# *Equilibria in Three Body Problems:*

# *stability, invariant tori and connections*

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*Bellaterra, Maig del 2004*

Certifico que la present memòria ha estat realitzada per l'Anna Samà Camí, i constitueix la seva Tesi per a aspirar al grau de Doctor en Matemàtiques per la Universitat Autònoma de Barcelona.

Bellaterra, Maig de 2004

Firmat: Dra. Regina Martínez Barchino

*Va per tu, bitxo*

# Contents

# Introduction

Celestial Mechanics is devoted to study the motion of planets, asteroids, satellites, .... Its starting point can be considered in the XVII century when Johannes Kepler formulated the laws of the motion of the planets in *Astronomia Nova* (1609). In the year 1687 Newton gave in his work *Philosophiae Naturalis Principia Mathematica* the formulation of the principal object of study of Celestial Mechanics: the $n$–body problem. This problem studies the motion of $n$ particles system under their mutual attraction, governed by the Newton Gravitational Law. Even though the formulation of the equations that describes the $n$–body problem is easy, it is difficult to solve them. In fact, the only case completely solved is the 2–body problem. All the efforts to solve explicitly the equations for $n \geq 3$ have failed. Poincaré showed that the main difficulty comes from the existence of small divisors. In his very famous *Méthodes Nouvelles de la Mécanique Céleste* (1899), Poincaré starts the study of the problem from a qualitative point of view. Actually, qualitative methods play a very important role in the study of differential equations.

However, some special solutions of the $n$–body problem are known: the homographic solutions. For these solutions the configuration of the particles is preserved for any time. This can only be achieved for the so called central configurations. It is well–known that for the Planar Three Body Problem there exists three collinear central configurations, where the masses are located on a straight line, and two triangular ones, in which the masses are on the vertices of an equilateral triangle. Concerning the number and type of central configurations for $n \geq 4$ only partial results are known.

On the other hand for many applications one can make several assumptions which simplify the mathematical problem. The Restricted Three Body Problem (RTBP) is one of the most used models to get a first insight in many applications. In this problem the main assumption is to suppose that one of the bodies has infinitessimal mass, in such a way that it does not influence the motion of the other two bodies, called primaries. Then, one can assume that the primaries move on a solution of the Kepler problem. The Restricted Three Body Problem tries to explain the motion of the infinitessimal mass influenced by the gravitational forces exerted by the primaries. The most interesting cases for the applications correspond to elliptic orbits of the primaries. If the eccentricity, $e$, of these orbits is zero then we obtain the Circular Restricted Three Body Problem and for $e \in (0,1)$ the Elliptic Restricted Three Body Problem.

In a rotating system of coordinates the Planar Circular Restricted Three Body Problem is described by a Hamiltonian system with two degrees of freedom ([Sz.]). For this problem it is well–known that there exists three collinear equilibrium points $L_{1,2,3}$ and two triangular ones $L_{4,5}$. The collinear equilibrium points are of centre–saddle type. Let be $C_{L_i}$ the value of the Jacobi constant at the equilibrium $L_i$. The Lyapunov theorem ([S.M.],[Ms.]) ensures the existence of a family of periodic orbits born at the equilibrium. This family can be parametrized by the Jacobi constant in such a way that for a fixed level of the Jacobi constant such that $C_{L_i} - C$ is small enough, the periodic orbit is the unique bounded orbit that for any time it remains in a small neighbourhood of the equilibrium point. Moreover these orbits are hyperbolic. They have stable and unstable 2–dimensional invariant manifolds of codimension 1 once the Jacobi constant has been fixed. Using these invariant manifolds it is possible to give a classification of the orbits passing through a small neighbourhood of the collinear equilibrium points ([Co.2],[McG.1]). The existence of transversal homoclinic orbits to the periodic Lyapunov orbit has been studied in [L.M.S.] for different values of the mass parameter and the Jacobi constant. This allows to introduce a symbolic dynamics ([L.M.S.],[Ms.2]) which gives the existence of orbits passing through different regions of the phase space. The applicability of these orbits to space missions have been studied in [K.L.M.R.].

In this work we distinguish mainly three parts. In the first one we study some questions related to the stability of homographic solutions. The second part is devoted to the Spatial RTBP. For that problem we study the existence of heteroclinic/homoclinic connections to the invariant tori contained in the centre manifold of the Spatial RTBP. Finally we study the applicability of KAM theorem to the centre manifold of the collinear equilibrium points in the Planar Three Body Problem. Next we introduce these three topics.

# Homographic solutions

We consider the Planar Three Body Problem with homogeneous potential of degree $-\alpha$, $0 < \alpha < 2$, of the following type

$$U(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \frac{m_1 m_2}{\|\mathbf{q}_1 - \mathbf{q}_2\|^\alpha} + \frac{m_1 m_3}{\|\mathbf{q}_1 - \mathbf{q}_3\|^\alpha} + \frac{m_2 m_3}{\|\mathbf{q}_2 - \mathbf{q}_3\|^\alpha}.$$

Notice that if $\alpha = 1$ we get the Newtonian potential. One can generalize the homographic solutions introduced for the Newtonian potential to the general case $(0 < \alpha < 2)$. These solutions can be written as equilibrium points of a periodic Hamiltonian system with 6 degrees of freedom. To this end one should introduce a change of variable which depends quasiperiodically (periodically if $\alpha = 1$) on time.

As we are interested in the stability of these solutions it will be necessary to compute the eigenvalues of the monodromy matrix. To reduce 2 degrees of freedom we use first the integrals of the centre of masses. At this point the linearized system for homographic solutions has order 8. Then we show that one can write this system as two four dimensional uncoupled system. The monodromy matrix of one of these systems has 1 as eigenvalue with multiplicity four. So, in order to obtain the non trivial characteristic multipliers we need to study the other four dimensional system,

$$\dot{\mathbf{x}} \;=\; A(t)\mathbf{x}, \tag{1}$$

where $t$ is the true anomaly in the Newtonian case. Beyond the degree of homogeneity $-\alpha$, this system depends on two parameters: $\beta$, that depends on the masses, and $e$, some generalized eccentricity. We note that the parameter $\beta$ is different for collinear and triangular cases.

When $e$ equals zero, system (1) has constant coefficients and the characteristic exponents, or equivalently the stability parameters, are trivially computed. As $e$ increases some bifurcation can appear. Furthermore as $e$ goes to 1, we get in the limit case a matrix $A(t)$ in (1) which has a singularity at $t = 0$.

Our purpose is to study the stability of systems which generalize in some sense the behaviour of the linearized homographic case for $e$ near 0 and $e$ near 1. So, we consider linear systems of the following type

$$\dot{\mathbf{x}} \;=\; A(t,e)\mathbf{x}, \qquad A(t,e) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 G_1(t,e) & 0 & 0 & -2 \\ 0 & \lambda_2 G_2(t,e) & 2 & 0 \end{pmatrix}, \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^4$, $\lambda_1, \lambda_2$ are real parameters, $e \in [0,1)$, and $G_1$, $G_2$ are periodic functions in $t$, depending on $e$. We shall study the stability for $e \gtrsim 0$ and $e \lesssim 1$. In any case, we shall make different hypothesis on $G_1$ and $G_2$ that will be satisfied in particular in the homographic case.

A system like (2) has several applications. One of them is the study of the stability for the equilibria of some Mechanical systems. Moreover, system (2) can be obtained as a first variational system on a periodic solution of an autonomous system. Chapters 2 and 3 are devoted to study the stability parameters of (2) for $e > 0$ small enough and for $e \lesssim 1$, respectively.

In chapter 2 we study the stability of system (2) for $e \gtrsim 0$ in the case that $G_1, G_2$ are even periodic functions of $t$ and analytic in $e$. First of all, in section 2.2 we study the trivial case $e = 0$ where some resonant points are found. As $e$ increases some bifurations can appear giving rise to some regions in the parameters spaces $\lambda_1, \lambda_2, e$ with different stability character. To study the boundary surfaces of that regions we use the Normal Form technique. In [B.S.1] this method was used to study the resonant tongues for Hill's quasiperiodic equation depending on two parameters, and in [B.S.2] to the unfolding of Mathieu–like equations in the periodic case.

In this work to study the boundary surfaces we concentrate mainly in the d'Alembert case, that is we assume that for $G_1$ and $G_2$ the $k$th harmonic has an amplitude which is at least of order $k$ in $e$. This is in fact a very common situation in mechanical systems. For example, it occurs in the study of the stability of families of periodic orbits which born at an equilibrium point from purely imaginary eigenvalues. Assuming d'Alembert property we distinguish between single and double resonances. The most interesting case is the second one. Under non degeneracy conditions in a neighbourhood of a double resonance, by changing the parameters one can get regions of any type for $e \gtrsim 0$.

In chapter 3 we study the stability of system (2) for $e \lesssim 1$. We assume $G_1 = G_2$ with some singularity for $e = 1$ at $t = 0$. The main result in this chapter is an asymptotic formula for the stability parameters. We use a kind of blow up technique to see the limit case as an heteroclinic connection.

In chapter 4 we use the results on chapters 2 and 3 in order to study the stability of the homographic solutions of the Planar Three Body Problem. In this case, the parameters $\lambda_1, \lambda_2$ depend on a unique mass parameter $\beta$. So, the bifurcation diagram is represented in the plane $(\beta, e)$ for fixed $\alpha$. We compute the resonant parameters at $e = 0$ for any $\alpha$. However we concentrate mainly in the Newtonian case. Using the Normal Form Method developed in chapter 2, we get the resonant tongues born at $e = 0$ up to a given order.

In the collinear case, $\beta \in (0, 7)$ for the physical problem. However, mathematically we can consider $\beta > 0$. Resonances are obtained at $e = 0$ for frequencies $\frac{k}{2}$, $k \geq 3$. If $k = 2n$ no bifurcation takes place for $e > 0$. If $k = 2n + 1$ resonant tongues $\mathcal{T}_{\frac{2n+1}{2}}$ are born at $e = 0$. Despite only $\mathcal{T}_{\frac{3}{2}}$, $\mathcal{T}_{\frac{5}{2}}$ are the unique tongues which emanate from $e = 0$ for $\beta \in (0, 7)$, all the other tongues $\mathcal{T}_{\frac{2n+1}{2}}$ enter this range of $\beta$ for values of $e$ in $(0, 1)$. The width of $\mathcal{T}_{\frac{3}{2}}$, $\mathcal{T}_{\frac{5}{2}}$ is of order 3 and 5 in $e$ respectively. Moreover the asymptotic formula for $e$ near 1 predicts that all these tongues accumulate at $\beta = \frac{1}{8}$ as $e$ goes to 1. This behaviour agrees with the numerical computations done for any $e \in (0,1)$.

Concerning the triangular case, for $0 < \beta < 1$ and $e = 0$ the system is elliptic–elliptic and only one resonant tongue $\mathcal{T}$ born at $\beta = \dfrac{3}{4}$ is found. It defines an elliptic–hyperbolic region in the plane $(\beta, e)$. The width is of order 1 in $e$. The behaviour for $\beta = \dfrac{3}{4}$ and $e \gtrsim 0$ was studied by G. Roberts (see [R.]). By expanding the monodromy matrix in series on $e$, he proves the existence of an elliptic–hyperbolic region for this value of $\beta$ and for $e$ small enough. The method used in [R.] is not useful in the collinear case because the computations are hard. This is due to the fact that in the collinear case the width of tongues is of bigger order in $e$ and so, one needs to compute at least the third order terms in $e$ of the monodromy matrix of the linearized system on the collinear solution.

# Spatial RTBP

Chapter 5 is devoted to the study of homoclinic orbits to the centre manifold of $L_2$ in the Spatial Restricted Three Body Problem. It is well–known that $L_2$ is a centre–centre–saddle equilibrium point. Then, it has one–dimensional stable and unstable invariant manifolds, and a four–dimensional centre manifold. In a neighbourhood of $L_2$ there exist the well–known families of planar and vertical periodic orbits of Lyapunov. These families of periodic orbits have two–dimensional stable and unstable manifolds. Moreover, in the centre manifold there exist invariant tori, with three dimensional stable and unstable manifolds. For the dynamics on the centre manifold see [J.M.], [G.M.]. The intersection of the unstable manifold of one torus in the centre manifold and the stable manifold of another torus give heteroclinic orbits from the first torus to the other one. If we consider the stable and the unstable manifold of the same torus, then we obtain homoclinic orbits to this torus. All these homoclinic and heteroclinic orbits are homoclinic orbits to the centre manifold of $L_2$. In order to get heteroclinic (or homoclinic) orbits we follow the main ideas developed in [L.M.S.] for the Planar RTBP. We compute up to a given order the intersection of the unstable invariant manifold of a given torus with the section $y = 0$ at the other side of the bigger primary. To do that we consider the Spatial RTBP as a perturbation of the 3–dimensional Hill's problem in a neighbourhood of the equilibrium point and then as a perturbation of the Spatial Synodic Two Body Problem. The stable manifold is obtained from the unstable one using the symmetries of the problem.

We give also some estimates on the difference in action space for two tori in order to have an heteroclinic connection. This allows us to construct heteroclinic chains. In particular from invariant tori close to the planar periodic orbit to invariant tori close to the vertical one in a neighbourhood of the $L_2$ point.

## Planar Three Body Problem

Finally, in chapter 6 we study the existence of invariant tori on the centre manifold of the collinear equilibrium points in the Planar Three Body Problem with Newtonian potential. To this end we do the following steps. First, we perform some canonical transformations to write the Hamiltonian in normal form. Then we reduce the Hamiltonian to the centre manifold. After that, we check, by numerical evaluation of the coefficients of the normal form up to order 4, the nondegeneracy conditions of KAM theorem. The results presented in section 6.4 show that both conditions (either isoenergetic or not) are satisfied for any values of the masses in the triangle of masses.

The linearized system on a collinear equilibrium point has eigenvalues $\pm\lambda$, $\pm\mathrm{i}$, $\pm\mathrm{i}\omega$, $\lambda, \omega \in \mathbb{R}^+$. Then the collinear points are of centre–centre–saddle type. Up to order 4 it is proved that we only need to take into account the resonance $2:1$. The corresponding resonant masses describe a curve in the triangle of masses. Therefore, for resonant masses it is expected to get resonant monomials of order three in the normal form of the Hamiltonian. We prove in section 6.3 that this is not the case. In fact, we prove that the coefficients of these monomials are different from zero for general masses but they become zero for resonant masses, and also in the symmetrical case $m_1 = m_3$. The existence of the homographic solutions allows us to compute analytically, in an easy way, the coefficients of the resonant monomials of order three. These coefficients have $(\omega - 2)$ as a factor. The results given in chapter 6 are published in [M.S.].

# Agraïments

Primer de tot vull agrair molt sincerament a la meva directora Regina Martínez per l'interès , el temps i l'esforç dipositat per tal que aquest treball fos possible. També vull agrair–li la seva comprensió i ajuda en alguns moments difícils que he passat durant la realització d'aquest treball.

Agraeixo especialment a Carles Simó pels seus valuosos comentaris i per l'ajut en la part numèrica.

També vull fer especial atenció als meus companys de despatx actuals (els més freqüents), Gerard Ascensi, Miquel Brustenga i Ramon Flores, i a la Laura Prat, 'antiga' companya de despatx, i de batalles. També agraeixo al col·lectiu dels dinars del Bar de Lletres per totes les estones passades. Voldria anomenar també als amics de ses Illes, Chus i Toni, a la noia de Lleida, Maite, i d'altres persones de l'Autònoma que han estat especials per mi, com la Margarida, la $\chi\alpha\rho\alpha$ (espero no haver–me equivocat a l'escriure–ho), . . ..

Vull agrair al Departament de Matemàtiques de la Universitat Autònoma de Barcelona per l'atorgació d'una beca.

Fóra del marc laboral, vull agrair enormement als meus amics de Barcelona, Sabadell, Cerdanyola, Begur, Castellar del Vallès, Sant Cugat, . . . per tots els bons moments.

Agraeixo molt a la meva família tot l'ajut i confiança dipositats, amb especial emfàsi a la meva mare, per què sempre que va poder va ser on la necessitava. Vull també esmentar al meu pare que m'ha estat recolçant i animant en la realització d'aquest treball.

I, sobretot, agraeixo enormement al Patxi per tota la paciència que ha tingut amb mi, la seva confiança dipositada, . . .

# Chapter 1

# Homographic solutions in the Planar Three Body Problem

## 1.1 Equations of motion

Consider three positive point masses $m_1, m_2, m_3$ moving in an inertial coordinate system in $\mathbb{R}^2$. Let us denote by $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \in \mathbb{R}^2$ the position vector of the $j$th particle of mass $m_j$, $j = 1, 2, 3$ (see figure 1.1).

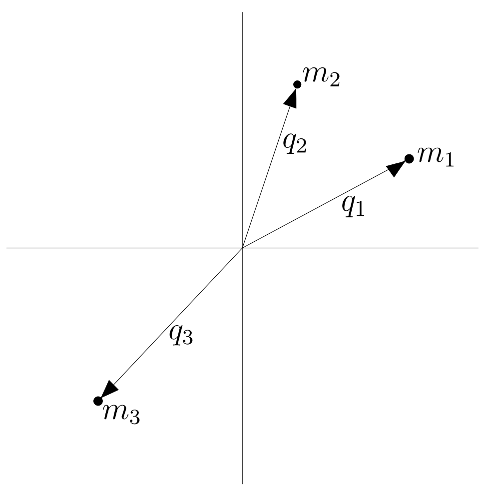



Figure 1.1: Masses $m_1, m_2, m_3$ and the respective position vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$.

Assume that the only forces acting on the masses are described by an homogeneous potential of the form

$$U(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3) = \frac{m_1 m_2}{\|\mathbf{q}_1-\mathbf{q}_2\|^\alpha} + \frac{m_1 m_3}{\|\mathbf{q}_1-\mathbf{q}_3\|^\alpha} + \frac{m_2 m_3}{\|\mathbf{q}_2-\mathbf{q}_3\|^\alpha}, \tag{1.1}$$

where $0 < \alpha < 2$. The purpose of the Planar Three Body Problem with homogeneous potential $U$ is to describe the motion of these three masses.

Applying Newton's second law yields the equations of motion

$$m_i \mathbf{q}_i'' = \sum_{j=1, j\neq i}^{3} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{\|\mathbf{q}_i - \mathbf{q}_j\|^{\alpha+2}} = \frac{\partial U}{\partial \mathbf{q}_i}, \qquad i = 1,2,3 \tag{1.2}$$

where $U(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)$ is defined in (1.1) and $' = \dfrac{d}{dt}$.

If $\alpha = 1$ we obtain the equations of the Planar Three Body Problem with Newtonian potential, that is, the only forces acting on the three masses are their mutual gravitational attraction.

Let us define $\Delta_{ij} = \{\mathbf{q} = (\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3) \in \mathbb{R}^6 | \mathbf{q}_i = \mathbf{q}_j\}$ for $1 \le i < j \le 3$, and $\Delta = \cup_{1\le i<j\le 3} \Delta_{ij}$. $\Delta$ is said to be the collision set. It is clear that $U$ is not defined for $\mathbf{q} \in \Delta$.

Let $\mathbf{q} = (\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3) \in \mathbb{R}^6 \setminus \Delta$ and $M = \mathrm{diag}(m_1,m_1,m_2,m_2,m_3,m_3)$. Then, equations (1.2) can be written as

$$M\mathbf{q}'' = \nabla U(\mathbf{q}).$$

If we define $\mathbf{p} = (\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) \in \mathbb{R}^6$ by $\mathbf{p} = M\mathbf{q}'$, then $\mathbf{p}_j = m_j \mathbf{q}_j'$ and $\mathbf{p}_j$ is the momentum of the $j$th particle. In variables $(\mathbf{q},\mathbf{p})$ the equations of motion

$$\begin{aligned} \mathbf{q}' &= M^{-1}\mathbf{p}, \\ \mathbf{p}' &= \nabla U(\mathbf{q}), \end{aligned} \tag{1.3}$$

form a Hamiltonian system with 6 degrees of freedom, with Hamiltonian function

$$\mathcal{H}(\mathbf{q},\mathbf{p}) = \frac{1}{2}\mathbf{p}^T M^{-1}\mathbf{p} - U(\mathbf{q}). \tag{1.4}$$

It is well known that there exist six first integrals for the Planar Three Body Problem. These integrals can be used to reduce the order of the system.

First, we can assume that the centre of masses is fixed at the origin. Then

$$\begin{aligned} m_1\mathbf{q}_1 + m_2\mathbf{q}_2 + m_3\mathbf{q}_3 &= 0, \\ \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 &= 0, \end{aligned} \tag{1.5}$$

obtaining 4 integrals for the system. Therefore, we can reduce the system in 4 equations. We also have that the angular momentum is a first integral for system (1.3). Then, if we fix the angular momentum

$$\mathbf{q}_1 \wedge \mathbf{p}_1 + \mathbf{q}_2 \wedge \mathbf{p}_2 + \mathbf{q}_3 \wedge \mathbf{p}_3 \quad = \quad \mathbf{c},$$

one can reduce in one the dimension of the system.

Moreover, the Hamiltonian function (1.4) is a first integral.

*Remark* 1.1.1. Let be $m := m_1 + m_2 + m_3$ and $\mu_j = \dfrac{m_j}{m}$, $j = 1, 2, 3$. After the change of variables $\mathbf{Q} = m^{-\frac{1}{\alpha+2}}\mathbf{q}$, $\mathbf{P} = m^{-\frac{\alpha+3}{\alpha+2}}\mathbf{p}$, we get the same equations (1.3) for $\mathbf{Q}$, $\mathbf{P}$ with $\mu_j$ instead of $m_j$, $j = 1, 2, 3$. So, we can assume $m_1 + m_2 + m_3 = 1$.

As $m_1 + m_2 + m_3 = 1$, we can represent the set of admissible masses as points in a triangle defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \quad = \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} m_1 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} m_2 + \begin{pmatrix} \dfrac{1}{2} \\ \dfrac{\sqrt{3}}{2} \end{pmatrix} m_3,$$

where $(x, y)$ denote the usual coordinates in $\mathbb{R}^2$. This relation represents a triangle in which the vertex $(0, 0)$, $(1, 0)$ and $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$ correspond to $m_1 = 1$, $m_2 = 1$ and $m_3 = 1$, respectively (see figure 1.2).

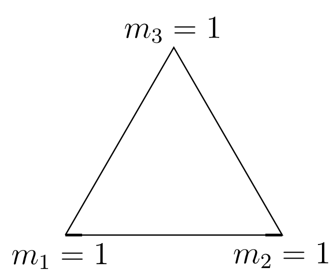



Figure 1.2: Triangle of masses

*Notation* 1.1.2. $I_n$ stands for the identity matrix of order $n$, $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ is the $2n \times 2n$ skew symmetric matrix, and $K_{2n}$ is the $2n \times 2n$ diagonal matrix defined as $K_{2n} = \mathrm{diag}(J_2, \ldots, J_2)$.

## 1.2 Homographic solutions

In this section we describe briefly central configurations and homographic solutions for the Planar Three Body Problem with homogeneous potential of degree $-\alpha$, $0 < \alpha < 2$. For details, see [M.H.]. Homographic solutions for a $n$–body problem are remarkable solutions for the simplicity of motions that describe. This kind of solutions are obtained from central configurations. A central configuration is a configuration in the position space such that for any mass $m_j$ its acceleration, $\ddot{\mathbf{q}}_j$, is proportional to the position, $\mathbf{q}_j$.

We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by considering $\mathbf{q}_j$ as complex numbers. We seek for solutions of (1.2) of the form

$$\mathbf{q}_j(t) \quad = \quad z(t)\mathbf{q}_{c_j}, \qquad j = 1, 2, 3, \tag{1.6}$$

where $\mathbf{q}_{c_j}$ are complex constant numbers and $z(t)$ is a complex valued function (the complex product is considered in the right hand part of (1.6)). Notice that the product by a complex number is a rotation followed by an homotecy, that is, an homography. So, a solution like (1.6) is called **homographic solution**.

If we substitute the expression of these special solutions into (1.2) and we rearrange the terms, yields

$$\|z(t)\|^{\alpha+2} z''(t) z(t)^{-1} m_i \mathbf{q}_{c_i} \quad = \quad \alpha \sum_{j=1, j\neq i}^{3} \frac{m_i m_j (\mathbf{q}_{c_j} - \mathbf{q}_{c_i})}{\|\mathbf{q}_{c_i} - \mathbf{q}_{c_j}\|^{\alpha+2}}.$$

As the right–hand part of last expression is constant, left–hand part needs also to be constant. Then, we obtain

$$z''(t) \quad = \quad -\lambda \frac{z(t)}{\|z(t)\|^{\alpha+2}}, \tag{1.7}$$

where $\lambda$ is a constant such that

$$-\lambda m_i \mathbf{q}_{c_i} \quad = \quad \alpha \sum_{j=1, j\neq i}^{3} \frac{m_i m_j (\mathbf{q}_{c_j} - \mathbf{q}_{c_i})}{\|\mathbf{q}_{c_i} - \mathbf{q}_{c_j}\|^{\alpha+2}}, \qquad i = 1, 2, 3. \tag{1.8}$$

Therefore, (1.6) is a solution of the Planar Three Body Problem (1.2) if and only if $z(t)$ is a solution of (1.7) and $\mathbf{q}_c = (\mathbf{q}_{c_1}, \mathbf{q}_{c_2}, \mathbf{q}_{c_3})$ satisfies (1.8). Equation (1.7) is the Kepler problem with homogeneous potential of degree $-\alpha$, $0 < \alpha < 2$. The solutions of this equation are discussed in the appendix A.

A solution $\mathbf{q}_c \in \mathbb{R}^6 \setminus \Delta$ of (1.8) is called a **central configuration**.

In the special case when $z(t)$ is a circular solution of (1.7) and $\mathbf{q}_c$ is a central configuration, the solution is also called a **relative equilibrium** because it becomes an equilibrium solution in a rotating coordinate system. For a relative equilibrium the three bodies rotate as a rigid body.

Equations (1.8) is a nontrivial system of nonlinear equations. This system can be written as

$$-\lambda M\mathbf{q}_c \quad = \quad \nabla U(\mathbf{q}_c). \tag{1.9}$$

Then, on a central configuration, the acceleration is proportional to the position.

If we consider the scalar product with $\mathbf{q}_c$ in the equation above, using Euler's Theorem about homogeneous functions (that stays that if $F : \mathbb{R}^m \longrightarrow \mathbb{R}$ is a $\mathcal{C}^1$ homogeneous function of degree $\beta$ then the equality $\mathbf{x}^T\nabla F(\mathbf{x}) = \beta F(\mathbf{x})$ holds) we obtain $\lambda = \alpha\dfrac{U(\mathbf{q}_c)}{\mathbf{q}_c^T M\mathbf{q}_c} > 0$.

(1.9) is invariant under rotations and under any uniform scaling. That is, if $\mathbf{q}_c$ is a central configuration and we introduce $\mathbf{s} \in \mathbb{R}^6$ as $\mathbf{q}_c = kR(\theta)\mathbf{s}$ where

$$R(\theta) = \mathrm{diag}(R_1(\theta), R_1(\theta), R_1(\theta))$$

is a $6 \times 6$ matrix, $\theta \in [0, 2\pi]$, $R_1(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and $k \in \mathbb{R}$, then

$$-\tilde{\lambda}M\mathbf{s} \quad = \quad \nabla U(\mathbf{s}),$$

with $\tilde{\lambda} = \lambda k^{\alpha+2}$. That is, $\mathbf{s}$ is a central configuration.

We will say that two central configurations are similar if they differ in a rotation or in an uniform scaling. Then, when counting central configurations one only counts similarity classes.

*Remark* 1.2.1. It is not restrictive to assume that $\lambda = 1$. This is due to the fact that if we introduce new variables $\mathbf{Q} = \lambda^{\frac{1}{\alpha+2}}\mathbf{q}$, $\mathbf{P} = \lambda^{-\frac{\alpha}{2(2+\alpha)}}\mathbf{p}$ and scale the time by $\lambda^{\frac{1}{2}}t$, the equations of motion are (1.3) and equation (1.9) holds with $\lambda = 1$.

Summarizing, assume that $\mathbf{q}_c$ is a central configuration. Then for any $z(t)$ solution of the Kepler Problem with homogeneous potential (1.7), an homographic solution of (1.2) is obtained as

$$\mathbf{q}(t) \quad = \quad z(t)\mathbf{q}_c.$$

We shall consider bounded homographic solutions. So, let $z(t)$ be a bounded solution of (1.7). It can be written (see appendix A) as $z(t) = r(t)e^{\mathrm{i}f(t)}$ where $r$ satisfies

$$r'' \quad = \quad -\frac{dV}{dr}(r), \tag{1.10}$$

being $V(r) = -\dfrac{1}{\alpha r^\alpha} + \dfrac{\omega^2}{2r^2}$ the effective potential of the associated Kepler problem and

$$f(t) \quad = \quad \int_0^t \frac{\omega}{r(s)^2} ds. \tag{1.11}$$

We note that $\omega$ is the angular momentum for the Kepler problem (1.7). In the Newtonian case, $f$ is the true anomaly.

We shall denote the energy of (1.10) by

$$E_K \quad = \quad \frac{(r')^2}{2} + V(r). \tag{1.12}$$

$V(r)$ has a minimum at $r^* = \omega^{\frac{2}{2-\alpha}}$ and $V(r^*) = -\left(\dfrac{2-\alpha}{2\alpha}\right)\omega^{-\frac{2\alpha}{2-\alpha}}$. For a fixed value $\omega > 0$, if $E_K$ is such that $V(r^*) < E_K < 0$, we get a periodic solution of (1.10) and then a bounded solution of (1.7) as $z(t) = r(t)e^{\mathrm{i}f(t)}$. On the level energy $E_K = V(r^*)$, (1.10) has an equilibrium point at $r = r^*$. Then, $z(t) = r^* e^{\mathrm{i}f(t)}$ is a circular solution of (1.7) and the corresponding homographic solution is a relative equilibrium.

It is well known from Euler and Lagrange that in the Newtonian case there exist three collinear central configurations with the three masses on a line and two triangular ones with the masses on the vertex of an equilateral triangle. From these central configuration, and by (1.6), we obtain the collinear and triangular homographic solutions, respectively. Figure 1.3 shows an elliptic and a circular collinear homographic solutions for the Newtonian Planar Three Body Problem.

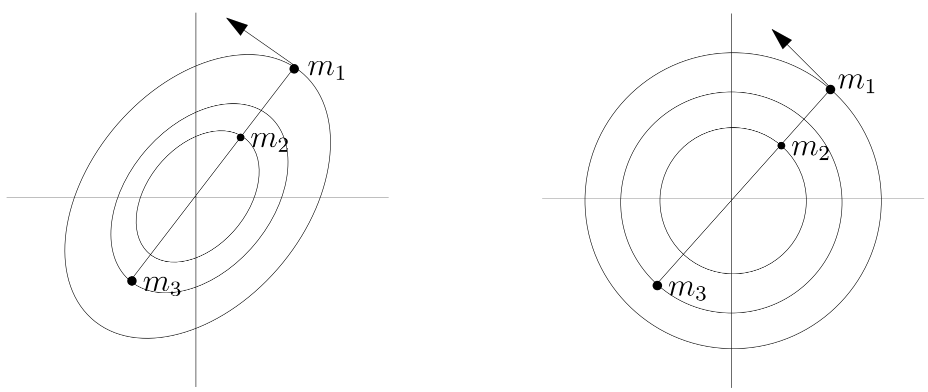



Figure 1.3: An elliptic and a circular collinear homographic solution for the Newtonian Planar Three Body Problem

In the homogeneous case there also exist three collinear homographic solutions and two triangular ones, as we shall see in section 1.5.

## 1.3 The homographic solutions as equilibrium points

Homographic solutions satisfy the equations of the Planar Three Body Problem with homogeneous potential, that is, a Hamiltonian system with six degrees of freedom. In this section we perform a suitable change of coordinates in such a way that the homographic solutions become equilibrium points of a periodic Hamiltonian system with six degrees of freedom. Moreover, we shall use the integrals of the centre of masses in order to reduce the order of the system by four.

We introduce a rotating and pulsating coordinate system through

$$\mathbf{q}(t) = r(t)\Omega(f(t))\boldsymbol{\zeta}(t),$$

where $\Omega = \mathrm{diag}(\Omega_1, \Omega_1, \Omega_1)$, $\Omega_1(f) = \begin{pmatrix} \cos f & -\sin f \\ \sin f & \cos f \end{pmatrix}$, and $r(t)$, $f(t)$ as defined in section 1.2, that is, for a given $\omega > 0$, $r(t)$ is a bounded solution of (1.10) and $f(t)$ is defined in (1.11). Using $f$ as independent variable the new system can be written as

$$\begin{aligned} \dot{\boldsymbol{\zeta}} &= K_6\boldsymbol{\zeta} + M^{-1}\boldsymbol{\eta}, \\ \dot{\boldsymbol{\eta}} &= \nabla\mathcal{V}(\boldsymbol{\zeta}) + K_6\boldsymbol{\eta}, \end{aligned} \tag{1.13}$$

where $\dot{}$ stands for the derivative with respect to $f$, $\boldsymbol{\eta}$ is the conjugate variable of $\boldsymbol{\zeta}$ and

$$\mathcal{V}(\boldsymbol{\zeta}) = \frac{r^{2-\alpha}}{\omega^2}U(\boldsymbol{\zeta}) + \frac{1}{2}\left(\frac{r^{2-\alpha}}{\omega^2} - 1\right)\boldsymbol{\zeta}^T M\boldsymbol{\zeta}.$$

In these variables, the homographic solutions are equilibrium points of system (1.13).

The Hamiltonian in the new variables becomes

$$H(\boldsymbol{\zeta}, \boldsymbol{\eta}) = \frac{1}{2}\boldsymbol{\eta}^T M^{-1}\boldsymbol{\eta} - \boldsymbol{\zeta}^T K_6\boldsymbol{\eta} - \mathcal{V}(\boldsymbol{\zeta}).$$

We note that $(\boldsymbol{\zeta}^*, \boldsymbol{\eta}^*)$ is an equilibrium of (1.13) if and only if

$$\begin{aligned} \nabla U(\boldsymbol{\zeta}^*) &= -M\boldsymbol{\zeta}^*, \\ \boldsymbol{\eta}^* &= -MK_6\boldsymbol{\zeta}^*, \end{aligned}$$

that is, $\boldsymbol{\zeta}^*$ is a central configuration. If we recover the initial variables we get an homographic solution $\mathbf{q}(t) = r(t)\Omega(f(t))\boldsymbol{\zeta}^*$.

The system (1.13) is a periodic (due to the presence of $r(f)$) Hamiltonian system with six degrees of freedom. A first reduction of (1.13) is done using the integrals of the centre of masses (1.5). We introduce new variables

$$\begin{array}{ll} \mathbf{u}_i = \boldsymbol{\zeta}_i - \boldsymbol{\zeta}_3, & \mathbf{v}_i = \boldsymbol{\eta}_i, \quad i = 1, 2, \\ \mathbf{u}_3 = \boldsymbol{\zeta}_3, & \mathbf{v}_3 = \boldsymbol{\eta}_1 + \boldsymbol{\eta}_2 + \boldsymbol{\eta}_3. \end{array}$$

The equations of motion become

$$\begin{aligned} \dot{\mathbf{u}}_1 &= J_2\mathbf{u}_1 + \alpha_1\mathbf{v}_1 + \frac{1}{m_3}\mathbf{v}_2 - \frac{1}{m_3}\mathbf{v}_3, \\ \dot{\mathbf{u}}_2 &= J_2\mathbf{u}_2 + \frac{1}{m_3}\mathbf{v}_1 + \alpha_2\mathbf{v}_2 - \frac{1}{m_3}\mathbf{v}_3, \\ \dot{\mathbf{u}}_3 &= J_2\mathbf{u}_3 + \frac{1}{m_3}(\mathbf{v}_3 - \mathbf{v}_1 - \mathbf{v}_2), \\ \dot{\mathbf{v}}_1 &= \frac{\partial}{\partial \mathbf{u}_1}\mathcal{V}(\tilde{\mathbf{u}}) + J_2\mathbf{v}_1, \\ \dot{\mathbf{v}}_2 &= \frac{\partial}{\partial \mathbf{u}_2}\mathcal{V}(\tilde{\mathbf{u}}) + J_2\mathbf{v}_2, \\ \dot{\mathbf{v}}_3 &= J_2\mathbf{v}_3, \end{aligned}$$

where $\tilde{\mathbf{u}} = (\mathbf{u}_1^T, \mathbf{u}_2^T, \mathbf{u}_3^T)^T$ and

$$\mathcal{V}(\tilde{\mathbf{u}}) = \frac{r^{2-\alpha}}{\omega^2}\mathcal{U}(\mathbf{u}_1, \mathbf{u}_2) + \frac{1}{2}\left(\frac{r^{2-\alpha}}{\omega^2} - 1\right)\tilde{\mathbf{u}}^T\tilde{C}\tilde{\mathbf{u}},$$

with

$$\mathcal{U}(\mathbf{u}_1, \mathbf{u}_2) = \frac{m_1m_2}{\|\mathbf{u}_1 - \mathbf{u}_2\|^\alpha} + \frac{m_1m_3}{\|\mathbf{u}_1\|^\alpha} + \frac{m_2m_3}{\|\mathbf{u}_2\|^\alpha}, \tag{1.14}$$

$$\tilde{C} = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}, \qquad C = m_1m_2\begin{pmatrix} \alpha_2m_3I_2 & -I_2 \\ -I_2 & \alpha_1m_3I_2 \end{pmatrix}, \tag{1.15}$$

$\alpha_1 = \dfrac{m_1 + m_3}{m_1m_3}$, $\alpha_2 = \dfrac{m_2 + m_3}{m_2m_3}$. From the integral of the centre of masses (1.5) we get $\mathbf{v}_3 = 0$. The equations for $\mathbf{u}_i, \mathbf{v}_i$, $i = 1, 2$, do not depend on $\mathbf{u}_3$. So, we can reduce to consider the following system

$$\begin{aligned} \dot{\mathbf{u}} &= K_4\mathbf{u} + C^{-1}\mathbf{v}, \\ \dot{\mathbf{v}} &= \nabla\mathcal{V}(\mathbf{u}) + K_4\mathbf{v}, \end{aligned} \tag{1.16}$$

where $\mathbf{u} = (\mathbf{u}_1^T, \mathbf{u}_2^T)^T$, $\mathbf{v} = (\mathbf{v}_1^T, \mathbf{v}_2^T)^T$ and

$$\mathcal{V}(\mathbf{u}) = \frac{r^{2-\alpha}}{\omega^2}\mathcal{U}(\mathbf{u}) + \frac{1}{2}\left(\frac{r^{2-\alpha}}{\omega^2} - 1\right)\mathbf{u}^T C\mathbf{u},$$

with $\mathcal{U}(\mathbf{u})$ and $C$ given in (1.14) and (1.15), respectively. We note that we can recover $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$ from $\mathbf{u}_1$, $\mathbf{u}_2$ using

$$\begin{pmatrix} m_1\boldsymbol{\zeta}_1 \\ m_2\boldsymbol{\zeta}_2 \end{pmatrix} = C\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}.$$

Then, $\boldsymbol{\zeta}_3$ is obtained using the integral of the centre of masses (1.5) as $\boldsymbol{\zeta}_3 = -\dfrac{1}{m_3}(m_1\boldsymbol{\zeta}_1 + m_2\boldsymbol{\zeta}_2)$.

