible if $\delta G_i < m = \max\{\delta P, \delta Q\}$ for $i = 1, 2$. Due to the fact that $\delta G_i = l - 1 + \sum_{i=1}^{p}\delta f_i$, we get that $l + \sum_{i=1}^{p}\delta f_i \geq m + 1$.

Since P and Q are coprime, if $\sum_{i=1}^{p} \delta f_i + l = m+1$ we have that there is a constant $\lambda \in \mathbb{C} \setminus \{0\}$ such that $P = -\lambda G_2$ and $Q = \lambda G_1$. Doing the change of time $t \to (1/\lambda)t$ the first part of statement (a) is proved. Now we shall show the second part of statement (a).

The algebraic curve $f_k = 0$ is invariant for the vector field (3.2) with cofactor

$$
\begin{aligned}
K_k \;=\; & \left(\prod_{l=1}^{p} f_l^{n_l}\right) \sum_{i=1}^{p} \lambda_i (f_{ix} f_{ky} - f_{iy} f_{kx}) \left(\prod_{\substack{j=1\\ j\neq i,k}}^{p} f_j\right) + (g_x f_{ky} - g_y f_{kx}) \left(\prod_{\substack{j=1\\ j\neq k}}^{p} f_j\right) \\
& + g \sum_{i=1}^{p} n_i (f_{iy} f_{kx} - f_{ix} f_{ky}) \left(\prod_{\substack{j=1\\ j\neq i,k}}^{p} f_j\right).
\end{aligned}
$$

The vector field (3.2) has divergence

$$
\operatorname{div} X = -\left(\prod_{l=1}^{p} f_l^{n_l}\right)_x \sum_{i=1}^{p} \lambda_i \left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{iy} + \left(\prod_{l=1}^{p} f_l^{n_l}\right)_y \sum_{i=1}^{p} \lambda_i \left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{ix} +
$$

$$
\left(\prod_{l=1}^{p} f_l^{n_l}\right) \sum_{i=1}^{p} \lambda_i \left(\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right)_y f_{ix} - \left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right)_x f_{iy} \right) +
$$

$$
g_x \sum_{i=1}^{p} n_i \left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{iy} - g_y \sum_{i=1}^{p} n_i \left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{ix} +
$$

$$
g\sum_{i=1}^{p} n_i\left(\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right)_x f_{iy}-\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right)_y f_{ix}\right)+
$$

$$
\left(\prod_{j=1}^{p} f_j\right)_y g_x-\left(\prod_{j=1}^{p} f_j\right)_x g_y,
$$

or equivalently,

$$
\begin{aligned}
\mathrm{div}\ X \quad = \quad & \sum_{i,k=1}^{p} n_k\lambda_i(f_{ky}f_{ix}-f_{kx}f_{iy})\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right)\left(\prod_{\substack{l=1\\ l\neq k}}^{p} f_l^{n_l}\right)f_k^{n_k-1}+\\
& \left(\prod_{l=1}^{p} f_l^{n_l}\right)\sum_{j,k=1}^{p}\lambda_i\left(f_{jy}f_{ix}-f_{jx}f_{iy}\right)\left(\prod_{\substack{k=1\\ k\neq i,j}}^{p} f_k\right)+\\
& \sum_{i=1}^{p} n_i(g_xf_{iy}-g_yf_{ix})\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right)+\\
& g\sum_{i=1}^{p} n_i\sum_{\substack{j=1\\ j\neq i}}^{p}\left(f_{jx}f_{iy}-f_{jy}f_{ix}\right)\left(\prod_{\substack{k=1\\ k\neq i,j}}^{p} f_k\right)+\\
& \sum_{i=1}^{p}\left(g_xf_{iy}-g_yf_{ix}\right)\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right),
\end{aligned}
$$

and it is easy to check that

$$\sum_{r=1}^{p} K_r + \sum_{r=1}^{p} n_r K_r = \text{div } X.$$

Therefore, by Theorem 1.7(b), $R_1 = \left(f_1 \cdots f_p f_1^{n_1} \cdots f_p^{n_p}\right)^{-1}$ is an integrating factor of the vector field (3.2).

Suppose that $l + \sum_{i=1}^{p} \delta f_i > m + 1$. Since P and Q are coprime, from (3.4) we have that there is a polynomial A such that $G_1 = AQ$ and $G_2 = -AP$. So, dividing G_1 and G_2 by A we obtain the polynomial vector field (P, Q) of degree m. This completes the proof of statement (b), and consequently of Theorem 3.1. ∎

Remark 3.6.

We shall show that the second part of statement (a) of Theorem 3.1 cannot be extended to integrating factors of the form (3.1) with $g \neq 0$. The system

$$\begin{aligned} \dot{x} &= x(x + y + 1), \\ \dot{y} &= y(x + y), \end{aligned} \tag{3.5}$$

has the two invariant algebraic curves $f_1 = x = 0$ and $f_2 = y = 0$, and the exponential factor $F = \exp\left(-(1 + x)/y\right)$ with cofactors $K_1 = x+y+1$, $K_2 = x+y$ and $L = 1$, respectively. Since $-K_1 + K_2 + L = 0$, by Theorem 1.7(a) system (3.5) has the first integral $H = f_1^{-1} f_2 F$. Doing simple computations we observe that system (3.5) can be written into the form (3.2) with $\lambda_1 = -1$, $\lambda_2 = 1$, $n_1 = 0$ and $n_2 = 1$. We also note that the polynomials P and Q are coprime.

Since the divergence of system (3.5) is div $= 1 + 3x + 3y$ and we have that $K_1 + K_2 \neq$ div and $K_1 + K_2 + L \neq$ div, by Theorem 1.7(d) there is no integrating factors of the form $(f_1 f_2)^{-1}$ or $\left(f_1 f_2 \exp F\right)^{-1}$. So, although system (3.5) can be written into the form (3.2), the second part of statements (a) of Theorem 3.1 cannot be extended to integrating factors of the form (3.1) with $g \neq 0$. However, since $K_1 + 2K_2 =$ div, this system has the integrating factor $R_1 = f_1^{-1} f_2^{-2}$. ∎

Proof of Theorem 3.5: Assume that the assumptions of Theorem 3.5 hold. Suppose that $\sum_{i=1}^{p} \delta f_i = m+1$. Then, by Theorem 2.4(b) it follows that the polynomial vector field satisfying the assumptions of Theorem 3.5 is of the form (3.3), and by Proposition 2.5 it has the first integral $f_1^{\lambda_1} \cdots f_p^{\lambda_p}$.

Now we shall prove the converse statement. Suppose that the polynomial vector field satisfying the assumptions of Theorem 3.5 has the first integral $f_1^{\lambda_1} \cdots f_p^{\lambda_p}$. So, for this first integral $l = 0$, using the notation of Theorem 3.1. Then, by Corollary 3.2 we have that $\sum_{i=1}^{p} \delta f_i \geq m+1$. Since all the invariant algebraic curves $f_i = 0$ are generic, by Theorem 2.4, it follows that $\sum_{i=1}^{p} \delta f_i \leq m+1$. Hence, $\sum_{i=1}^{p} \delta f_i = m+1$, and the proof of the theorem is completed. ∎

3.3 The examples

First, we provide three examples of a first integral satisfying statement (a) of Theorem 1.14.

Example 3.7.

The Darboux function $H = y^{-3} \exp(3x^3/y)$ is of the form (3.1) with $f_1 = y$, $\lambda_1 = -3$, $n_1 = 1$ and $g = 3x^3$. Then, the l defined in Theorem 3.1 satisfies $l = 3$. Therefore, since $l + \sum_{i=1}^{p} \delta f_i = 4$, and the polynomial vector field given by (3.2) is $X = 3(y + x^3, 3x^2 y)$ with $m = 3$, it follows that H and X satisfy statement (a) of Theorem 3.1. ∎

Example 3.8.

The Darboux function $H = (x^2 + y^2) \exp(2y)$ is of the form (3.1) with $f_1 = x + iy$, $f_2 = x - iy$, $\lambda_1 = \lambda_2 = 1$, $n_1 = n_2 = 0$ and $g = 2y$. Then, the $l = 1$. Therefore, since $l + \sum_{i=1}^{p} \delta f_i = 3$, and the polynomial vector field given by (3.2) is $X = 2(-y - x^2 - y^2, x)$ with $m = 2$, we have that H and X satisfy statement (a) of Theorem 1.14, because X has the first integral H and the integrating factor $1/(x^2 + y^2)$. ∎

The next first integral and its corresponding polynomial vector field provide examples satisfying Theorem 3.1(a) and Theorem 3.5.

Example 3.9.

The Darboux function $H = xy(x - 1 + y/3)$ is of the form (3.1) with $f_1 = x$, $f_2 = y$, $f_3 = x - 1 + y/3$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $n_1 = n_2 = n_3 = 0$ and $g = 0$. Then, the $l = 0$. Therefore, since $l + \sum_{i=1}^{p} \delta f_i = 3$, and the polynomial vector field given by (3.2) is $X = (x(1 - x - 2y/3), y(-1 + 2x + y/3))$ with $m = 2$, we get that H and X satisfy statement (a) of Theorem 3.1, because X has the first integral H and the integrating factor $1/H$. Additionally, this is an example satisfying Theorem 3.5. ∎

Now we shall provide two examples satisfying statement (b) of Theorem 3.1.

Example 3.10.

The Darboux function $H = y^{-4}(x^3 + x^4 + y^4)$ is of the form (3.1) with $f_1 = y$, $f_2 = x^3 + x^4 + y^4$, $\lambda_1 = -4$, $\lambda_2 = 1$, $n_1 = n_2 = 0$ and $g = 0$. Then, the $l = 0$. Therefore, $l + \sum_{i=1}^{p} \delta f_i = 5$, and the polynomial vector field given by (3.2) is $(P, Q) = (4x^3(1+x), x^2(3+4x)y)$ with P and Q non–coprime. So, $X = (4x(1+x), y(3+4x))$ with $m = 2$ is the polynomial vector field satisfying statement (b) of Theorem 3.1. ∎

Example 3.11.

The Darboux function $H = (x + 1)^{-2}(y - x^2) \exp(-1/(x + 1))$ is of the form (3.1) with $f_1 = x + 1$, $f_2 = y - x^2$, $\lambda_1 = -2$, $\lambda_2 = 1$, $n_1 = 1$, $n_2 = 0$ and $g = -1$. Then, $l = 1$. Therefore, $l + \sum_{i=1}^{p} \delta f_i = 4$, and the polynomial vector field given by (3.2) is $X = (-(x + 1)^2, -2x - y - 3x^2 - 2xy)$ with $m = 2$ satisfying statement (b) of Theorem 3.1. ∎

Chapter 4

Polynomial systems and Darboux integrating factors

4.1 Introduction

In this chapter we study the following inverse problem of the Darboux theory of integrability, what are the polynomial vector fields in $\mathbb{C}^2$ having the Darboux function

$$R(x,y) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp(g/(f_1^{n_1} \cdots f_p^{n_p})) \tag{4.1}$$

as a *Darboux integrating factor*?

The main results of this chapter are given in Theorems 4.1, 4.2 and 4.3. We organize them as follows.

The first theorem provides a connection between the degree of the invariant algebraic curves and the number of them in order to decide about the kind of the Darboux integrability and so improves statement (e) of Darboux Theorem 1.7.

Theorem 4.1. *Suppose that a polynomial vector field $X = (P,Q)$ of degree m, with P and Q coprime, admits p irreducible invariant algebraic curves $f_i = 0$ with cofactors K_i for $i = 1, \ldots, p$; q exponential factors $\exp(g_j/h_j)$ with cofactors L_j for $j = 1, \ldots, q$; and r independent singular points (x_k, y_k) such that $f_i(x_k, y_k) \neq 0$ for $i = 1, \ldots, p$ and for $k = 1, \ldots, r$. Then, the irreducible factors of the polynomials*

h_j are some f_i's and we can write

$$\left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1}\cdots\left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q}=\exp\left(\frac{\mu_1 g_1}{h_1}+\cdots+\frac{\mu_q g_q}{h_q}\right)=\exp\left(\frac{g}{f_1^{n_1}\cdots f_p^{n_p}}\right),$$

where $\mu_1,\cdots,\mu_q\in\mathbb{C}$, $n_1,\cdots,n_p\in\mathbb{N}\bigcup\{0\}$ and the polynomial g of $\mathbb{C}[x,y]$ is coprime with f_i if $n_i\neq 0$. We denote by $l=\max\left\{\sum_{i=1}^{p} n_i\delta f_i,\delta g\right\}$.

If $p+q+r=m(m+1)/2$, $l+\sum_{i=1}^{p}\delta f_i<m+1$, and the r independent singular points are weak, then the (multi–valued) function

$$f_1^{\lambda_1}\cdots f_p^{\lambda_p}\left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1}\cdots\left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q} \tag{4.2}$$

for convenient $\lambda_i,\mu_j\in\mathbb{C}$ not all zero is an integrating factor of X.

The proof of Theorem 4.1 is given in Section 4.2.

We are interested in studying the polynomial differential systems which have a given Darboux function (4.1) as an integrating factor.

Theorem 4.2. *We consider the Darboux function $R(x,y)=f_1^{\mu_1}\cdots f_p^{\mu_p}\exp(g/(f_1^{n_1}\cdots f_p^{n_p}))$ with $f_1,\cdots,f_p$ irreducible polynomials in $\mathbb{C}[x,y]$, $\mu_1,\cdots,\mu_p\in\mathbb{C}$, $n_1,\cdots,n_p\in\mathbb{N}\bigcup\{0\}$ and the polynomial g of $\mathbb{C}[x,y]$ is coprime with the f_i's for $n_i\neq 0$. Let $X=(P,Q)$ be the vector field (3.2) with $\lambda_i=\mu_i+n_i+1$. Consider the polynomial vector field*

$$Y=(Y_0-Y_C)\left(\prod_{l=1}^{p} f_l^{n_l}\right)\left(\prod_{j=1}^{p} f_j\right)-CX, \tag{4.3}$$

where $Y_0=(A,B)$, A,B,C are arbitrary polynomials satisfying $(AH)_x+(BH)_y=0$ where $H=f_1^{\lambda_1}\cdots f_p^{\lambda_p}\exp(g/(f_1^{n_1}\cdots f_p^{n_p}))$ is a first integral of X and $Y_C=(-C_y,C_x)$. Then, the polynomial vector field Y has R as an integrating factor.

In Section 4.3 we prove Theorem 4.3 which is actually the main theorem of this chapter. With Theorem 4.3 we provide all polynomial systems having an integrating factor forming by one generic curve. In the proof of Theorem 4.3 we also provide an algorithm where we construct step by step such systems.

In general, a polynomial system having a Darboux integrating factor does not always have a Darboux first integral as we observe in Exemple 4.11, but always it has a Liouvillian first integral (see [48]). With Theorem 4.3 and in its proof we can guarantee the existence of additional curves or exponential factors which show the existence of a Darboux first integral. In the next theorem, we characterize all the polynomial vector fields having an integrating factor of the form f^λ with $f = 0$ a generic curve and $\lambda \in \mathbb{C}$. We denote $[\cdot]$ the integer part function.

Theorem 4.3. *Let $f = 0$ be an irreducible algebraic curve of degree k in $\mathbb{C}^2$ and let m be an integer such that $m \geq k - 1$. We denote by $n = [(m+1)/k]$. We assume that f satisfies the following generic conditions.*

(i) There are no points at which f and its first derivatives are all vanish.

(ii) The highest order terms of f have no repeated factors.

Then, any differential polynomial system $\dot{x} = P, \dot{y} = Q$ of degree m having the integrating factor f^λ with $\lambda \in \mathbb{C}$ can be written as follows.

(a) If $(m+1)/(n+1) \leq k < (m+1)/n$ and $\lambda \notin \{-1, -2, \cdots, -n\}$, then we have

$$\dot{x} = -\frac{1}{\lambda+1} f F_y - F f_y, \qquad \dot{y} = \frac{1}{\lambda+1} f F_x + F f_x, \tag{4.4}$$

where

$$F = D_1 + \sum_{i=2}^{n+1} \frac{\lambda+1}{(\lambda+2)(\lambda+3)\cdots(\lambda+i)} D_i f^{i-1},$$

and the D_i, for $i = 1, \cdots, n$, is a convenient polynomial given in Lemma 4.9, $D_{n+1} = \alpha_n \in \mathbb{C}$ if $k = (m+1)/n$, otherwise $D_{n+1} = 0$. Moreover, $\delta F + \delta f = m + 1$ and system (4.4) has the Darboux first integral $H(x,y) = F^{\frac{1}{\lambda+1}} f$.

(b) If $(m+1)/(n+1) \leq k < (m+1)/n$ and $\lambda \in \{-1, -2, \cdots, -n\}$, then we have

$$\begin{aligned} \dot{x} &= -\beta f^{-\lambda-1} f_y - (\lambda+1) g f_y - f g_y, \\ \dot{y} &= \beta f^{-\lambda-1} f_x + (\lambda+1) g f_x + f g_x, \end{aligned} \tag{4.5}$$

where

$$g/f^{-\lambda-1} = \sum_{i=1}^{-\lambda-1} (-1)^{-\lambda-i} \frac{D_{-\lambda-i}}{i(i+1)\cdots(-\lambda-1)f^i} + (-1)^{-\lambda} \frac{G_{-\lambda} + F_{-\lambda+1} f}{(-\lambda-1)!},$$

$D_{-\lambda-i}, G_{-\lambda}$ and $F_{-\lambda+1}$ are convenient polynomials given in Lemma 4.10 and $\beta \in \mathbb{C}$. Moreover, $\delta(g/f^{-\lambda-1})+\delta f = m+1$ and system (4.5) has the Darboux first integral $H = f^\beta \exp\left(\frac{g}{f^{-\lambda-1}}\right)$.

By Proposition 1.2 a polynomial differential system of degree m having an invariant algebraic curve $f = 0$ of degree k satisfying the assumptions (i) and (ii) of Theorem 4.3 satisfies that $m + 1 \geq k$. So, the condition $m \geq k - 1$ of the statement of Theorem 4.3 is not restrictive.

Note that Theorem 4.3 says that any polynomial system having a Darboux integrating factor formed by one generic curve always has a Darboux first integral.

In Section 4.4 we present the examples 4.14 and 4.15 of Theorem 4.3.

We note that system (4.4) is a particular case of system (3.2) when $g = 0$ and $n_i = 0$ for all $i = 1, \cdots, p$. System (4.5) appears when some of the invariant algebraic curves (perhaps also the line at infinity) have multiplicity larger than 1, see for more details [21].

We observe in Theorem 4.2 that the family of the vectors fields Y having a Darboux integrating factor is a very general family depending on the arbitrary polynomials A, B, C, but in Theorem 4.2 we cannot determine all the polynomial vector fields having as integrating factor the given Darboux function as we did in Theorem 4.3 for the particular integrating factors of the form f^λ with $\lambda \in \mathbb{C}$.

By Theorem 4.3 we note that when a polynomial system has a Darboux integrating factor given by a generic curve, then it appears either an additionally invariant algebraic curve, or an exponential factor in such a way that $\delta F + \delta f = m+1$, or $\delta(g/f^{-\lambda-1}) + \delta f = m + 1$, respectively. Here, we have used the notations of Theorem 4.3.

In Section 4.4 the reader could find various examples of the previous theorems.

4.2 An integrating factor formed by arbitrary curves

Proof of Theorem 4.1: Assume that the assumptions of Theorem 4.1 hold. By Theorem 1.7(e) the function (4.2) is either a first integral, or an integrating factor of X. But, from Corollary 3.2 the function (4.2) cannot be a first integral of X because $l + \sum_{i=1}^{p} \delta f_i < m + 1$. Hence, the proof is completed. ∎

We note that for $Y_0 = (0, 0)$ the vector field (4.3) coincides with the vector field (3.2) having additionally $C = 0$ as an invariant algebraic curve.

Now we present a way to choose the polynomials A, B and C which appears in the statement of Theorem 4.2.

The polynomials A and B of Theorem 4.2 are arbitrary polynomials satisfying the condition $(AH)_x + (BH)_y = 0$, or equivalently the vector field (A, B) has

$$H(x, y) = f_1^{\mu_1+n_1+1} \cdots f_p^{\mu_p+n_p+1} \exp(g/(f_1^{n_1} \cdots f_p^{n_p}))$$

as an integrating factor. Hence, in order to construct the vector field Y defined in (4.3) which has $R(x, y) = f_1^{\mu_1} \cdots f_p^{\mu_p} \exp(g/(f_1^{n_1} \cdots f_p^{n_p}))$ as an integrating factor, first we need to guarantee the existence of such a vector field (A, B). Consider

$$H_2(x, y) = f_1^{\mu_1+2n_1+2} \cdots f_p^{\mu_p+2n_p+2} \exp(g/(f_1^{n_1} \cdots f_p^{n_p}))$$

and take (A, B) the vector field defined in (3.2) having the first integral H_2. Additionally, the arbitrary polynomial C can be chosen trying that the vector field Y has an appropriate degree. In Section 5 we present Example 4.13 of how to construct such a vector field (4.3).

Proof of Theorem 4.2: In order to prove that $R = f_1^{\mu_1} \cdots f_p^{\mu_p} F$ with $F = \exp\left(g/\left(f_1^{n_1} \cdots f_s^{n_p}\right)\right)$ is a Darboux integrating factor of the polynomial vector field Y defined in (4.3) we must prove that $YR + R\text{div}(Y) = 0$. We have that

$$YR = (AR_x + BR_y + C_yR_x - C_xR_y) \left(\prod_{i=1}^{p} f_i \right) \left(\prod_{l=1}^{p} f_l^{n_l} \right) - CXR.$$

Let R_1 be the integrating factor of X given in Theorem 3.1. We note that $R = HR_1$ is another integrating factor of the vector field X given by (3.2). Therefore, the last relation can be written into the form

$$YR = (AR_x + BR_y + C_yR_x - C_xR_y)\left(\prod_{i=1}^{p} f_i\right)\left(\prod_{l=1}^{p} f_l^{n_l}\right) + CR\,\mathrm{div}(X). \quad (4.6)$$

Since $(AH)_x + (BH)_y = 0$ and since $H = R/R_1$ we have that

$$AR_x + BR_y + (A_x + B_y)R - \frac{R}{R_1}(AR_{1x} + BR_{1y}) = 0. \quad (4.7)$$

Now we calculate

$$\begin{aligned}
\mathrm{div}(Y) \;&=\; (A_x + B_y)\left(\prod_{i=1}^{p} f_i\right)\left(\prod_{l=1}^{p} f_l^{n_l}\right)\\
&+\; \left(A\left(\prod_{i=1}^{p} f_i\right)_x + B\left(\prod_{i=1}^{p} f_i\right)_y\right)\left(\prod_{l=1}^{p} f_l^{n_l}\right)\\
&+\; \left(A\left(\prod_{l=1}^{p} f_l^{n_l}\right)_x + B\left(\prod_{l=1}^{p} f_l^{n_l}\right)_y\right)\left(\prod_{i=1}^{p} f_i\right)\\
&+\; \left(C_y\left(\prod_{i=1}^{p} f_i\right)_x - C_x\left(\prod_{i=1}^{p} f_i\right)_y\right)\left(\prod_{l=1}^{p} f_l^{n_l}\right)\\
&+\; \left(C_y\left(\prod_{l=1}^{p} f_l^{n_l}\right)_x - C_x\left(\prod_{l=1}^{p} f_l^{n_l}\right)_y\right)\left(\prod_{i=1}^{p} f_i\right)\\
&-\; (C_xP + C_yQ) - C\,\mathrm{div}(X).
\end{aligned}$$

Consequently, we have

$$
\begin{aligned}
\operatorname{div}(Y) &= (A_x + B_y)\left(\prod_{i=1}^{p} f_i\right)\left(\prod_{l=1}^{p} f_l^{n_l}\right) \\
&- \frac{1}{R_1}(AR_{1x} + BR_{1y})\left(\prod_{l=1}^{p} f_l^{n_l}\right)\left(\prod_{i=1}^{p} f_i\right) \\
&- \sum_{i=1}^{p}(n_i+1)(C_x f_{iy} - C_y f_{ix})\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right)\left(\prod_{l=1}^{s} f_l^{n_l}\right) \\
&+ \sum_{i=1}^{p}(\mu_i + n_i + 1)(C_x f_{iy} - C_y f_{ix})\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right)\left(\prod_{l=1}^{p} f_l^{n_l}\right) \\
&- g\sum_{i=1}^{p} n_i(C_x f_{iy} - C_y f_{ix})\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) \\
&+ (C_x g_y - C_y g_x)\left(\prod_{i=1}^{p} f_i\right) - C\ \operatorname{div}(X).
\end{aligned}
$$

We have

$$
\begin{aligned}
\operatorname{div}(Y) &= (A_x + B_y)\left(\prod_{i=1}^{p} f_i\right)\left(\prod_{l=1}^{p} f_l^{n_l}\right) \\
&- \frac{1}{R_1}(AR_{1x} + BR_{1y})\left(\prod_{l=1}^{p} f_l^{n_l}\right)\left(\prod_{i=1}^{p} f_i\right) \\
&+ \sum_{i=1}^{p}\mu_i(C_x f_{iy} - C_y f_{ix})\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right)\left(\prod_{l=1}^{p} f_l^{n_l}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \quad g \sum_{i=1}^{p} n_i (C_x f_{iy} - C_y f_{ix}) \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right) \\
&+ \quad (C_x g_y - C_y g_x) \left(\prod_{i=1}^{p} f_i \right) - C \operatorname{div}(X).
\end{aligned}
$$

Finally, multiply the above expression by R we get that

$$
\begin{aligned}
R \operatorname{div}(Y) \;=\;& (A_x + B_y) \left(\prod_{l=1}^{p} f_l^{n_l} \right) \left(\prod_{i=1}^{p} f_i \right) R \\
&- \frac{R}{R_1} (A R_{1x} + B R_{1y}) \left(\prod_{l=1}^{p} f_l^{n_l} \right) \left(\prod_{i=1}^{p} f_i \right) \\
&+ (C_x R_y - C_y R_x) \left(\prod_{l=1}^{p} f_l^{n_l} \right) \left(\prod_{i=1}^{p} f_i \right) - C \; R\operatorname{div}(X).
\end{aligned}
$$

Now, using the previous expression for $R \operatorname{div}(Y)$, the expressions (4.6) and (4.7), we obtain that $YR + R \operatorname{div}(Y) = 0$. Hence, the proof is completed. ∎

4.3 Darboux integrating factors formed by one generic curve

In order to prove Theorem 4.3 we shall need the following results.

Lemma 4.4. *Let A, B be polynomials in $\mathbb{C}[x, y]$ with max$\{\delta A, \delta B\} = n$ and such that $A_x + B_y = 0$. Then, there exist a unique $\widetilde{F} \in \mathbb{C}[x, y]$ with $\delta \widetilde{F} = n+1$ such that $\widetilde{F}$ has no constant term, and $A = -\widetilde{F}_y$ and $B = \widetilde{F}_x$. Of course, for an arbitrary constant $a \in \mathbb{C}$, $F = \widetilde{F} + a$ are all the solutions of $A_x + B_y = 0$.*

Proof: Let $A \in \mathbb{C}[x,y]$ with $\delta A \leq n$. Then, we write A as follows

$$A = A_0(x) + A_1(x)y + A_2(x)y^2 + \cdots + A_{n-1}(x)y^{n-1} + A_n y^n,$$

where $A_i(x) \in \mathbb{C}[x]$ for $i = 1, \cdots, n-1$ and $A_n \in \mathbb{C}$. We define the polynomial F as

$$F = -\int A dy = -A_0(x)y - A_1(x)\frac{y^2}{2} - \cdots - A_{n-1}(x)\frac{y^n}{n} - A_n\frac{y^{n+1}}{n+1} + g(x),$$

for some polynomial g. Then,

$$F_x = -A_{0x}y - A_{1x}\frac{y^2}{2} - \cdots - A_{n-1,x}\frac{y^n}{n} + g_x,$$

Since $B_y = -A_{1x} = -A_{0x} - A_{1x}y - A_{2x}y^2 - \cdots - A_{n-1,x}y^{n-1}$ we have that

$$B = -\int A_{1x} dy = -A_{0x}y - A_{1x}\frac{y^2}{2} - A_{2x}\frac{y^3}{3} - \cdots - A_{n-1,x}\frac{y^n}{n} + h(x),$$

for some $h \in \mathbb{C}[x]$ with $\delta h \leq n$. Now choosing $g = \int h(x)dx$ we have that $g \in \mathbb{C}[x]$ with $\delta g \leq n+1$, and so omitting the constant term of F we obtain the unique polynomial $\widetilde{F}$. Therefore, the proof of the lemma has been completed. ∎

Lemma 4.5. *Let $A, B, C, f \in \mathbb{C}[x,y]$ such that $\delta A, \delta B \leq m-k$, $\delta C \leq m-2k+1$, $\delta f = k$, and $\max\{\delta(A + fC_y), \delta(B - fC_x)\} = m-k$. Suppose that $A_x + B_y = f_yC_x - f_xC_y$. Then there exists $F \in \mathbb{C}[x,y]$ with $\delta F = m-k+1$ such that $A = -C_yf - F_y$ and $B = C_xf + F_x$.*

Proof: We have

$$0 = A_x + f_xC_y + B_y - f_yC_x = (A + fC_y)_x + (B - fC_x)_y.$$

From Lemma 4.4 there is $F \in \mathbb{C}[x,y]$ with $\delta F = m-k+1$ such that

$$A + fC_y = -F_y, \qquad B - fC_x = F_x,$$

and this completes the proof. ∎

Lemma 4.6. *Let A_1, B_1 and D_1 be polynomials. Consider the polynomial differential system of degree m*

$$\dot{x} = A_1 f - D_1 f_y, \quad \dot{y} = B_1 f + D_1 f_x, \tag{1a}$$

with $\delta A_1, \delta B_1 \leq m - k$, $\delta D_1 \leq m + 1 - k$ and $\delta f = k$. If system (1a) has f^λ for some $\lambda \in \mathbb{C}$ as an integrating factor, then the system

$$\dot{x} = (\lambda+1)A_1 + D_{1y}, \quad \dot{y} = (\lambda+1)B_1 - D_{1x}, \tag{1b}$$

of degree at most $m - k$ has $f = 0$ as an invariant algebraic curve and let $L_1 = A_{1x} + B_{1y}$. Then,

(a) system (1b) has the Darboux integrating factor $f^{\lambda+1}$ if $L_1 \neq 0$ and $\lambda \neq -1$,

(b) system (1b) is Hamiltonian if $L_1 = 0$ or $\lambda = -1$.

Proof: System (1a) has divergence equal to $\text{div}_1 = (A_1 + D_{1y})f_x + (B_1 - D_{1x})f_y + (A_{1x} + B_{1y})f$. We note that the algebraic curve $f = 0$ is invariant for system (1a) with cofactor $K = A_1 f_x + B_1 f_y$. Since f^λ is an integrating factor for system (1a), we have that (see Theorem **??**(d)) $\lambda K + \text{div}_1 = 0$, or equivalently

$$((\lambda+1)A_1 + D_{1y})\, f_x + ((\lambda+1)B_1 - D_{1x})\, f_y + (A_{1x} + B_{1y})f = 0. \tag{4.8}$$

System (1b) has divergence $\text{div}_2 = (\lambda+1)(A_{1x} + B_{1y})$. We note that due to relation (4.8) the algebraic curve $f = 0$ is also invariant for system (1b) with cofactor $L_1 = -(A_{1x} + B_{1y})$.

(a) We assume that $L_1 \neq 0$ and $\lambda \neq -1$. Then we have that $(\lambda+1)L_1 = -\text{div}_2$. Hence, system (1$b$) has $f^{\lambda+1}$ as an integrating factor.

(b) We now assume that $L_1 = 0$. System (1b) has f as a first integral. Since $A_{1x} + B_{1y} = 0$ from Lemma 4.4 there is $G \in \mathbb{C}[x,y]$ such that $A_1 = G_y$ and $B_1 = -G_x$. So, system (1b) becomes

$$\begin{aligned} \dot{x} &= (\lambda+1)G_y + D_{1y} = ((\lambda+1)G + D_1)_y, \\ \dot{y} &= -(\lambda+1)G_x - D_{1x} = -((\lambda+1)G + D_1)_x, \end{aligned} \tag{4.9}$$

therefore, it is a Hamiltonian system.

If $\lambda = -1$ we have that system $(1b)$ is a Hamiltonian system with Hamiltonian $H = D_1$. ∎

Proof of Theorem 4.3: Since $f = 0$ is an algebraic curve which satisfies all conditions of Theorem 2.4, all the non–zero polynomial vector fields of degree m having $f = 0$ as invariant algebraic curve can be written into the form (2.4) if $k < m+1$, or (2.5) if $k = m+1$.

The proof of Theorem 4.3 is organized as follows: Firstly, for simplicity we present with all the details the proof of the theorem in the first five cases. Secondly, in Lemma 4.9 we present the proof of statement (a) of Theorem 4.3. Lastly, in Lemma 4.10 we present the proof of statement (b) of Theorem 4.3.

Case 1: $k = m + 1$. Then, since all polynomial vector fields having $f = 0$ as invariant algebraic curve can be written into the form (2.5), the corresponding system can be written into the form (4.4) where $F \equiv$constant. Hence, Theorem 4.3 is proved in this case.

From now on we suppose that $k < m + 1$ and that the vector field is of the form (2.4), or equivalently of the form $(1a)$.

By Lemma 4.6, $f = 0$ is an invariant algebraic curve of system $(1b)$ of degree at most $m - k$. Therefore from Proposition 1.2 we have that $k \leq m - k + 1$ or equivalently $2k \leq m + 1$. We distinguish the following cases. ∎

Case 2: $(m + 1)/2 < k < m + 1$. Then, from Theorem 2.4(c), we have that the vector field $(1b)$ is identically zero. So, since we are in the assumptions of Lemma 4.6, from (4.8), we get that

$$(\lambda + 1)A_1 + D_{1y} = 0, \qquad (\lambda + 1)B_1 - D_{1x} = 0, \qquad A_{1x} + B_{1y} = 0.$$

If $\lambda \neq -1$, we can write

$$A_1 = -\frac{1}{\lambda + 1} D_{1y}, \qquad B_1 = \frac{1}{\lambda + 1} D_{1x}, \tag{4.10}$$

and $\delta D_1 = m+1-k$. Substituting A_1, B_1 in system $(1a)$ we get system (4.4) for $F = D_1$. We note that $F = D_1 = 0$ is also an invariant algebraic curve and $\delta f + \delta F = m+1$ and therefore system $(1a)$ has the Darboux first integral $H = F^{\frac{1}{\lambda+1}} f$.

Subcase $\lambda = -1$ then, from (4.10), $D_{1y} = 0$ and $D_{1x} = 0$ so $D_1 = \delta_1$ with $\delta_1 \in \mathbb{C}$. Since $A_{1x} + B_{1y} = 0$ from Lemma 4.4, there is $G \in \mathbb{C}[x,y]$ with $\delta G = m-k+1$ such that $A_1 = -G_y$ and $B_1 = G_x$. Hence, system $(1a)$ can be written as

$$\dot{x} = -G_y f - \delta_1 f_y, \qquad \dot{y} = G_x f + \delta_1 f_x. \tag{4.11}$$

We note that the algebraic curve $f=0$ is invariant for system (4.11) with cofactor $K = -(f_x G_y - f_y G_x)$. Additionally, system (4.11) has the invariant exponential factor $\exp(G)$ with cofactor $L = \delta_1(f_x G_y - f_y G_x) = -\delta_1 K$. Hence, system (4.11) has the first integral $H = f^{\delta_1} \exp(G)$. Taking $g/f^0 = G$ and $\beta = \delta_1$ system (4.11) it is of the normal form (4.5). So, Theorem 4.3 is proved in this case. ∎

Case 3: $k = (m+1)/2$. Note that this case only occurs if m is odd. Then, from Theorem 2.4(b) we have that for system $(1b)$ of degree $m-k$ holds $(m-k)+1 = k$ and so system $(1b)$ should be of the form

$$(\lambda+1)A_1 + D_{1y} = -\alpha f_y, \qquad (\lambda+1)B_1 - D_{1x} = \alpha f_x, \tag{4.12}$$

for some $\alpha \in \mathbb{C}$.

If $\lambda \neq -1$ we obtain that

$$A_1 = \frac{-\alpha f_y - D_{1y}}{\lambda+1}, \qquad B_1 = \frac{\alpha f_x + D_{1x}}{\lambda+1}.$$

Substituting A_1 and B_1 into system $(1a)$ we get

$$\dot{x} = -\frac{1}{\lambda+1}(\alpha f_y f + D_{1y} f) - D_1 f_y, \qquad \dot{y} = \frac{1}{\lambda+1}(\alpha f_x f + D_{1x} f) + D_1 f_x. \tag{4.13}$$

We note that system (4.13) for $\lambda \notin \{-1,-2\}$ has a second invariant algebraic $F = D_1 + \frac{\alpha}{\lambda+2} f = 0$ with cofactor $K_2 = f_x D_{1y} - f_y D_{1x}$. We also note for system

(4.13) that the invariant curve $f = 0$ has cofactor $K_1 = -\frac{1}{\lambda+1}(f_x D_{1y} - f_y D_{1x}) = -\frac{1}{(\lambda+1)} K_2$. Hence, system (4.13) has a Darboux first integral of the form $H = F^{\frac{1}{\lambda+1}} f$. Obviously, system (4.13) can be written into the normal form (4.4).

Subcase $\lambda = -1$. Then we have that

$$D_{1y} = -\alpha f_y, \qquad D_{1x} = -\alpha f_x,$$

and so $D_1 = -\alpha f - \delta_1$ with $\delta_1 \in \mathbb{C}$. Then, from (4.8) we get that $A_{1x} + B_{1y} = 0$ and therefore, from Lemma 4.4, there is $G \in \mathbb{C}[x, y]$ with $\delta G = m - k + 1$ such that $A_1 = G_y$ and $B_1 = -G_x$.

Substituting D_1, A_1 and B_1 into system $(1a)$ we have that

$$\dot{x} = \delta_1 f_y + (G + \alpha f)_y f, \qquad \dot{y} = -\delta_1 f_x - (G + \alpha f)_x f. \tag{4.14}$$

The invariant curve $f = 0$ in system (4.14) has cofactor $K = f_x G_y - f_y G_x$. Additionally, system (4.14) has the invariant exponential factor $\exp(G + \alpha f)$ with cofactor $L = \delta_1(f_y G_x - f_x G_y) = -\delta_1 K$, and therefore it has the first integral of the form $H = f^{\delta_1} \exp(G + \alpha f)$. For $g/f^0 = -(G + \alpha)$ and $\beta = -\delta_1$ system (4.14) is of the normal form (4.5).

Subcase $\lambda = -2$. Then system $(1a)$ becomes

$$\dot{x} = \alpha f_y f + D_{1y} f - D_1 f_y, \qquad \dot{y} = -\alpha f_x f - D_{1x} f + D_1 f_x. \tag{4.15}$$

System (4.15) has the exponential factor $\exp(D/f)$ with cofactor $L_1 = \alpha(f_y D_{1x} - f_x D_{1y}) = -\alpha K$ where K is the cofactor of the invariant algebraic curve $f = 0$. So, system (4.15) has the first integral $H = f^{-\alpha} \exp(D_1/f)$. System (4.15) is of the normal form (4.5) taking $g/f = -D_1/f$ and $\beta = -\alpha$. ∎

Case 4: $(m+1)/3 < k < (m+1)/2$. We have that system $(1a)$ has the integrating factor f^λ. Then, from Lemma 4.6(a) we have that the polynomial system $(1b)$ has the integrating factor $f^{\lambda+1}$. Additionally, from Theorem 2.4(a) the system $(1b)$ of degree $m - k$ having $f = 0$ as invariant algebraic curve can be written into the form

$$\dot{x} = A_2 f - D_2 f_y, \qquad \dot{y} = B_2 f + D_2 f_x, \tag{2a}$$

where A_2, B_2 are polynomials with $\delta A_2, \delta B_2 \leq m - 2k$ and D_2 is a polynomial such that $\delta D_2 \leq m - 2k + 1$. We note that system $(2a)$ has $f^{\lambda+1}$ as an integrating factor. So, applying Lemma 4.6 to system $(2a)$ we obtain system

$$\dot{x} = (\lambda + 2)A_2 + D_{2y}, \qquad \dot{y} = (\lambda + 2)B_2 - D_{2x}, \qquad (2b)$$

of degree at most $m - 2k$ which has $f = 0$ as invariant algebraic curve and an integrating factor of the form $f^{\lambda+2}$. From Proposition 1.2 we have that $k \leq m - 2k + 1$ and so $k \leq (m + 1)/3$ which is in contradiction with Case 4. So, system $(2b)$ is identically zero, and therefore $A_2 = -D_{2y}/(\lambda + 2)$ and $B_2 = D_{2x}/(\lambda + 2)$.

Since $(m + 1)/3 < k < (m + 1)/2$ is equivalent to $((m - k) + 1)/2 < k < (m - k) + 1$, then for system $(2a)$, working in a similar way as in Case 2, we have that for $\lambda \neq -2$ there is a polynomial $F_2 = D_2$ of degree $m - 2k + 1$ such that system $(2a)$ and consequently system $(1b)$ can be written as

$$\dot{x} = -\frac{1}{\lambda + 2} f F_{2y} - F_2 f_y, \qquad \dot{y} = \frac{1}{\lambda + 2} f F_{2x} + F_2 f_x,$$

and has the Darboux first integral $H_2 = F_2^{\frac{1}{\lambda+2}} f$. Since system $(1b)$ is equal to system $(2a)$ we get

$$\begin{aligned} (\lambda + 1)A_1 + D_{1y} &= -\frac{1}{\lambda + 2} f F_{2y} - F_2 f_y, \\ (\lambda + 1)B_1 - D_{1x} &= \frac{1}{\lambda + 2} f F_{2x} + F_2 f_x. \end{aligned} \qquad (4.16)$$

So for $\lambda \notin \{-1, -2\}$ we have

$$A_1 = -\frac{1}{\lambda + 1}\left(D_{1y} + \frac{1}{\lambda + 2} f F_{2y} + F_2 f_y\right), \quad B_1 = \frac{1}{\lambda + 1}\left(D_{1x} + \frac{1}{\lambda + 2} f F_{2x} + F_2 f_x\right),$$

Substituting A_1 and B_1 into system $(1a)$ which is of degree m we obtain that

$$\begin{aligned} \dot{x} &= -\frac{1}{\lambda + 1}\left(D_{1y} + \frac{1}{\lambda + 2} f F_{2y} + F_2 f_y\right) f - D_1 f_y, \\ \dot{y} &= \frac{1}{\lambda + 1}\left(D_{1x} + \frac{1}{\lambda + 2} f F_{2x} + F_2 f_x\right) f + D_1 f_x. \end{aligned} \qquad (4.17)$$

We note that system (4.17) has the invariant algebraic curve of degree at most $m+1-k$ $F = D_1 + \frac{1}{\lambda+2} F_2 f$ with cofactor $K_2 = f_x D_{1y} - f_y D_{1x} + \frac{1}{\lambda+2}(f_x F_{2y} - f_y F_{2x})$.

The invariant algebraic curve $f = 0$ has cofactor $K_1 = -\frac{1}{\lambda+1}K_2$. Hence, system (4.17) has the normal form (4.4) because it can be written as

$$\begin{aligned}
\dot{x} &= -\frac{1}{\lambda+1}\left(D_1 + \frac{1}{\lambda+2}F_2 f\right)_y f - \left(D_1 + \frac{1}{\lambda+2}F_2 f\right) f_y,\\
\dot{y} &= \frac{1}{\lambda+1}\left(D_1 + \frac{1}{\lambda+2}F_2 f\right)_x f + \left(D_1 + \frac{1}{\lambda+2}F_2 f\right) f_x.
\end{aligned}$$

If $A_{1x} + B_{1y} = 0$ then from Lemma 4.6(b) system $(1b)$ is a Hamiltonian system of the form (4.9) and has f as a first integral. Therefore, from Theorem 3.1(b) system $(1b)$ can be written as

$$\begin{aligned}
\dot{x} &= ((\lambda+1)G + D_1)_y &&= -D_2 f_y,\\
\dot{y} &= -((\lambda+1)G + D_1)_x &&= D_2 f_x,
\end{aligned}$$

for some $D_2 \in \mathbb{C}[x, y]$ with $\delta D_2 = m - 2k + 1$. Additionally, we have that $D_{2x} f_y - D_{2y} f_x = 0$ which means that system $\dot{x} = -D_{2y}, \dot{y} = D_{2x}$ has f as a first integral. From Proposition 1.2 we have that $k \le m - 2k + 1$ or equivalently $k \le (m + 1)/3$ which is in contradiction with Case 4. So, it must be $D_2 = \delta_2 \in \mathbb{C}$ and therefore $G_y = -(\delta_2 f + D_1)_y/(\lambda+1)$ and $G_x = (\delta_2 f + D_1)_x/(\lambda+1)$. Hence, system $(1a)$ can be rewritten into the form

$$\begin{aligned}
\dot{x} &= G_y f - D_1 f_y = -\delta_2 f f_y - \frac{1}{\lambda+1}D_{1y} f - D_1 f_y,\\
\dot{y} &= -G_x f + D_1 f_y = \delta_2 f f_x + \frac{1}{\lambda+1}D_{1x} f + D_1 f_x,
\end{aligned}$$

and setting $F = D_1 + \frac{1}{\lambda+2}\delta_2 f$ system $(1a)$ is of the normal form form (4.4) and has the Darboux first integral $H = f F^{\frac{1}{\lambda_1+1}}$.

Remark 4.7. *We note that if $L_1 = 0$ then system $(1b)$ is a Hamiltonian system and in this case system $(2a)$ is of the form*

$$\dot{x} = A_2 f - D_2 f_y, \qquad \dot{y} = B_2 f + D_2 f_x,$$

with $A_2 = B_2 = 0$ and $D_2 = \delta_2 \in \mathbb{C}$ and we can take $F_2 = \delta_2$ and so for $\lambda \ne -2$ the system has the Darboux first integral $H_2 = F_2^{\frac{1}{\lambda+2}} f$. System $(1a)$ has the additionally invariant algebraic curve $F = D_1 + \frac{1}{\lambda+2}F_2 f$ with $\delta F + \delta f = m + 1$, and it is of *the normal form (4.4). Hence, using the information about the degrees the case $L_1 = 0$ is included in the study whenever we apply Lemma 4.6(a). From now on, we will omit to study this case.*

Subcase $\lambda = -1$. We assume also that $L_1 \neq 0$. Then $A_2 = -D_{2y}$ and $B_2 = D_{2x}$, and so relation $A_{2x} + B_{2y} = 0$ holds and system $(2b)$ is Hamiltonian. System $(2a)$ can be written

$$\dot{x} = -D_{2y} f - D_2 f_y, \qquad \dot{y} = D_{2x} f + D_2 f_x,$$

and has the invariant algebraic curve $F_2 = D_2 = 0$ and the Darboux first integral $H_2 = D_2 f$. Since system $(1b)$ is identically equal to system $(2a)$ we get that

$$\begin{aligned} D_{1y} &= A_2 f - D_2 f_y = -F_{2y} f - F_2 f_y, \\ -D_{1x} &= B_2 f + D_2 f_x = F_{2x} f + F_2 f_x. \end{aligned}$$

So, $D_1 = -D_2 f - \delta_1$ for some $\delta_1 \in \mathbb{C}$. From relation (4.8), we get that $A_{1x} + B_{1y} = f_x F_{2y} - f_y F_{2x}$, and from Lemma 4.5, we have that there is $G \in \mathbb{C}[x, y]$ with $\delta G \leq m - k + 1$ such that $A_1 = F_{2y} f + G_y$ and $B_1 = -F_{2x} f - G_x$. So, system $(1a)$ can be written into the form

$$\dot{x} = \delta_1 f_y + (F_2 f + G)_y f, \qquad \dot{y} = -\delta_1 f_x - (F_2 f + G)_x f,$$

and we note that it has the Darboux first integral $H = f^{\delta_1} \exp(F_2 f + G)$. Obviously, system (1a) takes the normal form (4.5) for $g/f^0 = -(F_2 f + G)$ and $\beta = -\delta_1$.