The equilibria $(\mathbf{u}^*, \mathbf{v}^*)$ of this system are homographic solutions in these variables. An equilibrium point of system (1.16) must satisfy

$$\mathbf{v}^* = -CK_4\mathbf{u}^*. \tag{1.17}$$

Moreover, as $\nabla\mathcal{V}(\mathbf{u}^*) + K_4\mathbf{v}^* = 0$, we have

$$\frac{r^{2-\alpha}}{\omega^2}\nabla\mathcal{U}(\mathbf{u}) + \left(\frac{r^{2-\alpha}}{\omega^2} - 1\right)C\mathbf{u}^* + K_4\mathbf{v}^* = 0 \quad . \tag{1.18}$$

From (1.17) we obtain $K_4\mathbf{v}^* = C\mathbf{u}^*$ and therefore (1.18) becomes

$$\nabla\mathcal{U}(\mathbf{u}^*) = -C\mathbf{u}^*, \tag{1.19}$$

which together with (1.17) characterize the equilibria of (1.16).

## 1.4 The linearized system. Reduction of the order

We have seen that the homographic solutions are equilibrium points of a periodic system of differential equations. In this section we shall reduce the linearized system on an equilibrium point to a four dimensional system of differential equations. This reduction will be used in chapter 4 in order to study the stability of the homographic solutions.

The linearized system of (1.16) on an equilibrium $(\mathbf{u}^*, \mathbf{v}^*)$ is

$$\dot{\mathbf{y}} = \mathcal{A}\mathbf{y}, \qquad \mathcal{A} = \begin{pmatrix} K_4 & C^{-1} \\ \mathcal{D} & K_4 \end{pmatrix}, \tag{1.20}$$

where

$$\mathcal{D} = \frac{r^{2-\alpha}}{\omega^2} D\nabla\mathcal{U}(\mathbf{u}^*) + \left(\frac{r^{2-\alpha}}{\omega^2} - 1\right) C, \tag{1.21}$$

and $D$ denotes the differential.

We will see that system (1.20) can be written as two uncoupled systems of order four.

**Lemma 1.4.1.** *Let $\mathbf{u}^*$ be a solution of (1.19). System (1.20) can be written as two uncoupled linear systems of order four with matrices*

$$\mathcal{B}_1(f) = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ h_1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B}_2(f) = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ b_{11} & b_{21} & 0 & -1 \\ b_{12} & b_{22} & 1 & 0 \end{pmatrix}, \tag{1.22}$$

*where $h_1 = h_1(f) = (\alpha+2)\dfrac{r^{2-\alpha}}{\omega^2} - 1$, and*

$$b_{11} = \frac{r^{2-\alpha}}{\omega^2}(\gamma_{11}+1)-1, \quad b_{12} = \frac{r^{2-\alpha}}{\omega^2}\gamma_{12}, \quad b_{21} = \frac{r^{2-\alpha}}{\omega^2}\gamma_{21}, \quad b_{22} = \frac{r^{2-\alpha}}{\omega^2}(\gamma_{22}+1)-1,$$

*being $\gamma_{11}, \gamma_{12}, \gamma_{21}$ and $\gamma_{22}$ constant coefficients depending on $\mathbf{u}^*$.*

**Proof**

We introduce the following vectors

$$\mathbf{x}_1 = \begin{pmatrix} \mathbf{u}^* \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} \mathbf{0} \\ K_4 C\mathbf{u}^* \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} \mathbf{0} \\ C\mathbf{u}^* \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} K_4\mathbf{u}^* \\ \mathbf{0} \end{pmatrix}.$$

First, we shall show that the subspace $X$ of $\mathbb{R}^8$ spanned by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ is invariant under $\mathcal{A}$. Then, we will introduce also the skew–orthogonal complement in $\mathbb{R}^8$ of $X$ in order to uncouple the system (1.20).

In order to show that $X$ is invariant under $\mathcal{A}$ we need some relations between $D\nabla\mathcal{U}(u)$ and $\nabla\mathcal{U}(u)$. By Euler's Theorem for homogeneous functions we have $\nabla\mathcal{U}(\mathbf{u})^T\mathbf{u} = -\alpha\mathcal{U}(\mathbf{u})$. If we differenciate this equality, we obtain

$$D\nabla\mathcal{U}(\mathbf{u})\mathbf{u} = -(\alpha+1)\nabla\mathcal{U}(\mathbf{u}). \tag{1.23}$$

Moreover, due to the homogeneity of $\mathcal{U}(\mathbf{u})$ we obtain

$$D\nabla\mathcal{U}(\mathbf{u})K_4\mathbf{u} = K_4\nabla\mathcal{U}(\mathbf{u}). \tag{1.24}$$

So, if $\mathbf{u}^*$ is a central configuration from (1.23) and (1.24) and using (1.19) we get

$$D\nabla\mathcal{U}(\mathbf{u}^*)\mathbf{u}^* = (\alpha+1)C\mathbf{u}^*, \qquad D\nabla\mathcal{U}(\mathbf{u}^*)K_4\mathbf{u}^* = -K_4C\mathbf{u}^*. \tag{1.25}$$

Then

$$\mathcal{D}\mathbf{u}^* = \left[(\alpha+2)\frac{r^{2-\alpha}}{\omega^2} - 1\right] C\mathbf{u}^*, \qquad \mathcal{D}K_4\mathbf{u}^* = -CK_4\mathbf{u}^*, \tag{1.26}$$

where the second equality holds due to the fact that $K_4$ and $C$ commute. Using (1.26) we get easily

$$\mathcal{A}\mathbf{x}_1 = \mathbf{x}_4 + h_1\mathbf{x}_3, \quad \mathcal{A}\mathbf{x}_2 = \mathbf{x}_4 - \mathbf{x}_3, \quad \mathcal{A}\mathbf{x}_3 = \mathbf{x}_1 + \mathbf{x}_2, \quad \mathcal{A}\mathbf{x}_4 = -\mathbf{x}_1 - \mathbf{x}_2,$$

where $h_1 = (\alpha+2)\dfrac{r^{2-\alpha}}{\omega^2} - 1$. So, the four dimensional subspace, $X$, of $\mathbb{R}^8$ spanned by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ is invariant under $\mathcal{A}$ and the system (1.20) reduced to $X$ is given by the matrix $\mathcal{B}_1(f)$.

Let us denote by $W$ the four dimensional subspace of $\mathbb{R}^8$ spanned by $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$ where

$$\mathbf{w}_1 = \begin{pmatrix} C^{-1}\boldsymbol{\eta}_1 \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} C^{-1}\boldsymbol{\eta}_2 \\ \mathbf{0} \end{pmatrix}, \qquad \mathbf{w}_3 = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\eta}_1 \end{pmatrix}, \quad \mathbf{w}_4 = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\eta}_2 \end{pmatrix},$$

and

$$\boldsymbol{\eta}_1 = J_4\mathbf{u}^* + \gamma_1 K_4\mathbf{u}^*, \quad \boldsymbol{\eta}_2 = K_4\boldsymbol{\eta}_1, \qquad \gamma_1 = \frac{(\mathbf{u}^*)^T K_4 J_4 \mathbf{u}^*}{\|\mathbf{u}^*\|^2}. \tag{1.27}$$

We want to see that $W$ is the skew–orthogonal complement in $\mathbb{R}^8$ of $X$, that is, $W = \{\mathbf{w} \in \mathbb{R}^8 \,|\, \mathbf{w}^T J_8 \mathbf{x}_i = 0, i = 1, \ldots, 4\}$.

We have that

$$\boldsymbol{\eta}_1^T\mathbf{u}^* = -(\mathbf{u}^*)^T J_4\mathbf{u}^* - \gamma_1(\mathbf{u}^*)^T K_4\mathbf{u}^* = 0,$$

$$\boldsymbol{\eta}_2^T\mathbf{u}^* = (\mathbf{u}^*)^T J_4 K_4\mathbf{u}^* + \gamma_1(\mathbf{u}^*)^T K_4^2\mathbf{u}^* = 0.$$

Using these relations we find that $W$ is the subspace that we are looking for.

Next and last step is to find the system reduced to $W$.

We have that

$$\mathcal{A}\mathbf{w}_1 = \begin{pmatrix} K_4C^{-1}\boldsymbol{\eta}_1 \\ \mathcal{D}C^{-1}\boldsymbol{\eta}_1 \end{pmatrix}, \qquad \mathcal{A}\mathbf{w}_2 = \begin{pmatrix} K_4C^{-1}\boldsymbol{\eta}_2 \\ \mathcal{D}C^{-1}\boldsymbol{\eta}_2 \end{pmatrix},$$
$$\mathcal{A}\mathbf{w}_3 = \mathbf{w}_1 + \mathbf{w}_4, \qquad \mathcal{A}\mathbf{w}_4 = \mathbf{w}_2 - \mathbf{w}_3, \tag{1.28}$$

with

$$\mathcal{D}C^{-1}\boldsymbol{\eta}_j = \frac{r^{2-\alpha}}{\omega^2}(D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_j + \boldsymbol{\eta}_j) - \boldsymbol{\eta}_j, \quad j = 1, 2.$$

As $\mathbf{u}^*, K_4\mathbf{u}^*, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2$ span $\mathbb{R}^4$, we can write

$$D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_j = \gamma_{j1}\boldsymbol{\eta}_1 + \gamma_{j2}\boldsymbol{\eta}_2 + \gamma_{j3}\mathbf{u}^* + \gamma_{j4}K_4\mathbf{u}^*, \quad j = 1, 2, \tag{1.29}$$

for some constants $\gamma_{ji}$, $j = 1, 2$, $i = 1, \ldots, 4$.

Due to the symmetry of $D\nabla\mathcal{U}(\mathbf{u}^*)$ and $C$, from (1.25) we have that

$$(\mathbf{u}^*)^T D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_j = (\alpha+1)(\mathbf{u}^*)^T\eta_j = 0, \quad j = 1, 2$$

$$(K_4\mathbf{u}^*)^T D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_1 = (\mathbf{u}^*)^T CK_4C^{-1}\boldsymbol{\eta}_1 = (\mathbf{u}^*)^T\boldsymbol{\eta}_2 = 0,$$

$$(K_4\mathbf{u}^*)^T D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_2 = (\mathbf{u}^*)^T CK_4C^{-1}\boldsymbol{\eta}_2 = -(\mathbf{u}^*)^T\boldsymbol{\eta}_1 = 0.$$

Then, $D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_j = \gamma_{j1}\boldsymbol{\eta}_1 + \gamma_{j2}\boldsymbol{\eta}_2$, $j = 1, 2$, and

$$\begin{aligned}
\gamma_{11} &= \frac{1}{\|\boldsymbol{\eta}_1\|^2}\boldsymbol{\eta}_1^T D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_1, \qquad \gamma_{12} = \frac{1}{\|\boldsymbol{\eta}_2\|^2}\boldsymbol{\eta}_2^T D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_1,\\
\gamma_{21} &= \frac{1}{\|\boldsymbol{\eta}_1\|^2}\boldsymbol{\eta}_1^T D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_2, \qquad \gamma_{22} = \frac{1}{\|\boldsymbol{\eta}_2\|^2}\boldsymbol{\eta}_2^T D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_2.
\end{aligned} \tag{1.30}$$

A simple computation shows that

$$\begin{aligned}
\mathcal{D}C^{-1}\boldsymbol{\eta}_1 &= b_{11}\boldsymbol{\eta}_1 + b_{12}\boldsymbol{\eta}_2,\\
\mathcal{D}C^{-1}\boldsymbol{\eta}_2 &= b_{21}\boldsymbol{\eta}_1 + b_{22}\boldsymbol{\eta}_2.
\end{aligned}$$

Then, from (1.28),

$$\begin{aligned}
\mathcal{A}\mathbf{w}_1 &= \mathbf{w}_2 + b_{11}\mathbf{w}_3 + b_{12}\mathbf{w}_4, \qquad \mathcal{A}\mathbf{w}_2 = -\mathbf{w}_1 + b_{21}\mathbf{w}_3 + b_{22}\mathbf{w}_4,\\
\mathcal{A}\mathbf{w}_3 &= \mathbf{w}_1 + \mathbf{w}_4, \qquad\qquad\qquad\quad \mathcal{A}\mathbf{w}_4 = \mathbf{w}_2 - \mathbf{w}_3,
\end{aligned}$$

and the system (1.20) reduced to $W$ is defined by the matrix $\mathcal{B}_2(f)$. □

We shall see that, for any equilibrium $(\mathbf{u}^*, \mathbf{v}^*)$ of (1.16), 1 is an eigenvalue with multiplicity four of the monodromy matrix of the system defined by $\mathcal{B}_1(f)$. Then, the nontrivial characteristic exponents will be given by the system defined by $\mathcal{B}_2(f)$.

$\mathcal{B}_1(f)$ and $\mathcal{B}_2(f)$ depend on $f$ through the function $\dfrac{r^{2-\alpha}}{\omega^2}$ where $r$ is a solution of (1.10). We recall that for the levels of energy of interest one has $r(t)$ bounded. Its minimum tends to zero when $\omega$ does. Notice that in the Newtonian case ($\alpha = 1$) $\dfrac{r}{\omega^2} = \dfrac{1}{1 + e\cos f}$, where $e$ denotes the eccentricity of the orbit.

We introduce the function $g = \dfrac{\omega^{\frac{2}{2-\alpha}}}{r}$. Using (1.10) we obtain that

$$\ddot{g} = -g + g^{\alpha-1},$$

where we recall that $\dot{}$ stands for $\dfrac{d}{df}$. Then $g$ is a periodic solution of

$$\ddot{z} \;=\; -\frac{dU}{dz}(z), \quad \text{with} \quad U(z) = \frac{z^2}{2} - \frac{z^\alpha}{\alpha}. \tag{1.31}$$

We denote by $T$ the period of $g$ in time $f$.

$U$ has a positive zero at $\tilde{z} = \left(\dfrac{2}{\alpha}\right)^{\frac{1}{2-\alpha}}$, and a minimum at $z = 1$ attaining the value $U(1) = \dfrac{\alpha - 2}{2\alpha}$. Figure 1.4 shows some plots of $U$.

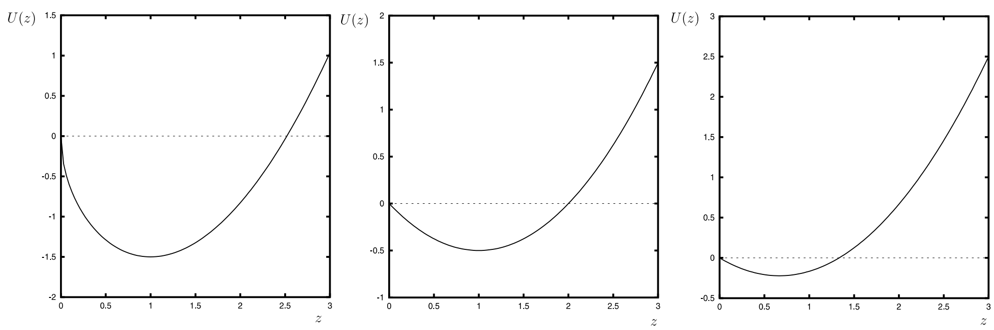



Figure 1.4: Plot of $U(z) = \dfrac{z^2}{2} - \dfrac{z^\alpha}{\alpha}$ for $\alpha = \dfrac{1}{2}, 1, \dfrac{3}{2}$, respectively

The energy of the problem (1.31) is

$$E = \frac{(\dot{z})^2}{2} + U(z). \tag{1.32}$$

Using $\dot{g} = -\omega^{\frac{\alpha}{2-\alpha}} r'$ (where we recall $' = \dfrac{d}{dt}$), we get

$$E \;=\; \omega^{\frac{2\alpha}{2-\alpha}} \left[\frac{1}{2}(r')^2 + \frac{\omega^2}{2r^2} - \frac{1}{\alpha r^\alpha}\right] = \omega^{\frac{2\alpha}{2-\alpha}} E_K, \tag{1.33}$$

where $E_K$ is given in (1.12).

Once a central configuration has been fixed, we get a family of homographic solutions for $E \in \left[-\dfrac{2-\alpha}{2\alpha}, 0\right)$. For the sake of simplicity we shall fix a value of $E_K$, say $E_K = -\dfrac{1}{2}$, and move $\omega$ in the following range

$$0 < \omega \leq \omega_c,$$

where $\omega_c$ satisfies

$$-\frac{1}{2}\omega_c^{\frac{2\alpha}{2-\alpha}} \;=\; -\frac{2-\alpha}{2\alpha}, \tag{1.34}$$

which corresponds to a relative equilibrium. In that case, $g \equiv 1$.

Let us define $g_0 = g(0)$ the minimum of $g$. It is clear that if $0 < \omega < \omega_c$, $0 < g_0 < 1$. Moreover, $g_0$ tends to zero as $\omega$ tends to zero. However, $\dfrac{r^{2-\alpha}}{\omega^2} = g^{\alpha-2}$. If we consider the linear systems defined by $\mathcal{B}_1(f)$ and $\mathcal{B}_2(f)$ as $\omega$ tends to 0, the limit systems have a singularity at $f = 0$. In this case we have an homagraphic solution going to collision.

*Remark* 1.4.2. If $\omega = \omega_c$ then $E = U(1) = \dfrac{\alpha - 2}{2\alpha}$ and (1.31) has an equilibrium point $z^* = 1$. Then, $g(f) \equiv 1$ and the function $h_1(f)$ defined in Lemma 1.4.1 becomes constant. In fact, $h_1 = \alpha + 1$. In this case the linear systems defined by $\mathcal{B}_1(f)$ and $\mathcal{B}_2(f)$ have constant coefficients. Furthermore, $r = \omega^{\frac{2}{2-\alpha}}$ and so, we have a circular solution of the Kepler problem.

**Lemma 1.4.3.** *For $0 < \omega \leq \omega_c$ the monodromy matrix $\mathcal{C}$ of the linear system*

$$\dot{\mathbf{U}} = \mathcal{B}_1(f)\mathbf{U} \tag{1.35}$$

*has the eigenvalue 1 with multiplicity four.*

**Proof**

In order to compute the monodromy matrix $\mathcal{C}$ of (1.35) it is necessary to integrate the variational equations. In this case, we need to solve the same system (1.35) with initial conditions the vectors $\mathbf{e}_j$, $j = 1, \ldots, 4$, of the canonical basis.

Let be $\mathbf{V} = (u_1, u_2, u_3, u_4)^T$. We denote by $\mathbf{V}_j(f)$, $j = 1, \ldots, 4$, the solution of (1.35) with initial condition $\mathbf{V}_j(0) = \mathbf{e}_j$. Then, $\mathcal{C} = (\mathbf{V}_1(T), \mathbf{V}_2(T), \mathbf{V}_3(T), \mathbf{V}_4(T))^T$ where $T$ is the period of $g$.

System (1.35) can be written as

$$\begin{aligned}
\dot{u}_1 &= u_3 - u_4, \\
\dot{u}_2 &= u_3 - u_4, \\
\dot{u}_3 &= h_1 u_1 - u_2, \\
\dot{u}_4 &= u_1 + u_2,
\end{aligned} \tag{1.36}$$

with $h_1(f) = (\alpha + 2)g^{\alpha-2} - 1$.

First we consider $\omega = \omega_c$. From remark 1.4.2 we know that the matrix $\mathcal{B}_1(f)$ is constant, and the solution of (1.31) for this value of $\omega$ corresponds to a circular solution $\mathbf{z}_c(t) = \omega^{\frac{2}{2-\alpha}} e^{if(t)}$ of the planar Kepler problem. Moreover, $h_1(f) = \alpha + 1$. The statement of the Lemma in this particular case follows due to the fact that the constant matrix $\mathcal{B}_1(f)$ has eigenvalues $0, 0, \pm \mathrm{i}\sqrt{2-\alpha}$ and the circular solution has period $\dfrac{2\pi}{\sqrt{2-\alpha}}$ (see appendix A).

Assume now that $0 < \omega < \omega_c$.

From the two first equations in (1.36) it is easy to see that $u_1 - u_2 = k$ with $k$ a real constant. Then, as $\ddot{u}_1 = \dot{u}_3 - \dot{u}_4$, $u_1$ satisfies the following equation

$$\ddot{u}_1 \quad = \quad [(\alpha+2)g^{\alpha-2} - 4]u_1 + 2k. \tag{1.37}$$

Once $u_1$ is obtained, $u_2$ is recovered from $u_2 = u_1 - k$ and $u_3$, $u_4$ are obtained by integration of the following equations

$$\begin{aligned} \dot{u}_3 &= [(\alpha+2)g^{\alpha-2} - 2]u_1 + k, \\ \dot{u}_4 &= 2u_1 - k. \end{aligned}$$

Now, we compute $\mathbf{V}_3, \mathbf{V}_4$. For these vectors, the initial conditions for $u_1$ and $u_2$ are $u_1(0) = u_2(0) = 0$. Then, we need to solve equation (1.37) with $k = 0$. It can be seen easily that $u_1(f) = cg\dot{g}$, $c$ a constant, is a solution of this equation.

As for $\mathbf{V}_3$ the initial conditions for $u_3$ and $u_4$ are $u_3(0) = 1, u_4(0) = 0$, we have that $\dot{u}_1(0) = 1$. Then

$$c \quad = \quad \frac{1}{g_0^\alpha - g_0^2}. \tag{1.38}$$

We note that $c$ is well defined due to the fact that for $0 < \omega < \omega_c$ one has $0 < g_0 < 1$.

A simple computation shows that

$$\mathbf{V}_3(f) \quad = \quad (cg\dot{g},\, cg\dot{g},\, 1 + c\frac{\alpha+2}{\alpha}(g^\alpha - g_0^\alpha) - c(g^2 - g_0^2),\, c(g^2 - g_0^2))^T.$$

Taking into account that for $\mathbf{V}_4$, $\dot{u}_1(0) = -1$, one can prove that

$$\mathbf{V}_4(f) \quad = \quad (-cg\dot{g},\, -cg\dot{g},\, -c\frac{\alpha+2}{\alpha}(g^\alpha - g_0^\alpha) + c(g^2 - g_0^2),\, 1 - c(g^2 - g_0^2))^T,$$

where $c$ is given in (1.38). As $g$ is $T$–periodic, $\mathbf{V}_3$ and $\mathbf{V}_4$ are also periodic with period $T$. Then,

$$\mathbf{V}_3(T) \quad = \quad (0,0,1,0)^T, \qquad \mathbf{V}_4(T) = (0,0,0,1)^T.$$

Now we look for $\mathbf{V}_2(f)$. The initial conditions $u_1(0) = 0$, $u_2(0) = 1$ imply that $k = -1$. Moreover, as $u_3(0) = 0$ and $u_4(0) = 0$, $\dot{u}_1(0) = 0$. So, we have to solve the equation (1.37) for $k = -1$ with initial conditions $u_1(0) = 0$, $\dot{u}_1(0) = 0$. Let us assume, for the moment being, that $u_1(T) = 0$. Therefore $u_2(T) = u_1(T) + 1 = 1$ and

$$\mathbf{V}_2(T) \quad = \quad (0, 1, u_3(T), u_4(T))^T.$$

The monodromy matrix has the following form

$$\mathcal{C} \;=\; \begin{pmatrix} * & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix},$$

where $*$ denotes some values that are not relevant. However the Liouville Theorem implies that $\det(\mathcal{C}) = 1$. Therefore, 1 is an eigenvalue of $\mathcal{C}$ with multiplicity four.

Our purpose now is to prove that $u_1(T) = 0$.

We introduce variables $x_1 = u_1$, $x_2 = \dot{u}_1$ and write the equation (1.37) for $k = -1$ as a non homogeneous linear system. So, we consider the initial value problem

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \tilde{h}(f)x_1 - 2, \qquad \tilde{h}(f) = [(\alpha+2)g^{\alpha-2} - 4], \\ x_1(0) &= 0, \quad x_2(0) = 0. \end{aligned} \tag{1.39}$$

We know that $\varphi_2(f) = (b_1(f), b_2(f))^T$ with

$$b_1(f) = cg\dot{g}, \qquad b_2(f) = c(\dot{g}^2 - g^2 + g^\alpha),$$

is a solution of the homogeneous system such that $\varphi_2(0) = (0,1)^T$. Moreover, it is periodic with period $T$. Then, $\varphi_2(T) = \varphi_2(0)$.

Let $\Phi(f)$ be the fundamental matrix of the homogeneous system such that $\Phi(0) = I_2$. We write

$$\Phi(f) \;=\; (\varphi_1(f), \varphi_2(f)),$$

where $\varphi_1(f) = (a_1(f), a_2(f))^T$ is a solution of the homogeneous system such that $\varphi_1(0) = (1,0)^T$. Using Liouville Theorem, $\det(\Phi(f)) = 1$ for any $f$. Then, $\det(\Phi(T)) = a_1(T) = 1$.

Let $\mathbf{x}(f) = (x_1(f), x_2(f))^T$ be the solution of the initial value problem (1.39). Using variation of parameters we can write

$$\mathbf{x}(f) \;=\; \Phi(f)\int_0^f \Phi^{-1}(s)\mathbf{r}(s)ds,$$

where $\mathbf{r}(s) = (0,-2)^T$. The first component of $\mathbf{x}(f)$ is

$$x_1(f) \;=\; a_1(f)\int_0^f 2b_1(s)ds - b_1(f)\int_0^f 2a_1(s)ds.$$

We recall that $b_1(T) = 0$. Then,

$$x_1(T) \quad = \quad 2\int_0^T b_1(s)ds = 2c\int_0^T g\dot{g}ds = c[g^2(T) - g^2(0)] = 0,$$

where the periodicity of $g(f)$ has been used. □

Let us consider now the linear system $\dot{\mathbf{U}} = \mathcal{B}_2(f)\mathbf{U}$. After Lemma 1.4.3 we know that the eigenvalues of the monodromy matrix of this system give us the non trivial characteristic multipliers.

We introduce $\mathbf{w} = M^{-1}\mathbf{U}$, where $M = \begin{pmatrix} I_2 & 0 \\ J_2 & I_2 \end{pmatrix}$, and then we can write our system as

$$\begin{aligned} \dot{\mathbf{w}} = B(f)\mathbf{w}, \qquad & B(f) = \begin{pmatrix} 0 & I_2 \\ \tilde{B} & -2J_2 \end{pmatrix}, \\ \tilde{B} = g^{\alpha-2} & \begin{pmatrix} \gamma_{11}+1 & \gamma_{21} \\ \gamma_{12} & \gamma_{22}+1 \end{pmatrix}. \end{aligned} \tag{1.40}$$

We recall that $g$ depends on the angular momentum $\omega$, and $\gamma_{11}$, $\gamma_{12}$, $\gamma_{21}$, $\gamma_{22}$ are constant depending on the solutions of (1.19), that is, on the central configurations. In the following section we shall solve the equation (1.19), and we will compute these constants $\gamma_{ij}$, $i,j = 1,2$, for the collinear and triangular central configurations.

## 1.5 Central configurations

In this section we study the solutions of the equation (1.19), that is, we seek central configurations. Our purpose is to compute the coefficients $\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}$ for any solution $\mathbf{u}^*$ of (1.19). Moreover, we will study some properties of these coefficients that will be useful in chapter 4 in order to study the stability of the homographic solutions.

We write (1.19) as

$$\begin{aligned} -\alpha\frac{m_1m_2(\mathbf{u}_1-\mathbf{u}_2)}{\|\mathbf{u}_1-\mathbf{u}_2\|^{\alpha+2}} - \alpha\frac{m_1m_3\mathbf{u}_1}{\|\mathbf{u}_1\|^{\alpha+2}} &= -m_1m_2(\alpha_2m_3\mathbf{u}_1 - \mathbf{u}_2), \\ \alpha\frac{m_1m_2(\mathbf{u}_1-\mathbf{u}_2)}{\|\mathbf{u}_1-\mathbf{u}_2\|^{\alpha+2}} - \alpha\frac{m_2m_3\mathbf{u}_2}{\|\mathbf{u}_2\|^{\alpha+2}} &= -m_1m_2(-\mathbf{u}_1 + \alpha_1m_3\mathbf{u}_2), \end{aligned} \tag{1.41}$$

where we know from section 1.3 that $\alpha_1 = \dfrac{m_1+m_3}{m_1m_3}$ and $\alpha_2 = \dfrac{m_2+m_3}{m_2m_3}$.

We look for the solutions $\mathbf{u}^*$ of this system of equations.

First we study the case when the three particles lie on a straight line. This kind of solutions are known as **collinear configurations**. We can assume that this line is the abscissa axis and we consider the masses ordered from left to right as $m_3$, $m_2$, $m_1$ (see figure 1.5). Other collinear configurations are obtained from this by a permutation of the masses. So, we can assume $\mathbf{u}^* = ((\mathbf{u}_1^*)^T, (\mathbf{u}_2^*)^T)^T$ with $\mathbf{u}_1^* = (u_1, 0)^T$ and $\mathbf{u}_2^* = (u_2, 0)^T$. We introduce $\rho$ and $a$ by $u_1 = a(\rho+1)$, $u_2 = a$.

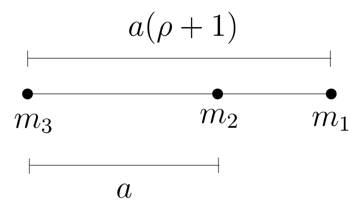



Figure 1.5: Coordinates $u_1$ and $u_2$ in a collinear configuration

If we substitute these expressions of $\mathbf{u}_1$ and $\mathbf{u}_2$ in (1.41) we obtain that $\rho > 0$ is the solution of the equation

$$\begin{aligned} & m_1[\rho^{\alpha+2} - (\rho+1)^{\alpha+2}] + m_3\rho^{\alpha+1}[(\rho+1)^{\alpha+2} - 1] + \\ & m_2(\rho+1)^{\alpha+1}(\rho^{\alpha+2} - 1) = 0, \end{aligned} \tag{1.42}$$

and

$$a^{\alpha+2} = \frac{\alpha[m_2(\rho+1)^{\alpha+1} + m_3\rho^{\alpha+1}]}{\rho^{\alpha+1}(\rho+1)^{\alpha+1}[m_3(\rho+1) + m_2\rho]}.$$

We note that $\rho = 1$ is the solution of (1.42) in the symmetric case in which $m_1 = m_3$. In the Newtonian case ($\alpha = 1$) (1.42) is the well known quintic equation for collinear configurations.

Now we look for non collinear solutions. We note that to solve the equation (1.19) is equivalent to compute the critical points of the function

$$\tilde{U}(\mathbf{u}) = \mathcal{U}(\mathbf{u}) + \frac{1}{2}\mathbf{u}^T C \mathbf{u}, \tag{1.43}$$

where $\mathcal{U}$ and $C$ are given in (1.14) and (1.15), respectively. We define $r_1 = \|\mathbf{u}_1 - \mathbf{u}_2\|$, $r_2 = \|\mathbf{u}_1\|$, $r_3 = \|\mathbf{u}_2\|$. Let us see that $\tilde{U}$ can ve written in terms of $r_1$, $r_2$ and $r_3$. A simple computation shows that

$$\mathbf{u}^T C \mathbf{u} = m_1 m_2(\alpha_2 m_3\|\mathbf{u}_1\|^2 - 2\mathbf{u}_1^T\mathbf{u}_2 + \alpha_1 m_3\|\mathbf{u}_2\|^2).$$

As $2\mathbf{u}_1^T\mathbf{u}_2 = \|\mathbf{u}_1\|^2 + \|\mathbf{u}_2\|^2 - \|\mathbf{u}_1 - \mathbf{u}_2\|^2$ we obtain that

$$\tilde{U}(\mathbf{u}) = m_1 m_2 \left(\frac{1}{r_1^\alpha} + \frac{r_1^2}{2}\right) + m_1 m_3 \left(\frac{1}{r_2^\alpha} + \frac{r_2^2}{2}\right) + m_2 m_3 \left(\frac{1}{r_3^\alpha} + \frac{r_3^2}{2}\right).$$

It is easy to see that a non collinear critical point of $\tilde{U}$ satisfies $r_1 = r_2 = r_3 = \alpha^{\frac{1}{\alpha+2}}$. In this case, the three masses are at the vertices of an equilateral triangle and $\|\mathbf{u}_1\| = \|\mathbf{u}_2\| = \|\mathbf{u}_1 - \mathbf{u}_2\| = \varrho$ where $\varrho = \alpha^{\frac{1}{\alpha+2}}$. These solutions are known as **triangular configurations**. There are two triangular configurations (see figure 1.6). We shall work with the first case in this figure. The other triangular configuration is obtained from this by changing the masses $m_1$ and $m_2$. We can assume that $\mathbf{u}_1 = \left(\frac{\varrho}{2}, \frac{\sqrt{3}}{2}\varrho\right)^T$ and $\mathbf{u}_2 = \left(-\frac{\varrho}{2}, \frac{\sqrt{3}}{2}\varrho\right)^T$.

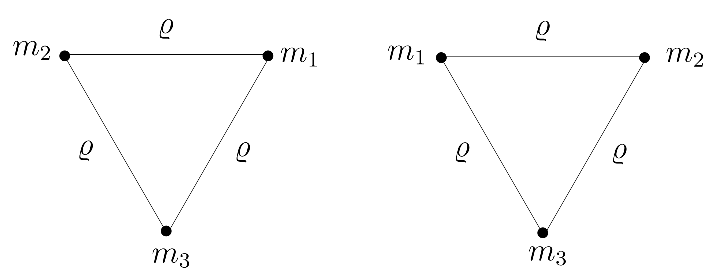



Figure 1.6: The triangular configurations in the Planar Three Body Problem with homogeneous potential of degree $-\alpha$, $0 < \alpha < 2$

We note that the three variables $r_1$, $r_2$ and $r_3$ are local coordinates near a non collinear configuration. Therefore, we have obtained all the non collinear solutions of the equation for central configurations of the Planar Three Body Problem with homogeneous potential of degree $-\alpha$, $0 < \alpha < 2$.

Now we want to compute the coefficients $\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}$ in (1.40). To this end we shall distinguish between a collinear and a triangular central configuration.

We fix $\mathbf{u}^* = ((\mathbf{u}_1^*)^T, (\mathbf{u}_2^*)^T)^T$, $\mathbf{u}_1^*, \mathbf{u}_2^* \in \mathbb{R}^2$ a central configuration. It is easy to check that $\|\boldsymbol{\eta}_1\|^2, \|\boldsymbol{\eta}_2\|^2$ and $\gamma_1$ in (1.27) can be computed through

$$\begin{aligned} \gamma_1 &= \frac{2(\mathbf{u}_1^*)^T J_2 \mathbf{u}_2^*}{\|\mathbf{u}_1^*\|^2 + \|\mathbf{u}_2^*\|^2}, \\ \|\boldsymbol{\eta}_1\|^2 &= [\|\mathbf{u}_1^*\|^2 + \|\mathbf{u}_2^*\|^2](1+\gamma_1^2) - 4\gamma_1 (\mathbf{u}_1^*)^T J_2 \mathbf{u}_2^*, \\ \|\boldsymbol{\eta}_2\|^2 &= \|\boldsymbol{\eta}_1\|^2. \end{aligned} \tag{1.44}$$

We begin with a triangular configuration. In this case, we have taken $\mathbf{u}_1 = \left(\frac{\varrho}{2}, \frac{\sqrt{3}}{2}\varrho\right)^T$ and $\mathbf{u}_2 = \left(-\frac{\varrho}{2}, \frac{\sqrt{3}}{2}\varrho\right)^T$. Using (1.44) we obtain that

$$\gamma_1 = \frac{\sqrt{3}}{2}, \qquad \|\boldsymbol{\eta}_1\|^2 = \|\boldsymbol{\eta}_2\|^2 = \frac{\varrho^2}{2}.$$

From the definition of $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ in (1.27) we have that these vectors can be written as $\boldsymbol{\eta}_1^T = (c_1, c_2, c_1, -c_2)$ and $\boldsymbol{\eta}_2^T = (c_2, -c_1, -c_2, -c_1)$, where $c_1 = \frac{\varrho}{4}$ and $c_2 = \frac{\sqrt{3}\varrho}{4}$. Moreover

$$D\nabla\mathcal{U}(\mathbf{u}^*) \;=\; \begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ a_2 & a_4 & 0 & a_5 \\ a_3 & 0 & a_6 & a_7 \\ 0 & a_5 & a_7 & a_8 \end{pmatrix}, \tag{1.45}$$

where $a_j$, $j = 1, \ldots, 8$ are constants depending on the masses and on $\varrho$. They are given in appendix B. Then after some trivial computations and using (1.30) we get the constants $\gamma_{ij}$ for the triangular configurations as

$$\begin{aligned} \gamma_{11} &= -1 + \frac{(\alpha+2)}{4}(m_1 + m_2 + 4m_3), \qquad \gamma_{22} = \alpha - \gamma_{11}, \\ \gamma_{12} &= \gamma_{21} = \frac{\alpha+2}{4}\sqrt{3}(m_2 - m_1). \end{aligned}$$

So, $\tilde{B}$ in (1.40) is a symmetric matrix.

Our purpose now it to write the linearized system on a triangular configuration as the system

$$\dot{\mathbf{x}} \;=\; A(f)\mathbf{x}, \tag{1.46}$$

with

$$A(f) = \begin{pmatrix} 0 & I_2 \\ \tilde{A}(f) & -2J_2 \end{pmatrix}, \qquad \tilde{A} = g^{\alpha-2}\Lambda \tag{1.47}$$

where $\Lambda$ is a $2 \times 2$ diagonal constant matrix. To this end, it will be useful the following remark.

*Remark* 1.5.1. Let us consider the system

$$\dot{\mathbf{w}} \;=\; B(f)\mathbf{w}, \tag{1.48}$$

with $B(f) = \begin{pmatrix} 0 & I_2 \\ \tilde{B}(f) & -2J_2 \end{pmatrix}$ and $\tilde{B} = g^{\alpha-2}\tilde{\Lambda}$ where $\tilde{\Lambda}$ is a $2 \times 2$ symmetric but not diagonal constant matrix. Let $P$ be an orthogonal matrix such that $P^{-1}\tilde{\Lambda}P$ is diagonal. We introduce new variables as $\mathbf{X} = A_1^{-1}\mathbf{w}$ with $A_1 = \mathrm{diag}(P, P)$. Then, (1.48) becomes

$$\dot{\mathbf{X}} \;=\; B_1(f)\mathbf{X},$$

with $B_1(f) = A_1^{-1}B(f)A_1$. We have that

$$B_1(f) \;=\; \begin{pmatrix} 0 & I_2 \\ \tilde{A}(f) & -2P^{-1}J_2P \end{pmatrix}$$

where $\tilde{A}(f) = g^{\alpha-2}P^{-1}\tilde{\Lambda}P$ is like in (1.47). Due to the orthogonality of $P$ it is easy to check that $P^{-1}J_2P = \pm J_2$, where the sign $+$ stands for $\det(P) = 1$ and $-$ for $\det(P) = -1$. Then, $B_1(f) = \begin{pmatrix} 0 & I_2 \\ \tilde{A}(f) & -2cJ_2 \end{pmatrix}$ with $c = \pm 1$.

We define $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$ by $x_1 = X_1$, $x_2 = X_2$, $X_3 = \dfrac{x_3}{c}$, $X_4 = \dfrac{x_4}{c}$. Scaling the time by a factor of $c$ the system in variables $\mathbf{x}$ is written as (1.46).

After this remark, we can write the linearized system on a triangular configuration as (1.46) by taking $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$ being $\lambda_1, \lambda_2$ the zeroes of

$$p(\lambda) = \lambda^2 - (\alpha+2)\lambda + \frac{\beta_t}{4}, \qquad \beta_t = 3\kappa(\alpha+2)^2, \tag{1.49}$$

where $\kappa = m_1m_2 + m_1m_3 + m_2m_3$. Then, we have that

$$\lambda_1 + \lambda_2 = \alpha + 2 \quad \text{and} \quad \lambda_1\lambda_2 = \frac{\beta_t}{4}. \tag{1.50}$$

*Remark* 1.5.2. We have that $3\kappa \leq 1$ is satisfied for all positive values of the masses in the triangle of masses. Moreover, the equality holds for $m_1 = m_2 = m_3$.

After this remark we have that the the zeroes of (1.49) are real. Moreover, it is clear that $\beta_t \in (0, (\alpha+2)^2]$.

Now we study the collinear central configurations. For $\mathbf{u}^*$ a collinear configuration we have that $(\mathbf{u}_1^*)^T J_2 \mathbf{u}_2^* = 0$. Then, from (1.44) we have $\gamma_1 = 0$ and from (1.27), $\boldsymbol{\eta}_1 = J_4\mathbf{u}^*$, $\boldsymbol{\eta}_2 = K_4\boldsymbol{\eta}_1$. In order to compute the coefficients $\gamma_{ij}$, $i, j = 1, 2$, we need to know the matrix $D\nabla\mathcal{U}(\mathbf{u}^*)$. It turns out that

$$D\nabla\mathcal{U}(\mathbf{u}^*) \;=\; \begin{pmatrix} a_1 & 0 & a_2 & 0 \\ 0 & a_3 & 0 & a_4 \\ a_2 & 0 & a_5 & 0 \\ 0 & a_4 & 0 & a_6 \end{pmatrix},$$

for some constants $a_j$, $j = 1, \dots, 6$, that depend on $m_1, m_2, m_3$ and $\rho$. From this expression it is easy to prove that $\gamma_{12} = \gamma_{21} = 0$. Therefore, the $2 \times 2$ matrix $\tilde{B}$ in (1.40) is diagonal. Moreover, after some computations we get the following expressions

$$\begin{aligned} \gamma_{11} &= -(\alpha+1)\gamma_{22}, \\ \gamma_{22} &= -1-\beta_c, \end{aligned}$$

where

$$\begin{aligned} \beta_c &= -1 - \frac{\alpha}{a^{\alpha+2}[1+(\rho+1)^2]}[(\rho+2)[(\rho+1)m_1+m_2]\rho^{-\alpha-2}+ \\ &\quad (\rho+1)[m_2\rho+m_3(\rho+1)]+(m_3-m_1\rho)(\rho+1)^{-\alpha-2}]. \end{aligned} \tag{1.51}$$

Therefore,

$$\tilde{B} = g^{\alpha-2}\begin{pmatrix} (\alpha+1)\beta_c+\alpha+2 & 0 \\ 0 & -\beta_c \end{pmatrix}.$$

Now we study some properties of the function $\beta_c$. We note that it depends on the three masses and on the solution of (1.42).

We consider the Newtonian case, that is, $\alpha = 1$. Then, equation (1.42) is the well–known Euler's quintic equation and can be written as

$$\begin{aligned} q(\rho) &:= \rho^5(m_2+m_3)+\rho^4(2m_2+3m_3)+\rho^3(m_2+3m_3)-\rho^2(3m_1+m_2)- \\ &\quad -\rho(3m_1+2m_2)-(m_1+m_2)=0. \end{aligned} \tag{1.52}$$

Moreover, $a^3 = -\dfrac{m_1}{\rho^2}+\dfrac{m_1}{(\rho+1)^2}+m_2+m_3$ and

$$\beta_c = \frac{m_1\left(1+\rho_1^{-1}+\rho_1^{-2}\right)+m_3\left(1+\rho_2^{-1}+\rho_2^{-2}\right)}{m_1+m_2\left(\rho_1^{-2}+\rho_2^{-2}\right)+m_3}, \tag{1.53}$$

where

$$\rho_1 = \frac{\rho}{\rho+1} \quad \text{and} \quad \rho_2 = \frac{1}{\rho+1}, \tag{1.54}$$

(see [S.M.]).

*Remark* 1.5.3. As we have assumed that $m_1+m_2+m_3 = 1$, fixed $m_2 \in (0,1)$ we can take $\beta_c$ in (1.53) as a function of $m_1$ and $m_3$. Then, if we write $\beta_c = \beta_c(m_1, m_3)$ we obtain that $\beta_c(m_1, m_3) = \beta_c(m_3, m_1)$ and then $\beta_c$ is symmetric with respect to $m_1 = m_3$. To see this, fixed $m_2 \in (0,1)$ it is only necessary to take into account that if $\rho(m_1, m_3)$ denotes the solution of the quintic (1.52) then $\rho(m_3, m_1) = \dfrac{1}{\rho(m_1, m_3)}$ where $\rho(m_3, m_1)$ is obtained from (1.52) by changing

$m_1$ by $m_3$ and viceversa. Then, the expressions of $\rho_1$ and $\rho_2$ in (1.54) satisfy $\rho_1(m_1, m_3) = \rho_2(m_3, m_1)$ and $\rho_2(m_1, m_3) = \rho_1(m_3, m_1)$. Therefore, $\beta_c(m_1, m_3) = \beta_c(m_3, m_1)$. We note that in the triangle of masses the values such that $m_1 = m_3$ correspond to the height $l$ of the triangle (see figure 1.7).

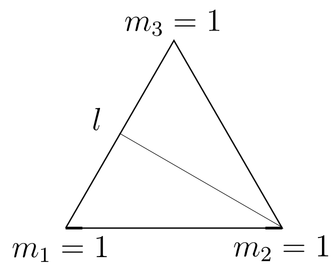



Figure 1.7: Line $m_1 = m_3$ in the triangle of masses

We note that $q(1) = 7(m_3 - m_1)$ and therefore, if $m_1 \leq m_3$ we have that the solution of the quintic equation satisfies $\rho \leq 1$ and the solution is $\rho = 1$ if and only if $m_1 = m_3$. Moreover, $q(0) = -(m_1 + m_2)$ and $\rho = 0$ if and only if $m_1 = m_2 = 0$.

For this problem we know the ranges of $\beta_c$ (see [M.S.]).

**Lemma 1.5.4.** *In the Newtonian case, for any positive masses, $\beta_c \in (0, 7)$. The values 0 and 7 are attained in the limit cases $m_2 = 1$ and $m_2 = 0$, $m_1 = m_3 = \dfrac{1}{2}$, respectively.*

**Proof**

We fix $m_1 \in (0, 1)$ and we take $m_3 = 1 - m_1 - m_2$. We have that $m_2 \in (0, 1 - m_1)$. From (1.53) we obtain

$$\beta_c = \frac{m_1(3\rho^2 + 3\rho + 1)(1 - m_1 - m_2)\rho^2(\rho^2 + 3\rho + 3)}{\rho^2 + m_2[(\rho + 1)^2(1 + \rho^2) - \rho^2]} \quad .$$

As $\rho$ is a solution of (1.52) we can consider $\rho$ as a function of $m_2$. Therefore $\beta_c$ is a function of $m_2$.

It has been proved that $\dfrac{d\beta_c}{dm_2}(m_2) < 0$ (see [M.S.]). Then, $\beta_c(m_2)$ is a strictly decreasing function of $m_2$.

Due to the symmetry of $\beta_c$ with respect to the line $m_1 = m_3$ in the triangle of masses, it is only necessary to consider $m_1 \leq m_3$, that is, $\rho \in (0, 1]$. Therefore, fixed a value of $m_1$ the monotonicity of $\beta_c$ implies that its maximum is taken when $m_2 = 0$ and the minimum at $m_1 = m_3$. In the case $m_1 = m_3$ we have that $\rho = 1$ and then $\beta_c = \dfrac{7(1 - m_2)}{1 + 7m_2}$. The minimum of this function is attained when $m_2 = 1$. In this case, $\beta_c = 0$.

For $m_2 = 0$ the quintic equation (1.52) can be written as $m_1(3\rho^2 + 3\rho + 1) = m_3\rho^3(\rho^2 + 3\rho + 3)$, and $\beta_c = (1 - m_1)(1 + \rho)(\rho^2 + 3\rho + 3)$. Using these equalities we obtain that

$$\beta_c = \frac{(\rho^2 + 3\rho + 3)(3\rho^2 + 3\rho + 1)}{\rho^4 + 2\rho^3 + \rho^2 + 2\rho + 1}.$$

It is easy to check that $\beta_c \leq 7$ and $\beta_c = 7$ if and only if $\rho = 1$. Then, the maximum in the triangle of masses is 7. This ends the proof of the lemma. $\square$

*Remark* 1.5.5. Numerically it can be seen that fixed $m_2 \in (0, 1)$, $\beta_c(m_1, m_3)$ has a maximum at $m_1 = m_3$. Figure 1.8 shows the numeric behaviour of function $\beta_c(m_1, m_3)$ taking into account that $\beta_c(m_1, m_3) = \beta_c(m_1)$.

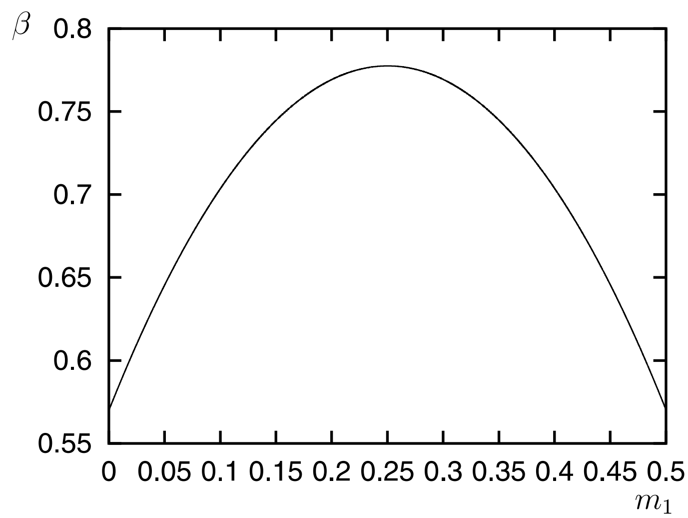



Figure 1.8: Graphic of the function $\beta_c$ in terms of $m_1$ for $m_2 = 0.5$

Now we consider the general case. Numerical computations shows that, fixed $m_2 \in (0, 1)$, the maximum of $\beta_c$ is also attained at $m_1 = m_3$ (see figure 1.9). Moreover, numerically it can be seen that $\beta_c$ is a decreasing function of $m_2$. Then, the maximum is attained at $m_1 = m_3$ and $m_2 = 0$. For these values of the masses the solution of (1.42) is $\rho = 1$ and $\beta_c = 2^{2+\alpha} - 1$. Therefore, we shall assume that $\beta_c \in (0, 2^{\alpha+2} - 1)$.

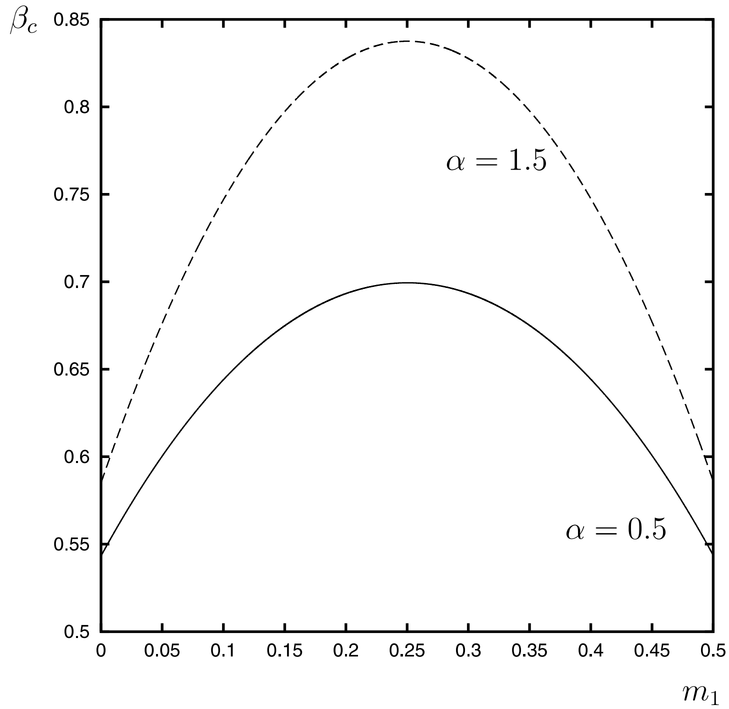



Figure 1.9: Graphic of $\beta_c$ in terms of $m_1$ for $m_2 = 0.5$ and $\alpha = 0.5$, $\alpha = 1.5$

## 1.6 About the study of the stability of the homographic solutions

In this section we shall see that the linearized system on an homographic solution can be included in a three parametric family of four dimensional periodic linear systems. This family can be written in a Hamiltonian formulation. Using the theory of Hamiltonian systems (see [M.H.]), we describe briefly how to study the stability parameters of these differential equations.

We consider the periodic linear system

$$\dot{\mathbf{x}} \;=\; A(f,e)\mathbf{x}, \tag{1.55}$$

with $\mathbf{x} \in \mathbb{R}^4$,

$$A(f,e) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 G_1(f,e) & 0 & 0 & -2 \\ 0 & \lambda_2 G_2(f,e) & 2 & 0 \end{pmatrix},$$

where $\lambda_1, \lambda_2$ are real parameters different from zero, $e \in [0,1)$, and $G_1(f,e)$, $G_2(f,e)$ are smooth functions. Let us assume that for $e = 0$, $G_1(f,0) = 1$, $G_2(f,0) = 1$.

We want to see that the linearized system on an homographic solution can be written as (1.55).

First we note that, as we have seen in last section, system (1.40) can be written as

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ g^{\alpha-2}\lambda_1 & 0 & 0 & -2 \\ 0 & g^{\alpha-2}\lambda_2 & 2 & 0 \end{pmatrix} \mathbf{x}, \tag{1.56}$$

where $\lambda_1$, $\lambda_2$ depends on the central configuration we are considering and $g$ is a periodic solution of (1.31). Table 1.1 shows the expression of $\lambda_1$ and $\lambda_2$ in both collinear and triangular case. We distinguish between the general and the Newtonian case.

| | Newtonian case | General case |
|---|---|---|
| Triangular configuration | $\lambda_1$, $\lambda_2$ zeroes of $p(\lambda) = \lambda^2 - 3\lambda + \frac{\beta_t}{4}$, $\beta_t = 27\kappa$ | $\lambda_1$, $\lambda_2$ zeroes of $p(\lambda) = \lambda^2 - (\alpha+2)\lambda + \frac{\beta_t}{4}$ $\beta_t = 3(\alpha+2)^2\kappa$ |
| Collinear configuration | $\lambda_1 = 2\beta_c + 3$ $\lambda_2 = -\beta_c$ $\beta_c \in (0, 2^{\alpha+2} - 1)$ | $\lambda_1 = (\alpha+1)\beta_c + \alpha + 2$ $\lambda_2 = -\beta_c$ $\beta_c \in (0, 7)$ |

Table 1.1: Values of $\lambda_1$, $\lambda_2$ for a triangular and a collinear central configuration.

We define a generalized eccentricity

$$e := \sqrt{1 + \frac{2\alpha}{2-\alpha} E_K \omega^{\frac{2\alpha}{2-\alpha}}},$$

where $E_K$ is defined in (1.12). We recall that we have normalized scales so that $E_K = -\frac{1}{2}$ and $\omega$ is such that $0 < \omega \leq \omega_c$ where $\omega_c$ is defined in (1.34). Then,

$$e = \sqrt{1 - \frac{\alpha}{2-\alpha}\omega^{\frac{2\alpha}{2-\alpha}}}, \qquad e \in [0,1). \tag{1.57}$$

Given a value of $\omega$, we have that $E$ is fixed by (1.33) and so, $g$ depends on $e$.

In the Newtonian case ($\alpha = 1$), $e$ is the eccentricity of the orbit. From remark 1.4.2 we know that the case $\omega = \omega_c$ corresponds to the relative equilibrium solution. For this value of $\omega$ we have that $e = 0$. On the other hand, we have seen in section 1.4 that when $\omega$ tends to 0 then $g_0$ tends to 0. So, for $e = 1$ system (1.56) has a singularity.

Then, it is clear that system (1.56) can be written as (1.55) by taking $G_1(f, e) = g^{\alpha-2}\lambda_1$ and $G_2(f, e) = g^{\alpha-2}\lambda_2$, where $\lambda_1$ and $\lambda_2$ are given in table 1.1.

System (1.55) can be written as a linear Hamiltonian system by the change of variables $\mathbf{y} = \tilde{M}\mathbf{x}$ with

$$\tilde{M} = \begin{pmatrix} I_2 & 0 \\ J_2 & I_2 \end{pmatrix}. \tag{1.58}$$

The Hamiltonian function is

$$H(\mathbf{y}, f) = \frac{1}{2}(y_3^2 + y_4^2) + y_1 y_4 - y_2 y_3 - V(y_1, y_2, f, e), \tag{1.59}$$

where $\mathbf{y} = (y_1, y_2, y_3, y_4)^T$ and

$$V(y_1, y_2, f, e) = [\lambda_1 G_1(f, e) - 1]\,\frac{y_1^2}{2} + [\lambda_2 G_2(f, e) - 1]\,\frac{y_2^2}{2}. \tag{1.60}$$

Let $\Phi(f)$ the fundamental matrix of (1.55) such that $\Phi(0) = I_4$. It is easy to check that

$$\Phi(f) = \tilde{M}^{-1}\Phi_1(f)\tilde{M}, \tag{1.61}$$

where $\Phi_1(f)$ is the fundamental matrix of the linear Hamiltonian system defined by (1.59). The symplectic character of $\Phi_1(f)$ implies that if $\tilde{\mu}$ is an eigenvalue of $\Phi(T)$ then $\tilde{\mu}^{-1}$ is also an eigenvalue (see [M.H.]). We denote by $\mu_1, \mu_1^{-1}, \mu_2, \mu_2^{-1}$ the eigenvalues of $\Phi(T)$ and define the stability parameters as

$$tr_i = \mu_i + \mu_i^{-1}, \qquad i = 1, 2.$$

The stability of the system (1.55) can be studied in terms of $\mathrm{tr}_j$, $j = 1, 2$. From the real character of (1.55), if $\mathrm{tr}_j \in \mathbb{C} \setminus \mathbb{R}$ for some $j = 1, 2$ then $\mathrm{tr}_{3-j} = \overline{\mathrm{tr}}_j$ where the bar stands for the complex conjugate. Therefore (1.55) is complex–saddle. We assume that $\mathrm{tr}_j \in \mathbb{R}$ for some $j = 1, 2$. If $|\mathrm{tr}_j| > 2$ then $\mu_j \in \mathbb{R}$ and one of the characteristic multipliers has modulus bigger that 1. In this case, (1.55) is unstable. If $|\mathrm{tr}_j| < 2$ then $\mu_j \in \mathbb{C}$ with $\|\mu_j\| = 1$. Bifurcations can occur when $|\mathrm{tr}_j| = 2$ for some $j = 1, 2$, and when $\mathrm{tr}_1 = \mathrm{tr}_2$.

In chapters 2 and 3 we shall study the stability parameters of the system (1.55) for a family of functions $G_j$, $j = 1, 2$. In chapter 2 we will analize the case $e \gtrsim 0$ and in chapter 3 the case $e \lesssim 1$ will be considered. In this last case, we shall introduce a parameter $\delta$ defined by $\delta = \dfrac{2-\alpha}{2\alpha}(1-e^2)$ and we will study the system for $\delta > 0$ small enough. Using the theory developed in these chapters, in chapter 4 we shall determine the stability parameters of the homographic solutions.

# Chapter 2

# Stability of a family of periodic linear systems: the perturbative case

## 2.1 Introduction

Let us consider the family of periodic linear systems

$$\dot{\mathbf{x}} = A(t,\varepsilon)\mathbf{x}, \tag{2.1}$$

with $\mathbf{x} \in \mathbb{R}^4$,

$$A(t,\varepsilon) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 G_1(t,\varepsilon) & 0 & 0 & -2 \\ 0 & \lambda_2 G_2(t,\varepsilon) & 2 & 0 \end{pmatrix},$$

$\lambda_1, \lambda_2$ are real parameters different from zero, $\varepsilon$ is a small positive parameter, and $G_i(t,\varepsilon) = 1 - F_i(t,\varepsilon)$, $i = 1,2$, where $F_i(t,\varepsilon)$, $i = 1,2$, are even functions, $T$–periodic in $t$ and analytic in $\varepsilon$, satisfying $F_i(t,0) = 0$. Then, we can write

$$F_1(t,\varepsilon) = \sum_{j\in\mathbb{N}} \varepsilon^j c_j(t), \qquad F_2(t,\varepsilon) = \sum_{j\in\mathbb{N}} \varepsilon^j d_j(t), \tag{2.2}$$

with $c_j(t), d_j(t)$ $T$–periodic even functions for $j \in \mathbb{N}$.

If $\varepsilon = 0$ then system (2.1) is linear with constant coefficients and one can obtain easily the stability and instability regions in the plane $(\lambda_1, \lambda_2)$. These regions are described in section 2.2. The goal of this chapter is to study the bifurcations for $\varepsilon$ small and positive.

In chapter 1 we have seen that (2.1) can be written as a linear Hamiltonian system with Hamiltonian function

$$H(\mathbf{y},t) \quad = \quad \frac{1}{2}(y_3^2+y_4^2)+y_1y_4-y_2y_3-V(y_1,y_2,t,\varepsilon), \tag{2.3}$$

where $\mathbf{y}=(y_1,y_2,y_3,y_4)^T$ and

$$V(y_1,y_2,t,\varepsilon) \quad = \quad [\lambda_1 G_1(t,\varepsilon)-1]\,\frac{y_1^2}{2}+[\lambda_2 G_2(t,\varepsilon)-1]\,\frac{y_2^2}{2}. \tag{2.4}$$

The analysis of system (2.1) has several applications. One of them is the study of the stability for the equilibria of mechanical systems defined by a Hamiltonian function of the form (2.3) with a potential $\mathcal{V}(y_1,y_2,t,\varepsilon)$ even in $t$ and such that the quadratic part in $y_1$ and $y_2$ has the form (2.4). In this case, the linearized system on the equilibrium point can be written as (2.1).

On the other hand, system (2.1) can be obtained as a first variational system on a periodic solution of an autonomous system. As we have seen in chapter 1, one example is given by the homographic solutions of the planar three body problem with homogeneous potential of order $-\alpha$, with $0<\alpha<2$, since after some reductions the linear stability of these orbits is given by the study of the non–autonomous linear system (1.40), that has the form (2.1). This application is developed in chapter 4.

As we have seen in chapter 1 the fundamental matrices of (2.1) and of the system associated to (2.3) have the same characteristic multipliers. Then, it is equivalent to study the linear stability of the two systems and we can apply the stability theory for Hamiltonian systems (see [M.H.]) to the system (2.1).

Using this theory, in order to study the linear stability of the system (2.1) it is only necessary to compute the stability parameters $\mathrm{tr}_j=\mu_j+\mu_j^{-1}$, $j=1,2$, where $\mu_j,\mu_j^{-1}$, $j=1,2$, are the characteristic multipliers of the system.

If $\mathrm{tr}_1,\mathrm{tr}_2\in\mathbb{C}\setminus\mathbb{R}$, then $\mathrm{tr}_2=\overline{\mathrm{tr}_1}$ and (2.1) is a complex–saddle.

Assume that $\mathrm{tr}_1,\mathrm{tr}_2\in\mathbb{R}$. If $|\mathrm{tr}_1|<2,|\mathrm{tr}_2|<2$, then (2.1) is elliptic–elliptic, if $|\mathrm{tr}_1|>2,|\mathrm{tr}_2|>2$, hyperbolic–hyperbolic and if one of the stability parameters has absolute value less than two and the other bigger than two then (2.1) is elliptic–hyperbolic.

We note that in our case the stability parameters depend on $\lambda_1,\lambda_2,\varepsilon$.

Now we explain briefly the methodology used in order to study the bifurcations of (2.1) for $\varepsilon>0$ small enough.

**Definition 2.1.1.** We say that $(\lambda_1,\lambda_2)=(a_1,a_2)$ is a **resonant point** for $\varepsilon=\varepsilon_0$ of the system (2.1) if for $(\lambda_1,\lambda_2,\varepsilon)=(a_1,a_2,\varepsilon_0)$, $|\mathrm{tr}_j|=2$ for some $j=1,2$ or $\mathrm{tr}_1=\mathrm{tr}_2$.

Let us consider $(\lambda_1, \lambda_2) = (a_1, a_2)$ a resonant point for $\varepsilon = 0$ . Our purpose is to study $\mathrm{tr}_1, \mathrm{tr}_2$ in a neighbourhood of $(a_1, a_2)$ for $\varepsilon > 0$ small enough. To this end, we introduce small parameters $\delta_1, \delta_2 \in \mathbb{R}$ and we shall consider $\lambda_j = a_j + \delta_j$, $j = 1, 2$.

We will apply the Normal Form techniques (see [B.S.2]) in order to detect changes in the stability. The idea is to perform some canonical transformations to cancel the time dependence up to high order in $\delta_1, \delta_2, \varepsilon$, if this is possible. In fact, we shall obtain the Floquet matrix up to a fixed order in $\delta_1, \delta_2, \varepsilon$.

The analysis of this Normal Form will give us the linear stability as well as the boundaries of the stability regions.

## 2.2 The case $\varepsilon = 0$

In this section we study the stability parameters of system (2.1) for $\varepsilon = 0$.

For $\varepsilon = 0$ we obtain the linear system with constant coeficients $\dot{\mathbf{x}} = \tilde{A}_0 \mathbf{x}$ where

$$\tilde{A}_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 & 0 & 0 & -2 \\ 0 & \lambda_2 & 2 & 0 \end{pmatrix}.$$

The characteristic polynomial of $\tilde{A}_0$ is

$$p(x) \quad = \quad x^4 - (\lambda_1 + \lambda_2 - 4)x^2 + \lambda_1 \lambda_2. \tag{2.5}$$

We note that a zero $\rho$ of (2.5) satisfies

$$2\rho^2 \quad = \quad \lambda_1 + \lambda_2 - 4 \pm \sqrt{(\lambda_1 + \lambda_2 - 4)^2 - 4\lambda_1\lambda_2}.$$

The stability parameters have different character depending on the region of the plane $(\lambda_1, \lambda_2)$ considered. Therefore, we distinguish on this plane the following regions

$$\begin{aligned}
\mathcal{R}_1 &= \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 | \lambda_1 \lambda_2 < 0\}, \\
\mathcal{R}_2 &= \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 | \lambda_1 \lambda_2 > 0, (\lambda_1 + \lambda_2 - 4)^2 > 4\lambda_1\lambda_2, \lambda_1 + \lambda_2 - 4 < 0\}, \\
\mathcal{R}_3 &= \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 | \lambda_1 \lambda_2 > 0, (\lambda_1 + \lambda_2 - 4)^2 < 4\lambda_1\lambda_2\}, \\
\mathcal{R}_4 &= \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 | \lambda_1 \lambda_2 > 0, (\lambda_1 + \lambda_2 - 4)^2 > 4\lambda_1\lambda_2, \lambda_1 + \lambda_2 - 4 > 0\}.
\end{aligned}$$

The boundaries of these regions are given by the coordinate axes and for the graphs of the functions $\lambda_2 = (\sqrt{\lambda_1} \pm 2)^2$. Figure 2.1 shows these regions and their boundaries in the plane $(\lambda_1, \lambda_2)$.

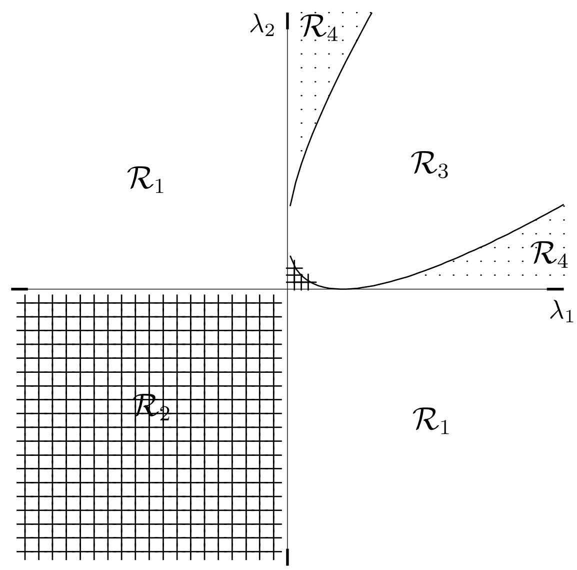



Figure 2.1: Regions $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$ in the plane $(\lambda_1, \lambda_2)$.

In $\mathcal{R}_1$, (2.5) has zeroes $\pm\alpha, \pm i\omega$ with $\alpha, \omega \in \mathbb{R}^+$. Therefore, $\mu_1 = e^{\alpha T}$ and $\mu_2 = e^{i\omega T}$. Then we have that $\mathrm{tr}_1 > 2$ and $|\mathrm{tr}_2| \leq 2$. Possible bifurcations for $\varepsilon > 0$ will be for $|\mathrm{tr}_2| = 2$.

In $\mathcal{R}_2$ the eigenvalues of $\tilde{A}_0$ have the form $\pm i\omega_1, \pm i\omega_2$ with $\omega_1, \omega_2 \in \mathbb{R}^+$ and $\omega_1 > \omega_2$. Therefore, $\mu_1 = e^{i\omega_1 T}$ and $\mu_2 = e^{i\omega_2 T}$, fact that implies $|\mathrm{tr}_1|, |\mathrm{tr}_2| \leq 2$. If $|\mathrm{tr}_1| = 2$ or $|\mathrm{tr}_2| = 2$ or $\mathrm{tr}_1 = \mathrm{tr}_2$, then bifurcations can be found for $\varepsilon > 0$.

In the region $\mathcal{R}_3$ the zeroes of (2.5) are $\pm\alpha \pm i\beta$ with $\alpha, \beta \in \mathbb{R}^+$. Therefore, the characteristic multipliers are $\mu_1 = e^{(\alpha+i\beta)T}$, $\mu_2 = e^{(\alpha-i\beta)T}$ and their inverses. We note that if $\beta T \neq n\pi$ for all $n \in \mathbb{N}$ then the stability parameters are complex, and in the other case are real. In this last case a bifurcation can occur since $\mathrm{tr}_1 = \mathrm{tr}_2$.

In $\mathcal{R}_4$, (2.5) has zeroes $\pm\alpha_1, \pm\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}^+$, $\alpha_1 \neq \alpha_2$. Therefore, $\mu_1 = e^{\alpha_1 T}$ and $\mu_2 = e^{\alpha_2 T}$. In this case, $|\mathrm{tr}_j| > 2$ for $j = 1, 2$ and no bifurcation can occur for $\varepsilon > 0$ small enough.

Now we study the stability parameters in the boundaries of these regions.

On the axis $\lambda_1$ one stability parameter is equal to two, and the other is $2\cos(\sqrt{4-\lambda_1}T)$ if $\lambda_1 < 4$ and bigger than 2 if $\lambda_1 > 4$. We obtain a symmetric behaviour on the axis $\lambda_2$.

If $\lambda_2 = (\sqrt{\lambda_1} - 2)^2$ then $\mathrm{tr}_1 = \mathrm{tr}_2$. In this case, if $0 < \lambda_1 < 4$ then $|\mathrm{tr}_1| = |\mathrm{tr}_2| \leq 2$ and $\mathrm{tr}_1 = \mathrm{tr}_2 > 2$ if $\lambda_1 > 4$. On $\lambda_2 = (\sqrt{\lambda_1} + 2)^2$, we obtain $\mathrm{tr}_1 = \mathrm{tr}_2 > 2$ if $\lambda_1 \neq 0$.

The points $(4,0),(0,4)$ in the plane $(\lambda_1,\lambda_2)$ correspond to degenerate cases in which 1 is a characteristic multiplier with multiplicity 4. Therefore, on these points we have $\mathrm{tr}_1 = \mathrm{tr}_2 = 2$.

We are interested in the study of the bifurcations for values of $(\lambda_1,\lambda_2)$ in the region $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$.

Now we study the curves in the plane $(\lambda_1,\lambda_2) \in \mathcal{R}$ formed for resonant points when $\varepsilon = 0$. We will refer to these curves as **resonant curves**.

We know that in $\mathcal{R}_1 \cup \mathcal{R}_2$ one has a resonant curve if $\omega T = n\pi$, $n \in \mathbb{N}$. We note that $p(i\omega) = \omega^4 + (\lambda_1 + \lambda_2 - 4)\omega^2 + \lambda_1\lambda_2$. Then the resonant curve is given by

$$(\lambda_1 + \omega^2)(\lambda_2 + \omega^2) = 4\omega^2, \quad \omega = \frac{n\pi}{T}, \;\; n \in \mathbb{N}. \tag{2.6}$$

In $\mathcal{R}_1$ we obtain a uniparametric family of resonant curves indexed by $n \in \mathbb{N}$. However, in $\mathcal{R}_2$ there are two uniparametric families of resonant curves of this type corresponding to $\omega_1$ and $\omega_2$, respectively. For one of them, $n \in \mathbb{N}$. The other family is defined for $n > \dfrac{2T}{\pi}$, $n \in \mathbb{N}$ if $\lambda_1 < 0$, and $n < \dfrac{2T}{\pi}$, $n \in \mathbb{N}$ if $\lambda_1 > 0$.

In $\mathcal{R}_2$ there are also resonant points for $\varepsilon = 0$ when $\omega_1 \pm \omega_2 = \dfrac{2n\pi}{T}$, $n \in \mathbb{N}$, which correspond to the case when $\mathrm{tr}_1 = \mathrm{tr}_2$. The corresponding resonant curves are

$$\lambda_2 \;\; = \;\; \lambda_1 + 4\left(1 - \frac{n^2\pi^2}{T^2}\right) \pm 4\sqrt{\lambda_1\left(1 - \frac{n^2\pi^2}{T^2}\right)}. \tag{2.7}$$

If $\lambda_1 > 0$, this uniparametric family of resonant curves is indexed by $n \leq \frac{T}{\pi}$, $n \in \mathbb{N}$, and if $\lambda_1 < 0$, the index are given by $n \geq \frac{T}{\pi}$, $n \in \mathbb{N}$.

Finally, if $(\lambda_1,\lambda_2) \in \mathcal{R}_3$, the resonant points are given when $T\beta = n\pi$ with $n \in \mathbb{N}$. A simple computation shows that the possible bifurcations in $\mathcal{R}_3$ are given on the uniparametric family of curves

$$\lambda_2 \;\; = \;\; (\sqrt{\lambda_1} \pm 2\sqrt{1-\beta^2})^2, \quad \beta = \frac{n\pi}{T}, \tag{2.8}$$

for $n \leq \dfrac{T}{\pi}$, $n \in \mathbb{N}$.

In figure 2.2 there are some examples of resonant curves in the different regions.

The red curves are of the form (2.6) and the green curves corresponds to both curves in (2.7) or (2.8), depending on the region they are. We note that the curves (2.8) in the parameter space $(\lambda_1,\lambda_2)$ are exactly the same as (2.7).

In region $\mathcal{R}_1$ there are only curves of the form (2.6) and there are no intersection between these curves. Then, in this region we only have single resonances.

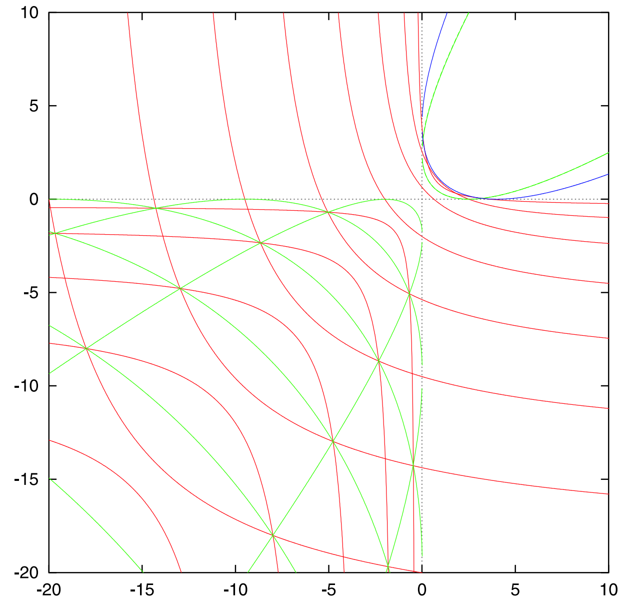



Figure 2.2: Some resonant curves in the plane $(\lambda_1, \lambda_2)$ for $T = \dfrac{2\pi}{\sqrt{2-\alpha}}$ with $\alpha = 0.5$. Color codes: Red for $|\mathrm{tr}_j| = 2$ for some $j = 1, 2$, Green for $|\mathrm{tr}_1| = |\mathrm{tr}_2|$, Blue for the boundary of region $\mathcal{R}_3$.

In region $\mathcal{R}_2$ we distinguish different intersections. The intersection between curves of the form (2.6) corresponds to a double resonance in which $\omega_1 = \dfrac{n_1\pi}{T}$ and $\omega_2 = \dfrac{n_2\pi}{T}$ for some $n_1, n_2 \in \mathbb{N}$. We note that if $n_1 \equiv n_2 (\mathrm{mod}\ 2)$ then $\omega_1 \pm \omega_2 = \dfrac{2n\pi}{T}$ for some $n \in \mathbb{N}$. In this case, the intersection point belongs also to a resonant curve (2.7).

In region $\mathcal{R}_3$ there are only single resonances.

Given a point $(a_1, a_2)$ on a resonant curve, that is, $(a_1, a_2)$ is a resonant point for $\varepsilon = 0$, we want to study if there is a bifurcation near $(a_1, a_2)$ for $\varepsilon \gtrsim 0$. To this end we shall introduce small parameters $\delta_1, \delta_2$, and we will take $\lambda_1 = a_1 + \delta_1$, $\lambda_2 = a_2 + \delta_2$. In section 2.3 we shall give the Normal Form of the Hamiltonian (2.3) in the regions $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$ up to a given order in $\delta_1, \delta_2, \varepsilon$. Then in section 2.4 we will study the conditions for bifurcation depending on the region $\mathcal{R}_1$, $\mathcal{R}_2$ or $\mathcal{R}_3$. In section 2.5 we shall study with more detail the boundaries of the resonant regions as well as the bifurcation diagram in the particular case that the periodic functions $F_j$, $j = 1, 2$, in (2.2) satisfies d'Alembert property. Section 2.6 is devoted to the proof of the Normal Form given in section 2.3.