Remark 4.8. *We note that if $L_1 = 0$ the proof is the same taking $F_2 \in \mathbb{C}$. From now on we will omit to study the case $L_1 = 0$.*

Subcase $\lambda = -2$. Then, since system $(2b)$ is identically zero, we get $D_{2y} = 0$, $D_{2x} = 0$, and therefore $D_2(x, y) = \delta_2$ with $\delta_2 \in \mathbb{C}$. Also we have that $A_{2x} + B_{2y} = 0$, and so, by Lemma 4.4, there is $G_2 \in \mathbb{C}[x, y]$ such that $A_2 = -G_{2y}$ and $B_2 = G_{2x}$. Hence, system $(2a)$ becomes

$$\dot{x} = -G_{2y} f - \delta_2 f_y, \qquad \dot{y} = G_{2x} f + \delta_2 f_x, \tag{4.18}$$

and has the exponential factor $\exp(G_2)$ and the first integral $H_2 = f^{\delta_2} \exp(G_2)$. Since system $(1b)$ is identically equal to system $(2a)$ and consequently to system (4.18), we obtain

$$A_1 = G_{2y} f + \delta_2 f_y + D_{1y}, \qquad B_1 = -G_{2x} f - \delta_2 f_x - D_{1x},$$

and hence system $(1a)$ takes the form

$$\dot{x} = (\delta_2 f_y + G_{2y} f) f + D_{1y} f - D_1 f_y, \qquad \dot{y} = -(\delta_2 f_x + G_{2x} f) - D_{1x} f + D_1 f_x,$$

and has the two invariant exponential factors $\exp(G_2)$ and $\exp(D_1/f)$, and the Darboux first integral $H = f^{\delta_2} \exp(D_1/f) \exp(G_2)$. Obviously, the last system is in the normal form (4.5) for $\beta = -\delta_2$ and $g/f^{n_1} = -(D_1 + G_2 f)/f$. ∎

Case 5: $k = (m+1)/3$. Then $k = (m-2k)+1$. Note that this case occurs when $m+1$ is a multiple of 3. System $(1b)$ of degree $m-k$ has $f=0$ as invariant algebraic curve. From Theorem 2.4(a) it can be written into the form $(2a)$. Since system $(1b)$ and consequently system $(2a)$ has $f^{\lambda+1}$ as an integrating factor, from Lemma 4.6, we obtain that system

$$\dot{x} = (\lambda+2) A_2 + D_{2y}, \qquad \dot{y} = (\lambda+2) B_2 - D_{2x}, \qquad (2b)$$

of degree $m-2k = k-1$ has $f=0$ as invariant algebraic curve and the integrating factor $f^{\lambda+2}$. From Theorem 2.4(b), all systems of degree $m-2k$ have $f=0$ as invariant algebraic curve are of the form

$$\dot{x} = -\alpha_2 f_y, \qquad \dot{y} = \alpha_2 f_x,$$

for some $\alpha_2 \in \mathbb{C}$. Hence, we have that

$$(\lambda+2) A_2 + D_{2y} = -\alpha_2 f_y, \qquad (\lambda+2) B_2 - D_{2x} = \alpha_2 f_x,$$

and so

$$A_2 = -\frac{1}{\lambda+2}(D_{2y} + \alpha f_2), \qquad B_2 = \frac{1}{\lambda+2}(D_{2x} + \alpha f_2).$$

Substituting A_2 and B_2 into system $(2a)$ we get

$$\dot{x} = -\frac{1}{\lambda+2}(D_{2y} + \alpha_2 f_y) f - D_2 f_y, \qquad \dot{y} = \frac{1}{\lambda+2}(D_{2x} + \alpha_2 f_x) f - D_2 f_x. \qquad (4.19)$$

Applying the same arguments as in Case 3 to system (4.19) of degree $m-2k=k-1$, and consequently to system $(2b)$ we have that for $\lambda\notin\{-2,-3\}$ there is an invariant algebraic curve $F_2=D_2+\dfrac{\alpha}{\lambda+3}f$, of degree at most $m-2k+1$ such that system (4.19) is of the normal form (4.4) because it can be written as

$$\dot{x}=-\frac{1}{\lambda+2}F_{2y}f-F_2f_y,\qquad \dot{y}=\frac{1}{\lambda+2}F_{2x}f+F_2f_x,$$

and so it has the first integral $H_2=F_2^{\frac{1}{\lambda+2}}f$ and $\delta F_2=m-2k+1$.

Similar to Case 3 we have that system $(1a)$ of degree m for $\lambda\notin\{-1,-2,-3\}$ has

$$F=D_1+\frac{1}{\lambda+2}F_2f=D_1+\frac{1}{\lambda+2}\left(D_2+\frac{\alpha}{\lambda+3}\right)f=0$$

as invariant algebraic curve and $\delta f+\delta F=m+1$ and takes the normal form (4.4).

Subcase $\lambda=-1$. Then $A_2=-\alpha_2f_y-D_{2y}$ and $B_2=\alpha_2f_x+D_{2x}$, so relation $A_{2x}+B_{2y}=0$ holds. System $(2a)$ becomes

$$\begin{aligned}\dot{x}&=-\left(\tfrac{\alpha_2}{2}f+D_2\right)_yf-\left(\tfrac{\alpha_2}{2}f+D_2\right)f_y,\\ \dot{y}&=\left(\tfrac{\alpha_2}{2}f+D_2\right)_xf+\left(\tfrac{\alpha_2}{2}f+D_2\right)f_x,\end{aligned}$$

and has the invariant algebraic curve $F_2=\frac{\alpha_2}{2}f+D_2$ and the Darboux first integral $H_2=F_2f$. Since system $(1b)$ is equal to system $(2a)$ we have

$$D_{1y}=-F_{2y}f-F_2f_y,\qquad -D_{1x}=F_{2x}f+F_2f_x,$$

and so $D_1=-F_2f-\delta_1$ for some $\delta_1\in\mathbb{C}$. From relation (4.8), we have that $A_{1x}+B_{1y}=f_xF_{2y}-f_yF_{2x}$. Then, from Lemma 4.5, we have that there is $G\in\mathbb{C}[x,y]$ with $\delta G\le m-k+1$ such that $A_1=F_{2y}f+G_y$ and $B_1=-F_{2x}f-G_x$. So, system $(1a)$ can be written as

$$\begin{aligned}\dot{x}&=\delta_1f_y+(F_2f+G)_yf,\\ \dot{y}&=-\delta_1f_x-(F_2f+G)_xf,\end{aligned}$$

and has the Darboux first integral $H=f^{\delta_1}\exp\left(F_2f+G\right)$, and therefore it is of the normal form (4.5) taking $g/f^0=-(F_2f+G)$ and $\beta=-\delta_1$.

Subcase $\lambda=-2$. Then from system $(2b)$ we obtain

$$D_{2y}=-\alpha_2f_y,\qquad D_{2x}=\alpha_2f_x,$$

and so there is $\delta_2 \in \mathbb{C}$ a such that $D_2 = -\alpha_2 f - \delta_2$. Since we have that $A_{2x} + B_{2y} = 0$ applying Lemma 4.4 there is $G_2 \in \mathbb{C}[x, y]$ such that $A_2 = G_{2y}$ and $B_2 = -G_{2x}$. Hence, system $(2a)$ becomes

$$\begin{aligned} \dot{x} &= \delta_2 f_y + (G_{2y} + \alpha_2 f_y) f = \delta_2 f_y + R_{2y} f, \\ \dot{y} &= -\delta_2 f_x - (G_{2x} + \alpha_2 f_x) f = -\delta_2 f_x - R_{2x} f, \end{aligned}$$

and has the invariant exponential factor $\exp(R_2) = \exp(G_2 + \alpha_2 f)$, and the Darboux first integral $H_2 = f^{\delta_2} \exp(R_2)$.

Since system $(1b)$ is equal to with system $(2a)$ we have that

$$A_1 = -\delta_2 f_y - R_{2y} f + D_{1y}, \qquad B_1 = \delta_2 f_x + R_{2x} f - D_{1x},$$

and so system $(1a)$ becomes

$$\begin{aligned} \dot{x} &= -(\delta_2 f_y + R_{2y} f) f + D_{1y} f - D_1 f_y, \\ \dot{y} &= (\delta_2 f_x + R_{2x} f) f - D_{1x} f + D_1 f_x, \end{aligned}$$

and has the two invariant exponential factors $\exp(R_2)$ and $\exp(D_1/f)$, and the Darboux first integral $H = f^{\delta_2} \exp(R_2) \exp(-D_1/f)$. Consequently, system $(1a)$ can be written into the normal form (4.5) taking $g/f^{n_1} = (R_2 f - D_1)/f$ and $\beta = \delta_2$.

Subcase $\lambda = -3$. Then

$$A_2 = \alpha_2 f_y + D_{2y}, \qquad B_2 = -\alpha_2 f_x - D_{2x},$$

and so system $(2a)$ becomes

$$\dot{x} = \alpha_2 f f_y + D_{2y} f - D_2 f_y, \qquad \dot{y} = -\alpha_2 f f_x - D_{2x} f + D_2 f_x,$$

and has the invariant exponential factor $\exp(D_2/f)$, and the Darboux first integral $H_2 = f^{\alpha_2} \exp(D_2/f)$. Since system $(2a)$ is equal to system $(1b)$, we have that

$$-2A_1 + D_{1y} = (\alpha_2 f_y + D_{2y}) f - D_2 f_y, \qquad -2B_1 - D_{1x} = (-\alpha_2 f_x - D_{2x}) f + D_2 f_x,$$

and therefore

$$2A_1 = -(\alpha_2 f_y + D_{2y}) f + D_2 f_y + D_{1y}, \qquad 2B_1 = (\alpha_2 f_x + D_{2x}) f - D_2 f_x - D_{1x}.$$

So, system $(1a)$ can be written into the form

$$\begin{aligned}\dot{x} &= \frac{1}{2}(-\alpha_2 f f_y - D_{2y} f + D_2 f_y + D_{1y}) f - D_1 f_y,\\ \dot{y} &= \frac{1}{2}(\alpha_2 f f_x + D_{2x} f - D_2 f_x - D_{1x}) f + D_1 f_x,\end{aligned}$$

and has the two invariant exponential factors $\exp(D_1/f^2)$ and $\exp(D_2/2f)$. Additionally, system $(1a)$ has the Darboux first integral

$$H = f^{\frac{\alpha_2}{2}} \exp\left(\frac{-D_1}{f^2}\right) \exp\left(\frac{D_2}{2f}\right).$$

For $g/f^2 = (-2D_1 + D_2 f^2)/(2f^2)$ and $\beta = \alpha_2/2$ system $(1a)$ has the normal form (4.5). ∎

Now, we present some notation.

We denote by (na) a system of degree $m-(n-1)k$ of the form

$$\dot{x} = A_n f - D_n f_y, \qquad \dot{y} = B_n f + D_n f_x,$$

with $\delta A_n, \delta B_n \leq m - nk, \delta f = k$ and $\delta D_n \leq m - nk + 1$ having $f = 0$ as invariant curve and an integrating factor of the form $f^{\lambda+n-1}$. This system according to Lemma 4.6 generates a system (nb) of degree $m - nk$ of the form

$$\dot{x} = (\lambda + n) A_n + D_{ny}, \qquad \dot{y} = (\lambda + n) B_n - D_{nx},$$

and has $f = 0$ as invariant algebraic curve and the integrating factor $f^{\lambda+n}$.

We consider the two sequences of systems (la) and (lb) having $f = 0$ as invariant algebraic curve and an integrating factor forming by f,

$$\begin{aligned}\dot{x} &= A_1 f - D_1 f_y,\\ \dot{y} &= B_1 f + D_1 f_x,\end{aligned}\qquad (1a)\qquad \begin{aligned}\dot{x} &= (\lambda+1)A_1 + D_{1y},\\ \dot{y} &= (\lambda+1)B_1 - D_{1x},\end{aligned}\qquad (1b)$$

$$\begin{aligned}\dot{x} &= A_2 f - D_2 f_y,\\ \dot{y} &= B_2 f + D_2 f_x,\end{aligned}\qquad (2a)\qquad \begin{aligned}\dot{x} &= (\lambda+2)A_2 + D_{2y},\\ \dot{y} &= (\lambda+2)B_2 - D_{2x},\end{aligned}\qquad (2b)$$

$$\cdots \qquad\qquad \cdots \qquad \cdots$$

$$\begin{aligned}\dot{x} &= A_{n-1} f - D_{n-1} f_y,\\ \dot{y} &= B_{n-1} f + D_{n-1} f_x,\end{aligned}\qquad (n-1,a)\qquad \begin{aligned}\dot{x} &= (\lambda+n-1)A_{n-1} + D_{n-1,y},\\ \dot{y} &= (\lambda+n-1)B_{n-1} - D_{n-1,x},\end{aligned}\qquad (n-1,b)$$

$$\begin{aligned}\dot{x} &= A_n f - D_n f_y,\\ \dot{y} &= B_n f + D_n f_x,\end{aligned}\qquad (na)\qquad \begin{aligned}\dot{x} &= (\lambda+n)A_n + D_{1y},\\ \dot{y} &= (\lambda+n)B_n - D_{1x},\end{aligned}\qquad (nb)$$

where $\delta A_i, \delta B_i \leq m - ik$ and $\delta D_i \leq m - ik + 1$.

Lemma 4.9. *We assume that the conditions of Theorem 4.3 hold and that* $(m+1)/(n+1) \leq k < (m+1)/n$ *and* $\lambda \notin \{-1, -2, \cdots, -n\}$. *Then, system* $(1a)$ *takes the normal form (4.4).*

Proof: For simplicity in the proof we distinguish the following cases.

Case A: $(m+1)/(n+1) < k < (m+1)/n$. Since $f = 0$ is an invariant algebraic curve for system (nb) of degree $m - nk$, then from Proposition 1.2 we have that $k \leq (m - nk) + 1$ which is in contradiction with the assumptions. Hence, system (nb) must be identically zero. So, for $\lambda \neq -n$ we have

$$A_n = -\frac{1}{\lambda+n} D_{ny}, \qquad B_n = \frac{1}{\lambda+n} D_{nx},$$

and therefore system (na) becomes

$$\dot{x} = -\frac{1}{\lambda+n} D_{ny} f - D_n f_y, \qquad \dot{y} = \frac{1}{\lambda+n} D_{nx} f + D_n f_x,$$

and has the invariant algebraic curve $F_n = D_n = 0$ with $\delta F_n = m - nk + 1$, and takes the normal form (4.4). So system (na) has the first integral $H_n = F_n^{\frac{1}{\lambda+n}} f$.

We note that system $(n-1,b)$ is equal to system (na), so we have

$$\begin{aligned}(\lambda+n-1)A_{n-1}+D_{n-1,y} &= -\frac{1}{\lambda+n}F_{ny}f-F_nf_y,\\ (\lambda+n-1)B_{n-1}-D_{n-1,x} &= \frac{1}{\lambda+n}F_{nx}f+F_nf_yx,\end{aligned}$$

and for $\lambda\neq-(n-1)$, we get

$$\begin{aligned}A_{n-1} &= -\frac{1}{\lambda+n-1}\left(D_{n-1,y}+\frac{1}{\lambda+n}F_{ny}f+D_{n-1}f_y\right),\\ B_{n-1} &= \frac{1}{\lambda+n-1}\left(D_{n-1,x}+\frac{1}{\lambda+n}F_{nx}f+D_{n-1}f_x\right).\end{aligned}$$

Substituting A_{n-1} and B_{n-1} into system (n-1,a) we obtain

$$\begin{aligned}\dot{x} &= -\frac{1}{\lambda+n-1}\left(D_{n-1,y}+\frac{1}{\lambda+n}F_{ny}f+D_{n-1}f_y\right)f-D_{n-1}f_y,\\ \dot{y} &= \frac{1}{\lambda+n-1}\left(D_{n-1,x}+\frac{1}{\lambda+n}F_{nx}f+D_{n-1}f_x\right)f+D_{n-1}f_x.\end{aligned}\tag{4.20}$$

System (4.20) of degree $m-(n-2)k$ has the invariant algebraic curve

$$F_{n-1}=D_{n-1}+\frac{1}{\lambda+n}F_nf,$$

and $\delta F_{n-1}\leq m+(n-1)k+1$. Moreover, for $\lambda\notin\{-(n-1),-n\}$ system (4.20) can be written into the normal form (4.4)

$$\dot{x}=-\frac{1}{\lambda+n-1}F_{n-1,y}f-F_{n-1}f_y,\qquad \dot{y}=\frac{1}{\lambda+n-1}F_{n-1,x}f+F_{n-1}f_x,$$

and has the Darboux first integral $H_{n-1}=F_{n-1}^{\frac{1}{\lambda+n-1}}f$ with $\delta F_{n-1}=m-(n-2)k+1$.

Working in a similar way we have that system $(n-2,a)$ can be written into the normal form (4.4) and has the first integral $H_{n-2}=F_{n-2}^{\frac{1}{\lambda+n-2}}f$ with $F_{n-2}=D_{n-2}+\frac{1}{\lambda+n-1}F_{n-1}f$ as an additional invariant algebraic curve and $\lambda\notin\{-(n-2),-(n-1),-n\}$.

Additionally, every term of the sequence of the vector fields (la) has the additional algebraic curve $F_l=D_l+\frac{1}{\lambda+l+1}F_{l+1}f$ for $l=1,\cdots,n$ and $\lambda\notin\{-1,\cdots,-n\}$.

Moreover, it has the Darboux first integral $H_l = F_l^{\frac{1}{\lambda+l}} f$. So, the first term of this sequence $(1a)$ of degree m has the invariant algebraic curve

$$
\begin{aligned}
F &= D_1 + \frac{1}{\lambda+2} F_2 f \\
&= D_1 + \frac{1}{\lambda+2}\left(D_2 + \frac{1}{\lambda+3} F_3 f\right) f \\
&= D_1 + \frac{1}{\lambda+2}\left(D_2 + \frac{1}{\lambda+3}\left(D_3 + \frac{1}{\lambda+4} F_4 f\right) f\right) f \\
&\cdots \\
&= D_1 + \frac{1}{\lambda+2} D_2 f + \frac{1}{(\lambda+2)(\lambda+3)} D_3 f^2 + \cdots + \frac{1}{(\lambda+2)(\lambda+3)\cdots(\lambda+n)} D_n f^{n-1} \\
&= D_1 + \sum_{i=2}^{n} \frac{\lambda+1}{(\lambda+2)(\lambda+3)\cdots(\lambda+i)} D_i f^{i-1}.
\end{aligned}
$$

Additionally, for $\lambda \notin \{-1, \cdots, -n\}$ system $(1a)$ is into the normal form (4.4) and has the Darboux first integral $H = F^{\frac{1}{\lambda+1}} f$. So, Theorem 4.3 is proved in this case.

Case B: $k = (m+1)/(n+1)$. Equivalently we have that $k = (m - nk) + 1$. In this case, system (nb) of degree $m - nk$ is of the form 2.4(b). Hence, we have

$$
(\lambda+n)A_n + D_{ny} = -\alpha_n f_y, \qquad (\lambda+n)B_n - D_{nx} = \alpha_n f_x,
$$

for some $\alpha_n \in \mathbb{C}$. Since, $\lambda \neq -n$ we can obtain A_n and B_n, and so system (na) can be written into the form

$$
\dot{x} = -\frac{1}{\lambda+n}(D_{ny} + \alpha_n f_y) f - D_n f_y, \qquad \dot{y} = \frac{1}{\lambda+n}(D_{nx} + \alpha_n f_x) f + D_n f_x, \tag{4.21}
$$

and has the invariant curve $F_n = D_n + \frac{\alpha_n}{\lambda+n+1} f$ of degree at most $m - nk + 1$. We note that system (4.21) can be written into the normal form (4.4)

$$
\dot{x} = -\frac{1}{\lambda+n} F_{ny} f - F f_y, \qquad \dot{y} = \frac{1}{\lambda+n} F_{nx} f + F f_x,
$$

and has the first integral $H_n = F_n^{\frac{1}{\lambda+n}} f$ with $\delta F_n = m - nk + 1$.

In a similar way we have that system $(n-1, a)$ can be written into the normal form (4.4) and has the Darboux first integral $H_{n-1} = F_{n-1}^{\frac{1}{\lambda+n-1}} f$ with $F_{n-1} = D_{n-1} + \frac{1}{\lambda+n} F_n f$ as an additional invariant algebraic curve. Furthermore, every term of the sequence of the vector fields (X_l) has the complementary algebraic curve $F_l = D_l + \frac{1}{\lambda+l+1} F_{l+1} f$ for $l = 1, \cdots, n-1$ and $\lambda \notin \{-1, \cdots, -n\}$. We note that for $\lambda \notin \{-1, \cdots, -n, -(n+1)\}$ the last term (na) of the sequence (la) has the algebraic curve $F_n = D_n + \frac{\alpha_n}{\lambda+n+1} f$. Moreover, every term (la) has the Darboux first integral $H_l = F_l^{\frac{1}{l+1}} f$. So, the last term of this sequence $(1a)$ of degree m has the invariant algebraic curve

$$
\begin{aligned}
F \quad &= \quad D_1 + \frac{1}{\lambda+2} F_2 f \\
&= \quad D_1 + \frac{1}{\lambda+2}\left(D_2 + \frac{1}{\lambda+3} F_3 f\right) f \\
&= \quad D_1 + \frac{1}{\lambda+2}\left(D_2 + \frac{1}{\lambda+3}\left(D_3 + \frac{1}{\lambda+4} F_4 f\right) f\right) f \\
&\cdots \\
&= \quad D_1 + \frac{1}{\lambda+2} D_2 f + \frac{1}{(\lambda+2)(\lambda+3)} D_3 f^2 + \cdots + \frac{1}{(\lambda+2)(\lambda+3)\cdots(\lambda+n)} D_n f^{n-1} \\
&+ \quad \frac{1}{(\lambda+2)(\lambda+3)\cdots(\lambda+n+1)} \alpha_n f^n \\
&= \quad D_1 + \sum_{i=2}^{n+1} \frac{\lambda+1}{(\lambda+2)(\lambda+3)\cdots(\lambda+i)} D_i f^{i-1},
\end{aligned}
$$

with $D_{n+1} = \alpha_n \in \mathbb{C}$. Additionally, for $\lambda \notin \{-1, \cdots, -(n+1)\}$ it is of the normal form (4.4) and it has the Darboux first integral $H = F^{\frac{1}{\lambda+1}} f$. So we prove case B.

With cases A and B we complete the proof of Theorem 4.3(a). ∎

Lemma 4.10. *We assume that the conditions of Theorem 4.3 hold and that* $(m+1)/(n+1) \leq k < (m+1)/n$ *and* $\lambda \in \{-1, -2, \cdots, -n\}$. *Then system* $(1a)$ *takes the normal form (4.5).*

Proof: For simplicity we distinguish the following cases.

Case A: $k = (m+1)/(n+1)$. Since $f = 0$ is an invariant algebraic curve of system (nb) of degree $m - nk$, and since in this case we have that $k = (m - nk) + 1$ then from Theorem 2.4(b) system (nb) can be written into the form

$$(\lambda + n)A_n + D_{ny} = -\alpha_n f_y, \qquad (\lambda + n)B_n - D_{nx} = \alpha_n f_x,$$

for some $\alpha_n \in \mathbb{C}$.

Subcase $\lambda = -n$. Then we have that $D_{ny} = -\alpha_n f_y$ and $D_{nx} = -\alpha_n f_x$, and therefore $D_n = -\alpha_n f - \delta_n$ for some $\delta_n \in \mathbb{C}$. Applying relation (4.8) to system (nb) we get that $A_{nx} + B_{ny} = 0$ and, from Lemma 4.4, we have that there is $G_n \in \mathbb{C}[x,y]$ with $\delta G_n \leq m - nk + 1$ such that $A_n = -G_{ny}$ and $B_n = G_{nx}$. Hence, system (na) becomes

$$\dot{x} = -(G_{ny} - \alpha_n f_y)f + \delta_n f_y, \qquad \dot{y} = (G_{nx} - \alpha_n f_x)f - \delta_n f_x,$$

and has the invariant exponential factor $\exp(G_n - \alpha_n f)$ with $\delta(G_n - \alpha_n f) = m - nk + 1$, and it has the Darboux first integral $H_n = f^{-\delta_n} \exp(G_n - \alpha_n f)$. Additionally, system (na) takes the normal form (4.5) for $\beta = -\delta_n$ and $g/f^0 = G_n - \alpha_n f$.

We note that system $(n-1, b)$ is equal to system (na) and so we have

$$-A_{n-1} + D_{n-1,y} = -(G_n - \alpha_n f)_y f + \delta_n f_y, \qquad -B_{n-1} - D_{n-1,x} = (G_n - \alpha_n f)_x f - \delta_n f_x.$$

Substituting A_{n-1} and B_{n-1} into system (n-1,a) we obtain

$$\begin{aligned} \dot{x} &= (-\delta_n f_y + (G_n - \alpha_n f)_y f)f + D_{n-1,y} f - D_{n-1} f_y, \\ \dot{y} &= -(-\delta_n f_x + (G_n - \alpha_n f)_x f)f - D_{n-1,x} f + D_{n-1} f_x. \end{aligned} \tag{4.22}$$

System (4.22) of degree $m-(n-2)k$ has the invariant exponential factors $\exp(-G_n + \alpha_n f)$ and $\exp(-D_{n-1}/f)$ and the Darboux first integral $H_{n-1} = f^{\delta_n} \exp(-D_{n-1}/f) \exp(-G_n + \alpha_n f)$. System (4.22) takes the normal form (4.5) for $\beta = \delta_n$ and $g/f = (-D_{n-1} - G_n f + \alpha_n f^2)/f$.

We note that system $(n-2, b)$ is equal to system (n-1,a), and so we have

$$\begin{aligned} -2A_{n-2} + D_{n-2,y} &= (-\delta_n f_y + (G_n - \alpha_n f)_y f)f + D_{n-1,y} f - D_{n-1} f_y, \\ -2B_{n-2} - D_{n-2,x} &= -(-\delta_n f_x + (G_n - \alpha_n f)_x f)f - D_{n-1,x} f + D_{n-1} f_x. \end{aligned}$$

Substituting A_{n-2} and B_{n-2} to system $(n-2,a)$ we get

$$\begin{aligned}\dot{x} &= \frac{1}{2}(D_{n-2,y} - (-\delta_n f_y + (G_n - \alpha_n)_y f)f - D_{n-1,y}f + D_{n-1}f_y) - D_{n-2}f_y,\\ \dot{y} &= \frac{1}{2}(-D_{n-2,x} + (-\delta_n f_x + (G_n - \alpha_n)_x f)f + D_{n-1,x}f - D_{n-1}f_x) + D_{n-2}f_x.\end{aligned} \tag{4.23}$$

System (4.23) of degree $m-(n-3)k$ has the invariant exponential factors $\exp((G_n - \alpha_n f)/2)$, $\exp\left(\frac{D_{n-1}}{2f}\right)$ and $\exp\left(\frac{-D_{n-2}}{2f^2}\right)$ and has the Darboux first integral

$$H_{n-2} = f^{\frac{-\delta_n}{2}} \exp\left(\frac{-D_{n-2}}{2f^2}\right) \exp\left(\frac{D_{n-1}}{2f}\right) \exp\left(\frac{G_n - \alpha_n f}{2}\right).$$

System (4.23) for $\beta = \delta_n/2$ and $g/f^2 = (G_n f^2 - \alpha_n f^3 + D_{n-1}f - D_{n-2})/(2f^2)$ takes the normal form (4.5).