## 2.3 Normal Form

In this section we reduce the Hamiltonian system associated to (2.3) to Normal Form.

Let us fix $(a_1, a_2) \in \mathcal{R}$ a point on a resonant curve, that is, $(\lambda_1, \lambda_2) = (a_1, a_2)$ is a resonant point for $\varepsilon = 0$. Let us take $\lambda_j = a_j + \delta_j$, $j = 1, 2$, with $|\delta_j|$, $j = 1, 2$, small enough.

If we take $\varepsilon = 0$ in (2.3) then the system associated to the Hamiltonian function is written as

$$\dot{\mathbf{y}} \;=\; A_0\mathbf{y}, \tag{2.9}$$

with

$$A_0 = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ a_1 - 1 & 0 & 0 & -1 \\ 0 & a_2 - 1 & 1 & 0 \end{pmatrix}. \tag{2.10}$$

We note that $A_0$ depends on $a_1$ and $a_2$.

The Hamiltonian function (2.3) can be written as

$$H(\mathbf{y}, t) \;=\; H_0(\mathbf{y}) + \tilde{H}(\mathbf{y}, t), \tag{2.11}$$

where

$$\begin{aligned} & H_0(\mathbf{y}) = -\frac{1}{2}\mathbf{y}^T J_4 A_0 \mathbf{y} = \\ = \;& \frac{1}{2}(y_3^2 + y_4^2) + y_1 y_4 - y_2 y_3 + (1 - a_1)\frac{y_1^2}{2} + (1 - a_2)\frac{y_2^2}{2}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} & \tilde{H}(\mathbf{y}, t) = -\frac{\delta_1}{2}y_1^2 - \frac{\delta_2}{2}y_2^2 + \\ & +(a_1 + \delta_1)\frac{y_1^2}{2}F_1(t; \varepsilon) + (a_2 + \delta_2)\frac{y_2^2}{2}F_2(t; \varepsilon). \end{aligned} \tag{2.13}$$

The Hamiltonian system associated to $H_0$ is a linear system with constant coefficients, and depends on $a_1$ and $a_2$. Once $a_1$ and $a_2$ are fixed the Hamiltonian (2.13) depends on three parameters, $\delta_1, \delta_2$ and $\varepsilon$.

Hamiltonian (2.11) admits the following symmetry.

**Lemma 2.3.1.** *The Hamiltonian (2.11) satisfies $H(\mathbf{y}, t) = H(\mathbf{y}, -t)$ and $H(L\mathbf{y}, t) = H(\mathbf{y}, t)$ for all $\mathbf{y} \in \mathbb{R}^4$ and $t \in \mathbb{R}$, where $L = diag(-1, 1, 1, -1)$.*

**Proof**

The equality $H(\mathbf{y},t) = H(\mathbf{y},-t)$ for all $\mathbf{y} \in \mathbb{R}^4, t \in \mathbb{R}$ is due to the fact that $F_j$, $j = 1,2$, are even functions.

To show that $H(L\mathbf{y},t) = H(\mathbf{y},t)$ for all $\mathbf{y} \in \mathbb{R}^4, t \in \mathbb{R}$ it is only necessary to take into account that if we change $y_1$ by $-y_1$ and $y_4$ by $-y_4$ the Hamiltonian does not change. □

### 2.3.1 Reduction to diagonal form

In this section we shall diagonalize the system associated to (2.12). In order to keep the Hamiltonian character of the system, it will be necessary to perform a symplectic change of coordinates. To this end, we will take into account the different $\mathcal{R}_j$ regions that defines $\mathcal{R}$.

We denote by $\pm\rho_1$, $\pm\rho_2$, the eigenvalues of $A_0$. In what follows, we will take $\rho_1 = \lambda, \rho_2 = i\omega$, $\lambda, \omega \in \mathbb{R}^+$ if $(a_1,a_2) \in \mathcal{R}_1$, $\rho_1 = i\omega_1, \rho_2 = i\omega_2$ with $\omega_1, \omega_2 \in \mathbb{R}^+$, $\omega_1 > \omega_2$ if $(a_1,a_2) \in \mathcal{R}_2$, and $\rho_1 = \alpha + i\beta, \rho_2 = \overline{\rho}_1$, $\alpha, \beta \in \mathbb{R}^+$, if $(a_1,a_2) \in \mathcal{R}_3$.

It is easy to check that

$$\mathbf{u}_\rho \quad = \quad (2\rho, a_1 - \rho^2, a_1 + \rho^2, \rho(a_1 - \rho^2 - 2))^T, \qquad (2.14)$$

is an eigenvector of eigenvalue $\rho$ of $A_0$.

Let us denote by $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{C}^4$ the eigenvectors corresponding to eigenvalues $\rho_1, \rho_2$, respectively.

Using the symmetry given by $L$ in lemma 2.3.1, and the fact that $A_0 L = -LA_0$, we obtain that $\mathbf{v}_1 := L\mathbf{u}_1$ and $\mathbf{v}_2 := L\mathbf{u}_2$ are eigenvectors of eigenvalues $-\rho_1, -\rho_2$, respectively.

If we restrict to the generic case, as $\rho_1 \pm \rho_2 \neq 0$, the eigenvectors $\mathbf{u}_1$ and $\mathbf{v}_1$ are $J$–orthogonal to $\mathbf{u}_2$ and $\mathbf{v}_2$. That is, $\mathbf{u}_1^T J\mathbf{u}_2 = 0$, $\mathbf{u}_1^T J\mathbf{v}_2 = 0$, $\mathbf{v}_1^T J\mathbf{u}_2 = 0$ and $\mathbf{v}_1^T J\mathbf{v}_2 = 0$. Moreover, a vector is $J$–orthogonal with itself.

We define the matrix $M$ as follows

$$M \quad = \quad (k_1\mathbf{u}_1, k_2\mathbf{u}_2, k_3\mathbf{v}_1, k_4\mathbf{v}_2), \qquad (2.15)$$

with $k_j \in \mathbb{C}$, $j = 1,\ldots,4$, satisfying $k_1k_3\mathbf{u}_1^T J\mathbf{v}_1 = 1$, $k_2k_4\mathbf{u}_2^T J\mathbf{v}_2 = 1$. Matrix $M$ is symplectic and defines a canonical change of variables as $\mathbf{y} = M\mathbf{z}$. Moreover, this change of coordinates transforms the system associated to (2.12) in diagonal form, that is, if $\mathcal{H}(\mathbf{z},t)$ denotes the transformed Hamiltonian, then

$$\mathcal{H}(\mathbf{z},t) \quad = \quad \mathcal{H}_0(\mathbf{z}) + \tilde{\mathcal{H}}(\mathbf{z},t), \qquad (2.16)$$

where

$$\mathcal{H}_0(\mathbf{z}) \quad = \quad \rho_1 z_1 z_3 + \rho_2 z_2 z_4, \qquad (2.17)$$

$\rho_1$ and $\rho_2$ being the eigenvalues of (2.10) and $\mathbf{z} = (z_1, z_2, z_3, z_4)^T$.

First, our purpose is to choose the constants $k_j$, $j = 1, \ldots, 4$, in an adequate way in order to have some relations between the new variables. To this end, it will be useful the next lemma.

**Lemma 2.3.2.** *1. If $(a_1, a_2) \in \mathcal{R}_1$ then $\mathbf{u}_1^T J \mathbf{v}_1 > 0$ if $a_1 > 0$, and $\mathbf{u}_1^T J \mathbf{v}_1 < 0$ if $a_1 < 0$. Moreover, $i\mathbf{u}_2^T J \mathbf{v}_2 > 0$.*

*2. If $(a_1, a_2) \in \mathcal{R}_2$ then $i\mathbf{u}_1^T J \mathbf{v}_1 > 0$ and, $i\mathbf{u}_2^T J \mathbf{v}_2 > 0$ if $a_1 < 0$, and $i\mathbf{u}_2^T J \mathbf{v}_2 < 0$ if $a_1 > 0$.*

**Proof**

Let us consider $\mathbf{u}_\rho$ in (2.14) an eigenvector of eigenvalue $\rho$ of $A_0$.

It is easy to check that

$$\mathbf{u}_\rho^T J \mathbf{v}_\rho = 2\rho\, q(a_1, a_2; \rho^2), \tag{2.18}$$

where

$$q(a_1, a_2; \rho^2) \;=\; -\rho^4 + 2a_1\rho^2 + 4a_1 - a_1^2. \tag{2.19}$$

Recall that $\rho$ is a solution of the characteristic polynomial of $A_0$, that is, $\tilde{p}(\rho) = 0$ where

$$\tilde{p}(x) \;=\; x^4 - (a_1 + a_2 - 4)x^2 + a_1 a_2.$$

Then, $-\rho^4 = (4 - a_1 - a_2)\rho^2 + a_1 a_2$, and therefore

$$q(a_1, a_2; \rho^2) \;=\; (4 + a_1 - a_2)\rho^2 + a_1 a_2 + 4a_1 - a_1^2.$$

Moreover

$$\rho^2 \;=\; \alpha_\pm, \quad \text{where} \quad \alpha_\pm = \frac{a_1 + a_2 - 4 \pm \sqrt{\Delta}}{2}, \quad \text{with} \quad \Delta = (a_1 + a_2 - 4)^2 - 4a_1 a_2.$$

We have that

$$q(a_1, a_2; \alpha_\pm) \;=\; -\frac{\sqrt{\Delta}}{2}[\sqrt{\Delta} \mp (4 + a_1 - a_2)],$$

where the sign $-$ stands for $\alpha_+$ and $+$ for $\alpha_-$.

If $a_1 > 0$ ($a_1 < 0$) we check that $(4 + a_1 - a_2)^2 > \Delta$ ($(4 + a_1 - a_2)^2 < \Delta$). Therefore, if $a_1 < 0$, one has $\sqrt{\Delta} \mp (4 + a_1 - a_2) > 0$ and then $q(a_1, a_2; \alpha_\pm) < 0$.

Furthermore, if $a_1 > 0$, as far as $(a_1, a_2) \in \mathcal{R}_1 \cup \mathcal{R}_2$, $4 + a_1 - a_2 > 0$ and then $\sqrt{\Delta} - (4 + a_1 - a_2) < 0$ and $\sqrt{\Delta} + (4 + a_1 - a_2) > 0$. So, $q(a_1, a_2; \alpha_+) > 0$ and $q(a_1, a_2; \alpha_-) < 0$.

1. Let us consider $(a_1, a_2) \in \mathcal{R}_1$. Then $\rho_1 = \lambda$, $\rho_2 = i\omega$, $\lambda, \omega \in \mathbb{R}^+$ and $\alpha_+ = \lambda^2$, $\alpha_- = -\omega^2$. Using (2.18), $\mathbf{u}_1^T J \mathbf{v}_1 = 2\lambda\, q(a_1, a_2; \alpha_+)$ and, $\mathbf{u}_1^T J \mathbf{v}_1 > 0$ if $a_1 > 0$ and $\mathbf{u}_1^T J \mathbf{v}_1 < 0$ if $a_1 < 0$. Moreover, $\mathrm{i}\mathbf{u}_2^T J \mathbf{v}_2 = -2\omega^2\, q(a_1, a_2; \alpha_-) > 0$.

2. Assume $(a_1, a_2) \in \mathcal{R}_2$. Then $\rho_1 = i\omega_1$, $\rho_2 = i\omega_2$, $\omega_1, \omega_2 \in \mathbb{R}^+$ with $\omega_1 > \omega_2$. In this case, $\alpha_- = -\omega_1^2$, $\alpha_+ = -\omega_2^2$. We get $\mathrm{i}\mathbf{u}_1^T J \mathbf{v}_1 = -2\omega_1^2\, q(a_1, a_2; \alpha_-) > 0$ and $\mathrm{i}\mathbf{u}_2^T J \mathbf{v}_2 = -2\omega_2^2\, q(a_1, a_2; \alpha_+)$. Then, $\mathrm{i}\mathbf{u}_2^T J \mathbf{v}_2 > 0$ if $a_1 < 0$ and $\mathrm{i}\mathbf{u}_2^T J \mathbf{v}_2 < 0$ if $a_1 > 0$.

□

We note that if $(a_1, a_2) \in \mathcal{R}_3$ then $\mathbf{u}_1^T J \mathbf{v}_1$ and $\mathbf{u}_2^T J \mathbf{v}_2$ are complex.
Then, we can do the following choice for the constants $k_j, j = 1, \ldots, 4$.

1. If $(a_1, a_2) \in \mathcal{R}_1$, we take
$$k_1 = \frac{1}{\sqrt{s\mathbf{u}_1^T J \mathbf{v}_1}}, \quad k_3 = sk_1, \quad k_2 = \frac{1}{\sqrt{i\mathbf{u}_2^T J \mathbf{v}_2}}, \quad k_4 = ik_2, \tag{2.20}$$

2. if $(a_1, a_2) \in \mathcal{R}_2$, we take
$$k_1 = \frac{1}{\sqrt{i\mathbf{u}_1^T J \mathbf{v}_1}}, \quad k_3 = ik_1, \quad k_2 = \frac{1}{\sqrt{si\mathbf{v}_2^T J \mathbf{u}_2}}, \quad k_4 = -sik_2, \tag{2.21}$$

3. if $(a_1, a_2) \in \mathcal{R}_3$, we take
$$k_1 = \frac{1}{\sqrt{\mathbf{u}_1^T J \mathbf{v}_1}}, \quad k_3 = k_1, \quad k_2 = \frac{1}{\sqrt{\mathbf{u}_2^T J \mathbf{v}_2}}, \quad k_4 = k_2, \tag{2.22}$$

where $s = \mathrm{sgn}(a_1)$.

After lemma 2.3.2, if $(a_1, a_2) \in \mathcal{R}_1 \cup \mathcal{R}_2$ then $k_1, k_2 \in \mathbb{R}$. If $(a_1, a_2) \in \mathcal{R}_3$ then the real character of $A_0$ implies that $\mathbf{u}_2 = \overline{\mathbf{u}}_1$, $\mathbf{v}_2 = \overline{\mathbf{v}}_1$ and therefore, $k_2 = \overline{k}_1$ (bar stands for complex conjugate).

From now on, $M$ will be the $4 \times 4$ symplectic matrix defined in (2.15) with $k_1, k_2, k_3$ and $k_4$ given in (2.20), (2.21) and (2.22) according to the region considered.

Let us define the following matrices
$$S_1 \;=\; M^{-1} L M, \quad S_2 = -J M^T J \overline{M}. \tag{2.23}$$

**Lemma 2.3.3.** *The new variable* $\mathbf{z}$ *satisfies* $\overline{\mathbf{z}} = \overline{S}_2 \mathbf{z}$, *and the following equalities hold*
$$\mathcal{H}(\mathbf{z}, t) = \mathcal{H}(S_1 \mathbf{z}, -t), \qquad \mathcal{H}(\mathbf{z}, t) = \overline{\mathcal{H}}(\overline{S}_2 \mathbf{z}, t), \tag{2.24}$$
*for all* $\mathbf{z} \in \mathbb{C}^4$, $t \in \mathbb{R}$.

*Moreover*

1. *if* $(a_1, a_2) \in \mathcal{R}_1$ *then* $S_1\mathbf{z} = (sz_3, iz_4, sz_1, -iz_2)^T$, $\overline{S}_2\mathbf{z} = (z_1, iz_4, z_3, iz_2)^T$,
2. *if* $(a_1, a_2) \in \mathcal{R}_2$ *then* $S_1\mathbf{z} = (iz_3, -isz_4, -iz_1, isz_2)^T$,
   $\overline{S}_2\mathbf{z} = (iz_3, -isz_4, iz_1, -isz_2)^T$,
3. *if* $(a_1, a_2) \in \mathcal{R}_3$ *then* $S_1\mathbf{z} = (z_3, z_4, z_1, z_2)^T$, $\overline{S}_2\mathbf{z} = (z_2, z_1, z_4, z_3)^T$.

**Proof**

The new variables $\mathbf{z} \in \mathbb{C}^4$ are defined by $\mathbf{y} = M\mathbf{z}$ where we recall that $\mathbf{y} \in \mathbb{R}^4$. Then

$$\mathbf{z} \quad = \quad M^{-1}\overline{\mathbf{y}} = -JM^T J\overline{M}\overline{\mathbf{z}} = S_2\overline{\mathbf{z}}, \tag{2.25}$$

where we have used that $M$ is a symplectic matrix and so, $M^{-1} = -JM^TJ$. Now $\overline{\mathbf{z}} = \overline{S}_2\mathbf{z}$ follows from (2.25).

From lemma 2.3.1,

$$\mathcal{H}(S_1\mathbf{z}, t) = H(MS_1\mathbf{z}, t) = H(MS_1M^{-1}\mathbf{y}, t) = H(L\mathbf{y}, t) = H(\mathbf{y}, t) = \mathcal{H}(\mathbf{z}, t),$$

and using the parity of $\mathcal{H}$ we get the first equality in (2.24).

Furthermore, $\mathcal{H}(\mathbf{z}, t)$ is real. Therefore

$$\mathcal{H}(\mathbf{z}, t) = \overline{\mathcal{H}(\mathbf{z}, t)} = \overline{\mathcal{H}}(\overline{\mathbf{z}}, t) = \overline{\mathcal{H}}(\overline{S}_2\mathbf{z}, t).$$

A simple computation gives

$$S_1 \quad = \quad \begin{pmatrix} 0 & \tilde{S}_1 \\ \tilde{S}_1^{-1} & 0 \end{pmatrix} \qquad \text{with} \qquad \tilde{S}_1 = \operatorname{diag}\left(\frac{k_3}{k_1}, \frac{k_4}{k_2}\right).$$

Then,

$$\tilde{S}_1 \quad = \quad \begin{cases} \operatorname{diag}(s, \mathrm{i}) & \text{if} \quad (a_1, a_2) \in \mathcal{R}_1, \\ \operatorname{diag}(\mathrm{i}, -\mathrm{i}s) & \text{if} \quad (a_1, a_2) \in \mathcal{R}_2, \\ I_2 & \text{if} \quad (a_1, a_2) \in \mathcal{R}_3, \end{cases}$$

where $I_2$ denotes the $2 \times 2$ identity matrix.

Now we compute $\overline{S}_2 = -J\overline{M}^T JM$.

In the region $\mathcal{R}_1$ we have that $\mathbf{u}_1, \mathbf{v}_1 \in \mathbb{R}^4$ and $\overline{\mathbf{u}}_2 = \mathbf{v}_2$. Then, we obtain

$$\overline{M}^T JM = \begin{pmatrix} 0 & 0 & k_1k_3\mathbf{u}_1^T J\mathbf{v}_1 & 0 \\ 0 & k_2^2\mathbf{v}_2^T J\mathbf{u}_2 & 0 & 0 \\ k_1\overline{k}_3\mathbf{v}_1^T J\mathbf{u}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4\overline{k}_4\mathbf{u}_2^T J\mathbf{v}_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{pmatrix} \tag{2.26}$$

In the region $\mathcal{R}_2$ we have $\overline{\mathbf{u}}_j = \mathbf{v}_j$, $j = 1, 2$. Then,

$$\begin{aligned}\overline{M}^T JM &= \mathrm{diag}(k_1^2 \mathbf{v}_1^T J\mathbf{u}_1, k_2^2 \mathbf{v}_2^T J\mathbf{u}_2, k_3\overline{k}_3 \mathbf{u}_1^T J\mathbf{v}_1, k_4\overline{k}_4 \mathbf{u}_2^T J\mathbf{v}_2) = \\ &= \mathrm{diag}(\mathrm{i}, -\mathrm{i}s, -\mathrm{i}, \mathrm{i}s). \end{aligned} \tag{2.27}$$

In $\mathcal{R}_3$ we get

$$\begin{aligned}\overline{M}^T JM &= \begin{pmatrix} 0 & 0 & 0 & \overline{k}_1 k_4 \mathbf{u}_2^T J\mathbf{v}_2 \\ 0 & 0 & k_3\overline{k}_3 \mathbf{u}_1^T J\mathbf{v}_1 & 0 \\ 0 & k_2\overline{k}_3 \mathbf{v}_2^T J\mathbf{u}_2 & 0 & 0 \\ k_1\overline{k}_4 \mathbf{v}_1^T J\mathbf{u}_1 & 0 & 0 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} 0 & B_1 \\ -B_1 & 0 \end{pmatrix}, \quad \text{with} \quad B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{2.28}$$

From (2.26), (2.27) and (2.28), $\overline{S}_2$ is easily computed at each region. □

### 2.3.2 The Normal Form in the different cases

In this section we apply the Normal Form techniques in order to simplify the Hamiltonian (2.16). To do this we need to distinguish the different regions.

In order to get the Normal Form we introduce $K$ as a conjugate variable of time $t$ and we consider the Hamiltonian

$$\mathcal{H}(\mathbf{z}, t, K) \quad = \quad \mathcal{H}_0(\mathbf{z}, K) + \tilde{\mathcal{H}}(\mathbf{z}, t), \tag{2.29}$$

where $\mathcal{H}(\mathbf{z}, K) = \mathcal{H}_0(\mathbf{z}) + K$ and $\mathcal{H}_0$ is given in (2.17).

Let be $\nu = \dfrac{T}{\pi}$ and $w = e^{\frac{2it}{\nu}}$. We can write the Hamiltonian as

$$\mathcal{H}(\mathbf{z}, w, K) \quad = \quad \mathcal{H}_0(\mathbf{z}, K) + \sum_{m=1}^{\infty} \mathcal{H}_m(\mathbf{z}, w), \tag{2.30}$$

where $\mathcal{H}_m(\mathbf{z}, w)$ contains terms of order $m$ in $\delta_1, \delta_2$ and $\varepsilon$. Moreover $\mathcal{H}_m(\mathbf{z}, w)$ is an homogeneous polynomial of degree 2 in $\mathbf{z}$ whose coefficients depend on $w$ and $w^{-1}$.

We can use Lie series method to perform some canonical transformations in order to cancel the time dependence on the Hamiltonian up to high order. This is done in section 2.6 using the Giorgilli–Galgani algorithm ([G.G.]). Then, if $\mathcal{N} = \mathcal{N}_0 + \mathcal{N}_1 + \mathcal{N}_2 + \ldots$ denotes the transformed Hamiltonian, we obtain

$$\mathcal{N}_m \quad = \quad \sum_{j=0}^{m} \mathcal{H}_{j,m-j}, \qquad \mathcal{H}_{m,j} = \sum_{l=1}^{j} \frac{l}{j} [G_l, \mathcal{H}_{m,j-l}], \qquad \mathcal{H}_{m,0} = \mathcal{H}_m, \tag{2.31}$$

| $\mathcal{N}_j$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\mathcal{N}_0$ | $\mathcal{H}_{0,0}$ | | | | | | |
| $\mathcal{N}_1$ | $\mathcal{H}_{1,0}$ | $\mathcal{H}_{0,1}$ | | | | | |
| $\mathcal{N}_2$ | $\mathcal{H}_{2,0}$ | $\mathcal{H}_{1,1}$ | $\mathcal{H}_{0,2}$ | | | | |
| $\mathcal{N}_3$ | $\mathcal{H}_{3,0}$ | $\mathcal{H}_{2,1}$ | $\mathcal{H}_{1,2}$ | $\mathcal{H}_{0,3}$ | | | |
| $\vdots$ | | | | | | | |
| $\mathcal{N}_m$ | $\mathcal{H}_{m,0}$ | $\mathcal{H}_{m-1,1}$ | $\dots$ | $\dots$ | $\dots$ | $\mathcal{H}_{1,m-1}$ | $\mathcal{H}_{0,m}$ |

Table 2.1: Functions involved in the computation of $\mathcal{N}_j$, $j = 0, \dots, m$

and $G_m$ is the solution of the homological equation

$$M_m + [G_m, \mathcal{H}_0] \quad = \quad R_m, \tag{2.32}$$

where

$$M_m = \sum_{j=0}^{m-1} \mathcal{H}_{m-j,j} + \sum_{l=1}^{m-1} \frac{l}{m}[G_l, \mathcal{H}_{0,m-l}],$$

and $R_m$ contains resonant terms of order $m$ in $\delta_1, \delta_2$ and $\varepsilon$.

Table 2.1 shows the functions involved in $\mathcal{N}_j$ up to order $m$, that is, up to $j = m$.

In what follows we shall denote the new variables, say $Z_j$, $j = 1, \dots, 4$, obtained by the canonical changes of variables involved in the normalization, as $z_j$, $j = 1, \dots, 4$, again. The next proposition gives the Normal Form depending on the region $\mathcal{R}_1, \mathcal{R}_2$ or $\mathcal{R}_3$. The proof is given in section 2.6.

**Proposition 2.3.4.** *Let us denote by $NF$ the Normal Form up to a fixed order in the small parameters $\delta_1, \delta_2, \varepsilon$.*

1. *If $(a_1, a_2) \in \mathcal{R}_1$ and $\nu\omega \in \mathbb{N}$, then*

$$\begin{aligned} NF \quad = \quad & K + \lambda z_1 z_3 + i\omega z_2 z_4 + \sigma_1 z_1 z_3 + i\sigma_2 z_2 z_4 + \\ & + \sigma_3 z_2^2 w^{-\nu\omega} - \sigma_3 z_4^2 w^{\nu\omega}, \end{aligned} \tag{2.33}$$

*where $\sigma_j \in \mathbb{R}$, $j = 1, \dots, 4$, depend on $\delta_1, \delta_2$ and $\varepsilon$.*

2. *If* $(a_1, a_2) \in \mathcal{R}_2$, *then*

$$NF = \begin{cases} N_0 + N_1 & \text{if } \nu\omega_1 \in \mathbb{N}, \quad \nu\omega_2 \notin \mathbb{N}, \\ N_0 + N_2 & \text{if } \nu\omega_1 \notin \mathbb{N}, \quad \nu\omega_2 \in \mathbb{N}, \\ N_0 + N_1 + N_2 & \text{if } \nu\omega_1 \in \mathbb{N}, \quad \nu\omega_2 \in \mathbb{N}, \\ & \quad \text{and} \quad \nu\omega_1 \not\equiv \nu\omega_2 (mod\ 2), \\ N_0 + N_3 & \text{if } \nu\omega_{hs} \in \mathbb{N}, \quad \nu\omega_{hd} \notin \mathbb{N}, \\ & \quad (\nu\omega_1 \notin \mathbb{N}, \quad \nu\omega_2 \notin \mathbb{N}), \\ N_0 + N_4 & \text{if } \nu\omega_{hd} \in \mathbb{N}, \quad \nu\omega_{hs} \notin \mathbb{N}, \\ & \quad (\nu\omega_1 \notin \mathbb{N}, \quad \nu\omega_2 \notin \mathbb{N}), \\ N_0 + N_1 + N_2 + N_3 + N_4 & \text{if } \nu\omega_1 \in \mathbb{N}, \quad \nu\omega_2 \in \mathbb{N} \\ & \quad \text{and} \quad \nu\omega_1 \equiv \nu\omega_2 (mod\ 2), \end{cases} \tag{2.34}$$

*where* $\omega_{hs} = \dfrac{\omega_1 + \omega_2}{2}$, $\omega_{hd} = \dfrac{\omega_1 - \omega_2}{2}$, *and*

$$\begin{aligned} N_0 &= K + i\omega_1 z_1 z_3 + i\omega_2 z_2 z_4 + i\sigma_1 z_1 z_3 + i\sigma_2 z_2 z_4, \\ N_1 &= \sigma_3 z_1^2 w^{-\nu\omega_1} - \sigma_3 z_3^2 w^{\nu\omega_1}, \\ N_2 &= \sigma_4 z_2^2 w^{-\nu\omega_2} - \sigma_4 z_4^2 w^{\nu\omega_2}, \\ N_3 &= \sigma_5 z_1 z_2 w^{-\nu\omega_{hs}} + s\sigma_5 z_3 z_4 w^{\nu\omega_{hs}}, \\ N_4 &= i\sigma_6 z_1 z_4 w^{-\nu\omega_{hd}} - is\sigma_6 z_2 z_3 w^{\nu\omega_{hd}}, \end{aligned} \tag{2.35}$$

*where* $\sigma_j \in \mathbb{R}$, $j = 1, \ldots, 6$ *depend on* $\delta_1, \delta_2, \varepsilon$, *and* $s = sgn(a_1)$.

3. *If* $(a_1, a_2) \in \mathcal{R}_3$ *and* $\nu\beta \in \mathbb{N}$ *then*

$$\begin{aligned} NF = {} & K + (\alpha + i\beta) z_1 z_3 + (\alpha - i\beta) z_2 z_4 + \sigma_1 z_1 z_3 + \overline{\sigma}_1 z_2 z_4 + \sigma_3 z_1 z_4 w^{-\nu\beta} + \\ & \sigma_3 z_2 z_3 w^{\nu\beta}, \end{aligned} \tag{2.36}$$

*where* $\sigma_1 \in \mathbb{C}$, $\sigma_3 \in \mathbb{R}$ *depend on* $\delta_1, \delta_2, \varepsilon$.

*Remark* 2.3.5. Proposition 2.3.4 gives the Normal Form up to a given order, say $n$, when $\lambda_1 = a_1 + \delta_1$, $\lambda_2 = a_2 + \delta_2$ and $(a_1, a_2)$ is a resonant point for $\varepsilon = 0$. The Normal Form can be written as

$$NF = N_0 + \mathcal{N}_n(w),$$

where

$$\begin{array}{lll} N_0 = K + (\lambda + \sigma_1) z_1 z_3 + \mathrm{i}(\omega + \sigma_2) z_2 z_4 & \text{if} & (a_1, a_2) \in \mathcal{R}_1, \\ N_0 = K + \mathrm{i}(\omega_1 + \sigma_1) z_1 z_3 + \mathrm{i}(\omega_2 + \sigma_2) z_2 z_4 & \text{if} & (a_1, a_2) \in \mathcal{R}_2, \\ N_0 = K + (\alpha + \mathrm{i}\beta + \sigma_1) z_1 z_3 + (\alpha - \mathrm{i}\beta + \overline{\sigma}_1) z_2 z_4 & \text{if} & (a_1, a_2) \in \mathcal{R}_3, \end{array}$$

and all the monomials in $\mathcal{N}_n(w)$ depend on $w$ and so, they are time dependent.

However, if $\varepsilon = 0$ the initial Hamiltonian (2.12) is autonomous. In this case, the Normal Form does not depend on $w$. Therefore, for the coefficients $\sigma_3$, $\sigma_4$, $\sigma_5$, $\sigma_6$ in Proposition 2.3.4 we have

$$\sigma_j = O(\varepsilon^k), \quad j = 3, \ldots, 6, \tag{2.37}$$

for some $k \geq 1$ which may depend on the index $j$.

Furthermore, $\sigma_1$ and $\sigma_2$ depend on $\delta_1, \delta_2, \varepsilon$. In fact $\sigma_1$ and $\sigma_2$ have terms of order 1 in $\delta_1, \delta_2$. That terms can be easily computed by taking into account the variation of the eigenvalues of the system when $\varepsilon = 0$ and we perturb $(a_1, a_2)$ by $(\delta_1, \delta_2)$. These terms will be explicitly computed in section 2.5.

## 2.4 Bifurcations

In this section we study the conditions for bifurcation when $\varepsilon > 0$ is small enough. We shall use the following notation for the different regions in the parameter space $(\lambda_1, \lambda_2, \varepsilon)$.

- EE (elliptic–elliptic) stands for a region such that $|\mathrm{tr}_j| < 2$, $j = 1, 2$,
- EH (elliptic–hyperbolic) corresponds to $|\mathrm{tr}_1| < 2$, $|\mathrm{tr}_2| > 2$ (or vice–versa),
- HH (hyperbolic–hyperbolic) corresponds to $\mathrm{tr}_j > 2$, $j = 1, 2$,
- CS (complex–saddle) stands for complex $\mathrm{tr}_1$, $\mathrm{tr}_2 = \overline{\mathrm{tr}}_1$.

We obtain the equations for the boundaries of the different bifurcation regions in terms of the coefficients of the Normal Form. In next section we shall concentrate in the d'Alembert case, that is, when the functions $F_j$, $j = 1, 2$, defined in (2.2) satisfy the d'Alembert property.

In order to obtain the boundaries of the bifurcation regions for $\varepsilon > 0$ small enough we study the Hamiltonian system associated to the Normal Form given in Proposition 2.3.4.

Let us take $(a_1, a_2) \in \mathcal{R}_1$ a resonant point for $\varepsilon = 0$.

For $\varepsilon > 0$, bifurcation occurs when a pair of characteristic multipliers on the unit circle collides and become real. In this case, system turns from EH to HH.

Normal Form (2.33) defines the following uncoupled linear system

$$\begin{aligned} \dot z_1 &= (\lambda + \sigma_1) z_1, & \dot z_2 &= i(\omega + \sigma_2) z_2 - 2\sigma_3 z_4 w^{\nu\omega}, \\ \dot z_3 &= -(\lambda + \sigma_1) z_3, & \dot z_4 &= -2\sigma_3 z_2 w^{-\nu\omega} - i(\omega + \sigma_2) z_4, \end{aligned} \tag{2.38}$$

where we assume $\nu\omega = n \in \mathbb{N}$.

The system for $z_1, z_3$ gives real characteristic exponents and, then, a stability parameter is greater than two. This gives an hyperbolic behavior.

In order to study the system for $z_2, z_4$ we perform the change of variables $u = z_2 w^{-\nu\omega/2}$, $v = z_4 w^{\nu\omega/2}$. Then, this system transforms in the following linear system with constant coefficients

$$\begin{aligned} \dot{u} &= i\sigma_2 u - 2\sigma_3 v, \\ \dot{v} &= -2\sigma_3 u - i\sigma_2 v. \end{aligned} \tag{2.39}$$

The eigenvalues of the above system are $\pm\sqrt{4\sigma_3^2 - \sigma_2^2}$. The bifurcation takes place when these eigenvalues cros zero. In this case, a transition EH$\leftrightarrow$HH occurs. For $\varepsilon > 0$ an instability region HH in the parameter space is created. The boundaries of this region up to a given order in $\delta_1, \delta_2, \varepsilon$ are defined by the equation

$$\sigma_2^2 - 4\sigma_3^2 = 0. \tag{2.40}$$

Now we consider $(a_1, a_2) \in \mathcal{R}_2$ a resonant point for $\varepsilon = 0$.

We study the general case in (2.34), that is, $NF = N_0 + N_1 + N_2 + N_3 + N_4$ where $N_i$, $i = 0, \ldots, 4$, are given in (2.35). The other cases in (2.34) are obtained by taking the suitable coefficients equal to zero. The linear system defined by $NF$ is the following.

$$\begin{aligned} \dot{z}_1 &= \mathrm{i}(\omega_1 + \sigma_1) z_1 - \mathrm{i}s\sigma_6 z_2 w^{\frac{\nu}{2}(\omega_1 - \omega_2)} - 2\sigma_3 z_3 w^{\nu\omega_1} + s\sigma_5 z_4 w^{-\frac{\nu}{2}(\omega_1 + \omega_2)}, \\ \dot{z}_2 &= \mathrm{i}\sigma_6 z_1 w^{-\frac{\nu}{2}(\omega_1 - \omega_2)} + \mathrm{i}(\omega_2 + \sigma_2) z_2 + s\sigma_5 z_3 w^{\frac{\nu}{2}(\omega_1 + \omega_2)} - 2\sigma_4 z_4 w^{\nu\omega_2}, \quad (2.41) \\ \dot{z}_3 &= -2\sigma_3 z_1 w^{-\nu\omega_1} - \sigma_5 z_2 w^{-\frac{\nu}{2}(\omega_1 + \omega_2)} - \mathrm{i}(\omega_1 + \sigma_1) z_3 - \mathrm{i}\sigma_6 z_4 w^{-\frac{\nu}{2}(\omega_1 - \omega_2)}, \\ \dot{z}_4 &= -\sigma_5 z_1 w^{-\frac{\nu}{2}(\omega_1 + \omega_2)} - 2\sigma_4 z_2 w^{-\nu\omega_2} + \mathrm{i}s\sigma_6 z_3 w^{\frac{\nu}{2}(\omega_1 - \omega_2)} - \mathrm{i}(\omega_2 + \sigma_2) z_4. \end{aligned}$$

We introduce new variables $u_1 = z_1 w^{-\frac{\nu}{2}\omega_1}$, $u_2 = z_2 w^{-\frac{\nu}{2}\omega_2}$, $v_1 = z_3 w^{\frac{\nu}{2}\omega_1}$, $v_2 = z_4 w^{\frac{\nu}{2}\omega_2}$. Then, system (2.41) becomes the following constant coefficients linear system

$$\begin{aligned} \dot{u}_1 &= \mathrm{i}\sigma_1 u_1 - \mathrm{i}s\sigma_6 u_2 - 2\sigma_3 v_1 + s\sigma_5 v_2, \\ \dot{u}_2 &= \mathrm{i}\sigma_6 u_1 + \mathrm{i}\sigma_2 u_2 + s\sigma_5 v_1 - 2\sigma_4 v_2, \\ \dot{v}_1 &= -2\sigma_3 u_1 - \sigma_5 u_2 - \mathrm{i}\sigma_1 v_1 - \mathrm{i}\sigma_6 v_2, \\ \dot{v}_2 &= -\sigma_5 u_1 - 2\sigma_4 u_2 + \mathrm{i}s\sigma_6 v_1 - \mathrm{i}\sigma_2 v_2. \end{aligned} \tag{2.42}$$

To study the bifurcations, we need to distinguish different cases.

1. $\nu\omega_1 \in \mathbb{N}$, $\nu\omega_2 \notin \mathbb{N}$.

In this case, $\sigma_4 = \sigma_5 = \sigma_6 = 0$ and (2.42) is an uncoupled system

$$\begin{aligned} \dot{u}_1 &= i\sigma_1 u_1 - 2\sigma_3 v_1, & \dot{u}_2 &= i\sigma_3 u_2, \\ \dot{v}_1 &= -2\sigma_3 u_1 - i\sigma_1 v_1, & \dot{v}_2 &= -i\sigma_3 v_2. \end{aligned}$$

The system in $u_2, v_2$ gives us a stability parameter $|\mathrm{tr}_2| < 2$ for $\delta_1, \delta_2, \varepsilon$ small enough. The study of the system in $u_1, v_1$ is analogous to (2.39) in $\mathcal{R}_1$. Then, we have a transition EE$\leftrightarrow$EH. The boundaries of the region EH are given by the equation

$$\sigma_1^2 - 4\sigma_3^2 \;=\; 0.$$

2. $\nu\omega_1 \notin \mathbb{N}$, $\nu\omega_2 \in \mathbb{N}$.

   This case is analogous to the previous one, obtaining an EH region, where their boundaries are defined by

$$\sigma_2^2 - 4\sigma_4^2 \;=\; 0.$$

3. $\nu\omega_1 \notin \mathbb{N}$, $\nu\omega_2 \notin \mathbb{N}$ and $\omega_{hs} \in \mathbb{N}$.

   In this case, $\sigma_3 = \sigma_4 = \sigma_6 = 0$ and (2.42) becomes an uncoupled system

$$\begin{aligned} \dot{u}_1 &= i\sigma_1 u_1 + s\sigma_5 v_2, & \dot{u}_2 &= i\sigma_2 u_2 + s\sigma_5 v_1, \\ \dot{v}_2 &= -\sigma_5 u_1 - i\sigma_2 v_2, & \dot{v}_1 &= -\sigma_5 u_2 - i\sigma_1 v_1. \end{aligned}$$

   The characteristic polynomials of the two uncoupled linear systems above are

$$x^2 \mp \mathrm{i}(\sigma_1 - \sigma_2)x + \sigma_1\sigma_2 + s\sigma_5^2, \tag{2.43}$$

   where $-$ stands for the system in $u_1, v_2$ and $+$ is for $u_2, v_1$.

   Let us define $\Delta = -[(\sigma_1 + \sigma_2)^2 + 4s\sigma_5^2]$ the discriminant of (2.43). The sign of $\Delta$ determines the character of the region.

   Figure 2.3 shows the character of the characteristic multipliers depending on the discriminant.

   The boundaries of the region CS are given by

$$(\sigma_1 + \sigma_2)^2 + 4s\sigma_5^2 \;=\; 0. \tag{2.44}$$

4. $\nu\omega_1 \notin \mathbb{N}$, $\nu\omega_2 \notin \mathbb{N}$ and $\omega_{hd} \in \mathbb{N}$.

   In this case system (2.42) becomes also an uncoupled system,

$$\begin{aligned} \dot{u}_1 &= i\sigma_1 u_1 - is\sigma_6 u_2, & \dot{v}_1 &= -i\sigma_1 v_1 - i\sigma_6 v_2, \\ \dot{u}_2 &= i\sigma_6 u_1 + i\sigma_2 u_2, & \dot{v}_2 &= is\sigma_6 v_1 - i\sigma_2 v_2. \end{aligned} \tag{2.45}$$

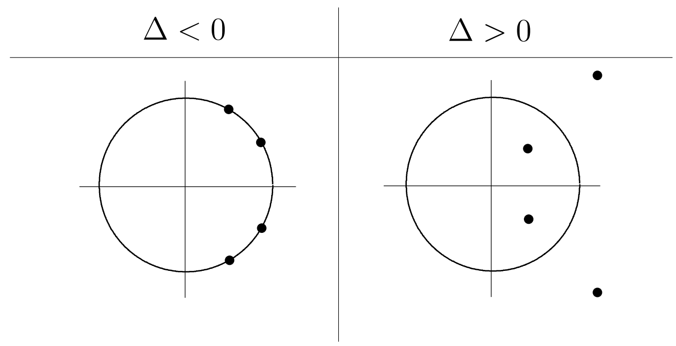



Figure 2.3: Characteristic multipliers in the case 3 according to the sign of $\Delta$

The characteristic polynomials of these uncoupled systems are

$$x^2 \mp \mathrm{i}(\sigma_1 + \sigma_2)x - s\sigma_6^2 - \sigma_1\sigma_2, \tag{2.46}$$

where $-$ stands for the system in $u_1, u_2$ and $+$ for $v_1, v_2$.

In this case, the discriminant of (2.46) is $\Delta = -(\sigma_1 - \sigma_2)^2 + 4s\sigma_6^2$. As before we get a transition EE $\leftrightarrow$ CS when the discriminant of (2.46) is zero (see Figure 2.3).

The boundaries of the new region CS are given by the equation

$$-(\sigma_1 - \sigma_2)^2 + 4s\sigma_6^2 \;\; = \;\; 0. \tag{2.47}$$

*Remark* 2.4.1. The equation (2.44) has no real solution if $s = 1$, that is $a_1 > 0$. So, there is no bifurcation for $\nu\omega_{hs} \in \mathbb{N}$ when $a_1 > 0$. In the same way, there is no bifurcation if $\nu\omega_{hd} \in \mathbb{N}$ when $a_1 < 0$. This fact is well known as a consequence of Krein theorem (see [K.]).

5. $\nu\omega_1 \in \mathbb{N}$, $\nu\omega_2 \in \mathbb{N}$ with different parity.

   In this case system (2.42) splits in two uncoupled systems

$$\begin{array}{ll} \dot{u}_1 = i\sigma_1 u_1 - 2\sigma_3 v_1, & \dot{u}_2 = i\sigma_2 u_2 - 2\sigma_4 v_2, \\ \dot{v}_1 = -2\sigma_3 u_1 - i\sigma_1 v_1, & \dot{v}_2 = -2\sigma_4 u_2 - i\sigma_2 v_2. \end{array} \tag{2.48}$$

   Let $\Delta_1 = \sigma_1^2 - 4\sigma_3^2$ and $\Delta_2 = \sigma_2^2 - 4\sigma_4^2$ be the discriminants of the characteristic polynomials of the uncoupled systems above. Figure 2.4 shows the behavior for the characteristic multipliers depending on the sign of $\Delta_1$ and $\Delta_2$.

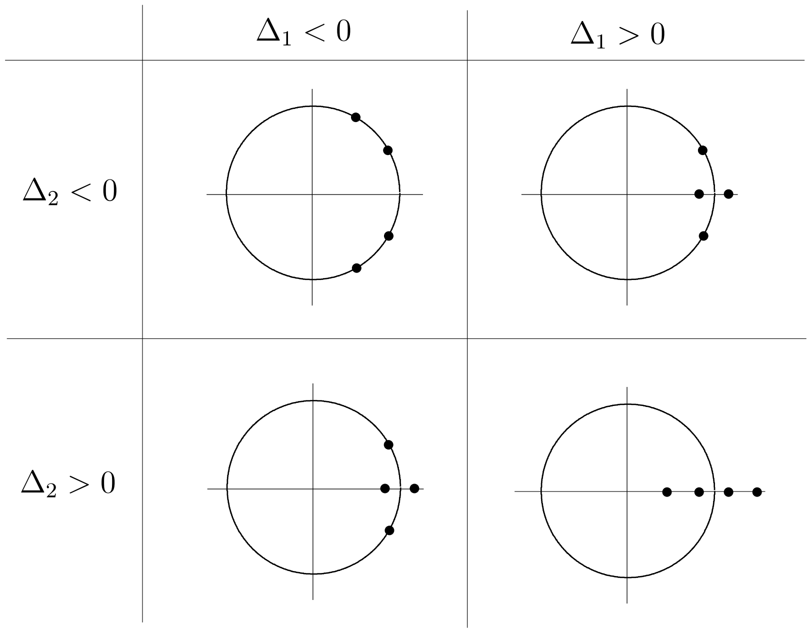



Figure 2.4: Characteristic multipliers according to the sign of $\Delta_1 = \sigma_1^2 - 4\sigma_3^2$ and $\Delta_2 = \sigma_2^2 - 4\sigma_4^2$

The boundaries of the EH region are given by

$$\sigma_1^2 - 4\sigma_3^2 = 0 \quad \text{or} \quad \sigma_2^2 - 4\sigma_4^2 = 0,$$

and the boundaries of the region HH are given by

$$\sigma_1^2 - 4\sigma_3^2 = 0 \quad \text{and} \quad \sigma_2^2 - 4\sigma_4^2 = 0.$$

The cases 1 to 5 are summarized in the following table.

| | | |
|---|---|---|
| $\nu\omega_1 \in \mathbb{N}, \nu\omega_2 \notin \mathbb{N}$ | EE↔EH | $\sigma_1^2 - 4\sigma_3^2 = 0$ |
| $\nu\omega_1 \notin \mathbb{N}, \nu\omega_2 \in \mathbb{N}$ | EE↔EH | $\sigma_2^2 - 4\sigma_4^2 = 0$ |
| $\nu\omega_1 \in \mathbb{N}, \nu\omega_2 \in \mathbb{N}$ | EE↔EH | $\sigma_1^2 - 4\sigma_3^2 = 0$ or $\sigma_2^2 - 4\sigma_4^2 = 0$ |
| with different parity | EE↔HH | $\sigma_1^2 - 4\sigma_3^2 = 0$ and $\sigma_2^2 - 4\sigma_4^2 = 0$ |
| $\nu\omega_1 \notin \mathbb{N}, \nu\omega_2 \notin \mathbb{N}$, $\frac{\nu}{2}(\omega_1 + \omega_2) \in \mathbb{N}$ | EE↔CS | $(\sigma_1 + \sigma_2)^2 + 4s\sigma_5 = 0$ |
| $\nu\omega_1 \notin \mathbb{N}, \nu\omega_2 \notin \mathbb{N}$, $\frac{\nu}{2}(\omega_2 - \omega_2) \in \mathbb{N}$ | EE↔CS | $(\sigma_1 - \sigma_2)^2 - 4s\sigma_6 = 0$ |

6. $\nu\omega_1 \in \mathbb{N}$, $\nu\omega_2 \in \mathbb{N}$ with the same parity.

   In this case, generically the coefficients $\sigma_j$, $j = 3, 4, 5, 6$ in (2.34) are different from zero. Now the system is not uncoupled.

However, due to his Hamiltonian character, the characteristic polynomial of (2.42) is

$$q(x) \quad = \quad x^4 + d_1 x^2 + d_2, \tag{2.49}$$

where

$$\begin{aligned} d_1 &= \sigma_1^2 + \sigma_2^2 - 4(\sigma_3^2 + \sigma_4^2) + 2s(\sigma_5^2 - \sigma_6^2), && (2.50)\\ d_2 &= D_1 D_2, && (2.51)\\ D_1 &= (\sigma_1 - 2s\sigma_3)(\sigma_2 + 2\sigma_4) + s(\sigma_5 + \sigma_6)^2, &&\\ D_2 &= (\sigma_1 + 2s\sigma_3)(\sigma_2 - 2\sigma_4) + s(\sigma_5 - \sigma_6)^2. && (2.52) \end{aligned}$$

Let

$$d_3 \quad = \quad d_1^2 - 4d_2, \tag{2.53}$$

the discriminant of (2.49). Then, the different possibilities for the character of the system, excluding boundary values, are

- EH if $d_2 < 0$,
- CS if $d_2 > 0$ and $d_3 < 0$,
- EE if $d_2 > 0$, $d_3 > 0$ and $d_1 > 0$,
- HH if $d_2 > 0$, $d_3 > 0$ and $d_1 < 0$.

Figure 2.5 represents these situations.

Finally we take $(a_1, a_2) \in \mathcal{R}_3$ a resonant point for $\varepsilon = 0$.
We recall that for $\varepsilon = 0$ the system (2.1) is in general saddle-complex.
The linear system defined by the Normal Form (2.36) is

$$\begin{aligned} \dot{z}_1 &= (\rho + \sigma_1) z_1 + \sigma_3 z_2 w^{\nu\beta},\\ \dot{z}_2 &= \sigma_3 z_1 w^{-\nu\beta} + (\overline{\rho} + \overline{\sigma}_1) z_2,\\ \dot{z}_3 &= -(\rho + \sigma_1) z_3 - \sigma_3 z_4 w^{-\nu\beta},\\ \dot{z}_4 &= -\sigma_3 z_3 w^{\nu\beta} - (\overline{\rho} + \overline{\sigma}_1) z_4, \end{aligned} \tag{2.54}$$

with $\rho = \alpha + i\beta$.

In order to study the stability parameters of system (2.54) we perform the change of variables $u_1 = z_1 w^{-\frac{\nu\beta}{2}}$, $u_2 = z_2 w^{\frac{\nu\beta}{2}}$, $v_1 = z_3 w^{\frac{\nu\beta}{2}}$, $v_2 = z_4 w^{-\frac{\nu\beta}{2}}$. We obtain the uncoupled linear system with constant coefficients

$$\begin{array}{ll} \dot{u}_1 = (\alpha + \sigma_1) u_1 + \sigma_3 u_2, & \dot{v}_1 = -(\alpha + \sigma_1) v_1 - \sigma_3 v_2,\\ \dot{u}_2 = \sigma_3 u_1 + (\alpha + \overline{\sigma}_1) u_2, & \dot{v}_2 = -\sigma_3 v_1 - (\alpha + \overline{\sigma}_1) v_2, \end{array} .$$

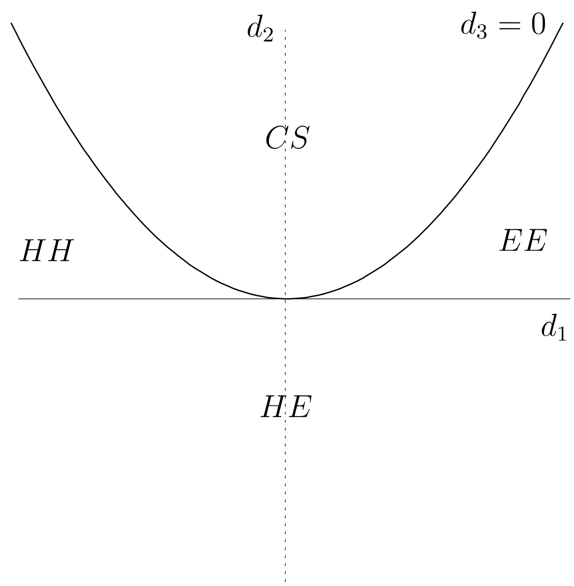



Figure 2.5: Stability regions in the $(d_1, d_2)$ plane

Let be $\Delta = \sigma_3^2 - \mathrm{Im}(\sigma_1)^2$. If $\Delta > 0$ then we have a pair of real eigenvalues and their opposites. Figure 2.6 shows the behavior of the characteristic multipliers in terms of $\Delta$.

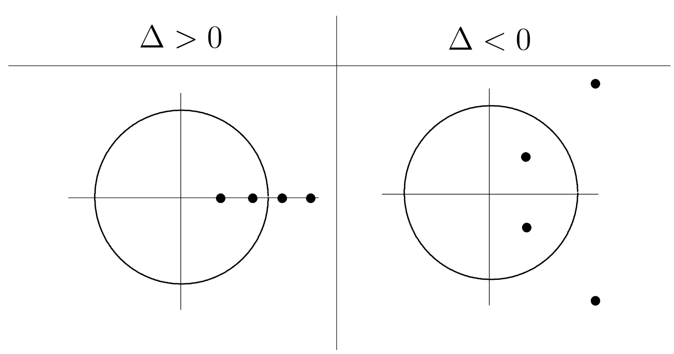



Figure 2.6: Characteristic multipliers according to the sign of $\Delta = \sigma_3^2 - \mathrm{Im}(\sigma_1)^2$

A transition HH↔CS occurs when $\Delta$ goes through 0. Then, the equations for the boundaries of the HH region are given by

$$\mathrm{Im}(\sigma_1) \quad = \quad \pm\sigma_3. \tag{2.55}$$

## 2.5 The d'Alembert case

In this section we study the resonant regions in the particular case that $F_j$ in (2.2) satisfy d'**Alembert property**, that is, functions $F_j$, $j = 1, 2$, are of the form

$$\sum_{m\geq 1} \varepsilon^m \sum_{l=0}^{m} c_{m,l} \cos\left(l\frac{2\pi t}{T}\right),$$

where $c_{m,l} \in \mathbb{R}$.

After remark 2.3.5 we know that for the coefficients $\sigma_j$, $j = 3, 4, 5, 6$, in the Normal Form, (2.37) is satisfied for $k \geq 1$. The d'Alembert property can be used to determine, under non degeneracy conditions, the order of these coefficients as follows.

Let us go back to the initial Hamiltonian (2.11) and consider the time dependent terms (see (2.13))

$$(a_1 + \delta_1)\frac{y_1^2}{2}F_1(t;\varepsilon) + (a_2 + \delta_2)\frac{y_2^2}{2}F_2(t;\varepsilon) \quad .$$

Let us consider a fixed $\sigma_j$, $j = 3, 4, 5, 6$, and assume that it is the coefficient of a resonant monomial $\mathbf{z}^{\mathbf{l}}$,

$$\sigma_j w^{\pm n}\mathbf{z}^{\mathbf{l}} \quad \text{with} \quad n \in \mathbb{N}, \tag{2.56}$$

where $\mathbf{z} = (z_1, z_2, z_3, z_4)^T$ and $\mathbf{l} = (l_1, l_2, l_3, l_4)$ with $l_j \in \mathbb{N}$ satisfying $l_1+l_2+l_3+l_4 = 2$. For a fixed $n \in \mathbb{N}$, the d'Alembert property implies that in $F_1$ and $F_2$ the terms in $\cos\left(n\frac{2\pi t}{T}\right)$ can be written as

$$\cos\left(n\frac{2\pi t}{T}\right)\varepsilon^n(c_{n,n} + O_1) = \frac{w^n + w^{-n}}{2}\varepsilon^n(c_{n,n} + O_1) \quad ,$$

where $w = e^{\frac{2\mathrm{i}t}{\nu}}$, $O_1$ contains terms of order at least 1 in $\varepsilon$, and $c_{n,n}$ is a coefficient eventually zero. We recall that in order to get the Normal Form, first we perform the change of variables $\mathbf{y} = M\mathbf{z}$ to (2.11). For the new Hamiltonian as given in (2.29) the terms in $w^n$, $w^{-n}$ are of order $n$ in $\varepsilon$. So, they only appear in $\mathcal{H}_m(\mathbf{z}, w)$ for $m \geq n$. Now it is not difficult to see that this property is preserved by the Giorgilli–Galgani algorithm. In fact, the terms $\varepsilon^n w^n$, $\varepsilon^n w^{-n}$ appear for the first time in the $n$–th row of table 1. Therefore, if $j = 3, 4, 5, 6$ for $\sigma_j$ in (2.56) we get

$$\sigma_j \quad = \quad c_j\varepsilon^n(1 + O_1),$$

where $c_j$ is a coefficient eventually zero and $O_1$ contains terms of order 1 in $\delta_1, \delta_2, \varepsilon$. We shall assume in the next as non degeneracy conditions that $c_j \neq 0$, $j = 3, 4, 5, 6$.

### 2.5.1 Single resonances

We shall consider resonant points $(a_1, a_2)$ which belong to a unique resonant curve. This kind of points are found at every region $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$.

We begin with $\mathcal{R}_1$ and assume that $(a_1, a_2)$ belongs to a resonant curve (2.6), that is,

$$\gamma_n(a_1, a_2) := (a_1 + \omega^2)(a_2 + \omega^2) - 4\omega^2 = 0, \quad \text{where} \quad \omega = \frac{n\pi}{T}, \tag{2.57}$$

for some $n \in \mathbb{N}$. From now on, $n$ is fixed.

The boundary surfaces which separate the EH and HH regions for $\varepsilon > 0$ are defined by (2.40). Our purpose is to give an estimation of the size of the HH region.

As we have seen in the beginning of this section the d'Alembert property implies that

$$\sigma_3 \quad = \quad c_3\varepsilon^n(1 + O_1), \tag{2.58}$$

where $c_3 \in \mathbb{R}$ and $O_1$ stands for terms of order 1 in $\delta_1, \delta_2, \varepsilon$. It is assumed that $c_3 \neq 0$. The following lemma gives the terms of $\sigma_2$ which are of order 1 in $\delta_1, \delta_2$.

**Lemma 2.5.1.** *Let $(a_1, a_2) \in \mathcal{R}_1$ be such that $\gamma_n(a_1, a_2) = 0$. Then, the dominant terms in the contribution of $\delta_1$ and $\delta_2$ to $\sigma_2$ are*

$$-\left[\frac{\omega^2 + a_2}{D(\omega)}\delta_1 + \frac{\omega^2 + a_1}{D(\omega)}\delta_2\right], \tag{2.59}$$

*where $D(\omega) = 2\omega[2\omega^2 + a_1 + a_2 - 4] \neq 0$.*

*Remark* 2.5.2. This lemma is also true if $F_j$, $j = 1, 2$, do not satisfy the d'Alembert property.

**Proof**

We use the hint given in remark 2.3.5, that is, assume $\varepsilon = 0$ and consider $\sigma_i = \sigma_i(\delta_1, \delta_2)$, $i = 1, 2$.

In the plane of parameters $\lambda_1, \lambda_2$, if $\lambda_1 = a_1 + \delta_1$, $\lambda_2 = a_2 + \delta_2$, the zeroes of the characteristic polynomial $p(x)$ defined in (2.5) are $\pm\Lambda(\delta_1, \delta_2) \in \mathbb{R}$, $\pm \mathrm{i}\Omega(\delta_1, \delta_2)$ where

$$\Lambda(\delta_1, \delta_2) := \lambda + \sigma_1(\delta_2, \delta_2), \qquad \Omega(\delta_1, \delta_2) := \omega + \sigma_2(\delta_1, \delta_2).$$

Here, $\pm\lambda$, $\pm \mathrm{i}\omega$, are the zeroes of $p(x)$ for $(\delta_1, \delta_2) = (0, 0)$. Then, we can write

$$\sigma_2(\delta_1, \delta_2) = \frac{\partial\Omega}{\partial\delta_1}(0, 0)\delta_1 + \frac{\partial\Omega}{\partial\delta_2}(0, 0)\delta_2 + \ldots,$$

and the partial derivatives are easily computed using that $p(\mathrm{i}\Omega) = 0$. □

In order to describe the boundary surfaces we shall consider perturbations of $(a_1, a_2)$ in an orthogonal direction to the resonant curve (2.57), that is, $\lambda_1 = a_1+\delta_1$, $\lambda_2 = a_2 + \delta_2$ with

$$\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \delta \nabla \gamma_n(a_1, a_2), \tag{2.60}$$

for some parameter $\delta$, being $|\delta|$ small enough. In this way, $\sigma_2$ and $\sigma_3$ depend on $\varepsilon$ and $\delta$. Moreover, (2.59) becomes

$$-\frac{\|\nabla\gamma_n(a_1, a_2)\|^2}{D(\omega)}\delta.$$

We remark that $\|\nabla\gamma_n(a_1, a_2)\| \neq 0$. Otherwise, we should have $\omega = 0$, but we know that $\omega \neq 0$ if $(a_1, a_2) \in \mathcal{R}_1$.

Therefore we can write

$$\sigma_2 = c\|\nabla\gamma_n(a_1, a_2)\|^2\delta + \phi_0(\varepsilon) + \delta\phi_1(\varepsilon) + \delta^2 f(\varepsilon, \delta),$$

where $c = -\dfrac{1}{D(\omega)}$, $\phi_0$ and $\phi_1$ are functions of order 1 in $\varepsilon$ and $f(\varepsilon, \delta)$ is of order 1 in $\varepsilon, \delta$.

Let us introduce the functions

$$f_1(\varepsilon, \delta) = \sigma_2 - 2\sigma_3, \qquad f_2(\varepsilon, \delta) = \sigma_2 + 2\sigma_3.$$

The boundaries of the HH region are defined by

$$f_1(\varepsilon, \delta) = 0, \qquad f_2(\varepsilon, \delta) = 0.$$

We have that $\sigma_2 = \sigma_3 = 0$ if $(\varepsilon, \delta) = (0, 0)$. Then, $f_1(0, 0) = f_2(0, 0) = 0$. Moreover,

$$\frac{\partial f_1}{\partial\delta}(0, 0) = \frac{\partial f_2}{\partial\delta}(0, 0) = c\|\nabla\gamma_n(a_1, a_2)\|^2 \neq 0.$$

Then, using the Implicit Function Theorem there exist $\delta_+(\varepsilon)$, $\delta_-(\varepsilon)$ analytic functions in $\varepsilon \gtrsim 0$ such that

$$\begin{aligned} f_1(\varepsilon, \delta_+(\varepsilon)) = \sigma_2(\delta_+(\varepsilon), \varepsilon) - 2\sigma_3(\delta_+(\varepsilon), \varepsilon) &= 0, \\ f_2(\varepsilon, \delta_-(\varepsilon)) = \sigma_2(\delta_-(\varepsilon), \varepsilon) + 2\sigma_3(\delta_-(\varepsilon), \varepsilon) &= 0. \end{aligned}$$

Therefore, in the direction of $\nabla\gamma_n(a_1, a_2)$, the boundaries of the HH region are given by

$$\lambda_1 = a_1 + \delta_+(\varepsilon), \qquad \lambda_2 = a_2 + \delta_-(\varepsilon),$$

for $\varepsilon > 0$ small enough.

**Proposition 2.5.3.** *Let be $(a_1, a_2) \in \mathcal{R}_1$ such that $\gamma_n(a_1, a_2) = 0$ for some $n \in \mathbb{N}$. Assume that $F_1$ and $F_2$ satisfy the d'Alembert property. If $c_3 \neq 0$ defined in (2.58) then the width $\delta_+(\varepsilon) - \delta_-(\varepsilon)$ of the HH region is of order $\varepsilon^n$ being the dominant term*

$$-\frac{8c_3\omega(2\omega^2 + a_1 + a_2 - 4)}{\|\nabla\gamma_n(a_1, a_2)\|^2}\varepsilon^n.$$

**Proof**

We have that

$$\begin{aligned} f_1(\varepsilon, \delta_+(\varepsilon)) &= c\|\nabla\gamma_n(a_1, a_2)\|^2\delta_+(\varepsilon) + \phi_0(\varepsilon) + \delta_+(\varepsilon)\phi_1(\varepsilon) + \\ &\quad + \delta_+(\varepsilon)^2 f(\varepsilon, \delta_+(\varepsilon)) - 2c_1\varepsilon^n(1 + g(\varepsilon, \delta_+(\varepsilon))) = 0, \end{aligned}$$

with $c = -\dfrac{1}{D(\omega)}$, $g(\varepsilon, \delta)$ a function of order 1 in $\varepsilon$, $\delta$, and

$$\begin{aligned} f_2(\varepsilon, \delta_-(\varepsilon)) &= c\|\nabla\gamma_1(a_1, a_2)\|^2\delta_+(\varepsilon) + \phi_0(\varepsilon) + \delta_-(\varepsilon)\phi_1(\varepsilon) + \\ &\quad + \delta_-(\varepsilon)^2 f(\varepsilon, \delta_+(\varepsilon)) + 2c_1\varepsilon^n(1 + g(\varepsilon, \delta_-(\varepsilon))) = 0. \end{aligned}$$

The difference $f_1(\varepsilon, \delta_-(\varepsilon)) - f_2(\varepsilon, \delta_+(\varepsilon))$ is

$$\begin{aligned} & c\|\nabla\gamma_n(a_1, a_2)\|^2(\delta_+(\varepsilon) - \delta_-(\varepsilon)) + (\delta_+(\varepsilon) - \delta_-(\varepsilon))\phi_1(\varepsilon) + \delta_+(\varepsilon)^2 f(\varepsilon, \delta_+(\varepsilon)) - \\ & \delta_-(\varepsilon)^2 f(\varepsilon, \delta_-(\varepsilon)) - 4c_1\varepsilon^n - 4c_1\varepsilon^n(g(\varepsilon, \delta_+(\varepsilon)) - g(\varepsilon, \delta_-(\varepsilon))) = 0. \end{aligned}$$

From this equation it is easy to obtain the dominant terms of $\delta_+(\varepsilon) - \delta_-(\varepsilon)$. $\square$

A similar analysis can be done in regions $\mathcal{R}_2$ and $\mathcal{R}_3$ in the case of a single resonance, that is, $(a_1, a_2)$ belongs to a unique resonant curve (2.6), (2.7) or (2.8). In any case we shall take $(\delta_1, \delta_2)$ as (2.60) for the corresponding resonant curve. We given explicitly the case $\mathcal{R}_3$ in the next proposition. We recall that Normal Form in a neighbourhood of $(a_1, a_2)$ is given by (2.36) and the boundaries of the HH region are defined by (2.55).

**Proposition 2.5.4.** *Let $(a_1, a_2) \in \mathcal{R}_3$, $a_1 \neq a_2$, be a point on a resonant curve (2.8) with $\beta = \dfrac{n\pi}{T}$ for some $n \in \mathbb{N}$. Assume that $F_1$ and $F_2$ satisfy d'Alembert property and $\sigma_3 = c_3\varepsilon^n(1 + O_1)$ with $c_3 \neq 0$. Then the width $\delta_+(\varepsilon) - \delta_-(\varepsilon)$ of the HH region is of order $\varepsilon^n$ being the dominant term*

$$\frac{2c_3}{c}\varepsilon^n, \quad \textit{where} \quad c = -\frac{\sqrt{a_2} - \sqrt{a_1}}{16\beta\sqrt{a_1 a_2}|\rho|^2}\left(\frac{(\sqrt{a_1} + \sqrt{a_2})^2}{2} + 2(1 - \beta^2)\right), \quad (2.61)$$

$\rho = \alpha + i\beta$.

**Proof**

Following the same ideas in the proof of lemma 2.5.1 it is not difficult to see that the contribution of $\delta_1$ and $\delta_2$ to $\sigma_1$ up to first order is

$$\frac{1}{2\rho W(\rho)}[(\rho^2-a_2)\delta_1+(\rho^2-a_1)\delta_2], \tag{2.62}$$

where $W(\rho)=2\rho^2-(a_1+a_2-4)=4\alpha\beta\mathrm{i}$. Using (2.60) and (2.62) we get that the terms of $\mathrm{Im}(\sigma_1)$ of order one in $\delta$ are $c\delta$ where $c$ is given in (2.61). Clearly $c\neq 0$ if $a_1\neq a_2$. There exist two analytic functions $\delta_+(\varepsilon),\delta_-(\varepsilon)$ which satisfy $\mathrm{Im}(\sigma_1)-\sigma_3=0$ and $\mathrm{Im}(\sigma_1)+\sigma_3=0$ respectively. Then, the proposition follows in a similar way as proposition 2.5.3 □

### 2.5.2 Double resonances

In this section we consider a double resonance, that is, $(a_1,a_2)$ is a resonant point which belongs to two or more resonant curves. Double resonances only occur at $\mathcal{R}_2$. So, we assume $(a_1,a_2)\in\mathcal{R}_2$ and

$$\nu\omega_j \;\;=\;\; n_j, \qquad j=1,2, \tag{2.63}$$

for some $n_1>n_2$ natural numbers. We shall consider the case $n_1\equiv n_2(\mathrm{mod}\ 2)$. The Normal Form is $N_0+N_1+N_2+N_3+N_4$ in (2.34). We want to discuss the possibilities for the bifurcations for sufficiently small $\varepsilon$ when we perturb a case that for $\varepsilon=0$ is totally elliptic and both frequencies are in resonance.

Resonant points are given by the zeroes of functions $d_1$, $d_2$ and $d_3=d_2^2-4d_1$ in (2.50), (2.51), (2.53), respectively. The analysis of the bifurcations amounts to study the composition of the maps

$$\mathcal{N} \;\;:\;\; (\lambda_1,\lambda_2,\varepsilon)\mapsto(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6),$$

and

$$\mathcal{P} \;\;:\;\; (\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6)\mapsto(d_1,d_2),$$

where $\mathcal{N}$ denotes the normalization map and $\mathcal{P}$ the characteristic polynomial of the Floquet matrix.

**Lemma 2.5.5.** *Let be $(a_1,a_2)\in\mathcal{R}_2$ and $\omega_1>\omega_2$ the frequencies obtained for $\varepsilon=0$. Then, the dominant terms in the contribution of $\delta_1$ and $\delta_2$ to $\sigma_1,\sigma_2$ are*

$$\mathcal{J}\begin{pmatrix}\delta_1\\ \delta_1\end{pmatrix} \qquad \textit{where} \quad \mathcal{J}=\begin{pmatrix}-\frac{\omega_1^2+a_2}{D_1} & -\frac{\omega_1^2+a_1}{D_1}\\ -\frac{\omega_2^2+a_2}{D_2} & -\frac{\omega_2^2+a_1}{D_2}\end{pmatrix},$$

*$D_1=2\omega_1[(a_1+a_2-4)+2\omega_1^2]\neq 0$, $D_2=2\omega_2[(a_1+a_2-4)+2\omega_2^2]\neq 0$. Moreover, the matrix $\mathcal{J}$ is regular if $\omega_1\neq\omega_2$ and $a_1\neq a_2$.*

**Proof**

It follows the same idea given in the proof of lemma 2.5.1. In fact, if we denote by $\mathrm{i}\Omega(\delta_1,\delta_2)=\mathrm{i}[\omega_1+\sigma_1(\delta_1,\delta_2)]$ and $\mathrm{i}\Omega_2(\delta_1,\delta_2)=\mathrm{i}[\omega_2+\sigma_2(\delta_1,\delta_2)]$, the zeroes of the characteristic polynomial $p(x)$, then $\mathcal{J}$ is the Jacobian of $\Omega_1,\Omega_2$. It is easy to check that $D_1D_2\neq 0$ if $(a_1,a_2)\in\mathcal{R}_2$. Furthermore,

$$|\mathcal{J}| \quad = \quad \frac{(\lambda_1-\lambda_2)(\omega_1^2-\omega_2^2)}{D_1D_2}.$$

□

After lemma 2.5.5 we can use $\sigma_1$ and $\sigma_2$ as parameters instead of $\delta_1,\delta_2$. Then bifurcations will be described in terms of $\sigma_1$ and $\sigma_2$.

As the functions $F_j$ in (2.2) satisfy d'Alembert property, we have,

$$\begin{aligned}
\sigma_3 &= m_1\varepsilon^{n_1}(1+O(\varepsilon,\delta_1,\delta_2)),\\
\sigma_4 &= m_2\varepsilon^{n_2}(1+O(\varepsilon,\delta_1,\delta_2)),\\
\sigma_5 &= m_3\varepsilon^{\frac{n_1+n_2}{2}}(1+O(\varepsilon,\delta_1,\delta_2)),\\
\sigma_6 &= m_4\varepsilon^{\frac{n_1-n_2}{2}}(1+O(\varepsilon,\delta_1,\delta_2)),
\end{aligned}$$

where $m_j$, $j=1,\ldots,4$, are real values. We shall assume non degeneracy conditions in the sense that $m_j\neq 0$, $j=1,\ldots,4$.

Our purpose in this section is to prove the following theorem.

**Theorem 2.5.6.** *Let $(a_1,a_2)\in\mathcal{R}_2$, $a_1\neq a_2$, and assume $\omega_j=n_j\frac{\pi}{T}$, $j=1,2$, $n_1>n_2$, $n_1\equiv n_2(\mathrm{mod}\ 2)$. In the d'Alembert case and if non degeneracy conditions are satisfied, one has*

*(i) if $n_1=3n_2$, then around $(\lambda_1,\lambda_2,\varepsilon)=(a_1,a_2,0)$, regions EE, EH and CS exist and a region HH has either $0$, $1$ or $2$ connected components.*

*(ii) If $n_1\neq 3n_2$, then the regions EE, EH, HH, CS are always present and no local changes in the topology of these domains occur.*

The proof of the theorem will be given at the end of this section as a summary of previous results.

First of all we study the magnitude of $\sigma_j$, $j=3,\ldots,6$.

We note that $n_1>\dfrac{n_1+n_2}{2}>n_2$. Then, $|\sigma_4|>>|\sigma_5|>>|\sigma_3|$ if $\varepsilon$ is sufficiently small. Moreover, $\dfrac{n_1-n_2}{2}<\dfrac{n_1+n_2}{2}$, fact that implies $|\sigma_6|>>|\sigma_5|$. Now, we distinguish different subcases.

1. If $n_1 > 3n_2$, then $n_1 > \dfrac{n_1+n_2}{2} > \dfrac{n_1-n_2}{2} > n_2$ and therefore $|\sigma_3| << |\sigma_5| << |\sigma_6| << |\sigma_4|$.
2. If $n_1 = 3n_2$, then $n_1 > \dfrac{n_1+n_2}{2} > \dfrac{n_1-n_2}{2} = n_2$ and therefore $|\sigma_3| << |\sigma_5| << |\sigma_4|$ and $\sigma_6$ is of the same order of magnitude of $\sigma_4$.
3. If $n_1 < 3n_2$, then $n_1 > \dfrac{n_1+n_2}{2} > n_2 > \dfrac{n_1-n_2}{2}$ and therefore $|\sigma_3| << |\sigma_5| << |\sigma_4| << |\sigma_6|$.

We introduce the following scaled parameters

$$\tilde{\sigma}_j = \frac{\sigma_j}{\sigma_4}, \quad j = 1,2,3,5, \quad A = \frac{\sigma_6}{\sigma_4}, \tag{2.64}$$

and we define $\mu := \varepsilon^{\frac{n_1-n_2}{2}}$.

We begin with the second case.

In this case, $\mu = \varepsilon^{n_2}$ and then

$$\tilde{\sigma}_3 = O(\mu^2), \quad \tilde{\sigma}_5 = O(\mu), \quad A = O(1).$$

Using the scalings we introduce new functions (see section 2.4, case 6)

$$\tilde{d}_1 = \frac{d_1}{\sigma_4^2}, \quad \tilde{D}_1 = \frac{D_1}{\sigma_4^2}, \quad \tilde{D}_2 = \frac{D_2}{\sigma_4^2}, \quad \tilde{d}_2 = \tilde{D}_1\tilde{D}_2, \quad \tilde{d}_3 = \tilde{d}_1^2 - 4\tilde{d}_2.$$

Let be $B := sA^2$. We can write these functions in terms of $\mu$ like

$$\begin{aligned}
\tilde{d}_1 &= \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - (4+2B) + O(\mu^2),\\
\tilde{D}_1 &= \tilde{\sigma}_1(\tilde{\sigma}_2+2) + B + O(\mu),\\
\tilde{D}_2 &= \tilde{\sigma}_1(\tilde{\sigma}_2-2) + B + O(\mu), \qquad \tilde{d}_2 = \tilde{D}_1\tilde{D}_2,\\
\tilde{d}_3 &= (\tilde{\sigma}_1^2 - \tilde{\sigma}_2^2 + 4)^2 - 4B[(\tilde{\sigma}_1+\tilde{\sigma}_2)^2 - 4] + O(\mu).
\end{aligned}$$

In order to study resonant points we need to compute the zeroes of $\tilde{d}_j, j = 1,2,3$ functions, as well as the intersections of the curves defined by $\tilde{d}_j = 0$, $j = 1,2,3$. We note that, up to terms of order $\mu$, $\tilde{d}_1, \tilde{D}_1, \tilde{D}_2, \tilde{d}_3$ depends on $B$. The idea is to study the bifurcation diagram in the plane $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ in terms of $B$. Notice that $B \neq 0$.

First we will assume that $\mu = 0$. We obtain the following result.

**Proposition 2.5.7.** *Assume that hypothesis in theorem 2.5.6 are satisfied and $n_1 = 3n_2$. Under the generic assumptions $m_2 \neq 0$, $m_4 \neq 0$ in the Normal Form and neglecting $\sigma_3, \sigma_5$ terms (i.e., setting $\mu = 0$) the unique changes in the bifurcation diagram are produced at $B = -1$ and $B = -\dfrac{27}{16}$.*

Figure 2.7 shows the bifurcation diagram for $\mu = 0$ in different cases. We note that, in particular, no HH regions exists if $B < -1$.

**Proof**

In this case

$$\begin{aligned}
\tilde{d}_1 &= \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - (4 + 2B), \\
\tilde{D}_1 &= \tilde{\sigma}_1(\tilde{\sigma}_2 + 2) + B, \\
\tilde{D}_2 &= \tilde{\sigma}_1(\tilde{\sigma}_2 - 2) + B, \\
\tilde{d}_3 &= (\tilde{\sigma}_1^2 - \tilde{\sigma}_2^2 + 4)^2 - 4B[(\tilde{\sigma}_1 + \tilde{\sigma}_2)^2 - 4].
\end{aligned}$$

The different stability regions are determined by the intersections of the zero sets of the functions above according to the figure 2.5.