Hence, for $\lambda = -n$ we have the sequence of systems (la) and their first integrals H_l:

$$\begin{aligned}
&(na) && H_n = f^{-\delta_n} \exp(G_n - \alpha_n f),\\
&(n-1,a) && H_{n-1} = f^{\delta_n} \exp\left(\frac{-D_{n-1}}{f}\right) \exp(-G_n + \alpha_n f),\\
&(n-2,a) && H_{n-2} = f^{\frac{-\delta_n}{2}} \exp\left(\frac{-D_{n-2}}{2f^2}\right) \exp\left(\frac{D_{n-1}}{2f}\right) \exp\left(\frac{G_n - \alpha_n f}{2}\right),\\
&(n-3,a) && H_{n-3} = f^{\frac{\delta_n}{3\cdot 2}} \exp\left(\frac{-D_{n-3}}{3f^3}\right) \exp\left(\frac{D_{n-2}}{3\cdot 2f^2}\right) \exp\left(\frac{-D_{n-1}}{3\cdot 2f}\right) \exp\left(\frac{-G_n + \alpha_n f}{3\cdot 2}\right),\\
&(n-4,a) && H_{n-4} = f^{\frac{-\delta_n}{4\cdot 3\cdot 2}} \exp\left(\frac{-D_{n-4}}{4f^4}\right) \exp\left(\frac{D_{n-3}}{4\cdot 3f^3}\right) \exp\left(\frac{-D_{n-2}}{4\cdot 3\cdot 2f^2}\right)\\
&&& \qquad \exp\left(\frac{D_{n-1}}{4\cdot 3\cdot 2f}\right) \exp\left(\frac{G_n - \alpha_n f}{4\cdot 3\cdot 2}\right),\\
&\dots && \dots\dots
\end{aligned}$$

$$(1a) \qquad H_1 = f^{(-1)^n \frac{\delta_n}{(n-1)!}} \exp\left(\frac{-D_1}{(n-1)f^{n-1}}\right) \exp\left(\frac{D_2}{(n-1)(n-2)f^{n-2}}\right)$$

$$\exp\left(\frac{-D_3}{(n-1)(n-2)(n-3)f^{n-3}}\right) \cdots \exp\left((-1)^{n-3}\frac{D_{n-3}}{(n-1)(n-2)\cdots 3f^3}\right)$$

$$\exp\left((-1)^{n-2}\frac{D_{n-2}}{(n-1)!f^2}\right) \exp\left((-1)^{n-1}\frac{D_{n-1}}{(n-1)!f}\right)$$
$$\exp\left((-1)^n\frac{G_n - \alpha_n f}{(n-1)!}\right).$$

Hence, system $(1a)$ for $\lambda = -n$ has the Darboux first integral

$$\begin{aligned} H_1 &= f^{(-1)^n \frac{\delta_n}{(n-1)!}} \left(\prod_{j=1}^{n-1} \exp\left((-1)^{n-j}\frac{D_{n-j}}{j.(j+1)\cdots(n-1)f^j}\right)\right) \\ &\exp\left((-1)^n\frac{G_n - \alpha_n f}{(n-1)!}\right), \end{aligned}$$

or equivalently $H_1 = f^\beta \exp\left(g/f^{n-1}\right)$ where

$$g/f^{n-1} = \sum_{j=1}^{n-1}(-1)^{n-j}\frac{D_{n-j}}{j.(j+1)\cdots(n-1)f^j} + (-1)^{n-1}\frac{G_n - \alpha_n f}{(n-1)!},$$

and $\beta = (-1)^n \frac{\delta_n}{(n-1)!}$.

Subcase $\lambda = -(n-1)$. We have that $A_n = -D_{ny} - \alpha_n f_y$ and $B_n = D_{nx} + \alpha_n f_x$. So, the vector field (na) of degree $m - (n-1)k$ becomes

$$\dot{x} = -(D_{ny} + \alpha_n f_y)f - D_n f_y, \qquad \dot{y} = (D_{nx} + \alpha_n f_x)f + D_n f_x,$$

and has the invariant algebraic curve $F_n = D_n + \alpha_n f = 0$ and the first integral $H_n = F_n f$.

The vector field $(n-1, b)$ is equal to the vector field (n, a). So, we have that

$$D_{n-1,y} = -F_{ny}f - F_n f_y, \qquad D_{n-1,x} = -F_{nx}f - F_n f_x,$$

and therefore $D_{n-1} = -F_n f - \delta_{n-1}$ for some $\delta_{n-1} \in \mathbb{C}$. Additionally, applying relation (4.8) to the vector field $(n-1, a)$ we get that

$$\begin{array}{lcl} D_{n-1,y} f_x - D_{n-1,x} f_y + (A_{n-1,x} + B_{n-1,y}) f & = & 0, \\ -F_{n,y} f_x f - F_{n,x} f_y f + (A_{n-1,x} + B_{n-1,y}) f & = & 0, \\ F_{n,y} f_x - D_{n,x} f_y + (A_{n-1,x} + B_{n-1,y}) & = & 0, \end{array}$$

and, from Lemma 4.5, there is G_{n-1} with $\delta G_{n-1} \leq m - (n-1)k + 1$ such that $A_{n-1} = F_{ny} f + G_{n-1,y}$ and $B_{n-1} = -F_{nx} f - G_{n-1,x}$. Hence, the vector field X_{n-1} can be written into the form

$$\begin{array}{lcl} \dot{x} & = & (F_{n,y} f + G_{n-1,y}) f + (F_n f + \delta_{n-1}) f_y, \\ \dot{y} & = & -(F_{n,x} f + G_{n-1,y}) f - (F_n f + \delta_{n-1}) f_x, \end{array}$$

and it has the exponential factor $\exp(-F_{n,y} f + G_{n-1})$ and the first integral $H_{n-1} = f^{-\delta_{n-1}} \exp(-G_{n-1} + F_{n,y} f)$.

Continuing in a similar way as in the previous cases we obtain that for $\lambda = -(n-1)$ the sequence of systems (na) have the first integrals H_n given in the following table

$$(na) \qquad H_n = F_n f, \qquad F_n = D_n + \alpha_n f,$$

$$(n-1, a) \qquad H_{n-1} = f^{-\delta_{n-1}} \exp\left(-(G_{n-1} + F_n f)\right),$$

$$(n-2, a) \qquad H_{n-2} = f^{\delta_{n-1}} \exp\left(-\frac{D_{n-2}}{f}\right) \exp\left(G_{n-1} + F_n f\right),$$

$$(n-3, a) \qquad H_{n-3} = f^{-\dfrac{\delta_{n-1}}{2}} \exp\left(\frac{-D_{n-3}}{2f^2}\right) \exp\left(\frac{D_{n-2}}{2f}\right) \exp\left(-\frac{G_{n-1} + F_n f}{2}\right),$$

$$(n-4, a) \quad H_{n-4} = \quad f^{\dfrac{\delta_{n-1}}{3 \cdot 2}} \exp\left(-\frac{D_{n-4}}{3f^3}\right) \exp\left(\frac{D_{n-3}}{3 \cdot 2f^2}\right) \exp\left(-\frac{D_{n-2}}{3 \cdot 2f}\right) \exp\left(\frac{G_{n-1} + F_n f}{3 \cdot 2}\right),$$

$$\cdots \qquad \cdots\cdots$$

$$(1a) \qquad H_1 = f^{(-1)^{n-1}\frac{\delta_{n-1}}{(n-2)!}} \exp\left(\frac{-D_1}{(n-2)f^{n-2}}\right) \exp\left(\frac{D_2}{(n-2)(n-3)f^{n-3}}\right)$$

$$\exp\left(\frac{-D_3}{(n-2)(n-3)(n-4)f^{n-4}}\right) \cdots \exp\left((-1)^{n-4}\frac{D_{n-4}}{(n-1)(n-2)\cdots 3f^3}\right)$$

$$\exp\left((-1)^{n-3}\frac{D_{n-3}}{(n-2)!f^2}\right) \exp\left((-1)^{n-2}\frac{D_{n-2}}{(n-2)!f}\right)$$

$$\exp\left((-1)^{n-1}\frac{G_{n-1}+F_nf}{(n-2)!}\right).$$

We note that in systems (na), it appears the additional invariant algebraic curve $F_n = D_n + \alpha_n f = 0$.

Hence for $\lambda = -(n-1)$ system $(1a)$ has the Darboux first integral

$$H_1 = f^{(-1)^{n-1}\frac{\delta_{n-1}}{(n-2)!}} \left(\prod_{j=1}^{n-2} \exp\left((-1)^{n-j-1}\frac{D_{n-j-1}}{j.(j+1)\cdots(n-2)f^j}\right)\right)$$
$$\exp\left((-1)^{n-1}\frac{G_{n-1}+F_nf}{(n-2)!}\right),$$

or equivalently, $H_1 = f^\beta \exp\left(g/f^{n-2}\right)$ where

$$g/f^{n-2} = \sum_{j=1}^{n-2}(-1)^{n-j-1}\frac{D_{n-j-1}}{j.(j+1)\cdots(n-2)f^j} + (-1)^{n-1}\frac{G_{n-1}+F_nf}{(n-2)!},$$

and $\beta = (-1)^{n-1}\frac{\delta_{n-1}}{(n-2)!}$.

Subcase $\lambda = -(n-2)$. We have that $2A_n = -D_{ny} - \alpha_n f_y$ and $2B_n = D_{nx} + \alpha_n f_x$. So, the vector field (na) of degree $m-(n-1)k$ becomes

$$\dot{x} = -\frac{1}{2}(D_{ny} + \alpha_n f_y)f - D_n f_y, \qquad \dot{y} = \frac{1}{2}(D_{nx} + \alpha_n f_x)f + D_n f_x,$$

and has the invariant algebraic curve $F_n = D_n + \alpha_n f = 0$ and the first integral $H_n = F_n^{\frac{1}{2}} f$.

Working in a similar way as in the previous cases we get

$$
\begin{array}{lll}
(na) & H_n = F_n^{\frac{1}{2}} f, & F_n = D_n + \alpha_n f, \\[2ex]
(n-1,a) & H_{n-1} = F_{n-1} f, & F_{n-1} = D_{n-1} + \dfrac{1}{2} F_n f, \\[2ex]
(n-2,a) & H_{n-2} = f^{-\delta_{n-2}} \exp\left(-(G_{n-2} + F_{n-1} f)\right), & \\[2ex]
(n-3,a) & H_{n-3} = f^{\delta_{n-2}} \exp\left(-\dfrac{D_{n-3}}{f}\right) \exp\left(G_{n-2} + F_{n-1} f\right), & \\[2ex]
(n-4,a) & H_{n-4} = f^{-\dfrac{\delta_{n-2}}{2}} \exp\left(-\dfrac{D_{n-4}}{2f^2}\right) \exp\left(\dfrac{D_{n-3}}{2f}\right) \exp\left(-\dfrac{G_{n-2} + F_{n-1} f}{2}\right), & \\[2ex]
(n-5,a) & H_{n-5} = f^{\dfrac{\delta_{n-2}}{3\cdot 2}} \exp\left(-\dfrac{D_{n-5}}{3f^3}\right) \exp\left(\dfrac{D_{n-4}}{3\cdot 2f^2}\right) \exp\left(-\dfrac{D_{n-3}}{3\cdot 2f}\right) & \\
 & \qquad \exp\left(\dfrac{G_{n-2} + F_{n-1} f}{3\cdot 2}\right), & \\[2ex]
\cdots & \cdots\cdots & \\
(1a) & H_1 = f^{(-1)^{n-2}\dfrac{\delta_{n-2}}{(n-3)!}} \exp\left(-\dfrac{D_1}{(n-3)f^{n-3}}\right) & \\
 & \qquad \exp\left(\dfrac{D_2}{(n-3)(n-2)f^{n-2}}\right) \exp\left(\dfrac{-D_3}{(n-3)(n-2)(n-1)f^{n-1}}\right) & \\
 & \qquad \cdots \quad \cdots \quad \cdots \quad \cdots & \\
 & \qquad \exp\left((-1)^{n-5}\dfrac{D_{n-5}}{(n-3)(n-2)\cdots 3f^3}\right) \exp\left((-1)^{n-4}\dfrac{D_{n-4}}{(n-3)!f^2}\right) & \\
 & \qquad \exp\left((-1)^{n-3}\dfrac{D_{n-3}}{(n-3)!f}\right) \exp\left((-1)^{n-2}\dfrac{G_{n-2} + F_{n-1} f}{(n-3)!}\right). &
\end{array}
$$

We observe that systems (na) and $(n-1,a)$ have the invariant algebraic curves $F_n = D_n + \alpha_n f = 0$ and $F_{n-1} = D_{n-1} + \frac{1}{2} F_n f = 0$, respectively.

System $(1a)$ for $\lambda = -(n-2)$ has the Darboux first integral

$$
\begin{aligned}
H_1 &= f^{(-1)^{n-2}\frac{\delta_{n-2}}{(n-3)!}} \left(\prod_{j=1}^{n-3} \exp\left((-1)^{n-j-2}\frac{D_{n-j-2}}{j.(j+1)\cdots(n-3)f^j}\right)\right) \\
&\quad \exp\left((-1)^{n-2}\frac{G_{n-2}+F_{n-1}f}{(n-3)!}\right),
\end{aligned}
$$

or equivalently, $H_1 = f^{\beta}\exp\left(g/f^{n-3}\right)$ where

$$
g/f^{n-3} = \sum_{j=1}^{n-3}(-1)^{n-j-2}\frac{D_{n-j-2}}{j.(j+1)\cdots(n-3)f^j} + (-1)^{n-2}\frac{G_{n-2}+F_{n-1}f}{(n-3)!},
$$

and $\beta = (-1)^{n-2}\frac{\delta_{n-2}}{(n-3)!}$.

Subcase $\lambda = -(n-j)$. We get

$$
\begin{array}{lll}
(na) & H_n = F_n^{\frac{1}{j}} f, & F_n = D_n + \alpha_n f, \\[2ex]
(n-1,a) & H_{n-1} = F_{n-1}^{\frac{1}{j-1}} f, & F_{n-1} = D_{n-1} + \dfrac{1}{j}F_n f, \\[2ex]
(n-2,a) & H_{n-2} = F_{n-2}^{\frac{1}{j-2}} f, & F_{n-2} = D_{n-2} + \dfrac{1}{j-1}F_{n-1} f, \\[2ex]
\cdots & \cdots\cdots & \\
(n-(j-1),a) & H_{n-(j-1)} = F_{n-(j-1)} f, & F_{n-(j-1)} = D_{n-(j-1)} + \dfrac{1}{2}F_{n-(j-2)} f, \\[2ex]
(n-j,a) & H_{n-j} = f^{-\delta_{n-j}} \exp\left(-(G_{n-j}+F_{n-(j-1)})f\right), &
\end{array}
$$

$$
(n-(j+1),a) \qquad H_{n-(j+1)} = f^{\delta_{n-j}} \exp\left(-\frac{D_{n-(j+1)}}{f}\right) \exp\left(G_{n-j}+F_{n-(j-1)}f\right),
$$

$$
\begin{aligned}
&(n-(j+2),a) \qquad H_{n-(j+2)} = f^{-\dfrac{\delta_{n-j}}{2}} \exp\left(-\frac{D_{n-(j+2)}}{2f^2}\right) \exp\left(\frac{D_{n-(j+1)}}{2f}\right) \\
&\qquad\qquad \exp\left(\frac{-(G_{n-j}+F_{n-(j-1)}f)}{2}\right),
\end{aligned}
$$

$\cdots \qquad \cdots\cdots$

$$
\begin{aligned}
(1a) \qquad H_1 = {} & f^{(-1)^{n-j}\dfrac{\delta_{n-j}}{(n-j-1)!}} \exp\left(\frac{-D_1}{(n-j-1)f^{n-j-1}}\right) \\
& \exp\left(\frac{D_2}{(n-j-1)(n-j-2)f^{n-j-2}}\right) \\
& \exp\left(\frac{-D_3}{(n-j-1)(n-j-2)(n-j-3)f^{n-j-3}}\right) \\
& \cdots \\
& \exp\left((-1)^{n-j-3}\frac{D_{n-j-3}}{(n-j-1)(n-j-2)\cdots 3f^3}\right) \\
& \exp\left((-1)^{n-j-2}\frac{D_{n-j-2}}{(n-j-1)!f^2}\right)
\end{aligned}
$$

$$
\exp\left((-1)^{n-j-1}\frac{D_{n-j-1}}{(n-j-1)!f}\right) \exp\left((-1)^{n-j}\frac{G_{n-j}+F_{n-(j-1)}f}{(n-j-1)!}\right).
$$

Therefore, system $(1a)$ for $\lambda = -(n-j)$ has the Darboux first integral

$$
\begin{aligned}
H_1 \;\; = {} & f^{(-1)^{n-j}\dfrac{\delta_{n-j}}{(n-j-1)!}} \left(\prod_{k=1}^{n-j-1} \exp\frac{(-1)^{n-j-k}D_{n-j-k}}{k(k+1)\cdots(n-j-1)f^k}\right) \\
& \exp\left((-1)^{n-j}\frac{G_{n-j}+F_{n-(j-1)}f}{(n-j-1)!}\right),
\end{aligned}
$$

or equivalently, $H_1 = f^\beta \exp\left(g/f^{n-j-1}\right)$, where

$$
g/f^{n-j-1} = \sum_{k=1}^{n-j-1} (-1)^{n-j-k}\frac{D_{n-j-k}}{k(k+1)\cdots(n-j-1)f^k} + (-1)^{n-j}\frac{G_{n-j}+F_{n-(j-1)}f}{(n-j-1)!},
$$

and $\beta = (-1)^{n-j}\frac{\delta_{n-j}}{(n-j-1)!}$. So we proved case A.

Case B: $(m+1)/(n+1) < k < (m+1)/n$. Since $f = 0$ is an invariant algebraic curve of system (nb) of degree $m - nk$, then from Proposition 1.2 we have that $k < (m - nk) + 1$ which is in contradiction with this case. Therefore system (nb) is identically equal to zero. The proof of this case follows directly of the proof of Case A for $\alpha_n = 0$. So we complete the proof of Lemma 4.10, and this also complete the proof of Theorem (4.3)(b). ∎

4.4 The examples

In the following we provide an example of a system which is Darboux integrable having a Darboux integrating factor. However, there is no Darboux first integral.

Example 4.11.

The system

$$\dot{x} = 1, \qquad \dot{y} = 2xy + y^2, \tag{4.24}$$

is Darboux integrable system because has the integrating factor $R = y^{-2}\exp(x^2)$. This result is not a new one, it appears in [12]. The authors proved that the only invariant curve of system (4.24) is the $y = 0$ and system (4.24) cannot have a Darboux first integral. Here, we prove the same using Theorem 3.1. We assume that system (4.24) has a Darboux first integral and we should get a contradiction. If system (4.24) has a Darboux first integral then this integral must be of the form (4.1). Since there is only one invariant curve $f = y = 0$ the fist integral (4.1) must be written as

$$H(x,y) = f^{\lambda}\exp\left(\frac{g}{f^n}\right), \tag{4.25}$$

with $\lambda \in \mathbb{R}$, $n \in \mathbb{N} \cup \{0\}$, $g \in \mathbb{C}[x,y]$ and g is coprime with f. If system (4.24) has the Darboux integral (4.25) then this system must be given by Theorem 3.1. We have that system (4.24) is quadratic, so we have $m = 2$. Let $l = \max\{\delta g, n\}$ and from Theorem 3.1 we have that $l \geq 2$.

If $l = 2$, then system (4.24) is of the normal form (3.2). So it must be written as

$$\dot{x} = -\lambda y^n + ng - yg_y, \qquad \dot{y} = yg_x. \tag{4.26}$$

Since systems (4.24) and (4.26) must be the same we get that $2x+y=g_x(x,y)$ or equivalently $g(x,y)=x^2+xy+G(y)$. Additionally, we have that $-\lambda y^n+ng-yg_y=1$ or equivalently $-\lambda y^n+n(x^2+xy+G(y))-(x+G'(y))y=1$ and so $-\lambda y^n+(n-1)xy-yG'(y)+nG(y)+nx^2=1$. Therefore, we get that n must be 0 and 1 simultaneously which is a contradiction.

If $l>2$ then from Theorem 3.1(c) system (4.24) is of the normal form (4.26) dividing its components by their common divisor $D(x,y)$, so

$$-\lambda y^n+ng-yg_y=D(x,y), \qquad yg_x=D(x,y)y(2x+y),$$

and therefore $g_x=(-\lambda y^n+ng-yg_y)(2x+y)$. Let $g(x,y)=g_0(x)+yg_1(x)+\cdots y^k g_k(x)$. Then

$$\begin{aligned} g_0'(x)+yg_1'(x)+\cdots+y^k g_k'(x) &= \big(-\lambda y^n-y(g_1(x)+2yg_2(x)+\cdots+ky^{k-1}g_k(x)) \\ &\quad +n(g_0(x)+yg_1(x)+\cdots y^k g_k(x))\big)(2x+y), \end{aligned} \tag{4.27}$$

and we get $g_0'(x)=2ng_0(x)x$. Hence it must be $g_0=C\exp(nx^2)$.

If $n\neq 0$ we get that $C=0$. Therefore, $g_0(x)=0$ which means that the polynomial g is not coprime with y. But this is a contradiction.

If $n=0$ then $g_0=C$. Now, we use the same arguments for computing $g_i(x)$ from (4.27) then and we get recursively that $g_i=0$ for all $i=1,\cdots,k$. So, $g(x)=C$ and this is a contradiction. Hence, system (4.24) cannot have a Darbouxian integral. It has a Liouvilian first integral. We provide an example satisfying Theorem 4.1.

Example 4.12.

The polynomial vector field $X=(x(y+1),-y(x+1))$ with $m=2$ has the invariant algebraic curve $f_1=x$ with cofactor $K_1=y+1$, the exponencial factor $\exp(x+y+1)$ with cofactor $L=x-y=-\mathrm{div}$, and the weak independent singular point $(-1,-1)$ which is not on $f_1=0$. Therefore, $l=1$, $p=q=r=1$, and consequently it satisfies $p+q+r=m(m+1)/2=3$ and $l+\sum_{i=1}^{p}\delta f_i=2<m+1=3$, and it has $f_1^0\exp(x+y+1)$ as integrating factor. Hence, X is an example of a polynomial vector field satisfying Theorem 4.1. We note that, from Theorem

1.7(a), there does not exist a first integral given by a Darboux function of the form $f_1^{\lambda_1}\exp(x+y+1)^{\mu_1}$. ∎

Now we present an example of Theorem 4.2.

Example 4.13.