We consider first the set of zeroes of $\tilde{d}_2$. The hyperbolas $\tilde{\sigma}_2 = \mp 2 - \dfrac{B}{\tilde{\sigma}_1}$ defined by $\tilde{D}_1 = 0$ and $\tilde{D}_2 = 0$ respectively have no self intersections. Moreover, $\tilde{d}_2 < 0 \Leftrightarrow -2 - \dfrac{B}{\tilde{\sigma}_1} < \tilde{\sigma}_2 < -2 + \dfrac{B}{\tilde{\sigma}_1}$. Therefore, the region $\tilde{d}_2 < 0$, which corresponds to an EH region, has 2 connected components. Figure 2.8 shows the boundaries of the region EH in the case $B > 0$. For $B < 0$ we get a symmetric picture respect the $\tilde{\sigma}_2$ axis.

In the region $\tilde{d}_2 > 0$ we can have the following behaviors:

- if $\tilde{d}_3 < 0 \rightarrow$ CS,
- if $\tilde{d}_3 > 0 \rightarrow \begin{cases} \text{if} \quad \tilde{d}_1 > 0 & \rightarrow \quad \text{EE} \\ \text{if} \quad \tilde{d}_1 < 0 & \rightarrow \quad \text{HH} \end{cases}$.

Now we consider the curve $\tilde{d}_3 = 0$. We note that the set of zeroes of $\tilde{d}_3$ is symmetric with respect to the origin. Self intersections are determined by the additional conditions $\dfrac{\partial \tilde{d}_3}{\partial \tilde{\sigma}_1} = 0$ and $\dfrac{\partial \tilde{d}_3}{\partial \tilde{\sigma}_2} = 0$. These equations only have common solutions for $B = -1$ and $B = 0$. If $B = -1$ there is a unique solution $(\tilde{\sigma}_1, \tilde{\sigma}_2) = (0, 0)$. If $B = 0$ we get $\tilde{\sigma}_1^2 - \tilde{\sigma}_2^2 + 4 = 0$. However we assume $B \neq 0$. So, only the origin when $B = -1$ gives a real self intersection for us.

In order to study the curve $\tilde{d}_3 = 0$ for any value of $B \neq 0$, it will be useful to introduce $z_1 := \tilde{\sigma}_1 + \tilde{\sigma}_2$ and $z_2 := \tilde{\sigma}_1 - \tilde{\sigma}_2$. Then, $\tilde{d}_3$ can be written as

$$\tilde{d}_3 = (z_1 z_2 + 4)^2 - 4B(z_1^2 - 4).$$

The following claims for the solutions of $\tilde{d}_3 = 0$ are trivially obtained. If $B < -1$, for any real value of $z_2$ there are two different solutions of $z_1$ (see figure 2.9 (a)). If $B = -1$, for any real value of $z_2$, $z_2 \neq 0$, there exist two different solutions of $z_1$. One of them is $z_1 = 0$, which corresponds to the straight line $\tilde{\sigma}_2 = -\tilde{\sigma}_1$ in the plane $\tilde{\sigma}_1, \tilde{\sigma}_2$. If $-1 < B < 0$, for any real value of $z_2$, $|z_2| > 2\sqrt{1+B}$ there are two different solutions for $z_1$. If $|z_2| = 2\sqrt{1+B}$, a double solution is obtained. If $B > 0$, for any real value of $z_1$, $|z_1| > 2$ we get two real solutions for $z_2 = 0$. In the cases $|z_1| = 2$ we get double solutions for $z_2$, that means, straight lines $\tilde{\sigma}_1 + \tilde{\sigma}_2 = \pm 2$ are tangent to $\tilde{d}_3 = 0$. Figure 2.9 shows the evolution of the region corresponding to $\tilde{d}_3 < 0$, that is the CS region, in the plane $\tilde{\sigma}_1, \tilde{\sigma}_2$.

Now we go to study the intersections of $\tilde{d}_2 = 0$ and $\tilde{d}_3 = 0$. This is equivalent to look at the intersections of $\tilde{d}_1 = 0$ and $\tilde{d}_2 = 0$. We recall that $\tilde{d}_2 = \tilde{D}_1 \tilde{D}_2$. So, we shall consider the intersections of

$$\tilde{d}_1 = 0, \qquad \tilde{D}_1 = 0. \tag{2.65}$$

Using the symmetry, the solutions of $\tilde{d}_1 = 0$, $\tilde{D}_2 = 0$ will be easily obtained.

The solutions of (2.65) are the intersection points of a circle of radius $4 + 2B$ and the hyperbola $\tilde{\sigma}_2 = -2 - \dfrac{B}{\tilde{\sigma}_1}$. We assume $B > -2$, otherwise (2.65) has no solutions. We shall do the following steps. First we look for the points $P_1, P_2$ in the hyperbola such that the distance to the origin has a relative minimum. Then, we shall determine the values of $B$ such that points $P_1, P_2$ are inside the circle of radius $4 + 2B$. We note that for any point $P_1$ or $P_2$ which satisfies that condition, there are two solutions of (2.65) , and using the symmetry two additional solutions of $\tilde{d}_1 = 0$, $\tilde{D}_2 = 0$ are obtained.

We begin by looking at the points in $\tilde{D}_1 = 0$ such that the distance to the origin is a relative minimum. To this end, we use a Lagrange multiplier $\rho$ with Lagrangian

$$L \;=\; \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - \rho \tilde{D}_1.$$

We get a minimum $(\tilde{\sigma}_{1,m}, \tilde{\sigma}_{2,m})$ for

$$\tilde{\sigma}_{1,m} = \frac{4\rho}{4-\rho^2}, \qquad \tilde{\sigma}_{2,m} = \frac{2\rho^2}{4-\rho^2}, \tag{2.66}$$

where $\rho$ satisfies

$$f_1(\rho, B) := (4-\rho^2)^2 + \frac{32\rho}{B} = 0,$$

or equivalently (note that $f_1(0, B) \neq 0$)

$$\frac{(4-\rho^2)^2}{\rho} = -\frac{32}{B}. \tag{2.67}$$

Figure 2.10 shows the graphic of $g_1(\rho) := \dfrac{(4-\rho^2)^2}{\rho}$. For any value of $B$, $B \neq 0$ (2.65) has two real solutions $\rho_1, \rho_2$ giving rise to points $P_1, P_2$, respectively, in the plane $\tilde{\sigma}_1, \tilde{\sigma}_2$. If $B > 0$ then $\rho_1 < -2 < \rho_2 < 0$ and, $0 < \rho_1 < 2 < \rho_2$ if $B < 0$. Now we study the sign of $\tilde{d}_1$ on $P_1, P_2$. Using (2.66) for $B \neq 0$ we get

$$\tilde{d}_1(\rho) := \tilde{d}_1(\tilde{\sigma}_{1,m}, \tilde{\sigma}_{2,m}) = -\frac{\rho B}{8}(4+\rho^2) - (4+2B) = -\frac{B}{8}\left[\rho(\rho^2+4) + \frac{8}{B}(4+2B)\right].$$

Let be $g_2(\rho) = \rho(\rho^2+4) + 16$. It is clear that for any value of $B \neq 0$ there is a unique $\rho$, $\rho_3$, such that $\tilde{d}_1(\rho_3) = 0$ (see figure 2.10). We are interested in the sign of $\tilde{d}_1(\rho_1)$ and $\tilde{d}_1(\rho_2)$. The figure 2.11 shows the graphics of $g_1(\rho)$ and $g_2(\rho)$ as well as $\rho_1, \rho_2, \rho_3$ for different values of $B$. If $B > 0$, $\rho_1 < \rho_3 < -2 < \rho_2 < 0$. Then, $\tilde{d}_1(\rho_1) > 0$ and $\tilde{d}_1(\rho_2) < 0$, that is, only $P_2$ is inside the circle. If $B < 0$ we distinguish three cases (see figure 2.11).

1. $B < -\dfrac{27}{16}$. Then $\rho_3 < \rho_1 < \rho_2$ and $\tilde{d}_1(\rho_i) > 0$, $i = 1, 2$. $P_1$ and $P_2$ are outside the circle, and (2.65) has no real solutions.

2. $-\dfrac{27}{16} < B < 0$. In this case $0 < \rho_1 < \rho_3 < \rho_2$ and then $\tilde{d}_1(\rho_1) < 0$, $\tilde{d}_1(\rho_2) > 0$. Only $P_1$ is inside the circle and (2.65) has two different solutions.

3. $B = -\dfrac{27}{16}$. Then $\tilde{d}_1(\rho_1) = 0$, $\tilde{d}_1(\rho_2) > 0$. (2.65) has a unique real solution, $(\tilde{\sigma}_1, \tilde{\sigma}_2) = \left(\dfrac{3}{4}, \dfrac{1}{4}\right)$.

We summarize the results above. Consider the set of points in the plane $\tilde{\sigma}_1, \tilde{\sigma}_2$ such that $\tilde{d}_2 > 0$ and $\tilde{d}_3 > 0$. According to the figure 2.5, if $d_1 < 0$ we get an HH region and if $d_1 > 0$ an EE region. We distinguish the following cases.

(a) $B < -\dfrac{27}{16}$. There is no HH region (see figure 2.7 (f)).

(b) $-\dfrac{27}{16} < B < -1$. There are 4 intersection points of $\tilde{d}_3 = 0$ and $\tilde{d}_2 = 0$. This gives an HH region which has two connected components (see figure 2.7 (d)).

(c) $-1 < B$. There are 4 intersection points of $\tilde{d}_3 = 0$ and $\tilde{d}_2 = 0$. However the HH region has one connected component (see figure 2.7 (a), (b)).

$\square$

Now we study the case $\mu \neq 0$, that is, we analize the effect of the neglected terms. We obtain the following result.

**Proposition 2.5.8.** *Assume that hypothesis in theorem 2.5.6 are satisfied and* $n_1 = 3n_2$. *Under the generic assumptions* $m_j \neq 0$, $j = 1, \ldots, 4$ *in the Normal Form (2.34) the unique changes in the bifurcation diagram are produced at* $B = -(1+\tilde{\sigma}_3)^2$ *and at* $B_\pm = -\dfrac{27}{16} \pm \dfrac{1}{2} sA\tilde{\sigma}_5 + O(\mu^2)$.

**Proof**

We know from Proposition 2.5.4 that in the case $\mu = 0$, bifurcations are produced at $B = -1$ due to self intersections of $\tilde{d}_3 = 0$ and, $B = -\dfrac{27}{16}$ when $\tilde{d}_1 = 0$ and $\tilde{d}_2 = 0$ have tangencies. We recall that in this case no self intersections of $\tilde{d}_2 = 0$ occurs.

Let us consider $\mu \neq 0$ small enough. In this case, self–intersections of $\tilde{d}_2 = 0$ can occur. Using (2.52) these will occur if

$$\begin{aligned}
\tilde{D}_1 &= (\tilde{\sigma}_1 - 2s\tilde{\sigma}_3)(\tilde{\sigma}_2 + 2) + s(\tilde{\sigma}_5 + A)^2 = 0,\\
\tilde{D}_2 &= (\tilde{\sigma}_1 + 2s\tilde{\sigma}_3)(\tilde{\sigma}_2 - 2) + s(\tilde{\sigma}_5 - A)^2 = 0.
\end{aligned}$$

Substracting this equations, we obtain that

$$\tilde{\sigma}_1 - s\tilde{\sigma}_2 O(\mu^2) + sAO(\mu) \quad = \quad 0.$$

If we substitute the relation obtained in $\tilde{D}_1 = 0$ it turns that

$$\tilde{\sigma}_2^2 O(\mu^2) + \tilde{\sigma}_2 O(\mu) + B + O(\mu^2) \quad = \quad 0.$$

Then, self–intersections of $\tilde{d}_2 = 0$ can occur, but outside a local neighbourhood of the origin on the $(\tilde{\sigma}_1, \tilde{\sigma}_2)$–plane. Hence, they should not be considered.

Concerning self–intersections of $\tilde{d}_3 = 0$, they are produced if

$$\tilde{d}_3 = 0, \qquad \frac{\partial \tilde{d}_3}{\partial \tilde{\sigma}_1} = 0, \qquad \frac{\partial \tilde{d}_3}{\partial \tilde{\sigma}_2} = 0 \quad .$$

If $\mu = 0$, the system above has the solution $(B, \tilde{\sigma}_1, \tilde{\sigma}_2) = (-1, 0, 0)$. The Jacobian with respect to $B$, $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ at that point is different from zero. Then, the Implicit Function Theorem ensures the preservation of the intersection which will occur for a value of $B$ equal, a priori, to $-1 + O(\mu)$ and with values $\tilde{\sigma}_1, \tilde{\sigma}_2 = O(\mu)$.

An elementary computation shows that the self–intersections of $\tilde{d}_3 = 0$ occurs exactly for $B = -(1 + \tilde{\sigma}_3)^2$ at $\tilde{\sigma}_1 = \tilde{\sigma}_2 = \tilde{\sigma}_5$. Furthermore, for that value of $B$, the line $\tilde{\sigma}_1 + \tilde{\sigma}_2 = 2\tilde{\sigma}_5$ is one of the components of $\tilde{d}_3 = 0$. Figure 2.12 shows an illustration.

It remains to study the modification of the tangencies of the zero sets of $\tilde{d}_1 = 0$ and $\tilde{d}_2 = 0$. We note that symmetry is lost for $\mu \neq 0$. So, one has to consider the cases $\tilde{d}_1 = 0$, $\tilde{D}_1 = 0$ and $\tilde{d}_1 = 0$, $\tilde{D}_2 = 0$ separately. Let us consider the first case. We have

$$\begin{aligned}
\tilde{d}_1 &= \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - (4 + 2B) + O(\mu^2) = 0,\\
\tilde{D}_1 &= \tilde{\sigma}_1(\tilde{\sigma}_2 + 2) + B + \nu + O(\mu^2) = 0,
\end{aligned}$$

where $\nu := 2sA\tilde{\sigma}_5 = O(\mu)$. Up to order $\mu$, $\tilde{d}_1 = 0$ is a cercle. Following the same steps as in the proof of Proposition 2.5.4, we look for the points of $\tilde{D}_1 = 0$ which are at minimum distance to the origin. Using the Lagrangian $L = \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - \rho\tilde{D}_1$ we get a minimum $(\tilde{\sigma}_{1,m}, \tilde{\sigma}_{2,m})$ as (2.66) where the Lagrange multiplier $\rho$ satisfies

$$\tilde{D}_1(\tilde{\sigma}_{1,m}, \tilde{\sigma}_{2,m}) \quad = \quad \frac{32\rho}{(4-\rho^2)^2} + B + \nu = 0.$$

However,

$$\tilde{d}_1(\tilde{\sigma}_{1,m}, \tilde{\sigma}_{2,m}) \quad = \quad \frac{16\rho^2}{(4-\rho^2)^2} + \frac{4\rho^4}{(4-\rho^2)^2} - (4+2B).$$

We must solve the following system

$$\begin{aligned} &32\rho + (B+\nu)(4-\rho^2)^2 = 0, \\ &16\rho^2 + 4\rho^4 - (4+2B)(4-\rho^2)^2 = 0. \end{aligned}$$

For $\mu = 0$, we have the solution $\rho = \dfrac{2}{3}$, $B = -\dfrac{27}{16}$. One step of Newton's Method around that solution gives the critical value of $B$

$$B_+ \quad = \quad -\frac{27}{16} + \frac{1}{2}sA\tilde{\sigma}_5 + O(\mu^2),$$

A similar study for $\tilde{d}_1 = 0$, $\tilde{D}_2 = 0$ gives a second critical value

$$B_- = -\frac{27}{16} - \frac{1}{2}sA\tilde{\sigma}_5 + O(\mu^2).$$

□

*Remark* 2.5.9. The geometrical interpretation is that the two narrow HH domains which in the figure 2.7 (f) disappear on the (b) plot ($B = -\dfrac{27}{16}$) when going from left to right, disappear for slightly different values of $B$ if $\mu \neq 0$. No further changes occur in the bifurcation diagram for $\varepsilon$ small enough in case 2).

**Proof of Theorem 2.5.6**

The item (i) follows from propositions 2.5.7 and 2.5.8.

To prove (ii) we study the cases 1. and 3.. To this end we use the same scalings as in case 2. We have that $A$ in (2.64) is of order $O(\varepsilon^{\frac{n_1-3n_2}{2}})$. Then, the case 1., $n_1 > 3n_2$, has the same characteristics than a very small value of $|B|$. In case 3., $n_1 < 3n_2$, has the same characteristics than a very large value of $|B|$. Then there are no changes in the bifurcation diagram. □

## 2.6 Proof of Proposition 2.3.4

In this section we shall prove the proposition 2.3.4.

Let $\mathcal{H}(\mathbf{z}, w, K)$ be the Hamiltonian defined in (2.29). Our purpose is to use the Giorgilli–Galgani algorithm to obtain the Normal Form for this Hamiltonian.

Let be $\mathcal{H}(\mathbf{z}, w) = \mathcal{H}(\mathbf{z}, w, K) - K$. We recall that $\mathcal{H}(\mathbf{z}, w)$ is an homogeneous polynomial of degree 2 in $\mathbf{z}$ whose coefficients depend on $w$ and $w^{-1}$.

It will be useful to introduce the following functions.

$$\begin{aligned} \mathcal{F}(\mathbf{z}, w) &= f_1 z_1^2 + f_2 z_2^2 + f_3 z_3^2 + f_4 z_4^2 + f_5 z_1 z_2 + f_6 z_1 z_3 + f_7 z_1 z_4 + f_8 z_2 z_3 + \\ &\quad + f_9 z_2 z_4 + f_{10} z_3 z_4, \end{aligned} \tag{2.68}$$

where $f_j = f_j(w)$, $j = 1, \ldots, 10$ can be written as

$$f(w) = \sum_{j \geq 0} (\tilde{c}_j w^j + \tilde{d}_j w^{-j}), \tag{2.69}$$

being the coefficients $\tilde{c}_j$, $\tilde{d}_j$ analytic functions on $\delta_1, \delta_2, \varepsilon$. Let us denote by $H_2^T$ the vector space of functions (2.68).

Given $\mathcal{F}(\mathbf{z}, w)$ in $H_2^T$, $\overline{\mathcal{F}}(\mathbf{z}, w)$ will be obtained from (2.68) by a substitution of $f_j$ by $\overline{f}_j = \overline{f_j(w)}$, for $j = 1, \ldots, 10$, where the bar stands for the complex conjugate.

From lemma 2.3.3 and taking into account that $w$ has been defined in section 2.3.2 as $w = e^{\frac{2it}{\nu}}$, we get

$$\mathcal{H}(\mathbf{z}, w) = \mathcal{H}(S_1 \mathbf{z}, w^{-1}). \tag{2.70}$$

Moreover, as far as $\mathcal{H}(\mathbf{z}, t)$ in (2.24) is an even function of $t$, we get

$$\mathcal{H}(\mathbf{z}, w) = \overline{\mathcal{H}}(\overline{S}_2 \mathbf{z}, w). \tag{2.71}$$

We shall see that these two symmetries will be preserved to the Normal Form. To do that we use the Giorgilli–Galgani algorithm (see appendix C) to get the Normal Form.

Let us denote by $NF$ the new Hamiltonian up to order $m$ in $\delta_1, \delta_2, \varepsilon$. Then $NF = \mathcal{N}_0 + \mathcal{N}_1 + \mathcal{N}_2 + \ldots + \mathcal{N}_m$ with $\mathcal{N}_k$ defined at (2.31), that is,

$$\mathcal{N}_k = \sum_{j=0}^{k} \mathcal{H}_{j,k-j}, \qquad \mathcal{H}_{k,j} = \sum_{l=1}^{j} \frac{l}{j} [G_l, \mathcal{H}_{k,j-l}], \qquad \mathcal{H}_{k,0} = \mathcal{H}_k,$$

and $G_k$ is the solution of the homological equation (2.32). In (2.32),

$$\mathcal{H}_0 = \mathcal{H}_0(\mathbf{z}, K) = \rho_1 z_1 z_3 + \rho_2 z_2 z_4 + K.$$

Functions $H_{i,j}, G_k, M_k$ belongs to $H_2^T$. We denote each term as

$$g = h\mathbf{z}^{\mathbf{l}}w^j, \qquad h = c\ \delta_1^{j_1}\delta_2^{j_2}\varepsilon^{j_3}, \tag{2.72}$$

where $c \in \mathbb{C}$ is a constant, $j_i \in \mathbb{Z}$, $j_i \geq 0$, $i = 1, 2, 3$, $j \in \mathbb{Z}$, and $\mathbf{z}^{\mathbf{l}} = z_1^{l_1} z_2^{l_2} z_3^{l_3} z_4^{l_4}$ with $l_k \in \mathbb{Z}$, $l_k \geq 0$, $k = 1, 2, 3, 4$ satisfying $l_1 + l_2 + l_3 + l_4 = 2$.

$\mathbf{z}^{\mathbf{l}}w^j$ as in (2.72) is a resonant monomial if $[\mathbf{z}^{\mathbf{l}}w^j, \mathcal{H}_0] = 0$, that is,

$$(l_1 - l_3)\rho_1 + (l_2 - l_4)\rho_2 + \frac{2ij}{\nu} \quad = \quad 0.$$

From this equation it is easy to get the following lemma.

**Lemma 2.6.1.** *$z_1z_3$, $z_2z_4$ are resonant terms for all $(a_1, a_2) \in \mathcal{R}$. Moreover,*

1. *if $(a_1, a_2) \in \mathcal{R}_1$ with $\omega\nu \in \mathbb{N}$ then $z_2^2 w^{-\nu\omega}$, $z_4^2 w^{\nu\omega}$;*
2. *if $(a_1, a_2) \in \mathcal{R}_2$ with*
   - *(a) $\omega_1\nu \in \mathbb{N}$ then $z_1^2 w^{-\nu\omega_1}$, $z_3^2 w^{\nu\omega_1}$,*
   - *(b) $\omega_2\nu \in \mathbb{N}$ then $z_2^2 w^{-\nu\omega_2}$, $z_4^2 w^{\nu\omega_2}$,*
   - *(c) $\dfrac{\nu}{2}(\omega_1 + \omega_2) \in \mathbb{N}$, $z_1z_2 w^{-\frac{\nu}{2}(\omega_1+\omega_2)}$, $z_3z_4 w^{\frac{\nu}{2}(\omega_1+\omega_2)}$,*
   - *(d) $\dfrac{\nu}{2}(\omega_1 - \omega_2) \in \mathbb{N}$, $z_1z_4 w^{-\frac{\nu}{2}(\omega_1-\omega_2)}$, $z_2z_3 w^{\frac{\nu}{2}(\omega_1-\omega_2)}$,*
3. *if $(a_1, a_2) \in \mathcal{R}_3$ and $\nu\beta \in \mathbb{N}$, $z_1z_4 w^{-\nu\beta}$, $z_2z_3 w^{\nu\beta}$,*

*are resonant monomials.*

Let $\mathcal{F}(\mathbf{z}, w)$ be in $H_2^T$.

**Definition 2.6.2.** $\mathcal{F}(\mathbf{z}, w)$ satisfies the $S_2$–property if

$$\mathcal{F}(\mathbf{z}, w) \quad = \quad \overline{\mathcal{F}}(\overline{S}_2\mathbf{z}, w), \tag{2.73}$$

for all $\mathbf{z} \in \mathbb{C}^4$, $w \in \mathbb{C}$, $|w| = 1$.

**Definition 2.6.3.** $\mathcal{F}(\mathbf{z}, w)$ satisfies the $S_1^+$–property if

$$\mathcal{F}(\mathbf{z}, w) \quad = \quad \mathcal{F}(S_1\mathbf{z}, w^{-1}), \tag{2.74}$$

for all $\mathbf{z} \in \mathbb{C}^4$, $w \in \mathbb{C}$, $|w| = 1$.

**Definition 2.6.4.** $\mathcal{F}(\mathbf{z}, w)$ satisfies the $S_1^-$–property if

$$\mathcal{F}(\mathbf{z}, w) \quad = \quad -\mathcal{F}(S_1\mathbf{z}, w^{-1}), \tag{2.75}$$

for all $\mathbf{z} \in \mathbb{C}^4$, $w \in \mathbb{C}$, $|w| = 1$.

**Lemma 2.6.5.** *The Normal Form up to order $m$, $NF$ satisfies the $S_2$–property.*

**Proof**

We have seen in (2.71) that the initial Hamiltonian satisfies the $S_2$–property. So, we only need to prove the following statements.

(i) The Poisson bracket preserves the $S_2$–property.

(ii) Assume that $M \in H_2^T$ satisfies the $S_2$–property and let $G$ be the solution of the homological equation

$$[G, \mathcal{H}_0] + M = 0.$$

Then, up to resonant terms, $G$ satisfies the $S_2$–property.

To prove (i) let us consider $\mathcal{F}, \mathcal{G} \in H_2^T$ satisfying the $S_2$–property. Let be $Q = [\mathcal{G}, \mathcal{F}]$. Using (2.73) we get

$$Q(\mathbf{z}, w) = \nabla\mathcal{G}(\mathbf{z}, w)^T J \nabla\mathcal{F}(\mathbf{z}, w) = \nabla\overline{\mathcal{G}}(\overline{S}_2\mathbf{z}, w)^T \;\; \overline{S}_2 J \overline{S}_2^T \;\; \nabla\mathcal{F}(\overline{S}_2\mathbf{z}, w).$$

However, using the definition (2.23) of $S_2$ and the symplectic character of $M$ we have that $\overline{S}_2 J \overline{S}_2^T = J$. Then,

$$Q(\mathbf{z}, w) = \overline{Q}(\overline{S}_2\mathbf{z}, w) \quad .$$

Now we prove (ii). Let be $\mathcal{D} = \mathrm{diag}(\rho_1, \rho_2, -\rho_1, -\rho_2)$ and the homological equation

$$M(\mathbf{z}, w) + \frac{\partial G}{\partial t}(\mathbf{z}, w) + \nabla G(\mathbf{z}, w)^T \mathcal{D}\mathbf{z} \;\; = \;\; 0. \tag{2.76}$$

We assume $M(\mathbf{z}, w) = \overline{M}(\overline{S}_2\mathbf{z}, w)$. From (2.76) and using that $\overline{\mathcal{D}S_2} = \overline{S}_2\mathcal{D}$ we get

$$\frac{\partial G}{\partial t}(\mathbf{z}, w) - \frac{\partial \overline{G}}{\partial t}(\overline{S}_2\mathbf{z}, w) + [\nabla G(\mathbf{z}, w)^T - \nabla\overline{G}(\overline{S}_2\mathbf{z}, w)^T\overline{S}_2]\mathcal{D}\mathbf{z} \;\; = \;\; 0.$$

Let us define $Y(\mathbf{z}, w) = G(\mathbf{z}, w) - \overline{G}(\overline{S}_2\mathbf{z}, w)$. Then,

$$\begin{aligned} [Y, \mathcal{H}_0] \;\; &= \;\; \frac{\partial Y}{\partial t}(\mathbf{z}, w) + \nabla Y(\mathbf{z}, w)^T \mathcal{D}\mathbf{z} = \\ &= \frac{\partial G}{\partial t}(\mathbf{z}, w) - \frac{\partial \overline{G}}{\partial t}(\overline{S}_2\mathbf{z}, w) + [\nabla G(\mathbf{z}, w)^T - \nabla\overline{G}(\overline{S}_2\mathbf{z}, w)^T\overline{S}_2]\mathcal{D}\mathbf{z} = 0. \end{aligned}$$

Therefore $Y(\mathbf{z}, w)$ only has resonant terms. □

**Lemma 2.6.6.** *The Normal Form $NF$ up to order $m$ of $\mathcal{H}(\mathbf{z}, w, K)$ satisfies the $S_1^+$–property.*

**Proof**

From (2.70) the initial Hamiltonian satisfies the $S_1^+$–property. So, we shall prove the following statements.

(i) If $\mathcal{F} \in H_2^T$ satisfies the $S_1^+$–property and $\mathcal{G} \in H_2^T$ satisfies the $S_1^-$–property, then $Q := [\mathcal{G}, \mathcal{F}]$ satisfies the $S_1^+$–property.

(ii) Assume that $M \in H_2^T$ satisfies the $S_1^+$–property. Let $G \in H_2^T$ be the solution of the homological equation

$$[G, \mathcal{H}_0] + M = 0.$$

Then, up to resonant terms, $G$ satisfies the $S_1^-$–property.

The proof of (i) follows the same steps as (i) in the proof of 2.6.5. However using the definition (2.23) of $S_1$ we get $S_1 J S_1^T = -J$. Then,

$$Q(\mathbf{z}, w) = -\nabla \mathcal{G}(S_1\mathbf{z}, w^{-1})^T \;\; S_1 J S_1^T \;\; \nabla \mathcal{F}(S_1\mathbf{z}, w^{-1}) = Q(S_1\mathbf{z}, w^{-1}) \quad .$$

To proof (ii) let us assume that $M$ satisfies the $S_1^+$ property. Then, $M(\mathbf{z}, w) = M(S_1\mathbf{z}, w^{-1})$. Taking into account the homological equation we get

$$\frac{\partial G}{\partial t}(\mathbf{z}, w) - \frac{\partial G}{\partial t}(S_1\mathbf{z}, w^{-1}) + [\nabla G(\mathbf{z}, w)^T - \nabla G(S_1\mathbf{z}, w^{-1})^T S_1]\mathcal{D}\mathbf{z} \quad = \quad 0,$$

where the equality $S_1\mathcal{D} = -\mathcal{D}S_1$ has been used.

Let us define $Y(\mathbf{z}, w) = G(\mathbf{z}, w) + G(S_1\mathbf{z}, w^{-1})$. Then,

$$[Y, \mathcal{H}_0] \quad = \quad \frac{\partial G}{\partial t}(\mathbf{z}, w) - \frac{\partial G}{\partial t}(S_1\mathbf{z}, w^{-1}) + [\nabla G(\mathbf{z}, w)^T + \nabla G(S_1\mathbf{z}, w^{-1})^T S_1]\mathcal{D}\mathbf{z} = 0.$$

Therefore $Y(\mathbf{z}, w)$ only has resonant terms.

As a consequence of (i) and (ii) we have that the functions $H_{i,j}$ which appear in the Giorgilli–Galgani algorithm satisfy the $S_1^+$–property, while $G_k$ satisfy the $S_1^-$–property. $\square$

Now we use the $S_2$ and $S_1^+$–properties of $NF$ to get the relations between the coefficients of $f_i(w)$ which appear in $NF$.

Let us consider $(a_1, a_2) \in \mathcal{R}_1$. The resonant terms are given in lema 2.6.1. So, we write the Normal Form as

$$\begin{aligned} NF(\mathbf{z}, w) \quad &= \quad K + \lambda z_1 z_3 + i\omega z_2 z_4 + a_6 z_1 z_3 + a_9 z_2 z_4 + \\ &\quad\quad + a_2 z_2^2 w^{-\nu\omega} + a_4 z_4^2 w^{\nu\omega}, \end{aligned}$$

for some constants $a_2, a_4, a_6, a_9$.

Using that $S_1\mathbf{z} = (sz_3, \mathrm{i}z_4, sz_1, -\mathrm{i}z_2)^T$ (lemma 2.3.3),

$$\begin{aligned} NF(S_1\mathbf{z}, w^{-1}) &= K + \lambda z_1 z_3 + i\omega z_2 z_4 + a_6 z_1 z_3 + a_9 z_2 z_4 + \\ &\quad -a_2 z_4^2 w^{\nu\omega} - a_4 z_2^2 w^{-\nu\omega}. \end{aligned}$$

The $S_1^+$–property for $NF$ implies that

$$a_4 = -a_2. \tag{2.77}$$

Moreover,

$$\begin{aligned} \overline{NF}(\overline{S}_2\mathbf{z}, w^{-1}) &= K + \lambda z_1 z_3 + i\omega z_2 z_4 + \overline{a}_6 z_1 z_3 - \overline{a}_9 z_2 z_4 + \\ &\quad -\overline{a}_2 z_4^2 w^{\nu\omega} - \overline{a}_4 z_2^2 w^{-\nu\omega}. \end{aligned}$$

The $S_2$–property implies that $a_6 \in \mathbb{R}$, $a_9$ is imaginary and

$$a_4 = -\overline{a}_2. \tag{2.78}$$

From (2.77) and (2.78), $a_2 \in \mathbb{R}$ and $a_4 = -a_2$. This proves (2.33).

Let be $(a_1, a_2) \in \mathcal{R}_2$. We consider the case for which the Normal Form contains all possible resonant terms and we write it as

$$\begin{gathered} NF(\mathbf{z}, w) = K + \mathrm{i}\omega_1 z_1 z_3 + \mathrm{i}\omega_2 z_2 z_4 + a_6 z_1 z_3 + a_9 z_2 z_4 + a_1 z_1^2 w^{-\nu\omega_1} + a_3 z_3^2 w^{\nu\omega_1} + \\ a_2 z_2^2 w^{-\nu\omega_2} + a_4 z_4^2 w^{\nu\omega_2} + a_5 z_1 z_2 w^{-\nu\omega_{hs}} + a_{10} z_3 z_4 w^{\nu\omega_{hs}} + \\ a_7 z_1 z_4 w^{-\nu\omega_{hd}} + a_8 z_2 z_3 w^{\nu\omega_{hd}}, \end{gathered}$$

where we recall that $\omega_{hs} = \dfrac{1}{2}(\omega_1 + \omega_2)$ and $\omega_{hd} = \dfrac{1}{2}(\omega_1 - \omega_2)$. Then,

$$\begin{gathered} NF(S_1\mathbf{z}, w^{-1}) = K + \mathrm{i}\omega_1 z_1 z_3 + \mathrm{i}\omega_2 z_2 z_4 + a_6 z_1 z_3 + a_9 z_2 z_4 - a_1 z_3^2 w^{\nu\omega_1} - \\ a_3 z_1^2 w^{-\nu\omega_1} - a_2 z_4^2 w^{\nu\omega_2} - a_4 z_2^2 w^{-\nu\omega_2} + s a_5 z_3 z_4 w^{\nu\omega_{hs}} + s a_{10} z_1 z_2 w^{-\nu\omega_{hs}} - \\ s a_7 z_2 z_3 w^{\nu\omega_{hd}} + s a_8 z_1 z_4 w^{-\nu\omega_{hd}}. \end{gathered}$$

Then,

$$a_3 = -a_1, \quad a_4 = -a_2, \quad a_{10} = s a_5, \quad a_8 = -s a_7.$$

In a similar way, using the $S_2$–property we get

$$a_6 = -\overline{a}_6, \quad a_9 = -\overline{a}_9, \quad a_3 = -\overline{a}_1, \quad a_4 = -\overline{a}_2, \quad a_{10} = s\overline{a}_5, \qquad a_8 = s\overline{a}_7.$$

Therefore $a_1, a_2, a_5 \in \mathbb{R}$, $a_6, a_7, a_9$ imaginary and $a_3 = -a_1$, $a_4 = -a_2$, $a_{10} = s a_5$, $a_8 = -s a_7$. This proves (2.34).

Finally, we take $(a_1, a_2) \in \mathcal{R}_3$. Then, $NF$ has the form

$$\begin{aligned} NF(\mathbf{z}, w) &= K + (\alpha + \mathrm{i}\beta)z_1z_3 + (\alpha - \mathrm{i}\beta)z_2z_4 + a_6z_1z_3 + a_9z_2z_4 + \\ &\quad a_7z_1z_4w^{-\nu\beta} + a_8z_2z_3w^{\nu\beta}. \end{aligned}$$

As $S_1\mathbf{z} = (z_3, z_4, z_1, z_2)^T$, we have

$$\begin{aligned} NF(S_1\mathbf{z}, w^{-1}) &= K + (\alpha + \mathrm{i}\beta)z_1z_3 + (\alpha - \mathrm{i}\beta)z_2z_4 + a_6z_1z_3 + a_9z_2z_4 + \\ &\quad a_8z_1z_4w^{-\nu\beta} + a_7z_2z_3w^{\nu\beta}. \end{aligned}$$

Therefore, due to the $S_1^+$–property, $a_7 = a_8$. Moreover,

$$\begin{aligned} \overline{NF}(\overline{S}_2\mathbf{z}, w^{-1}) &= K + (\alpha + \mathrm{i}\beta)z_1z_3 + (\alpha - \mathrm{i}\beta)z_2z_4 + \overline{a}_9z_1z_3 + \overline{a}_6z_2z_4 + \\ &\quad \overline{a}_8z_1z_4w^{-\nu\beta} + \overline{a}_7z_2z_3w^{\nu\beta}. \end{aligned}$$

By the $S_2$–property, $\overline{a}_9 = a_6$ and $\overline{a}_8 = a_7$.

Then, $a_7 = a_8$ with $a_7 \in \mathbb{R}$ and $\overline{a}_9 = a_6$. This proves (2.36).

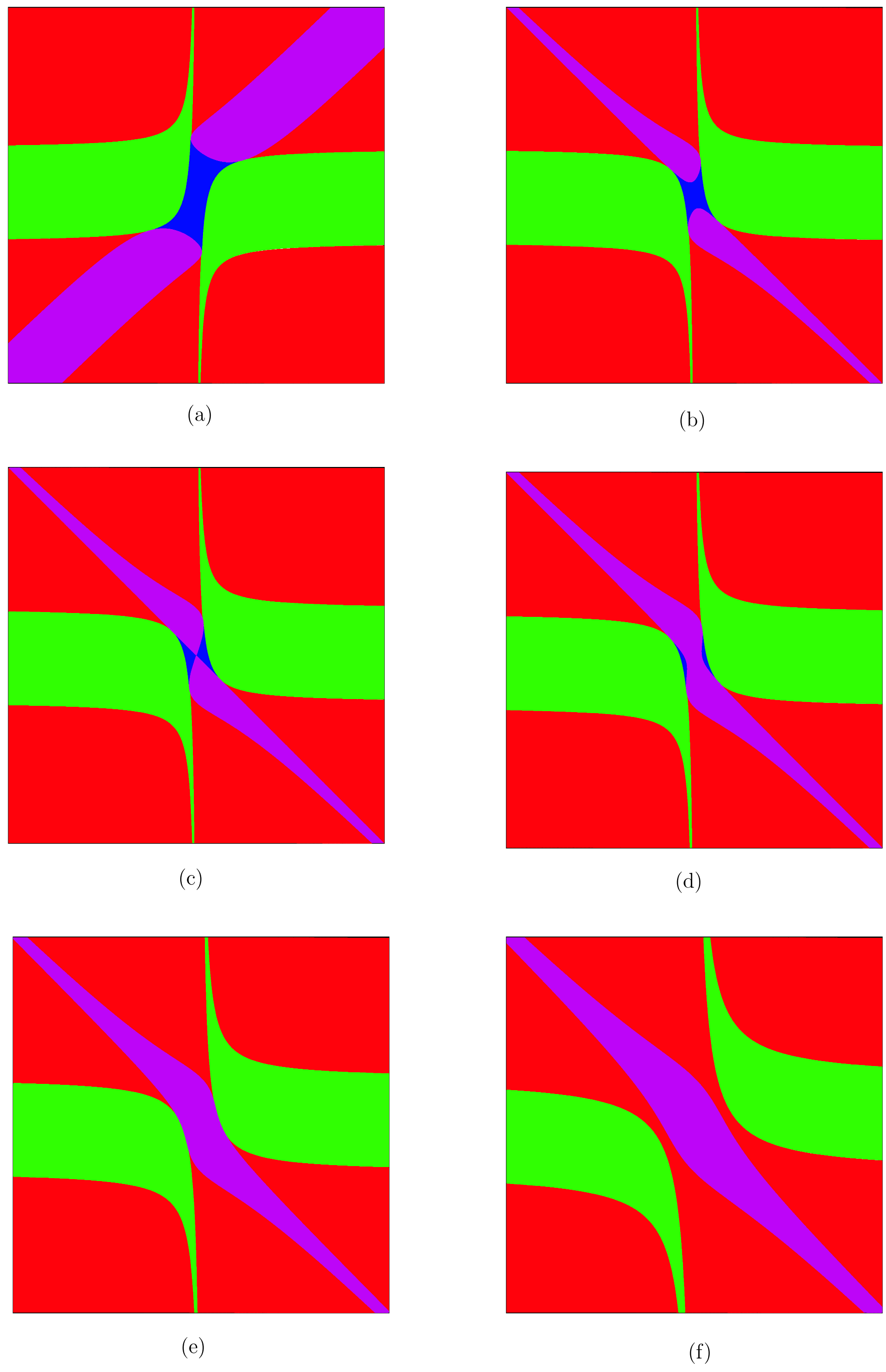



Figure 2.7: A sample of the bifurcation diagrams near double resonance in the d'Alembert case with $n_1 = 3n_2$ and $\mu = 0$. Values of $B$: (a) 1, (b) $-0.9$, (c) $-1$, (d) $-1.1$, (e) $-27/16$, (f) $-4$. Color codes: red for EE, green for HE, blue for HH, magenta for CS. The horizontal (resp. vertical) variable is $\tilde{\sigma}_1$ (resp. $\tilde{\sigma}_2$).

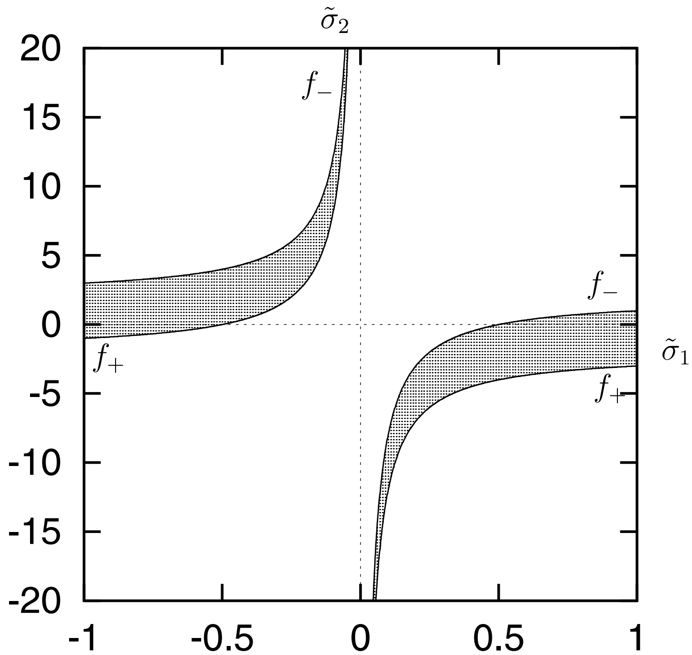



Figure 2.8: Region EH and their boundaries for $B = 1$ where $f_+(\tilde\sigma_1) = -2 - \dfrac{B}{\tilde\sigma_1}$ and $f_-(\tilde\sigma_1) = 2 - \dfrac{B}{\tilde\sigma_1}$

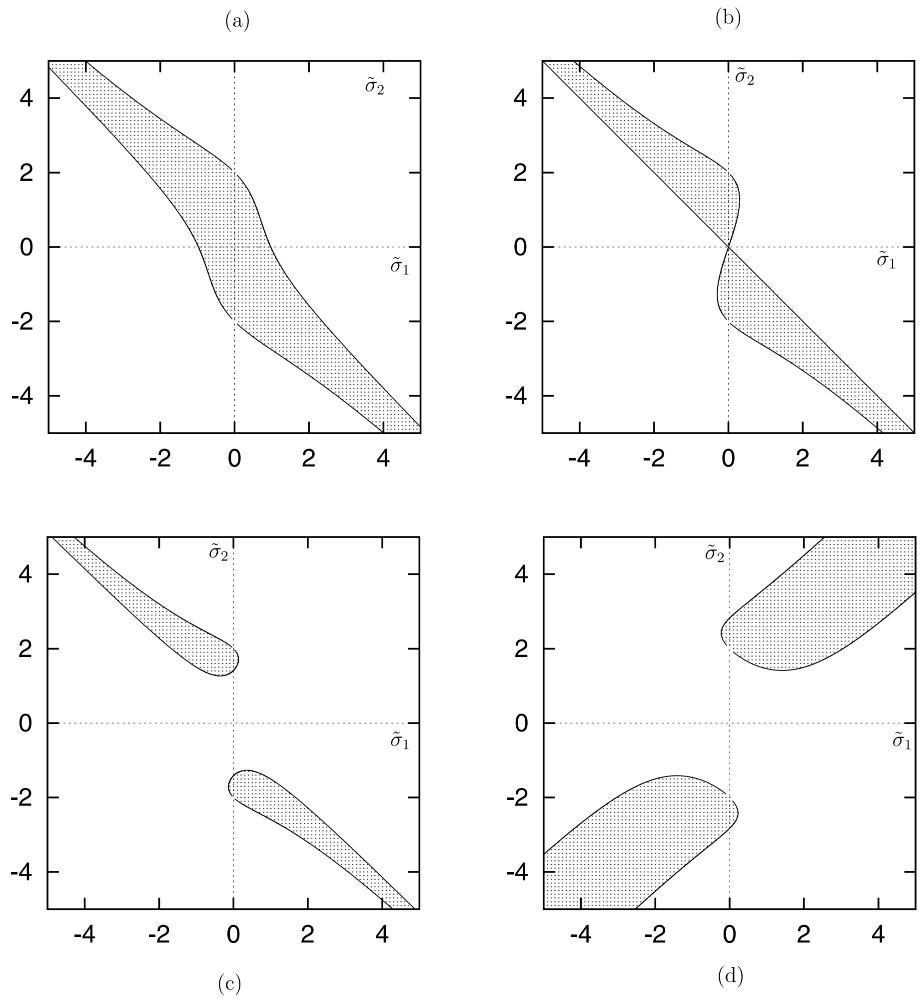



Figure 2.9: Graphic of the curve $\tilde{d}_3 = 0$ for $B = -2$, $B = -1$, $B = -0.5$ and $B = 1$, respectively. The dashed area corresponds to $\tilde{d}_3 < 0$.

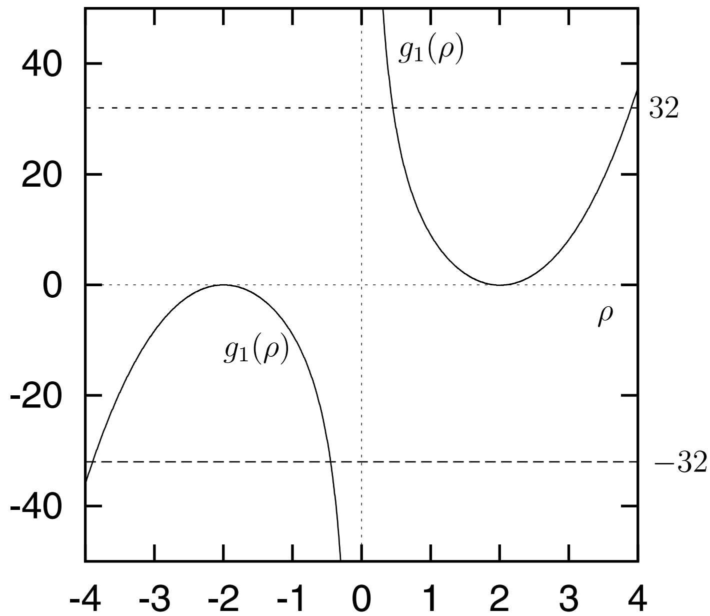



Figure 2.10: Graphic of $g_1(\rho)$ and intersections with $-\dfrac{32}{B}$ for $B = 1$, and $B = -1$

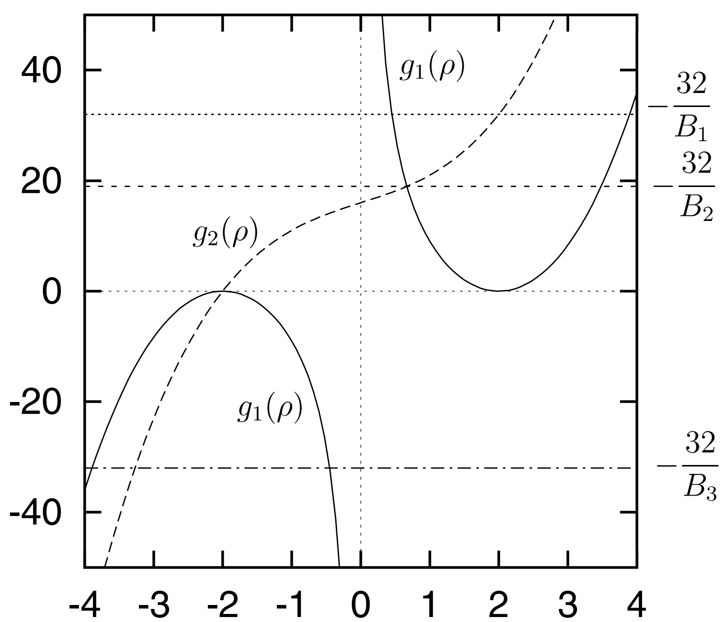



Figure 2.11: Graphic of $g_1(\rho)$, $g_2(\rho)$ and intersections with $-\dfrac{32}{B_j}$ for $B_1 = -1$, $B_2 = -\dfrac{27}{16}$ and $B_3 = 1$

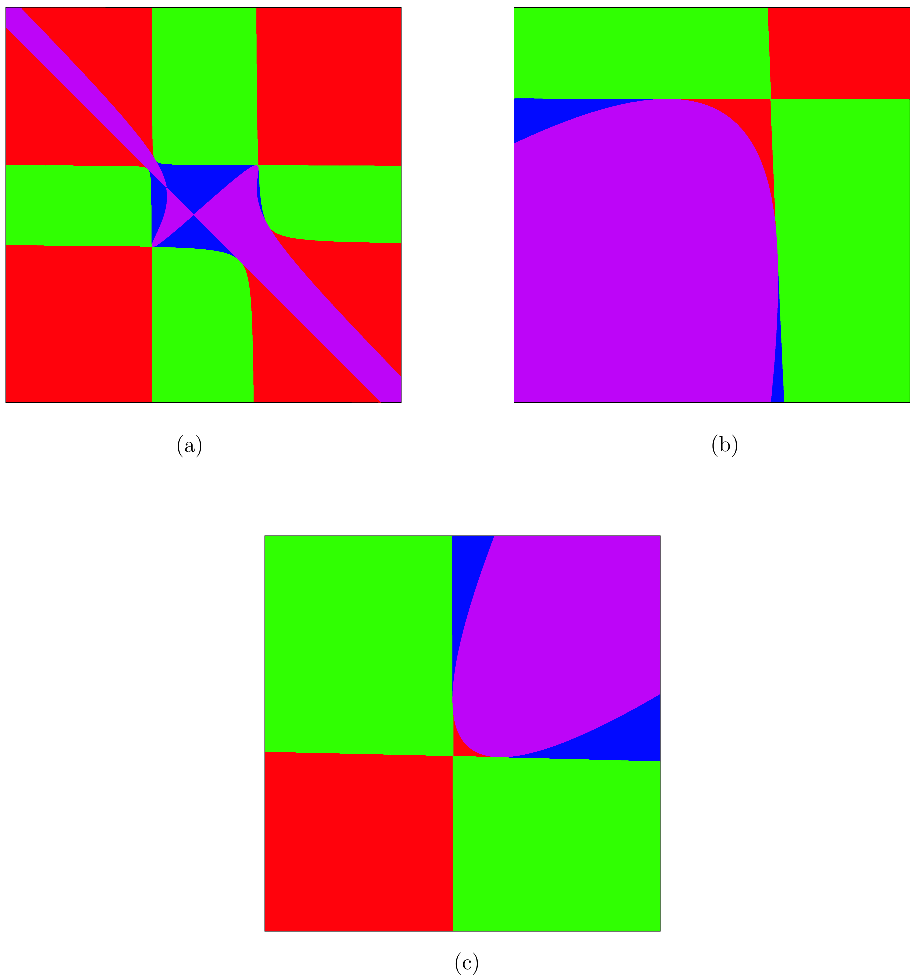



Figure 2.12: An example of self–intersections of $\tilde{d}_3 = 0$ in the general case. Scaled parameters used: $\tilde{\sigma}_3 = -1.3$, $\tilde{\sigma}_7 = -0.5$, $A = -0.3$, $s = -1$. Variables plotted as in figure 2.7. (b) plot shows a global view, (a) and (b) are magnifications. Up to 19 connected components can be seen.

# Chapter 3

# Stability of a family of periodic linear systems: the singular limit case

## 3.1 Introduction

We consider again a periodic linear system as the one given in (1.55), that is,

$$\dot{\mathbf{x}} \;=\; A(t)\mathbf{x}, \qquad A(t) = \begin{pmatrix} 0 & I_2 \\ \tilde{A}(t) & -2J_2 \end{pmatrix}, \qquad \tilde{A} = g^{\alpha-2}\mathrm{diag}(\lambda_1, \lambda_2), \quad (3.1)$$

where $0 < \alpha < 2$, $\lambda_1, \lambda_2$ are real parameters different from zero and $g = g(t;\delta)$ is a periodic function on $t$ which depends on a parameter $\delta \in [0, \delta_0]$ with $\delta_0$ small enough. Suppose $g(t,\delta) > 0$ for all $t$ and $g(0,\delta) \to 0$ for $\delta \to 0$. Therefore, the system (3.1) has a singularity at $t = 0$ for $\delta = 0$. Our purpose is to study the stability parameters of system (3.1) for small values of $\delta > 0$ under some hypothesis to be specified below.

Let $U(z) = z^\alpha V(z)$ be a real function defined on an open interval $(0, z_b)$ where $V(z)$ is an analytic function for $z > 0$ such that

(A1) *there exists* $z_a$, $0 < z_a < z_b$ *such that* $V(z_a) = 0$, $V(z) < 0$ *for all* $z \in (0, z_a)$ *and* $V_z(z) > 0$ *for all* $z \in (0, z_b)$.

(A2) $V(z) = \gamma + z^s V_1(z)$ *with* $\gamma < 0$, $s > \dfrac{2-\alpha}{2}$ *and* $V_1(z)$ *and analytic function on an open set* $J$, $J \supset [0, z_a]$.

In (A1), $V_z(z)$ stands for the derivative of $V(z)$ with respect to $z$.

*Remark* 3.1.1. We have that the only zeroes of $U(z)$ in $[0, z_a]$ are 0 and $z_a$.

Figure 3.1 shows the plot of $U(z)$ corresponding to the homographic system (1.31) for different values of $\alpha$.

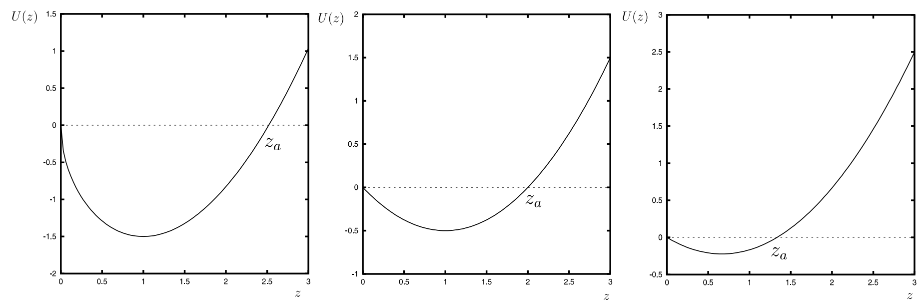



Figure 3.1: Plot of $U(z) = z^\alpha \left( -\frac{1}{\alpha} + \frac{z^{2-\alpha}}{2} \right)$ for values of $\alpha$ from left to right: $0 < \alpha < 1$, $\alpha = 1$ (Newtonian case), $1 < \alpha < 2$

Let us consider the conservative system

$$\ddot{z} = -U_z(z) \tag{3.2}$$

with $U(z)$ satisfying (A1) and (A2). $U_z(z)$ stands for the derivative with respect $z$. We denote the energy of (3.2) by

$$E = \frac{\dot{z}^2}{2} + U(z). \tag{3.3}$$

We shall assume the following hypothesis for $g(t; \delta)$

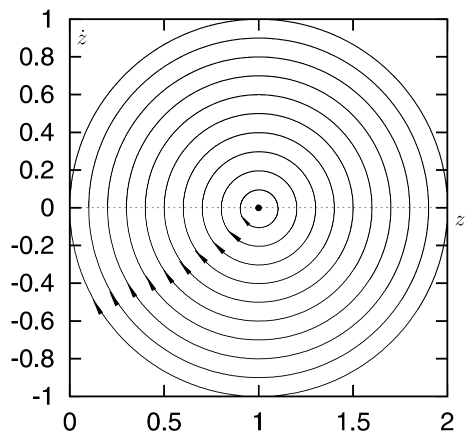
(B) *For $\delta > 0$, $g(t; \delta)$ is the periodic solution of (3.2) on the energy level $E = -\delta$ such that $g(0; \delta) = g_0$, $\dot{g}(0; \delta) = 0$ being $g_0$ the minimum of $g(t; \delta)$.*

Figure on the left in 3.2 shows the phase portrait of system (3.2) on the plane $(z, \dot{z})$ for $U(z) = z\left(-1 + \frac{z}{2}\right)$. Fixed $\delta > 0$, the figure on the right shows how we choose $g(t; \delta)$.

Note that for $\delta > 0$, $g(t; \delta)$ is periodic with period $T = T(\delta)$. Moreover, from (3.3) we have $-\delta = U(g_0)$. Then, $g_0 = \left(\frac{\delta}{|\gamma|}\right)^{\frac{1}{\alpha}} (1 + O(\delta^{\frac{s}{\alpha}}))$.

We remark that if $g$ satisfies property (B) then $g(t; \delta)$ is an even function on $t$.

As we have seen in chapter 1, the motivation to study the system (3.1) comes from the linear stability analysis of the homographic solutions. The corresponding variational equations of the linearized system on these solutions can be reduced



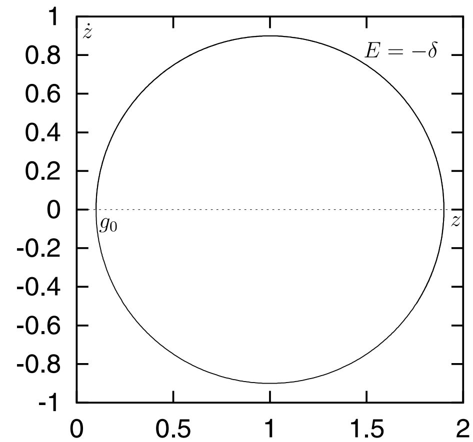




Figure 3.2: Phase portrait of (3.2) for $U(z) = z\left(-1+\dfrac{z}{2}\right)$

to a linear system of type (3.1). In particular the Newtonian case is obtained for $\alpha = 1$ and $U(z) = z\Big(-1+\dfrac{z}{2}\Big)$. In this case, $g(t;\delta) = 1 - e\cos t$, where $e$ is the eccentricity of the homographic solution, and $\delta = \dfrac{1-e^2}{2}$. The time $t$ is the true anomaly. The singularity of the equations is attained for $e = 1$.

To simplify the notation the dependence on the parameters of system (3.1) will not be explicitly written if there is no confusion. We shall use the same simplification for all linear systems and their corresponding monodromy matrices which appear in the chapter.

From section 1.6 system (3.1) can be written in Hamiltonian formulation. We recall that the Hamiltonian function is

$$H = \frac{1}{2}(y_3^2+y_4^2) + (-y_2y_3+y_1y_4) - \left(\lambda_1 g^{\alpha-2}-1\right)\frac{y_1^2}{2} - \left(\lambda_2 g^{\alpha-2}-1\right)\frac{y_2^2}{2}. \quad (3.4)$$

We shall give asymptotic formulae for these stability parameters. To do that the main point is to use some kind of blow up technique to see the limit case when $\delta$ tends to zero as a linear system on an heteroclinic connection.

We shall work, for $\delta > 0$, with a linear system without any singularity. So, we consider the change of variables $\mathbf{u} = S(t)\mathbf{x}$ where $S(t) = \mathrm{diag}(1,1,q,q)$ being $q = q(t;\delta)$ defined by $q = g^{\frac{2-\alpha}{2}}$, and we use time $\tau$ defined through $dt = q d\tau$. We remark that for $\delta > 0$, $S(t)$ is non singular for all $t$.

Then, the new system can be written as

$$\mathbf{u}' = B(\tau)\mathbf{u}, \qquad B(\tau) = q(\dot{S}+SA)S^{-1}, \quad (3.5)$$

where $'$ stands for the derivative with respect to $\tau$. The period $T$ of $g(t;\delta)$ in the new time $\tau$ will be denoted by $\mathcal{T}(\delta)$ or simply $\mathcal{T}$. In order to simplify the notation, in the following we shall write $q(t)$ instead of $q(t;\delta)$ if there is no confusion.

Let $\Psi(\tau)$ be the fundamental matrix of (3.5). We have that

$$\Phi(t) \quad = \quad S^{-1}(t)\Psi(\tau(t))S(0), \tag{3.6}$$

where $\Phi$ denotes the fundamental matrix of (3.1) such that $\Phi(0) = I_4$. Due to the $T$–periodicity of $S$ we get for the monodromy matrices the equality $\Phi(T) = S^{-1}(0)\Psi(\mathcal{T})S(0)$ and so, $\Phi(T)$ and $\Psi(\mathcal{T})$ have the same eigenvalues. Then, in order to obtain the stability parameters of system (3.1) it is only necessary to study the eigenvalues of $\Psi(\mathcal{T})$.

In order to compute the dominant terms of the traces of $\Psi(\mathcal{T})$ for values of $\delta$ near 0 it will be useful to study some properties of $q$. To this, we shall introduce in the next section an artificial planar system for the functions $q$ and $\dot{q}$ involved in $B(\tau)$.

## 3.2 An auxiliary planar system

In this section we study an artificial planar system that will be useful in order to study the stability parameters of system (3.1).

We define $Q(\tau) = -(2-\alpha)q^{-\frac{\alpha}{2-\alpha}}\dot{g}$ where $q = g^{\frac{2-\alpha}{2}}$ as before. We shall see that $q(\tau), Q(\tau)$ satisfies a planar system in time $\tau$.

It is easy to see that $q' = -\dfrac{1}{2}qQ$. Moreover,

$$Q' \quad = \quad \frac{\alpha}{2(2-\alpha)}Q^2 - (2-\alpha)q^{\frac{2-2\alpha}{2-\alpha}}\ddot{g}.$$

We recall that $g$ is a solution of the potential equation (3.2). Then, using the time $\tau$, $q(\tau), Q(\tau)$ is a solution of the following system

$$\begin{aligned} q' &= -\frac{1}{2}qQ, \\ Q' &= \frac{\alpha}{2(2-\alpha)}Q^2 + (2-\alpha)q^{\frac{2-2\alpha}{2-\alpha}}U_z(\hat{q}), \end{aligned} \tag{3.7}$$

where $\hat{q} = q^{\frac{2}{2-\alpha}}$, that is, $\hat{q} = g$.

As (3.3) is a first integral of (3.2) and $\hat{q}$ is a solution of this system, we have that

$$E \quad = \quad q^{\frac{2\alpha}{2-\alpha}}\left[\frac{Q^2}{2(2-\alpha)^2} + V(\hat{q})\right], \tag{3.8}$$

is a first integral of system (3.7).

From hypothesis (A2) we have that $U(z) = \gamma z^{\alpha} + O(z^{\alpha+s})$, with $s > \dfrac{2-\alpha}{2}$. Then,

$$q^{\frac{2-2\alpha}{2-\alpha}} U_z(\hat{q}) \;=\; \gamma\alpha + O(q^{\frac{2s}{2-\alpha}}), \quad \text{with} \quad \frac{2s}{2-\alpha} > 1.$$

Therefore the planar system (3.7) is well–defined at $q = 0$.

We study now the behavior of the orbits of system (3.7) for $E = 0$. On this energy level there are two equilibrium points $P_{\pm}$ with $(q, Q) = (0, \pm Q_p)$ where $Q_p = (2-\alpha)\sqrt{-2\gamma}$.

Taking into account that the Jacobian of the vector field that defines system (3.7) is

$$\begin{pmatrix} -\frac{1}{2}Q & -\frac{1}{2}q \\ (2-\alpha)O(q^{\frac{2s}{2-\alpha}-1}) & \frac{\alpha}{2-\alpha}Q \end{pmatrix}, \quad \text{with} \quad \frac{2s}{2-\alpha} > 1,$$

the eigenvalues of the linearized system at $P_{\pm}$ are $\mp\dfrac{Q_p}{2}$, $\pm Q_p\dfrac{\alpha}{2-\alpha}$. Then, $P_{\pm}$ are saddle points.

On the energy level $E = 0$ we distinguish also two orbits

$$\begin{aligned} \gamma_0 &= \{(q, Q) \in \mathbb{R}^2 \,|\, q = 0, |Q| < Q_p\} \qquad \text{and} \\ \gamma_+ &= \{(q, Q) \in \mathbb{R}^2 \,|\, q > 0, \frac{Q^2}{2(2-\alpha)^2} + V(\hat{q}) = 0\}. \end{aligned}$$

In a neighbourhood of $P_-$, $\gamma_+$ is given by

$$Q = G(q) = -(2-\alpha)\sqrt{-2V(\hat{q})}.$$

Then, $\dfrac{dG}{dq}(0) = 0$.

On $\gamma_0$ (3.7) reduces to

$$Q' \;=\; \frac{\alpha}{2(2-\alpha)}Q^2 + \alpha(2-\alpha)\gamma.$$

So, we get the following solution

$$q_{L_1}(\tau) \equiv 0, \quad Q_{L_1}(\tau) = -Q_p \tanh\left(\tfrac{\alpha}{2(2-\alpha)}Q_p\tau\right).$$

On $\gamma_+$, $\dfrac{Q^2}{2(2-\alpha)^2} + V(\hat{q}) = 0$. Then, the system on $\gamma_+$ is

$$\begin{aligned} q' &= -\frac{1}{2}qQ, \\ Q' &= (2-\alpha)\hat{q}V_z(\hat{q}). \end{aligned} \tag{3.9}$$

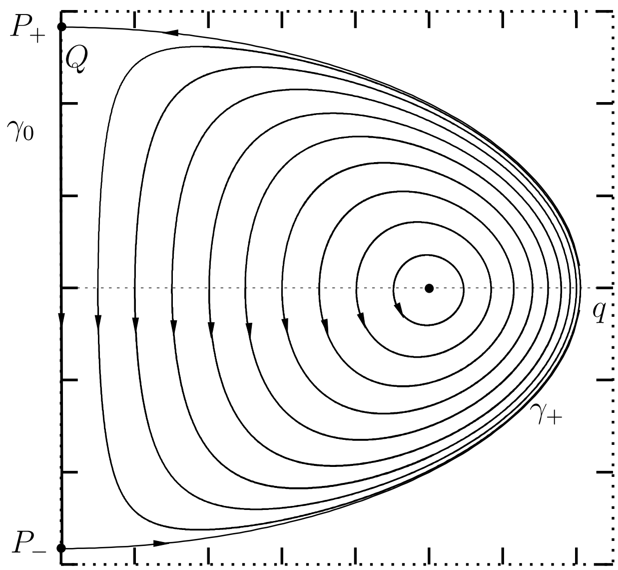



Figure 3.3: Phase portrait of system (3.7) in the newtonian case.

We shall denote by $q_{L_2}(\tau), Q_{L_2}(\tau)$ the solution of (3.9) such that $Q_{L_2}(0) = 0$.

Figure 3.3 shows the phase portrait of (3.7) for $U(z) = z\left(-1 + \dfrac{z}{2}\right)$ which corresponds to the homographic case for the newtonian potential.

As we are interested in $E = -\delta$ with $\delta > 0$ small enough, we need to study the behavior of the solutions of (3.7) near the heteroclinic connection defined by $\gamma_0$, $\gamma_+$ and the equilibrium points $P_\pm$.

To this end, we take $\varepsilon, \varepsilon_i$, $i = 1, \ldots, 4$, small enough, and we define the following sections

$$\begin{aligned}
\Sigma_0 &= \{(q,Q) | 0 < q < \varepsilon_1, Q = 0\}, \\
\Sigma_1 &= \{(q,Q) | 0 < q < \varepsilon_2, Q = -Q_p + \varepsilon\}, \\
\Sigma_2 &= \{(q,Q) | q = \varepsilon, |Q + Q_p| < \varepsilon_3\}, \\
\Sigma_3 &= \{(q,Q) | q_a - q < \varepsilon_4, Q = 0\}.
\end{aligned}$$

For a fixed value of $\varepsilon > 0$ sufficiently small we can take small enough $\varepsilon_i$ for $i = 1, \ldots, 4$, such that the Poincaré maps $\mathcal{P}_1 : \Sigma_0 \to \Sigma_1$, $\mathcal{P}_2 : \Sigma_1 \to \Sigma_2$ and $\mathcal{P}_3 : \Sigma_2 \to \Sigma_3$ be well defined.

Figure 3.4 shows these situation in the newtonian case.

We denote by $\tau_{L_1} > 0$ the time defined by $Q_{L_1}(\tau_{L_1}) = -Q_p + \varepsilon$, and $\tau_{L_2} > 0$ such that $q_{L_2}(-\tau_{L_2}) = \varepsilon$. That is, $\tau_{L_1}$ is the time needed for $(q_{L_1}, Q_{L_1})$ to go from $\Sigma_0$ to $\Sigma_1$, and $-\tau_{L_2}$ is the time used for the solution $(q_{L_2}, Q_{L_2})$ to travel backwards

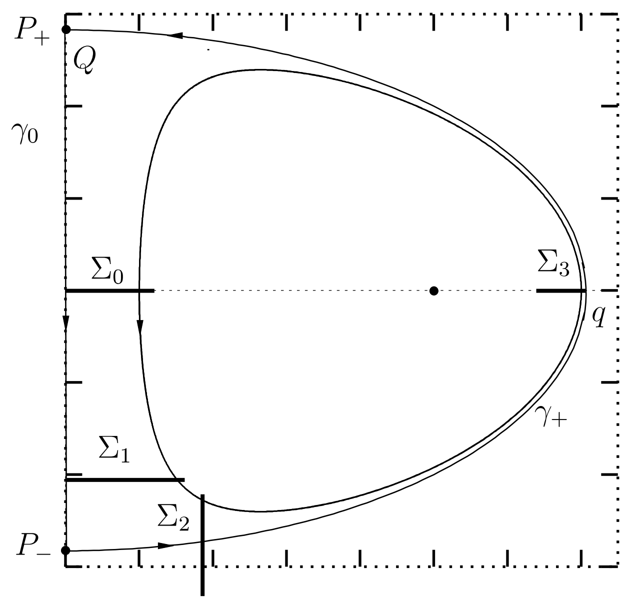



Figure 3.4: Poincaré sections

from $\Sigma_3$ to $\Sigma_2$. Note that $\tau_{L_1}$ and $\tau_{L_2}$ are finite and independent of $\delta$ once $\varepsilon$ is fixed.

Fixed a value of $\delta > 0$ small enough, we consider the solution of (3.7) with $E = -\delta$ such that $(q(0), Q(0)) \in \Sigma_0$. Taking into account that $Q(0) = 0$ and using the hypothesis (A) and the energy (3.8) we get that $q_0 = q(0) = \left(\frac{\delta}{|\gamma|}\right)^{\frac{2-\alpha}{2\alpha}} (1+O(\delta^{\frac{s}{\alpha}}))$.

Let $\tau_1$ be the smallest positive time such that $(q(\tau_1), Q(\tau_1)) \in \Sigma_1$. In a similar way we define $\tau_2$ such that $(q(\tau_2), Q(\tau_2)) \in \Sigma_2$. It is clear that $\tau_1$ and $\tau_2$ depend on $\delta$. Moreover, $\tau_1 \to \tau_{L_1}$ and $\frac{\mathcal{T}}{2} - \tau_2 \to \tau_{L_2}$ when $\delta \to 0$.

The following lemma gives bounds of $\tau_2 - \tau_1$.

**Lemma 3.2.1.** *Let $\epsilon > 0$ be a fixed value small enough. Then, for any $\delta > 0$ sufficiently small we have*

$$\frac{2}{Q_p + \epsilon} \ln\left(\frac{\epsilon}{q(\tau_1)}\right) \leq \tau_2 - \tau_1 \leq \frac{2}{Q_p - \epsilon} \ln\left(\frac{\epsilon}{q(\tau_1)}\right).$$

**Proof**

We have that $(q(\tau_1), Q(\tau_1)) \in \Sigma_1$ and $(q(\tau_2), Q(\tau_2)) \in \Sigma_2$. Taking $\delta$ small enough, for any $\tau \in [\tau_1, \tau_2]$ the following inequalities hold

$$-Q_p - \varepsilon \leq Q(\tau) \leq -Q_p + \varepsilon.$$

Multiplying these inequalities by $-\dfrac{1}{2}q(\tau)$ and using the first equation in (3.7) we get

$$\frac{1}{2}(Q_p-\varepsilon)q(\tau)\le q'(\tau)\le\frac{1}{2}(Q_p+\varepsilon)q(\tau). \tag{3.10}$$

We note that $q(\tau)$ is an increasing function in $[0,\tau_2]$. Then, $q(\tau)>0$ in $[\tau_1,\tau_2]$, and the inequalities above can be written as

$$\frac{1}{2}(Q_p-\varepsilon)\le\frac{q'(\tau)}{q(\tau)}\le\frac{1}{2}(Q_p+\varepsilon).$$

By integration of these inequalities one obtains

$$\frac{1}{2}(Q_p-\varepsilon)(\tau_2-\tau_1)\le\ln\left(\frac{\varepsilon}{q(\tau_1)}\right)\le\frac{1}{2}(Q_p+\varepsilon)(\tau_2-\tau_1).$$

Now, the statement on the lemma follows by a simple computation. □

The following lemma will be used in next sections.

**Lemma 3.2.2.** *Let $\epsilon>0$ small enough. For any $\delta>0$ sufficiently small we have*

*(a)* $\int_{\tau_1}^{\tau_2}q(\tau)d\tau\le\dfrac{2\epsilon}{Q_p-\epsilon}$.

*(b)* $\int_{\tau_1}^{\tau_2}|Q(\tau)+Q_p|d\tau\le c_0\epsilon$, *for some constant $c_0$.*

**Proof**

(a) From (3.10) in the proof of lemma 3.2.1, for $\tau\in[\tau_1,\tau_2]$ the following inequality holds

$$q(\tau)\ \le\ \frac{2}{Q_p-\varepsilon}q'(\tau).$$

We integrate between $\tau_1$ and $\tau_2$ the inequality above, obtaining

$$\int_{\tau_1}^{\tau_2}q(\tau)d\tau\le\frac{2}{Q_p-\varepsilon}(q(\tau_2)-q(\tau_1))\le\frac{2}{Q_p-\varepsilon}q(\tau_2).$$

As $q(\tau_2)=\varepsilon$, we obtain the desired result.

(b) The idea is to study the distance between the component $Q$ of a solution defined in $[\tau_1,\tau_2]$ and the unstable manifold of $P_-$ with $q>0$.

We consider a neighbourhood of $P_-$ defined by $|Q+Q_p|\le\varepsilon$, $0<q\le\varepsilon$. We introduce $\xi=Q+Q_p$ in order to translate the equilibrium point $P_-$ to the origin in the plane $(q,\xi)$. Let $W^{u,+}$ be the branch of the unstable invariant manifold of the origin with $q>0$. We know that in the plane $(q,Q)$ the unstable manifold of $P_-$ is given by $\gamma_+$, and then in a neighbourhood of $P_-$ is given by $Q = G(q) = -(2-\alpha)\sqrt{-2V(\hat{q})}$. Then, in the plane $(q, \xi)$, $W^{u,+}$ is given by the graphic of the function $F(q) = Q_p - (2-\alpha)\sqrt{-2V(\hat{q})}$, that is, $\xi = F(q)$.

We recall that $Q_p = (2-\alpha)\sqrt{-2\gamma}$ and, by the hypothesis (A2), $V(\hat{q}) = \gamma + z^s V_1(\hat{q})$. Then,

$$F(q) \quad = \quad Q_p - Q_p\sqrt{1 + z^s\frac{V_1(\hat{q})}{\gamma}}. \tag{3.11}$$

Therefore if $0 < q \leq \varepsilon$, $\varepsilon$ small enough,

$$|F(q)| \quad \leq \quad kq^{\frac{2s}{2-\alpha}}, \tag{3.12}$$

for some constant $k > 0$.

We define $y = \xi - F(q)$ for $0 < q \leq \varepsilon$. In this way, $W^{u,+}$ lies on the $q$ axis in the plane $(q, y)$ and our region of interest is a neighbourhood of $\mathbf{0}$ with $y \geq 0$. In that region using (3.7) and (3.11) we get the following equation for $y$

$$y' \quad = \quad -\frac{\alpha}{2-\alpha}Q_p y(1 + O(\varepsilon)).$$

So, there exists a constant $c_2 > 0$ such that

$$-\frac{\alpha}{2-\alpha}Q_p y(1 + c_2\varepsilon) \leq y' \leq -\frac{\alpha}{2-\alpha}Q_p y(1 - c_2\varepsilon).$$

By integration of the inequality on the right hand we obtain

$$\int_{\tau_1}^{\tau_2} y(\tau)d\tau \leq \frac{2-\alpha}{\alpha Q_p(1-c_2\varepsilon)}(y(\tau_1) - y(\tau_2)) \leq \frac{2-\alpha}{\alpha Q_p(1-c_2\varepsilon)}y(\tau_1),$$

where we have used that $y(\tau) \geq 0$ for all $\tau \in [\tau_1, \tau_2]$.

Using that $\dfrac{2s}{2-\alpha} > 1$, from (3.12) we get

$$|F(q)| \quad \leq \quad kq, \tag{3.13}$$

in a small neighbourhood of $P_-$. Then, $y(\tau_1) = Q(\tau_1) + Q_p - F(q(\tau_1)) \leq Q(\tau_1) + Q_p + |F(q(\tau_1))| \leq (k+1)\varepsilon$. Moreover, by integration of the inequality (3.13) between $\tau_1$ and $\tau_2$ and using (a),

$$\int_{\tau_1}^{\tau_2} |F(q(\tau))|d\tau \leq \frac{2k}{Q_p - \varepsilon}\varepsilon.$$

As $Q + Q_p = y + F(q)$ we have

$$\int_{\tau_1}^{\tau_2} |Q(\tau) + Q_p|d\tau \leq \int_{\tau_1}^{\tau_2}(y(\tau) + |F(q(\tau))|)d\tau \leq c_0\varepsilon,$$

for some constant $c_0 > 0$.