We are interested to construct a vector field Y given in Theorem 4.2 and defined by (4.3) and and has the integrating factor $R=(x^2+y^2-1)^3(x-1)^{-1}\exp\left(y/(x-1)^2\right)$. We have $f_1=x-1$, $f_2=x^2+y^2-1$, $g=y$, $n_1=2$, $n_2=0$, $\mu_1=-1$ and $\mu_2=3$. So, $\lambda_1=\mu_1+n_1+1=2$ and $\lambda_2=\mu_2+n_2+1=4$ and

$$H(x,y)=f_1^{\lambda_1}f_2^{\lambda_2}\exp g/(f_1^{n_1}f_2^{n_2})=(x-1)^2(x^2+y^2-1)^4\exp\frac{y}{(x-1)^2}.$$

Let $X=(x_p,x_q)$ be the vector field given by (3.2) which has H as a first integral. Since $n_2=0$ we have,

$$\begin{aligned}x_p &= f_1^{n_1}(-\lambda_1 f_2 f_{1y}-\lambda_2 f_1 f_{2y})+n_1 g f_2 f_{1y}-f_1 f_2 g_y,\\ x_q &= f_1^{n_1}(\lambda_1 f_2 f_{1x}+\lambda_2 f_1 f_{2x})-n_1 g f_2 f_{1x}+f_1 f_2 g_x,\end{aligned}$$

or equivalently,

$$\begin{aligned}x_p &= -8yx^3+24yx^2-24yx+8y-x^3-xy^2+x+x^2+y^2-1,\\ x_q &= 10x^4+2x^2y^2+24x^2-28x^3-4xy^2-4x+2y^2-2yx^2-2y^3+2y-2.\end{aligned}$$

We now need to construct a vector field (A,B) such that $(AH)_x+(BH)_y=0$. Consider

$$H_2(x,y)=f_1^{\mu_1+2n_1+2}f_2^{\mu_2+2n_2+2}\exp\frac{g}{f_1^{n_1}f_2^{n_2}}=(x-1)^5(x^2+y^2-1)^5\exp\frac{y}{(x-1)^2},$$

and take (A,B) to be the vector field of the form (3.2). Hence, since $n_2=0$ we obtain

$$\begin{aligned}A &= f_1^{n_1}(-(\mu_1+2n_1+2)f_2 f_{1y}-(\mu_2+2)f_1 f_{2y})+n_1 g f_2 f_{1y}-g_1 f_2 g_y,\\ B &= f_1^{n_1}((\mu_1+2n_1+2)f_2 f_{1x}+(\mu_2+2)f_1 f_{2x})-n_1 g f_2 f_{1x}+g_1 f_2 g_x,\end{aligned}$$

or equivalently,

$$\begin{aligned}A &= -10yx^3+30yx^2-30yx+10y-x^3-xy^2+x+x^2+y^2-1,\\ B &= 15x^4+5x^2y^2+30x^2-40x^3-10xy^2+5y^2-5-2yx^2-2y^3+2y.\end{aligned}$$

Therefore, for these specific A and B the vector field $Y = (y_p,\ y_q)$ having $R(x, y)$ as an integrating factor and defined by relation (4.3) is

$$y_p = (A + C_y) f_1^{n_1} f_1 f_2 - C x_p, \qquad y_q = (B - C_x) f_1^{n_1} f_1 f_2 - C x_q,$$

for an arbitrary polynomial C. For exemple take $C = x^3 - y + 1$ and we get

$$\begin{aligned} y_p \ = \ & -1 + y - 132yx^6 + 116yx^5 + 24yx^4 - 10yx^8 - 10y^3x^6 + 60yx^7 + 60y^3x^5 \\ & -150y^3x^4 + 200y^3x^3 - 150y^3x^2 + 59y^3x + 10y^2 - 7x^2 + 6x - 141yx^3 \\ & -35yx - 34xy^2 - 4x^3 + 37x^2y^2 + 12x^4 - 9y^3 + 4xy^4 + 4x^3y^4 - 10x^3y^2 \\ & +8x^5y^2 - 9x^4y^2 - 3x^6 - x^8 - 6x^5 + 4x^7 - 6x^2y^4 - 2x^6y^2 - x^4y^4 - y^4 \\ & +117yx^2, \end{aligned}$$

$$\begin{aligned} y_q \ = \ & -3 - 2y + 6yx^6 - 2yx^7 - 4y^3x^5 + 12y^3x^4 - 6y^3x^3 - 6y^3x^2 + 8y^3x + 10y^2 \\ & -2y^5x^3 + 6y^5x^2 - 6y^5x + 2y^5 + 19x - 20yx^3 + 28yx^2 - 10yx - 36xy^2 \\ & +29x^2y^2 + 198x^4 + 25xy^4 + 5x^5y^4 + 50x^3y^4 + 89x^3y^2 + 235x^5y^2 + 20x^7y^2 \\ & -237x^4y^2 - 103x^6 - 85x^8 - 100x^5 + 167x^7 + 15x^9 - 50x^2y^4 - 110x^6y^2 \\ & -25x^4y^4 - 7y^4 - 7x^2 - 101x^3 - 101x^3. \end{aligned}$$

Now we present two examples of Theorem 4.3.

Example 4.14.

Let $R = (x^2 + y^2 - 1)^{\frac{1}{3}}$. We are interesting to construct all the polynomial vector fields of degree $m = 3$ having the Darboux integrating factor R. We have that the curve $f = x^2 + y^2 - 1$ satisfies all the conditions of Theorem 4.3. We have that $k = 2$ and $k = (m + 1)/2$ so we are in Case 3 of Theorem 4.3. System (1a) is of the form

$$\dot{x} = A_1 f - D_1 f_y, \qquad \dot{y} = B_1 f + D_1 f_x,$$

with $\delta A_1, \delta B_1 \le 1$ and $\delta D_1 \le 2$. System (1b) of degree $m - k = 1$ is of the form

$$(\lambda + 1)A_1 + D_{1y} = -\alpha f_y, \qquad (\lambda + 1)B_1 - D_{1x} = \alpha f_x,$$

for some $\alpha \in \mathbb{C}$ and $\lambda = 1/3$. Following the proof of Case 3 of Theorem 4.3 we get that

$$A_1 = \frac{1}{\lambda + 1}(-\alpha f_y - D_{1y}), \qquad B_1 = \frac{1}{\lambda + 1}(-\alpha f_x + D_{1x}),$$

or equivalently,

$$A_1 = -\frac{3}{2}\alpha y - \frac{3}{4}D_{1y}, \qquad B_1 = \frac{3}{2}\alpha x + \frac{3}{4}D_{1x}.$$

Substituting A_1 and B_1 into system $(1a)$ we get

$$\begin{aligned} \dot{x} &= \left(-\frac{3}{2}\alpha y - \frac{3}{4}D_{1y}\right)(x^2+y^2-1) - 2yD_1, \\ \dot{y} &= \left(\frac{3}{2}\alpha x + \frac{3}{4}D_{1x}\right)(x^2+y^2-1) + 2xD_1, \end{aligned} \tag{$1a'$}$$

and these family describes all the polynomial systems having $R = (x^2+y^2-1)^{\frac{1}{3}}$ as Darboux integrating factor. Additionally, due to the form of the system $(1a')$ it appears the algebraic curve

$$F = D_1 + \frac{\alpha}{\lambda+2}f = D_1 + \frac{3}{7}\alpha(x^2+y^2-1),$$

and $\delta f + \delta F = m+1$. Hence, system $(1a')$ can be written into the form

$$\begin{aligned} \dot{x} &= -\frac{3}{4}\left(D_{1y} + \frac{6}{7}\alpha y\right)(x^2+y^2-1) - 2\left(D_1 + \frac{3}{7}\alpha(x^2+y^2-1)\right)y, \\ \dot{y} &= \frac{3}{4}\left(D_{1x} + \frac{6}{7}\alpha x\right)(x^2+y^2-1) + 2\left(D_1 + \frac{3}{7}\alpha(x^2+y^2-1)\right)x, \end{aligned}$$

which is the form (4.4) and has the Darboux first integral

$$H(x,y) = \left(D_1 + \frac{3}{7}\alpha(x^2+y^2-1)\right)^{\frac{3}{4}}(x^2+y^2-1).$$

Now we present a particular system of degree 3 having the integrating factor $R = (x^2+y^2-1)^{\frac{1}{3}}$.

The system

$$\begin{aligned} \dot{x} &= -18yx^2 - 14y^3 + 14y - \frac{3}{4}x^3 - \frac{11}{4}xy^2 + \frac{3}{4}x, \\ \dot{y} &= 21x^3 + 17xy^2 - 17x + \frac{11}{4}yx^2 + \frac{3}{4}y^3 - \frac{3}{4}y, \end{aligned} \tag{4.28}$$

has the algebraic curve $f = x^2+y^2-1 = 0$ invariant and has the integrating factor $R = f^{\frac{1}{3}}$. System (4.28) can be written as

$$\begin{aligned}\dot{x} &= (-12y - \frac{3}{4}x)(x^2+y^2-1) - 2(3x^2+y^2+xy-1)y,\\ \dot{y} &= (15x + \frac{3}{4}y)(x^2+y^2-1) + 2(3x^2+y^2+xy-1)x,\end{aligned}$$

where we detect that $A_1 = -12y - \frac{3}{4}x$, $B_1 = 15x + \frac{3}{4}y$ and $D_1 = 3x^2+y^2+xy-1$. We observe that for $\lambda = 1/3$ and $\alpha = 7$, we have that relations $A_1 = (-\alpha f_y - D_{1y})/(\lambda+1)$ and $B_1 = (\alpha f_x + D_{1x})/(\lambda+1)$ hold, and the curve

$$F = D_1 + \frac{1}{\lambda+2}\alpha(x^2+y^2-1) = 6x^2+4y^2+xy-4$$

of degree $\delta F = 2$ is invariant for system (4.28). We observe that $\delta f + \delta F = m+1$. System (4.28) can be rewritten into the form

$$\begin{aligned}\dot{x} &= -\frac{3}{4}(x^2+y^2-1)(x+8y) - 2(6x^2+4y^2+xy-4)y,\\ \dot{y} &= \frac{3}{4}(x^2+y^2-1)(12x+y) + 2(6x^2+4y^2+xy-4)x,\end{aligned}$$

and obviously it is in the normal form (4.4) and has the Darboux first integral

$$H(x,y) = (6x^2+4y^2+xy-4)^{\frac{3}{4}}(x^2+y^2-1).$$

Example 4.15.

Let $R = 1/(x^4+y^4-1)^2$. We are interesting to construct all the polynomial vector fields of degree $m = 8$ that are having the Darboux integrating factor $R = 1/(x^4+y^4-1)^2$. We have that the curve $f = x^4+y^4-1$ satisfies all the conditions of Theorem 4.3. We have that $k = 4$ and $(m+1)/3 < k < (m+1)/4$ and so we are in Case 4 of Theorem 4.3 for $\lambda = -2$. System $(2a)$ is of the form

$$\dot{x} = A_2 f - D_2 f_y, \qquad \dot{y} = B_2 f + D_2 f_x,$$

with $\delta A_2, \delta B_2 = m - 2k = 0$ so we can take $A_2 = a_2 \in \mathbb{C}$ and $B_2 = b_2 \in \mathbb{C}$. We also have that $\delta D_2 \leq m - 2k + 1 = 1$. Following the proof of Case 4 of Theorem 4.3 we get that $D_2 = \delta_2$ for some $\delta_2 \in C$ and we note that in this case relation

$A_{2x} + B_{2y} = 0$ do hold. We define $G_2 = b_2x - a_2y + c_2$ with $c_2 \in \mathbb{C}$ the polynomial satisfying $A_2 = -G_{2y}$ and $B_2 = G_{2x}$. System $(2a)$ becomes

$$\dot{x} = -a_2(x^2 + y^2 - 1) - 4\delta_2 y, \qquad \dot{y} = b_2(x^2 + y^2 - 1) + 4\delta_2 x,$$

and has the invariant exponential factor $\exp(G_2) = \exp(b_2x - a_2y + c_2)$, the integrating factor $R_2 = 1/(x^4 + y^4 - 1)$ and the Darboux first integral $H_2(x, y) = (x^4 + y^4 - 1)^{\delta_2} \exp(b_2x - a_2y + c_2)$. System $(1a)$ is of the form

$$\dot{x} = A_1 f - D_1 f_y, \qquad \dot{y} = B_1 f + D_1 f_x,$$

with $\delta A_1, \delta B_1 \leq 4$ and $\delta D_1 \leq 5$. Following the proof of Case 4 of Theorem 4.3 we get that

$$A_1 = G_{2y} + \delta_2 f_y + D_{1y}, \qquad B_1 = -G_{2x} - \delta_2 f_x - D_{1x},$$

or equivalently,

$$A_1 = -a_2(x^4 + y^4 - 1) + 4\delta_2 y^3 + D_{1y}, \qquad B_1 = -b_2(x^4 + y^4 - 1) - 4\delta_2 x^3 - D_{1x}.$$

Hence, system $(1a)$ becomes

$$\begin{aligned} \dot{x} &= (-a_2(x^4 + y^4 - 1) + 4\delta_2 y^3 + D_{1y}(x^4 + y^4 - 1) - 4y^3 D_1, \\ \dot{y} &= (-b_2(x^4 + y^4 - 1) - 4\delta_2 x^3 - D_{1x}(x^4 + y^4 - 1) + 4x^3 D_1, \end{aligned} \tag{1a'}$$

and this family describes all the polynomial systems of degree $m = 8$ having the Darboxian integrating factor $R = 1/(x^4 + y^4 - 1)^2$. Additionally, system $(1a')$ has the invariant exponential factors $\exp(-b_2x + a_2y - c_2)$, $\exp(-D1(x, y)/(x^4 + y^4 - 1))$ and the Darboux first integral

$$H(x, y) = (x^4 + y^4 - 1)^{-\delta_2} \exp(-b_2x + a_2y - c_2) \exp \frac{-D1(x, y)}{x^4 + y^4 - 1}.$$

We also note that $\delta f + \delta(-b_2x + a_2y - c_2) + \delta(-D1(x, y)/(x^4 + y^4 - 1)) = 9 = m + 1$. Additionally, for

$$\frac{g}{f} = \frac{(-b_2x + a_2y - c_2)(x^4 + y^4 - 1) - D_1(x, y)}{x^4 + y^4 - 1},$$

$\beta = \delta_2$ and $n_1 = 1$ and doing a simple calculation we get that system $(1a')$ takes the form (4.5).

Now we present a particular system which has the Darboux integrating factor $R = 1/(x^4 + y^4 - 1)$.

The system

$$\begin{aligned} \dot{x} &= 3x^8 + x^4y^4 - 6x^4 + 2y^8 - y^4 + 3 - 4y^3x^4 - 4y^7 - 4y^3x^5, \\ \dot{y} &= -3x^8 - 9x^4y^4 + 9x^4 - 2y^8 + 4y^4 - 2 + 4x^7 + 4x^3y^4 - 4x^3y^5, \end{aligned} \tag{4.29}$$

has the integrating factor $R = 1/(x^4 + y^4 - 1)^2$. So, we have that $f = x^4 + y^4 - 1$, $k = 4$, $\lambda = -2$ and $m = 8$. We notice that $(m+1)/3 < k < (m+1)/2$ and therefore we are in Case 4 of the proof of Theorem 4.3.

Doing a simple calculation we get that system (4.29) can be rewritten into the form $(1a)$

$$\begin{aligned} \dot{x} &= (3x^4 - 2y^4 - 4y^3 - 3)(x^4 + y^4 - 1) - 4(x^5 - y^5 + 1)y^3, \\ \dot{y} &= (-7x^4 - 2y^4 + 4x^3 + 2)(x^4 + y^4 - 1) + 4(x^5 - y^5 + 1)x^3, \end{aligned}$$

from where we detect that $A_1 = 3x^4 - 2y^4 - 4y^3 - 3$, $B_1 = -7x^4 - 2y^4 + 4x^3 + 2$ and $D_1 = x^5 - y^5 + 1$. According to the proof of Case 4 of Theorem 4.3 we have that

$$\begin{aligned} A &= G_{2y}f + \delta_2 f_y + D_{1y} = 3x^4 - 2y^4 - 4y^3 - 3, \\ B &= -G_{2x}f - \delta_2 f_x - D_{1x} = -7x^4 - 2y^4 + 4x^3 + 2, \end{aligned}$$

and so we get $G_2 = 2x + 3y + 1$. Since $A_2 = -G_{2y} = -3$, $B_2 = G_{2x} = 2$ and $D_2 = \delta_2 = -1$ we have that system $(2a)$ can be written into the form

$$A_2 f - D_2 f_y = 4y^3 - 3x^4 - 3y^4 + 3, \qquad B_2 f + D_2 f_x = -4x^3 + 2x^4 + 2y^4 - 2,$$

and has the integrating factor $R_2 = 1/(x^4 + y^4 - 1)$ and the Darboux first integral $H_2 = \exp(2x + 3y + 1)/(x^4 + y^4 - 1)$.

According to the proof of Case 4 of Theorem 4.3 system $(1a)$ can be rewritten into the form

$$\begin{aligned} (\delta_2 f_y + G_{2y}f)f + D_{1y}f - D_1 f_y &= (3x^4 + 3y^4 - 3 + 4\delta_2 y^3 + D_{1y})(x^4 + y^4 - 1) - 4D1y^3, \\ -(\delta_2 f_y + G_{2y}f)f - D_{1y}f + D_1 f_y &= (-2x^4 - 2y^4 + 2 - 4\delta_2 x^3 - D_{1x}(x^4 + y^4 - 1) + 4D_1x^3, \end{aligned}$$

and has the two invariant exponential factors $\exp(G_2) = \exp(-2x - 3y - 1)$, $\exp(D_1/f) = \exp\left(\frac{-(x^5 - y^5 + 1)}{x^4 + y^4 - 1}\right)$ and the Darboux first integral

$$H(x, y) = (x^4 + y^4 - 1)\exp(-2x - 3y - 1)\exp\frac{-(x^5 - y^5 + 1)}{x^4 + y^4 - 1}.$$

We note that the previous first integral could be calculated by relation (1.4)

$$H(x,y) = \int R(x,y)P(x,y)dy + f(x),$$

with the condition $H_x = -RQ$. However, sometimes it is not easy to get the complete expression of that integral. Our method provides a simplest way, (it is more algebraic) to calculate the first integrals whenever the conditions of Theorem 4.3 hold.

Chapter 5

Polynomial systems and generic Darboux integrating factors

5.1 Introduction

In this chapter we continue the study of the inverse problem analyzed in Chapter 4; i.e. given some kinds of Darboux function we characterize the polynomial vector fields which have such a function as an integrating factor.

Walcher provides in [50] the following theorem.

Theorem 5.1. *Let $f = f_1 \cdots f_p$ with $f_i \in \mathbb{C}[x, y]$ irreducible, and assume that the curve $f = 0$ has no singular points. Then $X = (P, Q)$ has the integrating factor $R = f^{-1}$ if and only if*

$$\begin{aligned} \dot{x} &= -\sum_{i=1}^{p} \alpha_i \frac{f}{f_i} f_{iy} + h_1 f, \\ \dot{y} &= \sum_{i=1}^{p} \alpha_i \frac{f}{f_i} f_{ix} + h_2 f, \end{aligned} \tag{5.1}$$

with $\alpha_i \in \mathbb{C}$ and $h = (h_1, h_2)$ is a divergence free vector field.

In the next corollary we prove a simpler version of Theorem 5.1.

Corollary 5.2. *Let $f = f_1 \cdots f_p = 0$ with $f_i \in \mathbb{C}[x, y]$ irreducible, and assume that the curve $f = 0$ has no singular points. Then $X = (P, Q)$ has the integrating factor $R = f^{-1}$ if and only if*

$$\begin{aligned} \dot{x} &= -\alpha \sum_{i=1}^{p} \frac{f}{f_i} f_{iy} - F_y f, \\ \dot{y} &= \alpha \sum_{i=1}^{p} \frac{f}{f_i} f_{ix} + F_x f, \end{aligned} \tag{5.2}$$

with $\alpha \in \mathbb{C}$ and $F \in \mathbb{C}[x, y]$.

We give the proof of Corollary 5.2 in Section 5.2.

The following Theorem is slightly related with Theorem 5.1.

Theorem 5.3. *We assume that the irreducible curves $f_1 = 0, \cdots, f_p = 0$ satisfying the generic conditions (i)-(v), i.e.*

(i) There are no points at which f_i and its first derivatives are all vanish.

(ii) The highest order terms of f_i have no repeated factors.

(iii) If two curves intersect at a point in the finite plane, they are transversal at this point.

(iv) There are no more than two curves $f_i = 0$ meeting at any point in the finite plane.

(v) There are no two curves having a common factor in the highest order terms.

Then the polynomial vector field $X = (P, Q)$ of degree m has the Darboux integrating factor $R = (f_1 \cdots f_p)^{-1}$ if and only if can be written into the following form

$$\begin{aligned} \dot{x} &= -\sum_{i=1}^{p} \alpha_i \left(\prod_{j=1}^{p} f_j \right) f_{iy} - F_y \prod_{j=1}^{p} f_j, \\ \dot{y} &= \sum_{i=1}^{p} \alpha_i \left(\prod_{j=1}^{p} f_j \right) f_{ix} + F_x \prod_{j=1}^{p} f_j, \end{aligned} \tag{5.3}$$

with $\alpha_i \in \mathbb{C}$, $F \in \mathbb{C}[x,y]$ *and* $\delta F \leq m - \sum_{i=1}^{p} \delta f_i$.

The proof of Theorem 5.3 is given in Section 5.3. We point out that Theorems 5.1 and 5.3 they use different assumptions.

Theorem 5.3 demands that every curve $f_1 = 0, \cdots, f_p = 0$ satisfy the generic condition (i), i.e. it asks that any curve $f_i = 0$ has no singular points. This is a different assumption from the one which is used in Theorem 5.1 where is demanded that the generic condition (i) holds for the product of the curves, i.e. asks that the curve $f = f_1 \cdots f_p = 0$ has no singular points. Hence, for example the curve $f = (x+y)(y+1)$ could not be used in Theorem 5.1 (because the point $(1,-1)$ is singular for $f = 0$), but the curves $f_1 = x + y$ and $f_2 = y + 1$ could be used in Theorem 5.3. Hence due to this different use of the generic condition (i) in the two theorems we remark that Theorems 5.1 and 5.3 are different.

In the following we want to characterize the polynomial vector fields in $\mathbb{C}^2$ having the following function

$$R(x,y) = f_1^{\lambda_1} \cdots f_p^{\lambda_p}, \tag{5.4}$$

as an integrating factor in the particular case where the curves $f_1, \cdots, f_p$ are generic and $\lambda_1, \cdots, \lambda_p \in \mathbb{C}$. In this case, the integrating factor (5.4) is called *generic Darboux integrating factor*. We provide this result in the next theorem.

Theorem 5.4. *We assume that the irreducible curves* $f_1 = 0, \cdots, f_p = 0$ *satisfying the generic conditions (i)–(v) and let* $k_1 = \delta f_1$ *and* $\gamma = \delta f_2 + \cdots + \delta f_p$. *We denote by* $n = [(m - \gamma + 1)/k_1]$. *If* $(m - \gamma + 1)/n \leq k_1 < (m - \gamma + 1)/(n+1)$ *with* $\lambda_1 \notin \{-1, -2, \cdots, -n\}$, *then the vector field* $X = (P, Q)$ *of degree* m *has the integrating factor of the form* $f_1^{\lambda_1} \cdots f_p^{\lambda_p}$ *if and only if*

$$\begin{aligned}
\dot{x} &= -\frac{1}{\lambda_1 + 1}\left(\prod_{i=1}^{p} f_i\right) F_y - \sum_{i=2}^{p} \frac{\lambda_i + 1}{\lambda_1 + 1} F f_1 \left(\prod_{i=2}^{p} f_i\right)_y - F\left(\prod_{i=2}^{p} f_i\right) f_{1y}, \\
\dot{y} &= \frac{1}{\lambda_1 + 1}\left(\prod_{i=1}^{p} f_i\right) F_x + \sum_{i=2}^{p} \frac{\lambda_i + 1}{\lambda_1 + 1} F f_1 \left(\prod_{i=2}^{p} f_i\right)_x + F\left(\prod_{i=2}^{p} f_i\right) f_{1x},
\end{aligned} \tag{5.5}$$

where

$$F = D_1 + \sum_{i=1}^{n-1} \frac{1}{(\lambda_1+1)\cdots(\lambda_1+i)} D_{i+1,1} {f_1}^i,$$

and the D_i, for $i = 1, \cdots, n$, are convenient polynomials given in Lemma 5.6. Moreover, $\delta F + \delta f = m + 1$ and system (5.5) has the Darboux first integral $H(x, y) = f_1^{\lambda_1+1} f_2^{\lambda_2+1} \cdots f_p^{\lambda_p+1} F$.

The proof of Theorem 5.4 is given in Section 5.4.

In Section 5.5 we comment Theorem 5.8 which is due to Walcher and we compare it with our results.

5.2 Proof of Corollary 5.2

Since $h = (h_1, h_2)$ is a divergence free vector field, we have that $h_{1x} + h_{1y} = 0$. Then, from Lemma 4.4, we have that there is $F_1 \in \mathbb{C}[x, y]$ such that $h_1 = -F_{1y}$ and $h_2 = F_{1x}$. So system (5.1) can be rewritten as

$$\begin{aligned} \dot{x} &= -\sum_{i=1}^{p} \alpha_i \frac{f}{f_i} f_{iy} - F_{1y} f, \\ \dot{y} &= \sum_{i=1}^{p} \alpha_i \frac{f}{f_i} f_{ix} + F_{1x} f. \end{aligned} \tag{5.6}$$

We should show that system (5.6) can be rewritten as (5.2). We note that

$$\left(-\sum_{i=1}^{p} (\alpha_i - \alpha) \frac{f_{iy}}{f_i}\right)_x + \left(\sum_{i=1}^{p} (\alpha_i - \alpha) \frac{f_{ix}}{f_i}\right)_y = 0,$$

and so, from Lemma 4.4, there is $F_2 \in \mathbb{C}[x, y]$ such that

$$\begin{aligned} -\sum_{i=1}^{p} (\alpha_i - \alpha) \frac{f_{iy}}{f_i} &= -F_{2y}, \\ \sum_{i=1}^{p} (\alpha_i - \alpha) \frac{f_{ix}}{f_i} &= F_{2x}. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} -\sum_{i=1}^{p} \alpha_i \frac{f_{iy}}{f_i} &= -\alpha \sum_{i=1}^{p} \frac{f_{iy}}{f_i} - F_{2y}, \\ \sum_{i=1}^{p} \alpha_i \frac{f_{ix}}{f_i} &= \alpha \sum_{i=1}^{p} \frac{f_{ix}}{f_i} + F_{2x}. \end{aligned}$$

Using the above relations, system (5.6) becomes

$$\begin{aligned} \dot{x} &= \left(-\alpha \sum_{i=1}^{p} \frac{f}{f_i} f_{iy} - F_{2y} f \right) - F_{1y} f, \\ \dot{y} &= \left(\alpha \sum_{i=1}^{p} \frac{f}{f_i} f_{ix} + F_{2x} f \right) + F_{1x} f, \end{aligned}$$

or equivalently,

$$\begin{aligned} \dot{x} &= -\alpha \sum_{i=1}^{p} \frac{f}{f_i} f_{iy} - (F_1 + F_2)_y f, \\ \dot{y} &= \alpha \sum_{i=1}^{p} \frac{f}{f_i} f_{ix} + (F_1 + F_2)_x f, \end{aligned}$$

and taking $F = F_1 + F_2$ we have shown that system (5.1) can be written as system (5.2) for some $\alpha \in \mathbb{C}$ and $F \in \mathbb{C}[x, y]$.

5.3 Proof of Theorem 5.3

We assume that the irreducible algebraic curves $f_1 = 0 \cdots, f_p = 0$ satisfies the generic conditions (i)-(v). Hence, from Theorem 2.4 we have that the polynomial vector field $X = (P, Q)$ can be written into the form (2.4), or equivalently,

$$X = Y_0 \prod_{i\,=\,1}^{p} f_i + \sum_{i\,=\,1}^{p} D_i \left(\prod_{\substack{j\,=\,1 \\ j\,\neq\,i}}^{p} f_j \right) X_{f_i}, \tag{5.7}$$

where $Y_0 = (A, B)$ and $A, B, D_i \in \mathbb{C}[x, y]$. We note that system (5.7) has divergence

$$\begin{aligned} \text{div} &= (A_{1x} + B_{1y}) \prod_{i=1}^{p} f_i + \sum_{i=1}^{p} (A_1 f_{ix} + B_1 f_{iy}) \prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \\ &+ \sum_{\substack{i=1 \\ j \neq i}}^{p} D_i (f_{ix} f_{jy} - f_{iy} f_{jx}) \prod_{\substack{k=1 \\ k \neq i,j}}^{p} f_k + \sum_{i=1}^{p} (f_{ix} D_{iy} - f_{iy} D_{ix}) \prod_{\substack{j=1 \\ j \neq i}}^{p} f_j, \end{aligned}$$

and the algebraic curve $f_i = 0$ is invariant of system (5.7) and has cofactor

$$K_i = (A_1 f_{ix} + B_1 f_{iy}) \prod_{\substack{j=1 \\ j \neq i}}^{p} f_j + \sum_{\substack{j=1 \\ j \neq i}}^{p} D_j (f_{jx} f_{iy} - f_{jy} f_{ix}) \prod_{\substack{k=1 \\ k \neq i,j}}^{p} f_k.$$

Since system (5.7) has the integrating factor $(f_1 \cdots f_p)^{-1}$ it must be

$$-\sum_{i=1}^{p} K_i = -\text{div}(P, Q),$$

or equivalently

$$\begin{aligned} 0 &= D_{1y} \left(\prod_{i=2}^{p} f_i \right) f_{1x} - D_{1x} \left(\prod_{i=2}^{p} f_i \right) f_{1y} \\ &+ \left[(A_{1x} + B_{1y}) \prod_{i=2}^{p} f_i + \sum_{i=2}^{p} (f_{ix} D_{iy} - f_{iy} D_{ix}) \prod_{\substack{j=2 \\ j \neq i}}^{p} f_j \right] f_1, \end{aligned} \tag{5.8}$$

and so for $i = 2, \cdots, p$ we get that $f_i | (f_{ix} D_{iy} - f_{iy} D_{ix})$ because f_i is irreducible. Hence, the polynomial system

$$\dot{x} = -D_{iy}, \qquad \dot{y} = D_{ix},$$

(if it is not the zero one) has the first integral D_i and the invariant curve $f_i = 0$. So, there are $E_i \in \mathbb{C}[x, y]$ and $c_i \in \mathbb{C}$ such that $E_i f_i = D_i - c_i$ for $i = 2, \cdots, p$.

Moreover, the system

$$\dot{x} = D_{1y} \prod_{i=2}^{p} f_i, \qquad \dot{y} = -D_{1x} \prod_{i=2}^{p} f_i, \tag{5.9}$$

due to relation (5.8) has the algebraic curve $f_1 = 0$ invariant and the first integral D_1. Additionally, system (5.9) has the Darboux integrating factors $(f_2 \cdots f_p)^{-1}$ and $(D_1 f_2 \cdots f_p)^{-1}$. Since D_1 is a first integral of system (5.9) and $f_1 = 0$ is an invariant curve of system (5.9) we have that there is $E_1 \in \mathbb{C}[x, y]$ such that $E_1 f_1 = D_1 - c_1$.

Hence, relation (5.8) can be rewritten as

$$\begin{aligned} 0 \; &= \; E_{1y} \left(\prod_{i=1}^{p} f_i \right) f_{1x} - E_{1x} \left(\prod_{i=1}^{p} f_i \right) f_{1y} \\ &+ \; \left[(A_{1x} + B_{1y}) \prod_{i=2}^{p} f_i + \sum_{i=2}^{p} (f_{ix} E_{iy} - f_{iy} E_{ix}) \prod_{j=2}^{p} f_j \right] f_1, \end{aligned}$$

or equivalently

$$0 \; = \; (A_{1x} + B_{1y}) \prod_{i=1}^{p} f_i + \sum_{i=1}^{p} (f_{ix} E_{iy} - f_{iy} E_{ix}) \prod_{j=1}^{p} f_j.$$

Simplifying the last relation we obtain

$$0 \; = \; A_{1x} + B_{1y} + \sum_{i=1}^{p} (f_{ix} E_{iy} - f_{iy} E_{ix}),$$

or equivalently

$$\left(A_1 - \sum_{i=1}^{p} E_i f_{iy} \right)_x + \left(B_1 + \sum_{i=1}^{p} E_i f_{ix} \right)_y = 0.$$

From Lemma 4.4 there is $F \in \mathbb{C}[x, y]$ such that

$$A_1 - \sum_{i=1}^{p} E_i f_{iy} = -F_y, \qquad B_1 + \sum_{i=1}^{p} E_i f_{ix} = F_x,$$

and so we can calculate the polynomials A_1 and B_1. Substituting A_1 and $D_1, \cdots, D_p$ into the first equation of system (5.7) we get

$$\dot{x} = \left(-F_y + \sum_{i=1}^{p} E_i f_{iy} \right) \prod_{j=1}^{p} f_j - \sum_{i=1}^{p} E_i f_{iy} \prod_{j=1}^{p} f_j - \sum_{i=1}^{p} c_i f_{iy} \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right),$$

or equivalently,

$$\dot{x} = -F_y \left(\prod_{j=1}^{p} f_j \right) - \sum_{i=1}^{p} c_i f_{iy} \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right).$$

Working in a similar way, the second equation of system (5.7) becomes

$$\dot{y} = F_y \left(\prod_{j=1}^{p} f_j \right) + \sum_{i=1}^{p} c_i f_{iy} \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right).$$

Therefore, system (5.9) is of the form (5.3) and has the Darboux first integral $H = f_1^{c_1} \cdots f_p^{c_p} \exp F$.

5.4 Proof of Theorem 5.4

In the following we assume that the curves $f_i = 0$ are generic for $i = 1, \cdots, p$ and we denote by $\gamma = \sum_{i=2}^{p} \delta f_i$ and let $k_1 = \delta f_1$ be the degree of the curve $f_1 = 0$.

By Theorem 2.4 we have that $k_1 + \gamma \leq m + 1$. Then, if $k_1 + \gamma = m + 1$, by Theorem 2.4(b) it follows Theorem 5.4 for $F = 1$. Hence, from now on we assume that $k_1 + \gamma < m + 1$.

Since the curves $f_i = 0$ for $i = 1, \cdots, p$ satisfy the conditions of Theorem 2.4(a) we have that any polynomial vector field Y having these curves as invariant algebraic curves takes the form

$$Y \;=\; Y_{01} \prod_{i=1}^{p} f_i + \sum_{i=1}^{p} D_i \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right) X_{f_i}, \qquad (1a)$$

where $Y_{01} = (A_1, B_1)$ and $A_1, B_1, D_i \in \mathbb{C}[x, y]$ with $\delta A_1, \delta B_1 \leq m - \gamma - k_1$ and $\delta D_i \leq m - \gamma - k_1 + 1$ for $i = 1, \cdots, p$. From now on we denote by

$$\begin{aligned} L_1 \;=\; & -(A_{1x} + B_{1y}) \prod_{i=2}^{p} f_i - \sum_{\substack{i=2 \\ i \neq j}}^{p} (\lambda_i + 1) D_j (f_{jx} f_{iy} - f_{jy} f_{ix}) \prod_{\substack{k=2 \\ k \neq i,j}}^{p} f_k \\ & - \sum_{i=2}^{p} (\lambda_i + 1)(A_1 f_{ix} + B_1 f_{iy}) \prod_{\substack{j=2 \\ j \neq i}}^{p} f_j - \sum_{i=2}^{p} (f_{ix} D_{iy} - f_{iy} D_{ix}) \prod_{\substack{j=2 \\ j \neq i}}^{p} f_j. \end{aligned}$$

Lemma 5.5. *Suppose that $p \geq 2$ and that f_i are irreducible polynomials in $C[x, y]$. We associate to system (1a) of degree m having the generic curves $f_1 = 0, \cdots, f_p = 0$ and the Darboux integrating factor $f_1^{\lambda_1} \cdots f_p^{\lambda_p}$ the system*

$$\begin{aligned} \tilde{Y_1} \;=\; & ((\lambda_1 + 1) Y_{01} - Y_{D_1}) \prod_{i=2}^{p} f_i - D_1 \sum_{i=2}^{p} (\lambda_i + 1) \left(\prod_{\substack{j=2 \\ j \neq i}}^{p} f_j \right) X_{f_i} \\ & + (\lambda_1 + 1) \sum_{i=2}^{p} \left(\prod_{\substack{j=2 \\ j \neq i}}^{p} f_j \right) D_i X_{f_i}, \end{aligned} \qquad (1b)$$

of degree $m - k_1$ which has $f_1 = 0, \cdots, f_p = 0$ as invariant algebraic curves. Then, only one of the following conditions holds.

- *(a) If $L_1 = 0$ then system (1b) must be the zero vector field.*
- *(b) If $L_1 \neq 0$ then system (1b) has the Darboux integrating factor $f_1^{\lambda_1+1} f_2^{\lambda_2} \cdots f_p^{\lambda_p}$. In particular, if $\lambda_1 = -1$, then system (1b) has the first integral $H = D_1 f_2^{\lambda_2+1} \cdots f_p^{\lambda_p+1}$ and the integrating factor $R_1 = (D_1 f_2 \cdots f_p)^{-1}$.*

Proof: The invariant algebraic curve $f_i = 0$ of system $(1a)$, has cofactor

$$K_i = (A_1 f_{ix} + B_1 f_{iy}) \prod_{\substack{j=1 \\ j \neq i}}^{p} f_j + \sum_{\substack{j=1 \\ j \neq i}}^{p} D_j (f_{jx} f_{iy} - f_{jy} f_{ix}) \prod_{\substack{k=1 \\ k \neq i,j}}^{p} f_k,$$

for $i = 1, \cdots, p$ respectively; and $\delta K_1, \cdots, \delta K_p \leq m - 1$.

Since system $(1a)$ has a Darboux integrating factor of the form $f_1^{\lambda_1} \cdots f_p^{\lambda_p}$ then, from Theorem 1.7(d), it must satisfy the relation

$$\sum_{i=1}^{p} \lambda_i K_i + \mathrm{div}_1 = 0, \tag{5.10}$$

where

$$\begin{aligned} \mathrm{div}_1 = {} & (A_{1x} + B_{1y}) \prod_{i=1}^{p} f_i + \sum_{i=1}^{p} (A_1 f_{ix} + B_1 f_{iy}) \prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \\ & + \sum_{\substack{i=1 \\ j \neq i}}^{p} D_i (f_{ix} f_{jy} - f_{iy} f_{jx}) \prod_{\substack{k=1 \\ k \neq i,j}}^{p} f_k + \sum_{i=1}^{p} (f_{ix} D_{iy} - f_{iy} D_{ix}) \prod_{\substack{j=1 \\ j \neq i}}^{p} f_j, \end{aligned}$$

is the divergence of system $(1a)$. Substituting $K_1, \cdots, K_p$ and div_1 in (5.10), we get that

$$\begin{aligned} 0 = {} & \left[((\lambda_1 + 1) A_1 + D_{1y}) \prod_{i=2}^{p} f_i + D_1 \sum_{i=2}^{p} (\lambda_i + 1) \left(\prod_{\substack{j=2 \\ j \neq i}}^{p} f_j \right) f_{iy} \right. \\ & \left. - (\lambda_1 + 1) \sum_{i=2}^{p} D_i \left(\prod_{\substack{j=2 \\ j \neq i}}^{p} f_j \right) f_{iy} \right] f_{1x} + \left[((\lambda_1 + 1) B_1 - D_{1x}) \prod_{i=2}^{p} f_i \right. \end{aligned}$$

$$
\begin{aligned}
&-D_1 \sum_{i=2}^{p} (\lambda_i+1)\left(\prod_{\substack{j=2\\ j\neq i}}^{p} f_j\right) f_{ix} + (\lambda_1+1)\sum_{i=2}^{p} D_i \left(\prod_{\substack{j=2\\ j\neq i}}^{p} f_j\right) f_{ix}\Bigg] f_{1y}\\
&+\Bigg[(A_{1x}+B_{1y})\prod_{i=2}^{p} f_i + \sum_{i=2}^{p} (\lambda_i+1) D_j (f_{jx}f_{iy}-f_{jy}f_{ix}) \prod_{\substack{k=2\\ k\neq i,j}}^{p} f_k\\
&+\sum_{i=2}^{p} (\lambda_i+1)(A_1 f_{ix}+B_1 f_{iy}) \prod_{\substack{j=2\\ j\neq i}}^{p} f_j + \sum_{i=2}^{p} (f_{ix}D_{iy}-f_{iy}D_{ix}) \prod_{\substack{j=2\\ j\neq i}}^{p} f_j\Bigg] f_1,
\end{aligned} \tag{5.11}
$$

where we have taken $Y_{01}=(A_1,B_1)$.

Hence, from relation (5.11), system (1b) of degree $m-k_1$ has $f_1=0$ as an invariant algebraic curve with cofactor L_1.

System (1b) has divergence equal to

$$
\begin{aligned}
\mathrm{div}_2 \;=\; & (\lambda_1+1)(A_{1x}+B_{1y})\prod_{i=2}^{p} f_i + (\lambda_1+1)\sum_{i=2}^{p} (A_1 f_{ix}+B_1 f_{iy}) \prod_{\substack{j=2\\ j\neq i}}^{p} f_j\\
&+ \sum_{i=2}^{p} \lambda_i (D_{1x}f_{iy}-D_{1y}f_{ix}) \prod_{\substack{j=2\\ j\neq i}}^{p} f_j\\
&+ D_1 \sum_{2\le i<k\le p} (\lambda_i-\lambda_k)(f_{kx}f_{iy}-f_{ky}f_{ix}) \prod_{\substack{j=2\\ j\neq i,k}}^{p} f_j\\
&- (\lambda_1+1)\sum_{i=2}^{p} (D_{ix}f_{iy}-D_{iy}f_{ix}) \prod_{\substack{j=2\\ j\neq i}}^{p} f_j\\
&- (\lambda_1+1)\sum_{\substack{i=2\\ i\neq j}}^{p} D_j (f_{ix}f_{jy}-f_{iy}f_{jx}) \prod_{\substack{k=2\\ k\neq i,j}}^{p} f_k.
\end{aligned}
$$

The algebraic curves $f_i = 0$, for $i = 2, \cdots, p$, are also invariant for system (1b) and have cofactors

$$
\begin{aligned}
L_i \;=\; & (\lambda_1 + 1)(A_1 f_{ix} + B_1 f_{iy}) \prod_{\substack{j=2 \\ j \neq i}}^{p} f_j + (D_{1y} f_{ix} - D_{1x} f_{iy}) \prod_{\substack{j=2 \\ j \neq i}}^{p} f_j \\
& -(\lambda_1 + 1) \sum_{\substack{j=2 \\ j \neq i}}^{p} D_j (f_{ix} f_{jy} - f_{iy} f_{jx}) \prod_{\substack{k=2 \\ k \neq i,j}}^{p} f_k \\
& +D_1 \sum_{\substack{j=2 \\ j \neq i}}^{p} (\lambda_j + 1)(f_{ix} f_{jy} - f_{iy} f_{jx}) \prod_{\substack{k=2 \\ k \neq i,j}}^{p} f_k .
\end{aligned}
$$

(a) If $L_1 = 0$ then we have that f_1 is a first integral for system (1b). Note that we cannot guarantee that system (1b) has its two components coprime. The algebraic curves $f_2 = 0, \cdots f_p = 0$ are also invariant for system (1b). Hence, for $i = 2, \cdots, p$ we have that $\{f_i = 0\} \subseteq \{f_1 - c = 0\}$ for some $c \in \mathbb{C}$. Therefore, from the Hilbert's Nullstellensatz relation we get that there exist a non negative integer N and $M \in \mathbb{C}[x, y]$ such that $(f_1 - c)^N = M f_i$. Since f_i are irreducible polynomials we get that $f_i | (f_1 - c)$ for all $i = 2, \cdots, p$. So $f_1 - c = A_i f_i$ for some $A_i \in \mathbb{C}[x, y]$, and this is a contradiction with the generic condition (v). Then, system (1b) must be the zero vector field. So, statement (a) is proved.

(b) We now assume that $L_1 \neq 0$. Then, it is easy to check that relation

$$(\lambda_1 + 1) L_1 + \sum_{i=2}^{p} \lambda_i L_i + \operatorname{div}_2 = 0,$$

always holds. Hence, system (1b) has the Darboux integrating factor $f_1^{\lambda_1+1} f_2^{\lambda_2} \cdots f_p^{\lambda_p}$.

If $\lambda_1 = -1$ then system (1b) is of the normal form (3.2) and has also the invariant curve $D_1 = 0$. Hence, due to Theorem 3.1(a) statement (b) follows directly. ∎

Proof of Theorem 5.4: Let $f_1 = 0$ be a generic curve of system (1b) of degree $m - k_1$. Since system (1b) has $f_2 = 0, \cdots, f_p = 0$ as invariant algebraic curves, then from Proposition 1.2, we have that $k_1 + \gamma \leq m - k_1 + 1$ and so $k_1 \leq (m + 1 - \gamma)/2$. We distinguish the following cases.

Case 1: $(m-\gamma+1)/2 < k_1 < m-\gamma+1$. Then, system (1b) must be identically equal to zero, i.e.

$$0 = ((\lambda_1+1)A_1 + D_{1y}) \prod_{i=2}^{p} f_i - \sum_{i=2}^{p} ((\lambda_1+1)D_i - (\lambda_i+1)D_1) \left(\prod_{\substack{j=2 \\ j\neq i}}^{p} f_j \right) f_{iy}. \tag{5.12}$$

Since f_i is irreducible we get that

$$f_i | [(\lambda_1+1)D_i - (\lambda_i+1)D_1].$$

Hence, there are $E_{1i} \in \mathbb{C}[x,y]$ such that

$$(\lambda_1+1)D_i - (\lambda_i+1)D_1 = E_{1i} f_i, \tag{5.13}$$

for all $i = 2, \cdots, p$. Since $\lambda_1 \neq -1$ then from relation (5.12) we get that

$$A_1 = -\frac{1}{\lambda_1+1} D_{1y} + \frac{1}{\lambda_1+1} \left(\sum_{i=2}^{p} E_{1i} f_{iy} \right),$$

and so the first equation of system $(1a)$ can be written into the following form

$$\dot{x} = \frac{-1}{\lambda_1+1} \left(\prod_{i=1}^{p} f_i \right) D_{1y} - D_1 \left(\prod_{i=2}^{p} f_i \right) f_{1y} - \sum_{i=2}^{p} \frac{\lambda_i+1}{\lambda_1+1} D_1 \left(\prod_{\substack{j=1 \\ j\neq i}}^{p} f_j \right) f_{iy}.$$

Working in a similar way with the second equation system $(1a)$ becomes

$$Y = \frac{1}{\lambda_1+1} \left(\prod_{i=1}^{p} f_i \right) Y_{D_1} + D_1 \left(\prod_{i=2}^{p} f_i \right) X_{f_1} + \sum_{i=2}^{p} \frac{\lambda_1+1}{\lambda_i+1} D_1 \left(\prod_{\substack{j=1 \\ j\neq i}}^{p} f_j \right) X_{f_i}.$$

System $(1a)$ has the additional invariant algebraic curve $F = D_1 = 0$ with $\delta F = m - \gamma - k_1 + 1$ and it is in the normal form (5.5). Additionally, it has the Darboux first integral $H = f_1 f_2^{\frac{\lambda_2+1}{\lambda_1+1}} \cdots f_p^{\frac{\lambda_p+1}{\lambda_1+1}} F^{\frac{1}{\lambda_1+1}}$.

Case 2: $k_1 = (m-\gamma+1)/2$. Since $f_1 = 0, \cdots f_p = 0$ are invariant algebraic curves for system (1b) from Proposition 1.2 we have that $k_1 + \gamma \leq m - k_1 + 1$ and, consequently in this case we get that $k_1 + \gamma = m - k_1 + 1$. Hence, by Theorem 2.4(b) there are $\alpha_i \in \mathbb{C}$ such that system (1b) can be written into the form

$$((\lambda_1+1)Y_{01} - Y_{D_1}) \prod_{i=2}^{p} f_i - \sum_{i=2}^{p} ((\lambda_1+1)D_i - (\lambda_i+1)D_1) \left(\prod_{\substack{j=2 \\ j \neq i}}^{p} f_j \right) X_{f_i}$$
$$= \sum_{i=1}^{p} \alpha_i \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right) X_{f_i}, \tag{5.14}$$

and so because of his form (see also from Proposition 2.5) system ($1b$) has the integrating factor $R_1 = (f_1 \cdots f_p)^{-1}$.

If $L_1 = 0$ then applying Lemma 5.5(a) we get that system (1b) must be the zero vector field and this can be studied by a similar way to Case 1.

If $L_1 \neq 0$ then from Lemma 5.5(b) we have that system ($1b$) has the integrating factor $R_2 = f_1^{\lambda_1+1} f_2^{\lambda_2} \cdots f_p^{\lambda_p}$. Hence, it has the first integral $H_2 = R_2/R_1 = f_1^{\lambda_1+2} f_2^{\lambda_2+1} \cdots {f_p}^{\lambda_p+1}$. Therefore, without loss of generality we can take $\alpha_1 = \lambda_1+2$ and $\alpha_i = \lambda_i + 1$ for $i = 2, \cdots, p$. Then, the first equation of system (1b) can be rewritten as

$$((\lambda_1+1)A_1 + D_{1y}) \prod_{i=2}^{p} f_i - \sum_{i=2}^{p} ((\lambda_1+1)D_i - (\lambda_i+1)D_1) \left(\prod_{\substack{j=2 \\ j \neq i}}^{p} f_j \right) f_{iy}$$
$$= -(\lambda_1+2) \left(\prod_{\substack{j=2 \\ j \neq i}}^{p} f_j \right) f_{1y} - \sum_{i=2}^{p} (\lambda_i+1) \left(\prod_{\substack{j=2 \\ j \neq i}}^{p} f_j \right) f_{1y}. \tag{5.15}$$

Since f_i is irreducible for every $i = 1, \cdots, p$ we get that

$$f_i | [(\lambda_1 + 1)D_i - (\lambda_i + 1)D_1 - (\lambda_i + 1)f_1].$$

Hence, there are $E_{1i} \in \mathbb{C}[x, y]$ such that

$$(\lambda_1 + 1)D_i - (\lambda_i + 1)D_1 - (\lambda_i + 1)f_1 = E_{1i} f_i, \tag{5.16}$$

for all $i = 2, \cdots, p$. Since $\lambda_1 \neq -1$ then from relation (5.15) we get that

$$A_1 = -\frac{1}{\lambda_1 + 1} D_{1y} + \frac{1}{\lambda_1 + 1} \sum_{i=2}^{p} E_{1i} f_{iy} - \frac{\lambda_1 + 2}{\lambda_1 + 1} f_{1y},$$

and so the first equation of system ($1a$) can be written into the following form

$$\begin{aligned} \dot{x} &= \frac{-1}{\lambda_1 + 1} \left(\prod_{i=1}^{p} f_i \right) D_{1y} - D_1 \left(\prod_{i=2}^{p} f_i \right) f_{1y} - \frac{\lambda_1 + 2}{\lambda_1 + 1} \left(\prod_{i=2}^{p} f_i \right) f_{1y} \\ &\quad - \sum_{i=2}^{p} \frac{\lambda_i + 1}{\lambda_1 + 1} D_1 \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right) f_{iy} - \sum_{i=2}^{p} \frac{\lambda_i + 1}{\lambda_1 + 1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right) f_{iy}. \end{aligned}$$

Working in a similar way with the second equation system ($1a$) becomes

$$\begin{aligned} Y &= \frac{1}{\lambda_1 + 1} \left(\prod_{i=1}^{p} f_i \right) Y_{D_1} + D_1 \left(\prod_{i=2}^{p} f_i \right) X_{f_1} + \frac{\lambda_1 + 2}{\lambda_1 + 1} \left(\prod_{i=2}^{p} f_i \right) X_{f_1} \\ &\quad + \sum_{i=2}^{p} \frac{\lambda_i + 1}{\lambda_1 + 1} D_1 \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right) X_{f_i} + \sum_{i=2}^{p} \frac{\lambda_i + 1}{\lambda_1 + 1} D_1 \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right) X_{f_i}. \end{aligned}$$

We note that system ($1a$) has the additional invariant algebraic curve $F = D_1 + f_1 = 0$ and $\delta F = m - \gamma - k_1 + 1$. We also note that system ($1a$) can be rewritten into the normal form (5.5) and it has the Darboux first integral $H = f_1 f_2^{\frac{\lambda_2+1}{\lambda_1+1}} \cdots f_p^{\frac{\lambda_p+1}{\lambda_1+1}} F^{\frac{1}{\lambda_1+1}}$.

Case 3: $(m-\gamma+1)/3 < k_1 < (m-\gamma+1)/2$. If $L_1 = 0$ then, from Lemma 5.5(a) we get that system $(1b)$ is identically equal to zero so that case has been studied in Case 1. Therefore we assume that $L_1 \neq 0$. Since system (1b) of degree $m-k_1$ has the generic invariant curves $f_i = 0$ for $i = 1, \cdots, p$, by Theorem 2.4(a) it can be written as

$$Y_2 \quad = \quad Y_{02} \prod_{i=1}^{p} f_i + \sum_{i=1}^{p} D_{2i} \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right) X_{f_i}, \qquad (2a)$$

with $Y_{02} = (A_2, B_2)$ where $A_2, B_2, D_{2i} \in \mathbb{C}[x,y]$ with $\delta A_2, \delta B_2 \leq m-\gamma-2k_1$ and $\delta D_{2i} \leq m-\gamma-2k_1+1$ for $i = 1, \cdots, p$. From Lemma 5.5(b) we have that system (1b) has the Darboux integrating factor $f_1^{\lambda_1+1} f_2^{\lambda_2} \cdots f_p^{\lambda_p}$, and so system $(2a)$ which is the same as system (1b) has that integrating factor. Then, applying again Lemma 5.5(b) to system $(2a)$ we can associate to system $(2a)$ the system

$$\tilde{Y}_2 = ((\lambda_1+2)Y_{02} - Y_{D_{21}}) \prod_{i=2}^{p} f_i + \sum_{i=2}^{p} ((\lambda_1+2)D_{2i} - (\lambda_i+1)D_{21}) \left(\prod_{\substack{j=2 \\ j \neq i}}^{p} f_j \right) X_{f_i}, \quad (2b)$$

which has $f_i = 0$, for $i = 1, \cdots, p$, as invariant algebraic curves. We note that system $(2b)$ has degree $m-2k_1$. From Proposition 1.2, we get that $k_1 \leq m-2k_1+1$ and so $k_1 \leq (m+1)/3$ which is in contradiction with Case 3. Hence, system $(2b)$ is identically zero. Therefore, we have that

$$0 \quad = \quad ((\lambda_1+2)A_2 + D_{21y}) \prod_{i=2}^{p} f_i - \sum_{i=2}^{p} ((\lambda_1+2)D_{2i} - (\lambda_i+1)D_{21}) \left(\prod_{\substack{j=2 \\ j \neq i}}^{p} f_j \right) f_{iy}.$$

Since $\lambda_1 \neq -2$ we get

$$A_2 = -\frac{1}{\lambda_1+2} D_{21y} + \frac{1}{\lambda_1+2} \frac{\displaystyle\sum_{i=2}^{p} ((\lambda_1+2)D_{2i} - (\lambda_i+1)D_{21}) \left(\prod_{\substack{j=2 \\ j \neq i}}^{p} f_j \right) f_{iy}}{\displaystyle\prod_{i=2}^{p} f_i},$$

and since the curves f_i are irreducible we get that

$$f_i|\left((\lambda_1+2)D_{2i}-(\lambda_i+1)D_1\right),$$

for $i=2,\cdots,p$. Hence, there are $E_{2i}\in\mathbb{C}[x,y]$ for $i=2,\cdots,p$ such that

$$(\lambda_1+2)D_{2i}-(\lambda_i+1)D_{21}=E_{2i}f_i. \tag{5.17}$$

So, substituting A_2 and using (5.17), system $(2a)$ can be written as

$$Y_2 = \frac{1}{\lambda_1+2}\left(\prod_{i=1}^{p} f_i\right)Y_{D_{21}}+D_{21}\left(\prod_{i=2}^{p} f_i\right)X_{f_1}+\sum_{i=2}^{p}\frac{\lambda_i+1}{\lambda_1+2}D_{21}\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right)X_{f_i},$$

and obviously has the invariant curve $F_2=D_{21}$ and the Darboux first integral $H_2=f_1f_2^{\frac{\lambda_2+1}{\lambda_1+2}}\cdots f_p^{\frac{\lambda_p+1}{\lambda_1+2}}F_2^{\frac{1}{\lambda_1+2}}$. Note that system (1b) is equal to the last system, and so we get that

$$\begin{aligned}
&((\lambda_1+1)A_1+D_{1y})\prod_{i=2}^{p} f_i-\sum_{i=2}^{p}((\lambda_1+1)D_i-(\lambda_i+1)D_1)\left(\prod_{\substack{j=2\\ j\neq i}}^{p} f_j\right)f_{iy}\\
&=\frac{-1}{\lambda_1+2}\left(\prod_{i=1}^{p} f_i\right)F_{2y}-F_2\left(\prod_{i=2}^{p} f_i\right)f_{1y}-\sum_{i=2}^{p}\frac{\lambda_i+1}{\lambda_1+2}F_2\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right)f_{iy},
\end{aligned}$$

and since the curves f_i are irreducible we have that

$$f_i|\left((\lambda_1+1)D_i-(\lambda_i+1)D_1-\frac{\lambda_i+1}{\lambda_1+2}F_2f_1\right),$$

for $i=2,\cdots,p$. Hence, there are $E_i\in\mathbb{C}[x,y]$ such that

$$(\lambda_1+1)D_i-(\lambda_i+1)D_1-\frac{\lambda_i+1}{\lambda_1+2}F_2f_1=E_if_i, \tag{5.18}$$

for $i=2,\cdots,p$. So, for $\lambda_1\notin\{-1,-2\}$ we have that

$$A_1=\frac{1}{\lambda_1+1}\left(-D_{1y}-\frac{1}{\lambda_1+2}f_1F_{2y}-F_2f_{1y}+\sum_{i=2}^{p}E_if_{iy}\right),$$

and consequently the first equation of system $(1a)$ becomes

$$
\begin{aligned}
\dot{x} &= \frac{1}{\lambda_1+1}\left(-D_{1y} - \frac{1}{\lambda_1+2} f_1 F_{2y} - F_2 f_{1y} + \sum_{i=2}^{p} E_i f_{iy},\right) \prod_{i=1}^{p} f_i \\
&\quad -D_1 \left(\prod_{i=2}^{p} f_i\right) f_{1y} - \sum_{i=2}^{p} D_i \left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{iy}.
\end{aligned}
$$

Using (5.18) we get

$$
\begin{aligned}
\dot{x} &= \frac{1}{\lambda_1+1}\left(-D_{1y} - \frac{1}{\lambda_1+2} f_1 F_{2y} - F_2 f_{1y} + \sum_{i=2}^{p} E_i f_{iy}\right) \prod_{i=1}^{p} f_i \\
&\quad -D_1 \left(\prod_{i=2}^{p} f_i\right) f_{1y} - \frac{1}{\lambda_1+1} \sum_{i=2}^{p} \left(E_i f_i + (\lambda_i+1) D_1 + \frac{\lambda_i+1}{\lambda_1+2} F_2 f_1\right) \left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{iy}.
\end{aligned}
$$

We note that the last system has the additional invariant algebraic curve

$$
F = D_1 + \frac{1}{\lambda_1+2} F_2 f_1,
$$

and takes the normal form (5.5). So, Theorem 5.4 is proved in Case 3.

Case 4: $k_1 = (m-\gamma+1)/3$. Since $f_1 = 0, \cdots, f_p = 0$ are invariant algebraic curves for system (2b) from Proposition 1.2 we have that $k_1 + \gamma \leq m - 2k_1 + 1$ and, consequently in this case we get that $k_1 + \gamma = m - 2k_1 + 1$. Hence, by Theorem 2.4(b) there are $\alpha_i \in \mathbb{C}$ such that system (2b) can be written into the form (2.5). Then arguing by a similar way to Case 2 we have that system $(2a)$ has the additional invariant algebraic curve $F_2 = D_{21} + f_1$ and has the Darboux first integral $H_2 = f_1 f_2^{\frac{\lambda_2+1}{\lambda_1+2}} \cdots f_p^{\frac{\lambda_p+1}{\lambda_1+2}} F_2^{\frac{1}{\lambda_1+2}}$. Continuing by a similar way to Case 3 we have that system $(1a)$ has the additional invariant algebraic curve $F = D_1 + \frac{1}{\lambda_1+2} F_2 f_1$ and takes the normal form (5.5).

Now, we present some notation.

We consider a system of degree $m-(l-1)k_1$ of the form

$$Y_{la} = Y_{0l} \prod_{i=1}^{p} f_i + \sum_{i=1}^{p} D_{li} \left(\prod_{\substack{j=1 \\ j\neq i}}^{p} f_j \right) X_{f_i}, \qquad (la)$$

with $\delta A_l, \delta B_l \leq m+\gamma-lk_1$ and $\delta D_l \leq m+\gamma-lk_1+1$ having the integrating factor $f_1^{\lambda_1+l-1}\cdots f_p^{\lambda_p}$. According to Lemma 5.5 we associate to system (la) the system

$$Y_{lb} = ((\lambda_1+l)Y_{0l} - Y_{D_{l1}}) \prod_{i=2}^{p} f_i + \sum_{i=2}^{p} ((\lambda_1+l)D_{li} - (\lambda_i+1)D_{l1}) \left(\prod_{\substack{j=2 \\ j\neq i}}^{p} f_j \right) X_{f_i}, \quad (lb)$$

of degree $m-lk_1$ having the integrating factor $f_1^{\lambda_1+l}\cdots f_p^{\lambda_p}$.

Lemma 5.6. *We assume that the conditions of Theorem 5.4 hold and that* $(m-\gamma+1)/(n+1) \leq k_1 < (m-\gamma+1)/n$. *Then system* $(1a)$ *takes the normal form (5.5).*

Proof: We consider the two sequences of systems (la) and (lb) having $f=0$ as invariant algebraic curve. System $(1b)$

can be rewritten as

$$Y_{2a} = Y_{02} \prod_{i=1}^{p} f_i + \sum_{i=1}^{p} D_{2i} \left(\prod_{\substack{j=1 \\ j\neq i}}^{p} f_j \right) X_{f_i}, \qquad (2a)$$

and has the associated system $(2b)$

$$Y_{2b} = ((\lambda_1+2)Y_{02} - Y_{D21}) \prod_{i=2}^{p} f_i + \sum_{i=2}^{p} ((\lambda_1+2)D_{2i} - (\lambda_i+1)D_{21}) \left(\prod_{\substack{j=2 \\ j\neq i}}^{p} f_j \right) X_{f_i},$$

respectively. In a similar way, we get the system

$$Y_{na} = Y_{0n} \prod_{i=1}^{p} f_i + \sum_{i=1}^{p} D_{ni} \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right) X_{f_i}, \qquad (na)$$

and its associated system (nb)

$$Y_{nb} = ((\lambda_1 + n)Y_{0n} - Y_{D_{n1}}) \prod_{i=2}^{p} f_i + \sum_{i=2}^{p} ((\lambda_1 + n)D_{ni} - (\lambda_i + 1)D_{n1}) \left(\prod_{\substack{j=2 \\ j \neq i}}^{p} f_j \right) X_{f_i},$$

where $\delta A_i, \delta B_i \leq m + \gamma - ik_1$ and $\delta D_i \leq m + \gamma - ik_1 + 1$. For simplicity, we distinguish the following two cases:

Case A: $(m - \gamma + 1)/(n + 1) < k_1 < (m - \gamma + 1)/n$. The generic curve $f_1 = 0$ is invariant for system (nb) of degree at most $m - nk_1$. Hence, from Proposition 1.2, we have that $k_1 \leq m - nk_1 + 1$ and therefore $k_1 \leq (m + 1)/(n + 1)$. Consequently, system (nb) must be identically equal to zero. So, for $\lambda_1 \neq -n$ we get that

$$A_n \quad = \quad -\frac{1}{\lambda_1 + n} D_{n1y} + \frac{1}{\lambda_1 + n} \frac{((\lambda_1 + n)D_{ni} - (\lambda_i + 1)D_{n1}) \left(\prod\limits_{\substack{j=2 \\ j \neq i}}^{p} f_j \right) f_{iy}}{\prod\limits_{i=2}^{p} f_i}.$$

Since f_i is irreducible we have that

$$f_i | \left((\lambda_1 + n)D_{ni} - (\lambda_i + 1)D_{n1} \right),$$

for $i = 2, \cdots, p$. Therefore, there are $E_{ni} \in \mathbb{C}[x, y]$ such that

$$(\lambda_1 + n)D_{ni} - (\lambda_i + 1)D_{n1} = E_{ni} f_i,$$

and so

$$A_n = -\frac{1}{\lambda_1 + n} D_{n1,y} + \frac{1}{\lambda_1 + n} \sum_{i=2}^{p} E_{ni} f_{iy}.$$

Substituting A_n into the first equation of system (na) we get

$$\begin{aligned}
\dot{x} &= -\frac{1}{\lambda_1+n}\left(\prod_{i=1}^{p} f_i\right) D_{n1,y} + \frac{1}{\lambda_1+n}\sum_{i=2}^{p} E_{ni}\left(\prod_{i=1}^{p} f_i\right) f_{iy} \\
&\quad - \sum_{i=1}^{p} D_{ni}\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{iy} \\
&= -\frac{1}{\lambda_1+n}\left(\prod_{i=1}^{p} f_i\right) D_{n1,y} + \frac{1}{\lambda_1+n}\sum_{i=2}^{p} E_{ni}\left(\prod_{i=1}^{p} f_i\right) f_{iy} \\
&\quad - D_{n1}\left(\prod_{i=2}^{p} f_i\right) f_{1y} - \sum_{i=2}^{p}\left(\frac{1}{\lambda_1+n}E_{ni}f_i + \frac{\lambda_i+1}{\lambda_1+n}D_{n1}\right)\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{iy} \\
&= -\frac{1}{\lambda_1+n}\left(\prod_{i=1}^{p} f_i\right) D_{n1,y} - \sum_{i=2}^{p}\frac{\lambda_i+1}{\lambda_1+n}D_{n1}\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{iy} - D_{n1}\left(\prod_{\substack{j=2\\ j\neq i}}^{p} f_j\right) f_{1y}.
\end{aligned}$$

Since, system $(n-1)b$ is equal to system (na), we have that

$$\begin{aligned}
&\left((\lambda_1+n-1)A_{n-1} + D_{(n-1)1,y}\right)\prod_{i=2}^{p} f_i \\
&- \sum_{i=2}^{p}\left((\lambda_1+n-1)D_{(n-1)i} - (\lambda_i+1)D_{(n-1)1}\right)\left(\prod_{\substack{j=2\\ j\neq i}}^{p} f_j\right) f_{iy} \\
&= -\frac{1}{\lambda_1+n}\left(\prod_{i=1}^{p} f_i\right) D_{n1,y} - \sum_{i=2}^{p}\frac{\lambda_i+1}{\lambda_1+n}D_{n1}\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{iy} - D_{n1}\left(\prod_{j=2}^{p} f_j\right) f_{1y},
\end{aligned}$$

and therefore for $\lambda_1 \notin \{-n, -(n-1)\}$ we have that

$$A_{n-1} = \frac{1}{\lambda_1+n-1}\left(-D_{(n-1)1y} - \frac{1}{\lambda_1+1} f_1 D_{n1,y} - D_{n1} f_{1y}\right)$$

$$+\frac{1}{\lambda_1+n-1}\frac{\sum\limits_{i=2}^{p}\left((\lambda_1+n-1)D_{(n-1)i} - (\lambda_i+1)D_{(n-1)1} - \frac{\lambda_i+1}{\lambda_1+n}D_{n1}f_1\right)}{\prod\limits_{i=2}^{p} f_i}\left(\prod_{\substack{j=2\\ j\neq i}}^{p} f_j\right) f_{iy}.$$

Since f_i is irreducible, we have that

$$f_i | \left((\lambda_1+n-1)D_{(n-1)i} - (\lambda_i+1)D_{(n-1)1} - \frac{\lambda_i+1}{\lambda_1+n}D_{n1}f_1\right),$$

and so there are $E_{(n-1)i} \in \mathbb{C}[x,y]$ such that

$$(\lambda_1+n-1)D_{(n-1)i} - (\lambda_i+1)D_{(n-1)1} - \frac{\lambda_i+1}{\lambda_1+n}D_{n1}f_1 = E_{(n-1)i}f_i,$$

for $i = 2, \cdots, p$. Hence, A_{n-1} becomes

$$A_{n-1} = \frac{1}{\lambda_1+n-1}\left(-D_{(n-1)1,y} - \frac{1}{\lambda_1+n} f_1 D_{n,1y} - D_{n1} f_{1y} + \sum_{i=2}^{p} E_{(n-1)i} f_{iy}\right).$$

Substituting A_{n-1} into system $(n-1, a)$ we have

$$\begin{aligned}
\dot{x} &= A_{n-1}\prod_{i=1}^{p} f_i - \sum_{i=1}^{p} D_{(n-1)i}\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{iy} \\
&= \frac{1}{\lambda_1+n-1}\left(-D_{(n-1)1,y} - \frac{1}{\lambda_1+n} f_1 D_{n1,y} - D_{n1} f_{1y}\right)\prod_{i=1}^{p} f_i \\
&\quad +\frac{1}{\lambda_1+n-1}\sum_{i=2}^{p} E_{(n-1)i}\left(\prod_{j=1}^{p} f_j\right) f_{iy} - \sum_{i=1}^{p} D_{(n-1)i}\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{iy}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\lambda_1+n-1}\left(-D_{(n-1)1,y}-\frac{1}{\lambda_1+n}f_1D_{n1,y}-D_{n1}f_{1y}\right)\prod_{i=1}^{p} f_i \\
&\quad +\frac{1}{\lambda_1+n-1}\sum_{i=2}^{p} E_{(n-1)i}\left(\prod_{j=1}^{p} f_j\right) f_{iy} - D_{(n-1)1}\left(\prod_{j=2}^{p} f_j\right) f_{1y} \\
&\quad \frac{1}{\lambda_1+n-1}\sum_{i=2}^{p}\left(-E_{(n-1)i}f_i-\frac{\lambda_i+1}{\lambda_1+n}D_{n1}f_1-(\lambda_i+1)D_{(n-1)1}\right)\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{iy} \\
&= \frac{1}{\lambda_1+n-1}\left(-D_{(n-1)1,y}-\frac{1}{\lambda_1+n}f_1D_{n1,y}-D_{n1}f_{1y}\right)\prod_{i=1}^{p} f_i - D_{(n-1)1}\left(\prod_{j=2}^{p} f_j\right) f_{1y} \\
&\quad +\frac{1}{\lambda_1+n-1}\sum_{i=2}^{p}\left(-\frac{\lambda_i+1}{\lambda_1+n}D_{n1}f_1-(\lambda_i+1)D_{(n-1)1}\right)\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{iy} \\
&= \frac{-1}{\lambda_1+n-1}\left(\sum_{i=1}^{p} f_i\right) F_{n-1,y} - F_{n-1}\left(\prod_{i=2}^{p} f_i\right) f_{1y} \\
&\quad -\frac{1}{\lambda_1+n-1}F_{n-1}\sum_{i=2}^{p}(\lambda_i+1)\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{iy},
\end{aligned}
$$

and setting

$$F_{n-1} = D_{n-1} + \frac{1}{\lambda_1+n}F_n f_1,$$

into the last system we get

$$
\begin{aligned}
\dot{x} &= \frac{1}{\lambda_1+n-1}\left(\sum_{i=1}^{p} f_i\right) F_{n-1,y} + \frac{1}{\lambda_1+n-1}F_{n-1}\sum_{i=2}^{p}(\lambda_i+1)\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right) f_{iy} \\
&\quad +F_{n-1}\left(\prod_{i=2}^{p} f_i\right) f_{1y}.
\end{aligned}
$$

In a similar way the second equation of system $(n-1,a)$ can be written

$$\begin{aligned}\dot y = & \frac{1}{\lambda_1+n-1}\left(\sum_{i=1}^{p} f_i\right)F_{n-1,x} + F_{n-1}\left(\prod_{i=2}^{p} f_i\right)f_{1x}\\ & +\frac{1}{\lambda_1+n-1}F_{n-1}\sum_{i=2}^{p}(\lambda_i+1)\left(\prod_{\substack{j=1\\ j\neq i}}^{p} f_j\right)f_{ix}.\end{aligned}$$

Hence, system $(n-1,a)$ of degree at most $m-nk_1$ has the additional invariant algebraic curve $F_{n-1}=0$ with $\delta F_{n-1}=m-\gamma-nk_1$. Additionally, system $(n-1,a)$ is written into the normal form (5.5) and has the Darboux first integral

$$H_{n-1}=f_1 f_2^{\frac{\lambda_2+1}{\lambda_1+n-1}}\cdots f_p^{\frac{\lambda_p+1}{\lambda_1+n-1}}F^{\frac{1}{\lambda_1+n-1}}.$$

Working in a similar way we have that system $(n-2,a)$ has the additional invariant algebraic curve $F_{n-2}=D_{n-2}+\frac{1}{\lambda_1+n-1}F_{n-1}f_1$, and can be written into the normal form (5.5) and has the Darboux first integral

$$H_{n-2}=f_1 f_2^{\frac{\lambda_2+1}{\lambda_1+n-2}}\cdots f_p^{\frac{\lambda_p+1}{\lambda_1+n-2}}F_{n-2}^{\frac{1}{\lambda_1+n-2}}.$$

Similarly, the sequence of systems (la) has the following invariant curves and Darboux first integrals:

$$(n,a)\qquad F_n=D_{n1},\qquad\qquad H_n=F_n^{\frac{1}{\lambda_1+n}}\prod_{i=2}^{p} f_i^{\frac{\lambda_i+1}{\lambda_1+n}}f_1$$

$$(n-1,a)\quad F_{n-1}=\left(D_{n-1,1}+\frac{1}{\lambda_1+n}F_n f_1\right),\qquad H_{n-1}=F_{n-1}^{\frac{1}{\lambda_1+n-1}}\prod_{i=2}^{p} f_i^{\frac{\lambda_i+1}{\lambda_1+n-1}}f_1$$

$$(n-2,a)\quad F_{n-2}=\left(D_{n-2,1}+\frac{1}{\lambda_1+n-1}F_{n-1} f_1\right),\quad H_{n-2}=F_{n-2}^{\frac{1}{\lambda_1+n-2}}\prod_{i=2}^{p} f_i^{\frac{\lambda_i+1}{\lambda_1+n-2}}f_1$$

$$\begin{array}{lll} \cdots & \cdots & \cdots \\ (n-i,a) & F_{n-i} = \left(D_{n-i,1} + \dfrac{1}{\lambda_1+n-i} F_{n-i} f_1\right), & H_{n-i} = F_{n-i}^{\frac{1}{\lambda_1+n-i}} \prod_{i=2}^{p} f_i^{\frac{\lambda_i+1}{\lambda_1+n-i}} f_1 \\ \cdots & \cdots & \cdots \end{array}$$

$$\begin{array}{lll} (2a) & F_2 = \left(D_{21} + \dfrac{1}{\lambda+2} F_3 f_1\right), & H_2 = F_2^{\frac{1}{\lambda_1+2}} \prod_{i=2}^{p} f_i^{\frac{\lambda_i+1}{\lambda_1+2}} f_1 \\ (1a) & F = \left(D_1 + \dfrac{1}{\lambda_1+1} F_2 f_1\right), & H = F^{\frac{1}{\lambda_1+1}} \prod_{i=2}^{p} f_i^{\frac{\lambda_i+1}{\lambda_1+1}} f_1. \end{array}$$

Hence system $(1a)$ has the additional invariant algebraic curve $F = 0$ given by the following expression

$$\begin{aligned} F &= D_1 + \frac{1}{\lambda_1+1} F_2 f_1 = D_1 + \frac{1}{\lambda_1+1}\left(D_{21} f_1 + \frac{1}{\lambda_1+2} F_3 f_1^2\right) \\ &= D_1 + \frac{1}{\lambda_1+1} D_{21} f_1 + \frac{1}{(\lambda_1+1)(\lambda_1+2)}\left(D_{31} + \frac{1}{\lambda_1+3} F_4 f_1\right) f_1^2 \\ &= D_1 + \frac{1}{\lambda_1+1} D_{21} f_1 + \frac{1}{(\lambda_1+1)(\lambda_1+2)} D_{31} f_1^2 + \frac{1}{(\lambda_1+1)(\lambda_1+2)(\lambda_1+3)} F_4 f_1^3 \\ &= \cdots \qquad \cdots \qquad \cdots \\ &= D_1 + \sum_{i=1}^{n-1} \frac{1}{(\lambda_1+1)\cdots(\lambda_1+i)} D_{i+1,1} {f_1}^i, \end{aligned}$$

with $\delta F = m-\gamma-k_1+1$. Additionally, system $(1a)$ can be written into the normal form (5.5).

Case B: $k_1 = (m-\gamma+1)/(n+1)$. We note that $f_1 = 0, \cdots, f_p = 0$ are generic curves and for system (nb) holds that $k_1 + \gamma = m - nk_1 + 1$ then, by Theorem

2.4(b) there are $\alpha_i \in \mathbb{C}$ such that system (nb) can be written into the form

$$((\lambda_1+1)Y_{0n} - Y_{D_{n1}}) \prod_{i=2}^{p} f_i - \sum_{i=2}^{p} ((\lambda_1+n)D_{ni} - (\lambda_i+1)D_{n1}) \left(\prod_{\substack{j=2 \\ j\neq i}}^{p} f_j \right) X_{f_i}$$
$$= \sum_{i=2}^{p} \alpha_i \left(\prod_{\substack{j=1 \\ j\neq i}}^{p} f_j \right) X_{f_i}, \tag{5.19}$$

and so from Proposition 2.5 system (nb) has the integrating factor $R_1 = (f_1 \cdots f_p)^{-1}$.

If $L_1 = 0$ then applying Lemma 5.5(a) we get that system (nb) must be the zero vector field and this can be studied by a similar way to Case A.

If $L_1 \neq 0$ then from Lemma 5.5(b) we have that system $(1b)$ has the integrating factor $R_2 = f_1^{\lambda_1+n} f_2^{\lambda_2} \cdots f_p^{\lambda_p}$. Hence, it has the first integral $R_2/R_1 = f_1^{\lambda_1+n+1} f_2^{\lambda_2+1} \cdots f_p{}^{\lambda_p+1}$. Therefore, without loss of the generality we can take $\alpha_1 = \lambda_1+n+1$ and $\alpha_i = \lambda_i+1$ for $i = 2, \cdots, p$. Then, the first equation of system (nb) can be rewritten as

$$((\lambda_1+n)A_{n1} + D_{n1,y}) \prod_{i=2}^{p} f_i - \sum_{i=2}^{p} ((\lambda_1+n)D_{ni} - (\lambda_i+1)D_{n1}) \left(\prod_{\substack{j=2 \\ j\neq i}}^{p} f_j \right) f_{iy}$$
$$= -(\lambda_1+n+1) \left(\prod_{\substack{j=2 \\ j\neq i}}^{p} f_j \right) f_{1y} - \sum_{i=2}^{p} (\lambda_i+1) \left(\prod_{\substack{j=2 \\ j\neq i}}^{p} f_j \right) f_{1y}. \tag{5.20}$$

Note that the curves $f_i = 0$ are irreducible so we get that

$$f_i | [(\lambda_1+n)D_{ni} - (\lambda_i+1)D_{n1} - (\lambda_i+1)f_1].$$

Hence, there are $E_{ni} \in \mathbb{C}[x,y]$ such that

$$(\lambda_1+n)D_{ni} - (\lambda_i+1)D_{1n} - (\lambda_i+1)f_1 = E_{ni} f_i, \tag{5.21}$$

for all $i = 2, \cdots, p$. Since $\lambda_1 \neq -n$ then from relation (5.20) we get that

$$A_n = -\frac{1}{\lambda_1 + n} D_{n1,y} + \frac{1}{\lambda_1 + n} \sum_{i=2}^{p} E_{ni} f_{iy} - \frac{\lambda_1 + n + 1}{\lambda_1 + 1} f_{1y},$$

and so the first equation of system (na) can be written into the following form

$$\begin{aligned} \dot{x} &= \frac{-1}{\lambda_1 + n} \left(\prod_{i=1}^{p} f_i \right) D_{n1,y} - D_{n1} \left(\prod_{i=2}^{p} f_i \right) f_{1y} - \frac{\lambda_1 + n + 1}{\lambda_1 + n} \left(\prod_{i=2}^{p} f_i \right) f_{1y} \\ &\quad - \sum_{i=2}^{p} \frac{\lambda_i + 1}{\lambda_1 + n} D_{n1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right) f_{iy} - \sum_{i=2}^{p} \frac{\lambda_i + 1}{\lambda_1 + n} \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right) f_{iy}. \end{aligned}$$

Working in a similar way with the second equation system $(1a)$ becomes

$$\begin{aligned} Y &= \frac{1}{\lambda_1 + n} \left(\prod_{i=1}^{p} f_i \right) Y_{D_{n1}} + D_{n1} \left(\prod_{i=2}^{p} f_i \right) X_{f_1} + \frac{\lambda_1 + n + 1}{\lambda_1 + n} \left(\prod_{i=2}^{p} f_i \right) X_{f_1} \\ &\quad + \sum_{i=2}^{p} \frac{\lambda_i + 1}{\lambda_1 + n} D_{n1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right) X_{f_i} + \sum_{i=2}^{p} \frac{\lambda_i + 1}{\lambda_1 + n} D_{n1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} f_j \right) X_{f_i}. \end{aligned}$$

We note that system (na) has the additional invariant algebraic curve $F_n = D_{n1} + f_1 = 0$ and $\delta F_n = m - \gamma - nk_1 + 1$. We also note that system (na) has the Darboux first integral $H = f_1 f_2^{\frac{\lambda_2+1}{\lambda_1+1}} \cdots f_p^{\frac{\lambda_p+1}{\lambda_1+1}} F^{\frac{1}{\lambda_1+1}}$.

By a similar way to Case A we prove that system $(1a)$ takes the normal form (5.5). ∎

Example 5.7.

We are interesting to construct all polynomial differential systems of degree $m = 7$ having the Darboux integrating factor $R = f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ where $f_1 = x^3 + y^3 - 1$, $f_2 = x^2 + xy + 1$ and $f_3 = y + 1$. We note that the curves $f_1 = 0$, $f_2 = 0$ and

$f_3 = 0$ satisfy the generic conditions (i)–(v) and so we are under the assumptions of Theorem 3.1. We have $k_1 = \delta f_1 = 3$, $\gamma = \delta f_2 + \delta f_3 = 2 + 1 = 3$ and $n = [(m-\gamma+1)/k_1] = [(7-3+1)/3] = 1$. So we are in Case–1 of the proof of Theorem 5.4. System $(1a)$ is of the form

$$
\begin{aligned}
\dot{x} &= A_1 f_1 f_2 f_3 - D_1 f_2 f_3 f_{1y} - D_2 f_1 f_3 f_{2y} - D_3 f_1 f_2 f_{3y},\\
\dot{y} &= B_1 f_1 f_2 f_3 + D_1 f_2 f_3 f_{1x} + D_2 f_1 f_3 f_{2x} + D_3 f_1 f_2 f_{3x},
\end{aligned}
$$

with $\delta A_1, \delta B_1 \leq 1$ and $\delta D_1, \delta D_2, \delta D_3 \leq 2$. Let $D_1 = \sum_{i,j=0}^{2} d_{ij} x^i y^j$ be a polynomial of degree 2. Since in this case system $(1b)$ is the zero vector field we have that relations (5.13) hold. Let $E_{12}(x,y) = E \in \mathbb{C}$ and $E_{13} = e_1 x + e_2 y + e_0 \in \mathbb{C}[x,y]$. According to relations (5.13) we can calculate the polynomials D_2 and D_3. So, we have

$$
\begin{aligned}
D_2 &= \frac{(\lambda_2+1)D_1 + E_{12} f_2}{\lambda_1+1}\\
&= \frac{(\lambda_2+1)(d_{20}x^2 + d_{11}xy + d_{02}y^2 + d_{10}x + d_{01}y + d_0) + E(x^2+xy+1)}{\lambda_1+1},
\end{aligned}
$$

$$
\begin{aligned}
D_3 &= \frac{(\lambda_3+1)D_1 + E_{13} f_2}{\lambda_1+1}\\
&= \frac{(\lambda_3+1)(d_{20}x^2 + d_{11}xy + d_{02}y^2 + d_{10}x + d_{01}y + d_0)}{\lambda_1+1}\\
&+ \frac{e_1 x(y+1) + e_2 y(y+1) + e_0(y+1)}{\lambda_1+1},
\end{aligned}
$$

and therefore we can calculate the polynomials A_1 and B_1. We obtain

$$
\begin{aligned}
A_1 &= -\frac{1}{\lambda_1+1} D_{1y} + \frac{1}{\lambda_1+1}\left(E_{12} f_{2y} + E_{13} f_{3y}\right)\\
&= -\frac{1}{\lambda_1+1}(d_{11}x + 2d_{02}y + d_{01} + E(x + e_1 x + e_2 y + e_0),
\end{aligned}
$$

$$
\begin{aligned}
B_1 &= \frac{1}{\lambda_1+1} D_{1x} - \frac{1}{\lambda_1+1}\left(E_{12} f_{2x} + E_{13} f_{3x}\right)\\
&= \frac{1}{\lambda_1+1}(2d_{20}x + d_{11}y + d_{10} - E(2x+y)).
\end{aligned}
$$

Substituting A_1, B_1, D_1, D_2 and D_3 into system $(1a)$ and doing a simple computation we have that system $(1a)$ can be written into the form

$$\begin{aligned}\dot{x} &= -f_2f_3Ff_{1y} - \frac{\lambda_2+1}{\lambda_1+1}f_1f_3Ff_{2y} - \frac{\lambda_3+1}{\lambda_1+1}f_1f_2Ff_{3y} - \frac{1}{\lambda_1+1}f_1f_2f_3F_y,\\ \dot{y} &= f_2f_3Ff_{1x} + \frac{\lambda_2+1}{\lambda_1+1}f_1f_3Ff_{2x} + \frac{\lambda_3+1}{\lambda_1+1}f_1f_2Ff_{3x} + \frac{1}{\lambda_1+1}f_1f_2f_3F_x,\end{aligned}$$

where $F = D_1$ is the additional invariant algebraic curve.

5.5 On a result due to Walcher

In [50] Walcher also proves the following theorem.

Theorem 5.8. *Let $f = f_1 \cdots f_p$ with $f_i \in \mathbb{C}[x,y]$ irreducible, and assume that the curve $f = 0$ has no singular points. Additionally, we assume that $(f_x, f_y) = 1$. Then $X = (P,Q)$ admits the integrating factor $R = \left(f_1^{\lambda_1}\cdots f_p^{\lambda_p}\right)^{-1}$ with λ_i positive integers if and only if*

$$\begin{aligned}\dot{x} &= -\sum_{i=1}^{p}\alpha_i f_1^{\lambda_1}\cdots f_i^{\lambda_i-1}\cdots f_p^{\lambda_p}f_{iy} - f_1^{\lambda_1}\cdots f_p^{\lambda_p}\left(\frac{F}{f_1^{\lambda_1-1}\cdots f_p^{\lambda_p-1}}\right)_y,\\ \dot{y} &= \sum_{i=1}^{p}\alpha_i f_1^{\lambda_1}\cdots f_i^{\lambda_i-1}\cdots f_p^{\lambda_p}f_{ix} + f_1^{\lambda_1}\cdots f_p^{\lambda_p}\left(\frac{F}{f_1^{\lambda_1-1}\cdots f_p^{\lambda_p-1}}\right)_x,\end{aligned} \tag{5.22}$$

with $\alpha_i \in \mathbb{C}$ and $F \in \mathbb{C}[x,y]$.

Under the assumptions of Theorem 5.8 polynomial systems having the Darboux integrating factor $R = \left(f_1^{\lambda_1}\cdots f_p^{\lambda_p}\right)^{-1}$ with λ_i positive integers are of the form (5.22). So, from Theorem 3.1(a), they have the Darboux first integral $H = f_1^{\alpha_1}\cdots f_p^{\alpha_p}\exp\left(\frac{F}{f_1^{\lambda_1-1}\cdots f_p^{\lambda_p-1}}\right)$ with $\alpha_i \in \mathbb{C}$ and $F \in \mathbb{C}[x,y]$.

We note that part of the statement of Theorem 4.3(b) can be obtained from Theorem 5.8 taking $p = 1$ and $\lambda_1 = -\lambda \in \mathbb{Z}_+$ and $g = F$. Note that, in Theorem 4.3(b) and in its proof we present an algorithm in order to construct the polynomial F, see also Example 4.15. Additionally, in Theorem 4.3(b) we also prove the following relation between the degrees: $\delta f + \delta\left(\frac{F}{f^{-\lambda_1-1}}\right) = m + 1$.

Chapter 6

Appendix

A planar vector field

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y} \tag{6.1}$$

is polynomial of degree n if P and Q are real polynomials in the variables x and y, and the maximum degree of P and Q is n.

A periodic orbit of a vector field X in $\mathbb{R}^2$ is a *limit cycle* if it is isolated in the set of all periodic orbits of X.

In 1900 Hilbert [30] in the second part of its 16–th problem proposed to find an estimation of the uniform upper bound for the number of limit cycles of all polynomial vector fields of a given degree, and also to study their distribution or configuration in the plane. This has been one of the main problems in the qualitative theory of planar differential equations in the XX century. The contributions of Écalle [23] and Ilyashenko [41] proving that any polynomial vector field has finitely many limit cycles have been the best results in this area. But until now it is not proved the existence of an uniform upper bound. This problem remains open even for the quadratic polynomial vector fields.

A limit cycle is *algebraic* of *degree* m if it is a contained in an irreducible algebraic curve of degree m.

Hilbert also asked about the possible distributions of the limit cycles of polynomial vector fields. Recently, it has been proved that any finite configuration of limit cycles is realizable by polynomial vector fields. More precisely, we say that a *configuration of limit cycles* is a finite set of disjoint simple closed curves of the plane pairwise disjoint. Two configurations of limit cycles are *(topologically) equivalent* if there is a homeomorphism of $\mathbb{R}^2$ applying one configuration into the other. We say that the vector field X *realizes* a given configuration of limit cycles if the set of all limit cycles of X is equivalent to that configuration. Recently, in [39] it is proved that any configuration of limit cycles is topologically realizable as algebraic limit cycles by a polynomial vector field of a convenient degree.

In [52] Winkel did the following conjecture about the algebraic limit cycles of polynomial vector fields.

Conjecture 6.1. *For a given algebraic curve $f = 0$ of degree $m \geqslant 4$ there is in general no polynomial vector field of degree less than $2m-1$ leaving invariant $f = 0$ and having exactly the ovals of $f = 0$ as limit cycles.*

We shall prove that this conjecture is not true.

Here we will work with the one–parameter family of irreducible algebraic curves

$$f = f(x, y) = \frac{1}{4} + x - x^2 + px^3 + xy + x^2y^2 = 0 \ , \tag{6.2}$$

of degree $m = 4$ with $0 < p < 1/4$. These curves have three connected components, one is an oval and each of the other two is homeomorphic to a straight line, see Figure 1. We note that the oval of $f = 0$ borns at the point $(2, -1/4)$ when $p = 1/4$. Then, when p decreases the oval increases its size and ends having infinite size at the irreducible curve $1/4 + x - x^2 + xy + x^2y^2 = 0$ when $p = 0$.

We must mention that the curve $f = 0$ has no singular points, i.e. there is no real solutions of the system $f = 0$, $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$.

First we will prove that the oval of the curve (6.2) is the unique limit cycle of a 13–parameter family of polynomial vector fields of degree 5. Since $2m-1 = 7 > 5$, this provides a counterexample to Conjecture 6.1. Many other counterexamples can be constructed changing the algebraic curve $f = 0$.

Our main result is the following one.

Figure 6.1: Algebraic limit cycle of degree 4.

Theorem 6.2. *Let* a, b, c, d, e *and* p *arbitrary real numbers. Then, the algebraic curve* $f = 0$ *given by (6.2) is invariant by the* 6*–parameter family of polynomial vector fields (6.1) of degree* 5 *given by*

$$
\begin{aligned}
P \;=\;& (be-cd)+[c+4(be-cd)-2a(d^2+e^2)]x-by+\\
& 4[c+(a+c)d-be]x^2-4[b+cd-(a+b)e]xy-\\
& 2[a+2c+2p(cd-be)]x^3+4[b+c-a(d^2+e^2)]x^2y-\\
& 2(a+2b)xy^2+4cp\,x^4+4(2ad-bp)x^3y-\\
& 4(cd-2ae-be)x^2y^2-4ax^4y+4cx^3y^2-4(a+b)x^2y^3,
\end{aligned}
$$

$$
\begin{aligned}
Q \;=\;& 2a(d^2+e^2)-bd-ce+[b-4((a+b+ad)d+(c+ae)e)]x+\\
& [c+2a(d^2-2e+e^2)]y-4(-c+ad+bd-2ae+ce)xy+\\
& 2[a+2b+2d(2a+b)+2ce+3ap(d^2+e^2)]x^2+2a(1-2e)y^2-\\
& 4(a+b+3adp+bdp+cep)x^3+2(a+2b-2c-6aep)x^2y+\\
& 4(-a+c+ad^2+ae^2)xy^2+2ay^3+\\
& 2(3a+2b)p\,x^4+4cp\,x^3y-2(4ad+2bd+2ce-3ap)x^2y^2-\\
& 8aexy^3+4(a+b)x^3y^2+4cx^2y^3+4axy^4.
\end{aligned}
$$

Moreover, if $ac \neq 0$, $0 < p < 1/4$ *and the point* (d, e) *is in the interior of the bounded region limited by the oval of* $f = 0$*, then the unique limit cycle of this vector field is the algebraic one formed by the oval of* $f = 0$*.*

We note that if we do an affine transformation of the polynomial differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, where P and Q are the ones given in the statement of Theorem 6.2, and a rescaling of the independent variable, then the polynomial vector fields of degree 5 associated to the new differential systems form a 13–parameter family providing a counterexample to Conjecture 6.1.

In fact a weaker counterexample formed by an 8–parameter family of quadratic polynomial vector fields follows from the next theorem proven in [5].

Theorem 6.3. *The quadratic polynomial differential system*

$$\begin{aligned} \dot{x} &= 2(1 + 2x - 2px^2 + 6xy) \ , \\ \dot{y} &= 8 - 3p - 14px - 2pxy - 8y^2 \ , \end{aligned}$$

with $0 < p < 1/4$ *possesses the irreducible invariant algebraic curve* $f = 0$*. Moreover, if* $0 < p < 1/4$*, then the unique limit cycle of this system is the algebraic one formed by the oval of* $f = 0$*.*

The following result due to Giacomini, Llibre and Viano [27], will play a main role in our proof of Theorem 6.2. Here, we provide an easier and direct proof, which also appears in Llibre and Rodríguez [39].

Theorem 6.4. *Let* X *be a* C^1 *vector field defined in the open subset* U *of* $\mathbb{R}^2$*. Let* $V : U \to \mathbb{R}$ *be an inverse integrating factor of* X*. If* γ *is a limit cycle of* X*, then* γ *is contained in* $\mathbf{\Sigma} = \{(x, y) \in U : V(x, y) = 0\}$*.*

Proof: Due to the existence of the inverse integrating factor V defined in U, we have that the vector field X/V is Hamiltonian in $U \setminus \mathbf{\Sigma}$. Since the flow of a Hamiltonian vector field preserves the area and in a neighborhood of a limit cycle a flow does not preserve the area, the theorem follows. ∎

A straightforward computation shows that the algebraic curve $f = 0$ given by (6.2) is invariant by the polynomial vector field X whose components P and Q are given in the statement of Theorem 6.2. In fact, the cofactor K of $f = 0$ is

$$\begin{aligned} K &= 4[be - cd + (c - bd + 2cd - 2be - ce)x + (be - b - cd)y + \\ &\quad (b - 2c - 3cdp + 3bep)x^2 + 2(b + c)xy - by^2 + \\ &\quad 3cpx^3 - (2bd + 2ce + 3bp)x^2y - 2(cd - be)xy^2 + \\ &\quad 2bx^3y + 4cx^2y^2 - 2bxy^3]. \end{aligned}$$

Now the key point in the proof of Theorem 6.2 is to show that the unique limit cycle of X is the oval γ contained in $f = 0$ for $0 < p < 1/4$ when $ac \neq 0$ and (d, e) is a point contained in the interior of the bounded region limited by γ. In order to prove that, first with another easy computation we check that

$$V = f \cdot [(x - d)^2 + (y - e)^2],$$

and

$$H = 2a \log f + 2b \log[(x - d)^2 + (y - e)^2] - 4c \arg[(x - d) + i(y - e)],$$

are the inverse integrating factor and its associated Hamiltonian for our polynomial vector field X.

Since V is polynomial, V is defined in the whole $\mathbb{R}^2$. Therefore, by Theorem 6.4 and since $V(x, y) = 0$ if and only if $(x, y) \in \{f = 0\} \cup \{(d, e)\}$, it follows that if the vector field X has some limit cycle, this must be the oval γ of $f = 0$. Now, we shall prove that this oval is a limit cycle. Hence, Theorem 6.2 will be proved.

We observe that P and Q can be written as

$$\begin{aligned} P &= -2af_2f_3\frac{\partial f_1}{\partial y} - 2(b + ic)f_1f_3\frac{\partial f_2}{\partial y} - 2(b - ic)f_1f_2\frac{\partial f_3}{\partial y}, \\ Q &= 2af_2f_3\frac{\partial f_1}{\partial x} + 2(b + ic)f_1f_3\frac{\partial f_2}{\partial x} + 2(b - ic)f_1f_2\frac{\partial f_3}{\partial x}, \end{aligned} \tag{6.3}$$

where $i = \sqrt{-1}$, $f_1 = f$, $f_2 = x - d + i(y - e)$ and $f_3 = x - d - i(y - e)$.

Since $f = 0$ is an invariant algebraic curve of the vector field X, the oval γ is formed by solutions of X. Now we shall prove that on the oval γ there are no singular points of X and, therefore, γ will be a periodic orbit. Assume that (x_0, y_0) is a singular point of X contained on the oval γ; i.e., $P(x_0, y_0) = Q(x_0, y_0) = f(x_0, y_0) = 0$. From (6.3) we have that

$$\begin{aligned} P(x_0, y_0) &= -2af_2(x_0, y_0)f_3(x_0, y_0)\frac{\partial f}{\partial y}(x_0, y_0) = 0, \\ Q(x_0, y_0) &= 2af_2(x_0, y_0)f_3(x_0, y_0)\frac{\partial f}{\partial x}(x_0, y_0) = 0. \end{aligned}$$

Since $a \neq 0$ and $f_2(x_0, y_0)f_3(x_0, y_0) = (x_0 - d)^2 + (y_0 - e)^2 \neq 0$, we obtain that $\dfrac{\partial f}{\partial x}(x_0, y_0) = 0$ and $\dfrac{\partial f}{\partial y}(x_0, y_0) = 0$. This is not possible, otherwise the point (x_0, y_0) would be a singular point of the algebraic curve $f = 0$, and this curve has no singular points when $0 < p < 1/4$. Hence, the oval γ is a periodic orbit of the vector field X. Now, we shall prove that γ will be a limit cycle, and this will complete the proof of Theorem 6.2.

We define the first integral $\overline{H}$ of X as follows

$$\overline{H} = e^H = f^{2a}[(x-d)^2 + (y-e)^2]^{2b} e^{-4c\arg[(x-d)+i(y-e)]}.$$

Then we note that the oval γ and the point (d, e) are in the level $\overline{H}(x, y) = 0$, and that they are the unique orbits of X in this level. Now suppose that γ is not a limit cycle. Then, there is a periodic orbit $\gamma' = \{(x(t), y(t)) : t \in \mathbb{R}\}$ different from γ and sufficiently close to γ such that the bounded component B limited by γ' contains the point (d, e).

As γ' is different from γ, there exists $h \neq 0$ such that

$$\overline{H}(x(t), y(t)) = f^{2a}(x(t), y(t))[(x(t)-d)^2 + (y(t)-e)^2]^{2b} e^{-4c\theta(t)} = h,$$

where $\theta(t) = \arg[(x(t)-d) + i(y(t)-e)]$. The function $f^{2a}(x(t), y(t))[(x(t)-d)^2 + (y(t)-e)^2]^{2b}$ is bounded on γ'. Clearly, since the point (d, e) is in the bounded region limited by γ' the angle $\theta(t)$ tends to either $+\infty$ or $-\infty$, when $t \to +\infty$. Since $c \neq 0$, this fact is in contradiction with equality $\overline{H}(x(t), y(t)) = h \neq 0$. Consequently, we have proved that γ is a limit cycle. In short, Theorem 6.2 is proved. ∎

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