□

## 3.3 Asymptotic formulae for the stability parameters

In this section we will prove the main result of this chapter, that give us asymptotic formulae for the stability parameters of system (3.1). Some auxiliar results are proven in sections 3.4 and 3.5.

**Theorem 3.3.1.** *Let us consider the system (3.1) where $g(t;\delta)$ satisfies the hypothesis (B). Let be $\hat{\lambda} = \gamma\frac{(2-\alpha)^2}{8}$ where $\gamma$ is defined in (A2). We assume that $\lambda_1, \lambda_2$ satisfy $\lambda_1 > \lambda_2 > \hat{\lambda}$ or, $\lambda_1 > \hat{\lambda} > \lambda_2$. Let be $\beta_j = \sqrt{1 - \frac{\lambda_j}{\hat{\lambda}}}$, $j = 1, 2$. Then we have the following asymptotic behaviour for the stability parameters when $\delta$ goes to 0*

$$\begin{aligned}
\log tr_1 &= k_1 - \frac{2-\alpha}{2\alpha}\beta_1 \log\delta(1+o(1)) + \ldots, \\
\log tr_2 &= k_2 - \frac{2-\alpha}{2\alpha}\beta_2 \log\delta(1+o(1)) + \ldots, \qquad \text{if} \quad \lambda_2 > \hat{\lambda}, \\
tr_2 &= k_3 + k_4\cos[k_5 - \gamma_2(1+o(1))\log(\delta)] + \ldots, \qquad \text{if} \quad \lambda_2 < \hat{\lambda},
\end{aligned} \tag{3.14}$$

*provided that some coefficient, $d_1$, is different from zero. In the last case $\gamma_2$ denotes $\frac{\beta_2}{i}$. The coefficients $k_j$, $j = 1, \ldots, 5$ are constants. The coefficient $k_4$ is different from zero if some coefficient $e_3 \neq 0$.*

The coefficients $d_1$ and $e_3$ will be defined in lemma 3.3.5. They depend on the particular potential $U(z)$ as well as on the parameters $\lambda_1$ and $\lambda_2$.

About the hypothesis in theorem 3.3.1 we have $\beta_1 \in \mathbb{R}^+$ and in the case $\beta_2 \in \mathbb{R}^+$, $\beta_1 > \beta_2$. These assumptions will give a dominant term depending on $\beta_1$ in the stability parameters. As we will see in chapter 4, these hyphotesis will be satisfied in the case of homographic solutions.

We note that asymptotic formulae (3.14) gives $\mathrm{tr}_1 > 2$ if $\delta$ is small enough. Furthermore, if $\beta_2 > 0$ then $\mathrm{tr}_2 > 2$ and the system is hyperbolic–hyperbolic. In the case $\beta_2 = \gamma_2\mathrm{i}$, $\mathrm{tr}_2$ oscillates between the values $k_3 + k_4$ and $k_3 - k_4$ as $\delta$ tends to 0. Therefore it can cros the lines $\mathrm{tr}_2 = 2$ and $\mathrm{tr}_2 = -2$ infinitely many times as $\delta$ tends to zero depending on the values of $k_3 + k_4$ and $k_3 - k_4$. In particular, if $k_3 - k_4 < -2$ and $k_3 + k_4 > 2$, $\mathrm{tr}_2 = -2$ for a sequence $\delta_i \to 0$, and we found intervals $(\delta_{2i}, \delta_{2i-1})$ with $\mathrm{tr}_2 < -2$, that is, hyperbolic-elliptic intervals. This will be the case for the collinear homographic solutions to be studied in chapter 1. A similar things occurs if $k_3 + k_4 > 2$ and $k_3 - k_4 < 2$.

In order to prove Theorem 3.3.1 we need to study the stability parameters of (3.1). As we have seen in section 3.1 this is equivalent to study the stability of (3.5).

First we begin by writing the monodromy matrix $\Psi(\mathcal{T})$ in terms of the transition matrix in a half period.

**Lemma 3.3.2.** *The equality*

$$\Psi(\mathcal{T}) \quad = \quad \frac{q_0}{q_a} G_0 \Psi \left(\frac{\mathcal{T}}{2}\right)^T G_M \Psi \left(\frac{\mathcal{T}}{2}\right), \tag{3.15}$$

*holds, where*

$$G_0 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -2q_0 \\ 0 & 1 & -2q_0 & 0 \end{pmatrix}, \quad G_M = \begin{pmatrix} 0 & -2q_a & -1 & 0 \\ -2q_a & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{3.16}$$

$q_0 = q(0)$ *and* $q_a = q(T/2)$.

**Proof**

Let $\Phi(t)$ be the fundamental matrix of the system (3.1) such that $\Phi(0) = I_4$.We begin showing that the following equality holds

$$\Phi(T) = \mathcal{F}^{-1}\Phi(T/2)^T \mathcal{F}\Phi(T/2) \tag{3.17}$$

where $\mathcal{F} = \begin{pmatrix} 0 & -2 & -1 & 0 \\ -2 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

Let

$$\dot{\mathbf{y}} \quad = \quad B_H(t)\mathbf{y} \tag{3.18}$$

the linear periodic Hamiltonian system with Hamiltonian function (3.4).

Now we prove that the fundamental matrix $\Phi_1(t)$ of this system such that $\Phi_1(0) = I_4$ satisfies the relation

$$\Phi_1(T) \quad = \quad -J_4\Phi_1\left(-\frac{T}{2}\right)^T J_4\Phi_1\left(\frac{T}{2}\right).$$

As $\Phi_1(t)$ is a fundamental matrix of a $T$–periodic system, then $\Phi_1(t+T)$ is also a fundamental matrix. Then, there exists a non–singular constant matrix $C$ such that $\Phi_1(t+T) = \Phi_1(t)C$. If we take $t = t_0$ then

$$\Phi_1(t+T) \quad = \quad \Phi_1(t)\Phi_1(t_0)^{-1}\Phi_1(t_0+T).$$

Then, if $t = 0$ and $t_0 = -\dfrac{T}{2}$, $\Phi_1(T) = \Phi_1\left(-\dfrac{T}{2}\right)^{-1}\Phi_1\left(\dfrac{T}{2}\right)$.

As $\Phi_1(t)$ is a symplectic matrix for all $t$, $\Phi_1(t)^{-1} = -J_4\Phi_1(t)^T J_4$. Taking $t = -\dfrac{T}{2}$, one obtain the desired relation.

The hypothesis (B) implies that $g(t)$ is an even function of $t$. Then, $B_H(t)$ is an even function of $t$. A simple computation shows that

$$B_H(t) = \begin{pmatrix} -J_2 & I_2 \\ \tilde{A}(t) - I_2 & -J_2 \end{pmatrix}.$$

Then, $LB_H(t)L = -B_H(t)$ where $L = \mathrm{diag}(-1, 1, 1, -1)$, and the parity of $B_H$ yields

$$LB_H(t)L = -B_H(-t). \tag{3.19}$$

Let us take define $\tilde{\Phi}(t) := L\Phi_1(-t)$. Using the property (3.19) it is easy to check have that $\tilde{\Phi}$ is also a fundamental matrix of the Hamiltonian system (3.18) associated to (3.4). Therefore, $\Phi_1(-t) = L\Phi_1(t)L$. In particular, $\Phi_1\left(-\dfrac{T}{2}\right) = L\Phi_1\left(\dfrac{T}{2}\right)L$. Then,

$$\Phi_1(T) = -JL\Phi_1\left(\frac{T}{2}\right)^T LJ\Phi_1\left(\frac{T}{2}\right),$$

and taking into account (1.61), that is, $\Phi(t) = \tilde{M}^{-1}\Phi_1(t)\tilde{M}$, where $\tilde{M}$ is defined in (1.58), we obtain

$$\Phi(T) = -(\tilde{M}^{-1})^T J_4 L(\tilde{M}^T)^{-1}\Phi\left(\frac{T}{2}\right)^T \tilde{M}^T LJ_4\tilde{M}\Phi\left(\frac{T}{2}\right).$$

From this equality and denoting $\mathcal{F} = \tilde{M}^T LJ_4\tilde{M}$, we obtain (3.17). From this relation and using (3.6) we have that

$$\Psi(\mathcal{T}) = S(0)\mathcal{F}^{-1}S(0)\Psi\left(\frac{\mathcal{T}}{2}\right)^T S^{-1}\left(\frac{\mathcal{T}}{2}\right)\mathcal{F}S^{-1}\left(\frac{\mathcal{T}}{2}\right)\Psi\left(\frac{\mathcal{T}}{2}\right).$$

The statement of the lemma holds taking $G_0 = \frac{1}{q_0}S(0)\mathcal{F}^{-1}S(0)$ and $G_M = q_a S^{-1}\left(\frac{\mathcal{T}}{2}\right)\mathcal{F}S^{-1}\left(\frac{\mathcal{T}}{2}\right)$. $\square$

Our purpose now is to get an expression for $\Psi\left(\frac{\mathcal{T}}{2}\right)$ which allows us to compute the dominant terms of the traces of $\Psi(\mathcal{T})$ for values of $\delta$ near 0. To do that we shall use the planar system (3.7) in order to split $\Psi\left(\frac{\mathcal{T}}{2}\right)$ in three matrices each obtained from (3.5) following the heteroclinic connections of that planar system, that is, in a neighbourhood of $\gamma_0$, $P_-$ and $\gamma_+$, respectively.

For a fixed value of $\delta > 0$ small enough, let $(q(\tau), Q(\tau))$ be the solution of (3.7) for $E = -\delta$ such that $q(0) = q_0$, $Q(0) = 0$, being $q_0$ the minimum of $q(\tau)$. Then the matrix $B(\tau)$ in (3.5) can be written as

$$B(\tau) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 & 0 & -Q(\tau)/2 & -2q(\tau) \\ 0 & \lambda_2 & 2q(\tau) & -Q(\tau)/2 \end{pmatrix} =: B_a(q, Q). \tag{3.20}$$

From now on we consider

$$\dot{\mathbf{y}} \;=\; B(\tau)\mathbf{y}.$$

Let $\tilde{\Psi}(\tau_b, \tau_a)$ be the transition matrix of (3.5) from $\tau_a$ to $\tau_b$, that is, $\tilde{\Psi}(\tau_b, \tau_a)$ is the fundamental matrix of (3.5) satisfying $\tilde{\Psi}(\tau_a, \tau_a) = I_4$ evaluated at $\tau = \tau_b$.

Then, we can write

$$\Psi\left(\frac{\mathcal{T}}{2}\right) = \tilde{\Psi}\left(\frac{\mathcal{T}}{2}, \tau_2\right)\tilde{\Psi}(\tau_2, \tau_1)\tilde{\Psi}(\tau_1, 0). \tag{3.21}$$

We note that $\tau_1$ and $\tau_2$ (as defined in section 3.2) and also $\dfrac{\mathcal{T}}{2}$ depend on $\delta$.

Our purpose is to approximate the transition matrices involved in (3.21) by simpler ones. First, we shall give approximations of $\tilde{\Psi}(\tau_1, 0)$ and $\tilde{\Psi}\left(\dfrac{\mathcal{T}}{2}, \tau_2\right)$ in (3.21) by the transition matrices for the system (3.5) along $\gamma_0$ and $\gamma_+$, respectively.

Using (3.20) we define

$$B_{L_1}(\tau) = B_a(0, Q_{L_1}(\tau)), \qquad B_{L_2}(\tau) = B_{L_2}(q_{L_2}(\tau), Q_{L_2}(\tau)), \tag{3.22}$$

where we recall that $(0, Q_{L_1}(\tau))$, $(q_{L_2}(\tau), Q_{L_2}(\tau))$ are the solutions of (3.7) corresponding to $\gamma_0$, $\gamma_+$, respectively.

Let $Z_1(\tau)$ be the fundamental matrix of

$$\mathbf{u}' \;=\; B_{L_1}(\tau)\mathbf{u}, \tag{3.23}$$

such that $Z_1(0) = I_4$.

*Remark* 3.3.3. The system (3.23) only depends on $\lambda_1$, $\lambda_2$, $\alpha$ and $\lambda$. It does not depend on the particular form of the function $V_1(z)$ defined in (A2).

We denote by $Z_2(\tau)$ the fundamental matrix of

$$\mathbf{u}' \;=\; B_{L_2}(\tau)\mathbf{u}, \tag{3.24}$$

such that $Z_2(-\tau_{L_2}) = I_4$.

For a fixed value of $\varepsilon > 0$ small enough $\mathcal{P}_1$ and $\mathcal{P}_3$ are diffeomorphisms. So, we can write

$$\tilde{\Psi}(\tau_1, 0) = Z_1(\tau_{L_1}) + \Delta_1, \qquad \tilde{\Psi}\left(\frac{\mathcal{T}}{2}, \tau_2\right) = Z_2(0) + \Delta_2,$$

for some matrices $\Delta_1$, $\Delta_2$ with $\|\Delta_1\| = O(q_0)$, $\|\Delta_2\| = O(q_{L_2}(0) - q_0) = O(\delta)$. We recall that $q_0 = O(\delta^{\frac{2-\alpha}{2\alpha}})$.

This relations give us approximation of the matrices $\tilde{\Psi}(\tau_1, 0)$ and $\tilde{\Psi}\left(\frac{\mathcal{T}}{2}, \tau_2\right)$ by the fundamental matrices of the system (3.5) on $\gamma_0$ and $\gamma_+$.

Now we want to obtain an approximation of the transition matrix $\tilde{\Psi}(\tau_2, \tau_1)$

We consider the system (3.5) in a neighbourhood of the equilibrium point $P_-$, and we write

$$B(\tau) \quad = \quad B_p + B_1(\tau),$$

$$B_1(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & B_{11} \end{pmatrix} \quad \text{with} \quad B_{11} = \begin{pmatrix} -\frac{1}{2}(Q + Q_p) & -2q \\ 2q & -\frac{1}{2}(Q + Q_p) \end{pmatrix},$$

where

$$B_p \quad = \quad B_a(0, -Q_p),$$

that is the matrix of (3.20) evaluated on $P_-$. We note that $B_p$ does not depend on $\delta$.

The characteristic polynomial of $B_p$ is

$$p(x) \quad = \quad x^2 \left(\frac{Q_p}{2} - x\right)^2 + (\lambda_1 + \lambda_2) x \left(\frac{Q_p}{2} - x\right) + \lambda_1 \lambda_2.$$

Performing the change of variables $y = \dfrac{Q_p}{4} - x$, the equation for the eigenvalues transforms in a biquadratic equation. Then, it is easy to compute the roots of $p$ and one obtains that the eigenvalues of $B_p$ are

$$\rho_1^{\pm} = \frac{Q_p}{4}(1 \pm \beta_1), \qquad \rho_2^{\pm} = \frac{Q_p}{4}(1 \pm \beta_2), \tag{3.25}$$

where $\beta_i = \sqrt{1 - \dfrac{8\lambda_i}{\gamma(2-\alpha)^2}}$, $i = 1, 2$. We note that under the hypothesis of Theorem 3.3.1, $\beta_1 \neq 0$, $\beta_2 \neq 0$ and $\beta_1 \neq \beta_2$. Hence the eigenvalues of $B_p$ are also differents.

The associated eigenvectors are $(1,0,\rho_1^{\pm},0)^T$ and $(0,1,0,\rho_2^{\pm})^T$, respectively. Let $P$ be the matrix

$$P = \begin{pmatrix} I_2 & I_2 \\ P_3 & P_4 \end{pmatrix}, \qquad P_3 = \mathrm{diag}(\rho_1^+, \rho_2^+), \qquad P_4 = \mathrm{diag}(\rho_1^-, \rho_2^-). \tag{3.26}$$

$P$ is nonsingular and

$$P^{-1} B_p P = \frac{Q_p}{4} I + \bar{D}$$

where $\bar{D} = \dfrac{Q_p}{4}\mathrm{diag}(\beta_1, \beta_2, -\beta_1, -\beta_2)$. We introduce a new variable

$$w = \exp\left(-\frac{Q_p}{4}(\tau - \tau_1)\right) P^{-1} y$$

and we get the following system for $w$

$$w' = (\bar{D} + P^{-1} B_1(\tau) P) w. \tag{3.27}$$

Let $W(\tau)$ be the fundamental matrix of (3.27) such that $W(\tau_1) = I$. A simple computation shows that

$$\tilde{\Psi}(\tau, \tau_1) = \exp\left(\frac{Q_p}{4}(\tau - \tau_1)\right) P W(\tau) P^{-1}.$$

Then, from (3.21), we get

$$\Psi\left(\frac{\mathcal{T}}{2}\right) = \sigma \tilde{\Psi}\left(\frac{\mathcal{T}}{2}, \tau_2\right) P W(\tau_2) P^{-1} \tilde{\Psi}(\tau_1, 0), \tag{3.28}$$

where $\sigma = \exp\left(\dfrac{Q_p}{4}(\tau_2 - \tau_1)\right)$.

Next lemma give us an approximation of $W(\tau)$ by a simpler matrix.

**Lemma 3.3.4.** *Let $\epsilon > 0$ be small enough. If $\delta > 0$ is sufficiently small we have for all $\tau \in [\tau_1, \tau_2]$*

$$W(\tau) = (I_4 + \Delta(\tau)) \mathcal{D}(\tau)(I_4 + R),$$

*where $\mathcal{D}(\tau) = diag(e^{\nu_1(\tau-\tau_1)}, e^{\nu_2(\tau-\tau_1)}, e^{-\nu_1(\tau-\tau_1)}, e^{-\nu_2(\tau-\tau_1)})$, $\nu_i = \dfrac{Q_p}{4}\beta_i$, $i = 1, 2$, and $\Delta(\tau)$ is a matrix such that $\|\Delta(\tau)\| \leq c_1 \varepsilon$ for any $\tau \in [\tau_1, \tau_2]$ and $R$ is a constant matrix such that $\|R\| \leq c_2 \varepsilon$, for some constants $c_1, c_2$, uniformly in $\delta$.*

The proof of this lemma is given in section 3.4.

We note that $W(\tau)$ can be approximated by a diagonal matrix for $\varepsilon > 0$ small enough.

Up to now, we have approximated the transition matrices $\tilde{\Psi}(\tau_1, 0)$, $\tilde{\Psi}\left(\frac{\mathcal{T}}{2}, \tau_1\right)$ by $Z_1(\tau_{L_1})$, $Z_2(0)$, respectively, and $W(\tau_2)$ by $\mathcal{D}(\tau_2)$. Then, using (3.28), we have that

$$\Psi\left(\frac{\mathcal{T}}{2}\right) = \sigma\left[Z_2(0)P\mathcal{D}(\tau_2)P^{-1}Z_1\tau_{L_1}\right](I_4 + \Delta_3),$$

where $\|\Delta_3\| = O(\varepsilon, q_0, \delta)$. We remark that we are assuming that

$$\|Z_2(0)P\mathcal{D}(\tau_2)P^{-1}Z_1(\tau_{L_1})\|$$

has the same order of magnitud as the product of norms. We shall see that this is the case if the coefficient $d_1 \neq 0$.

Now, using (3.15), we get

$$\Psi(\mathcal{T}) = \frac{q_0}{q_a}\sigma^2\mathcal{M}(I_4 + \mathcal{O}), \qquad \mathcal{M} = A_1\mathcal{D}A_2\mathcal{D}A_3, \tag{3.29}$$

where $A_1 = G_{00}A_3^T$, $A_2 = P^T Z_2(0)^T G_M Z_2(0)P$, $A_3 = P^{-1}Z_1(\tau_{L_1})$, $\mathcal{D} = \mathcal{D}(\tau_2)$, $G_{00} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$, $C = \mathrm{diag}(-1, 1)$, and $\mathcal{O}$ stands for a matrix with contains terms of order $\varepsilon$, $q_0$ and $\delta_0$. The same remark concerning the product of norms holds for (3.29). We note also that matrices $A_1$, $A_2$ and $A_3$ are independent of $\delta$.

Now we want to obtain the stability parameters of (3.5). We recall that $\Phi(T)$ has the same eigenvalues that $\Psi(\mathcal{T})$. Moreover, as $\Phi(T)$ is symplectic, the characteristic polynomial of $\Psi(\mathcal{T})$ is

$$p(x) = x^4 + a_1x^3 + a_2x^2 + a_1x + 1,$$

where

$$\begin{aligned} a_1 &= -(\mathrm{tr}_1 + \mathrm{tr}_2), \\ a_2 &= 2 + \mathrm{tr}_1\mathrm{tr}_2. \end{aligned} \tag{3.30}$$

Then, the stability parameters can be obtained from $a_1$ and $a_2$ as the zeroes of $x^2 + a_1x + a_2 - 2$. To estimate the dominant terms of these coefficients we shall use the matrix $\mathcal{M}$.

Let us denote by

$$q(x) = x^4 + b_3x^3 + b_2x^2 + b_1x + b_0,$$

the characteristic polynomial of $\mathcal{M}$. We denote $k = \frac{q_a}{q_0\sigma^2}$. From (3.29),

$$\det(\Psi(\mathcal{T}) - xI_4) = \frac{1}{k^4}\det(\mathcal{M} + \mathcal{O} - kxI_4).$$

Using the equality above we obtain the following relation between $a_1, a_2$ and $b_0, \dots, b_3$,

$$a_1 = \frac{b_3}{k}, \quad a_2 = \frac{b_2}{k^2}, \quad a_1 = \frac{b_1}{k^3}, \quad \frac{b_0}{k^4} = 1. \tag{3.31}$$

From (3.30) and (3.31) we get that the stability parameters $\mathrm{tr}_1$ and $\mathrm{tr}_2$ satisfy, up to order 1 in $\varepsilon$, the quadratic equation

$$k^2\tilde{t}^2 + kb_3\tilde{t} + b_2 - 2k^2 \quad = \quad 0. \tag{3.32}$$

We are interested in the dominant terms of $b_2$ and $b_3$. The following lemma give us these terms.

**Lemma 3.3.5.** *Let $\epsilon > 0$ be small enough. Assume that $\lambda_1$ and $\lambda_2$ satisfy the hypothesis of teorem 3.3.1. Then*
*(a) There exist some constants $d_i$, $i = 1, \dots 5$ such that*

$$-b_3 = d_1\sigma^{2\beta_1} + d_2\sigma^{2\beta_2} + d_3\sigma^{-2\beta_1} + d_4\sigma^{-2\beta_2} + d_5. \tag{3.33}$$

*The coefficient $d_1$ is the product of two constants, $d_1 = d_n d_g$ with $d_n$ depending on $\lambda_1, \lambda_2, \alpha$ and $\gamma$ but not on the function $V_1$ defined in section 3.1. $d_g$ depends also on $V_1$. If $\lambda_1$ and $\lambda_2$ are different from zero, then $d_n \neq 0$.*
*(b) The coefficient $b_2$ does not contain terms in $\sigma^{\pm 4\beta_1}$ nor $\sigma^{\pm 4\beta_2}$, that is the dominant terms are*

$$b_2 = e_1\sigma^{2\beta_1} + e_2\sigma^{2\beta_2} + e_3\sigma^{2(\beta_1+\beta_2)} + e_4\sigma^{2(\beta_1-\beta_2)} + \dots \tag{3.34}$$

*for some constants $e_1, e_2, e_3, \dots$ The coefficient $e_3$ is the product of two constants $e_3 = e_n e_g$ where $e_n$ depends on $\lambda_1, \lambda_2, \alpha$ and $\gamma$ but not on the function $V_1$ defined in section 3.1. $e_g$ depends also on $V_1$. If $\lambda_1$, $\lambda_2$ are different from zero then $e_n \neq 0$.*

*Moreover, if $\lambda_2 < \gamma\dfrac{(2-\alpha)^2}{8}$, then $d_4 = \bar{d}_2$, and $e_4 = \bar{e}_3$ where the bar stands for the complex conjugate.*

The proof of this lemma will be given in section 3.5.

Now, the stability parameters are obtained by solving the quadratic equation (3.32). The solutions of this equation are

$$\tilde{t} \quad = \quad \frac{-b_3 \pm \sqrt{d}}{2k},$$

where $d = k[b_3^2 - 4b_2 + 8k^2]$.

We assume that the constant $d_1$ in Lemma 3.3.5 is different from zero. If $\lambda_1$, $\lambda_2$ satisfy the hypothesis of Theorem 3.3.1 then either $\beta_1 > \beta_2 > 0$ or $\beta_1 > 0$ and

$\beta_2 = \gamma_2 \mathrm{i}$, $\gamma_2 \in \mathbb{R}$. In any case, the dominant term in $-b_3$ is $d_1\sigma^{2\beta_1}$. Therefore, we have that the dominant terms of $d$ are

$$d = d_1^2 \sigma^{4\beta_1} \tilde{d},$$

where $\tilde{d} = 1 + 2n_1\sigma^{-2\beta_1} + 2n_2\sigma^{-2(\beta_1-\beta_2)} + 2n_3\sigma^{-2(\beta_1+\beta_2)} + \ldots$, and

$$n_1 = \frac{d_1 d_5 - 2e_1}{d_1^2}, \quad n_2 = \frac{d_1 d_2 - 2e_3}{d_1^2}, \quad n_3 = \frac{d_1 d_4 - 2e_4}{d_1^2}.$$

Then,

$$\mathrm{tr}_1 = \frac{-b_3 + d_1\sigma^{2\beta_1}\sqrt{\tilde{d}}}{2k} = \frac{q_0}{q_a}\sigma^2 (d_1\sigma^{2\beta_1} + \ldots),$$

and

$$\mathrm{tr}_2 = \frac{-b_3 - d_1\sigma^{2\beta_1}\sqrt{\tilde{d}}}{2k}.$$

As $\sqrt{\tilde{d}} = 1 + n_1\sigma^{-2\beta_1} + n_2\sigma^{-2(\beta_1-\beta_2)} + n_3\sigma^{-2(\beta_1+\beta_2)} + \ldots$, the dominant terms in $\mathrm{tr}_2$ are

$$\mathrm{tr}_2 = \frac{1}{2k}[(d_2 - d_1 n_2)\sigma^{2\beta_2} + (d_4 - d_1 n_3)\sigma^{-2\beta_2} + d_5 - d_1 n_1 + \ldots]. \quad (3.35)$$

If $\beta_2$ is real the dominant term is given by $(d_2 - d_1 n_2)\sigma^{2\beta_2}$ and then

$$\mathrm{tr}_2 = \frac{q_0}{q_a}\sigma^2 \left(\frac{e_3}{d_1}\sigma^{2\beta_2} + \ldots\right).$$

If $\beta_2$ is imaginary, all the terms written in (3.35) are of the same order. From lemma 3.3.5 we have that in this case $d_4 = \overline{d}_2$ and $e_4 = \overline{e}_3$. Then,

$$\mathrm{tr}_2 = \frac{q_0}{q_a}\sigma^2 \left(\frac{e_1}{d_1} + 2\mathrm{Re}\left(\frac{e_3}{d_1}\sigma^{2\beta_2}\right) + \ldots\right).$$

Summarizing, we have that if $d_1 \neq 0$, then the stability parameters are obtained as

$$\mathrm{tr}_1 = \frac{q_0}{q_a}\sigma^2 (d_1\sigma^{2\beta_1} + \ldots), \quad (3.36)$$

and

$$\mathrm{tr}_2 = \frac{q_0}{q_a}\sigma^2 \left(\frac{e_3}{d_1}\sigma^{2\beta_2} + \ldots\right), \quad \text{if} \quad \beta_2 > 0$$

$$\mathrm{tr}_2 = \frac{q_0}{q_a}\sigma^2 \left(\frac{e_1}{d_1} + 2\mathrm{Re}\left(\frac{e_3}{d_1}\sigma^{2\beta_2}\right) + \ldots\right), \quad \text{if} \quad \beta_2 = \gamma_2 \mathrm{i}, \gamma_2 \in \mathbb{R}.$$

Now, using lemma 3.2.1 the result on the theorem 3.3.1 follows.

*Remark* 3.3.6. We recall that matrices $A_1$, $A_2$ and $A_3$ in (3.29) do not depend on $\delta$, so their norms are finite. Therefore $\|A_1\|\|\mathcal{D}\|\|A_2\|\|\mathcal{D}\|\|A_3\|$ depends mainly on $\|\mathcal{D}\|^2$ for $\delta > 0$ small enough. Furthermore $\mathcal{D} = \mathcal{D}(\tau_2)$ is a diagonal matrix and so, $\|\mathcal{D}\|^2$ is of the order or $\sigma^{2\beta_1}$. However, if $d_1 \neq 0$ from (3.36) we have that $\mathrm{tr}_1$ is of order $\sigma^{2\beta_1}$. This gives an estimation of the spectral radius of $\mathcal{M}$. Using that $\|\mathcal{M}\|$ is bounded from below by the spectral radius we conclude that it is of the same order of magnitude of the product of norms and then (3.29) holds.

## 3.4 Proof of lemma 3.3.4

In order to prove lemma 3.3.4 it will be useful the following lemma.

**Lemma 3.4.1.** *Let us consider the system*

$$\mathbf{x}' = D\mathbf{x} + C(t)\mathbf{x}, \tag{3.37}$$

*where $D$ is a diagonal matrix $n \times n$ and $C(t)$ is a continuous matrix in $t$. Assume that there exists some constant $\hat{\epsilon} < 1/4$ such that*

$$\int_0^{\hat{t}} \|C(s)\| ds < \hat{\epsilon}. \tag{3.38}$$

*Let $\lambda$ be an eigenvalue of $D$ and $\mathbf{v}$ an eigenvector corresponding to $\lambda$. Then, there exists a solution, $\varphi(t)$, of (3.37) such that*

$$\|e^{-\lambda t}\varphi(t) - \mathbf{v}\| \leq \|\mathbf{v}\| \frac{3\hat{\epsilon}}{1 - 3\hat{\epsilon}},$$

*for all $t \in [0, \hat{t}]$.*

**Proof**

Let us consider $\lambda_1, \ldots, \lambda_n$ the eigenvalues of $D$. That is, $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Assume that $\mathrm{Re}(\lambda_j) < \mathrm{Re}(\lambda)$ for $0 < j \leq k < n$, $\mathrm{Re}(\lambda_j) > \mathrm{Re}(\lambda)$ for $k+1 \leq j \leq m < n$ and $\mathrm{Re}(\lambda_j) = \mathrm{Re}(\lambda)$ for $m+1 \leq j \leq n$.

We denote by $D_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_k)$ the $k \times k$ diagonal matrix obtained from $D$ such that their eigenvalues have real part less than $\mathrm{Re}(\lambda)$. Analogously, we define $D_2 = \mathrm{diag}(\lambda_{k+1}, \ldots, \lambda_m)$ the $(m-k) \times (m-k)$ diagonal matrix with eigenvalues with real part bigger than $\mathrm{Re}(\lambda)$, and $D_3 = \mathrm{diag}(\lambda_{m+1}, \ldots, \lambda_n)$ a $(n-m) \times (n-m)$ diagonal matrix, which has eigenvalues with the same real part as $\lambda$.

It could be possible that $D_1$ or $D_2$ does not contain any term, but it is clear that $\lambda$ is in $D_3$. We take $\lambda_n = \lambda$.

We have that $D = \mathrm{diag}(D_1, D_2, D_3)$. Let us denote by $X(t)$ the fundamental matrix of the system $\mathbf{x}' = D\mathbf{x}$ with $X(0) = I_n$. Then, $X(t) = e^{Dt} =$

$\mathrm{diag}(e^{D_1 t}, e^{D_2 t}, e^{D_3 t})$. We can write $X(t) = X_1(t) + X_2(t) + X_3(t)$ where $X_1(t) = \mathrm{diag}(e^{D_1 t}, 0, 0)$, $X_2(t) = \mathrm{diag}(0, e^{D_2 t}, 0)$, $X_3(t) = \mathrm{diag}(0, 0, e^{D_3 t})$.

$\tilde{X}_j(t) = e^{D_j t}$ is the fundamental matrix of the system

$$\mathbf{x}_j' \;=\; D_j \mathbf{x}_j,$$

satisfying $\tilde{X}_j(0)$ the identity matrix, $j = 1, 2, 3$. We note that $\mathbf{x}_1 \in \mathbb{R}^k$, $\mathbf{x}_2 \in \mathbb{R}^{m-k}$, $x_3 \in \mathbb{R}^{n-m}$.

Then $e^{-\lambda t}\tilde{X}_j(t)$ is a fundamental matrix of the system

$$\mathbf{x}_j' \;=\; \tilde{A}_j \mathbf{x}_j,$$

where $\tilde{A}_j = D_j - \lambda Id$, and $Id$ denotes the identity matrix of suitable dimension. $\tilde{A}_1$ ($\tilde{A}_2$) has eigenvalues with negative (positive) real part. Then, there exists a positive constant $a > 0$ such that $\|e^{-\lambda t}\tilde{X}_1(t)\| \le e^{-at}$ for all $t \ge 0$, and $\|e^{-\lambda t}\tilde{X}_2(t)\| \le e^{at}$ for all $t \le 0$.

Now we consider $\tilde{A}_3$, that has eigenvalues with zero real part. Let us write $\lambda = \alpha + \mathrm{i}\beta$. Then, $\lambda_j = \alpha + i\beta_j$, $j = m+1, \ldots, n$, and

$$e^{-\lambda t}\tilde{X}_3(t) = \mathrm{diag}(e^{\mathrm{i}(\beta_{m+1}-\beta)t}, \ldots, e^{\mathrm{i}(\beta_{n-1}-\beta)t}, 1).$$

Therefore $\|e^{-\lambda t}\tilde{X}_3(t)\| = 1$.

As $\|e^{-\lambda t}X_j(t)\| = \|e^{-\lambda t}\tilde{X}_j(t)\|$ for $j = 1, 2, 3$ we get $\|e^{-\lambda t}X_1(t)\| \le e^{-at}$ for all $t \ge 0$, $\|e^{-\lambda t}X_2(t)\| \le e^{at}$ for all $t \le 0$, for a constant $a > 0$, and $\|e^{-\lambda t}X_3(t)\| = 1$.

It is easy to check that the solution $\varphi(t)$ of the integral equation

$$\begin{aligned}\varphi(t) \;&=\; e^{t\lambda}\mathbf{v} + \int_0^t X_1(t-s)C(s)\varphi(s)ds - \int_t^{\hat{t}} X_2(t-s)C(s)\varphi(s)ds \\ &\quad - \int_t^{\hat{t}} X_3(t-s)C(s)\varphi(s)ds\end{aligned}$$

is a solution of (3.37).

We want to obtain the solution of the integral equation above. To this end we use an iterative scheme with $\varphi_0(t) \equiv 0$. For $m \ge 1$ we define

$$\begin{aligned}\varphi_m(t) \;&=\; e^{t\lambda}\mathbf{v} + \int_0^t X_1(t-s)C(s)\varphi_{m-1}(s)ds - \int_t^{\hat{t}} X_2(t-s)C(s)\varphi_{m-1}(s)ds \\ &\quad - \int_t^{\hat{t}} X_3(t-s)C(s)\varphi_{m-1}(s)ds.\end{aligned}$$

Then, for all $t \in [0, \hat{t}]$ the following inequalities hold

$$\|\varphi_m(t) - \varphi_{m-1}(t)\| \;\le\; \|\mathbf{v}\|(3\hat{\epsilon})^{m-1}|e^{t\lambda}| \le \|\mathbf{v}\|(3\hat{\epsilon})^{m-1}|e^{\hat{t}\lambda}|, \tag{3.39}$$

$$\|e^{-t\lambda}\varphi_m(t) - \mathbf{v}\| \;\le\; \|\mathbf{v}\|\sum_{k=1}^{m-1}(3\hat{\epsilon})^k. \tag{3.40}$$

The inequalities above will be proved now by induction.

For (3.39) is clear that

$$\|\varphi_1(t) - \varphi_0(t)\| = \|\varphi_1(t)\| \le \|\mathbf{v}\||e^{\lambda t}|.$$

Assume that (3.39) is true for $m-1$. We note that

$$\begin{aligned}\varphi_m(t) - \varphi_{m-1}(t) \quad = \quad & e^{t\lambda}\left\{\int_0^t e^{-\lambda(t-s)}X_1(t-s)C(s)e^{-s\lambda}(\varphi_{m-1}(s) - \varphi_{m-2}(s))ds\right. \\ & - \int_t^{\hat{t}} e^{-\lambda(t-s)}X_2(t-s)C(s)e^{-s\lambda}(\varphi_{m-1}(s) - \varphi_{m-2}(s))ds \\ & \left. - \int_t^{\hat{t}} e^{-\lambda(t-s)}X_3(t-s)C(s)e^{-s\lambda}(\varphi_{m-1}(s) - \varphi_{m-2}(s))ds\right\}.\end{aligned}$$

Then, using the inequalities for $\|e^{-\lambda t}X_j(t)\|$, $j = 1, 2, 3$, one obtains

$$\begin{aligned}&\|\varphi_m(t) - \varphi_{m-1}(t)\| \le |e^{t\lambda}|\left\{\int_0^t e^{-a(t-s)}\|C(s)\|\|\mathbf{v}\|(3\hat{\epsilon})^{m-2}ds + \right. \\ &\left. + \int_t^{\hat{t}} e^{a(t-s)}\|C(s)\|\|\mathbf{v}\|(3\hat{\epsilon})^{m-2}ds + \int_t^{\hat{t}} \|C(s)\|\|\mathbf{v}\|(3\hat{\epsilon})^{m-2}ds\right\} \le \\ &\le |e^{t\lambda}|\|\mathbf{v}\|(3\hat{\epsilon})^{m-1}.\end{aligned}$$

This proves (3.39).

The proof of (3.40) for $m = 1$ is trivial. For $m \ge 2$, first we note that

$$\begin{aligned}&e^{-t\lambda}\varphi_2(t) - \mathbf{v} = \int_0^t e^{-\lambda(t-s)}X_1(t-s)C(s)\mathbf{v}ds - \\ &\int_t^{\hat{t}} e^{-\lambda(t-s)}X_2(t-s)C(s)\mathbf{v}ds \quad - \int_t^{\hat{t}} e^{-\lambda(t-s)}X_3(t-s)C(s)\mathbf{v}ds.\end{aligned}$$

Then

$$\begin{aligned}&\|e^{-t\lambda}\varphi_2(t) - \mathbf{v}\| \le \int_0^t e^{-a(t-s)}\|C(s)\|\|\mathbf{v}\|ds + \int_t^{\hat{t}} e^{a(t-s)}\|C(s)\|\|\mathbf{v}\|ds \\ &+ \int_t^{\hat{t}} \|C(s)\|\|\mathbf{v}\|ds \le 3\hat{\epsilon}\|\mathbf{v}\|.\end{aligned}$$

For the general step we get

$$\begin{aligned}&e^{-t\lambda}\varphi_m(t) - \mathbf{v} = \int_0^t e^{-\lambda(t-s)}X_1(t-s)C(s)(e^{-s\lambda}\varphi_{m-1}(s) - \mathbf{v})ds \\ &- \int_t^{\hat{t}} e^{-\lambda(t-s)}X_2(t-s)C(s)(e^{-s\lambda}\varphi_{m-1}(s) - \mathbf{v})ds \\ &- \int_t^{\hat{t}} e^{-\lambda(t-s)}X_3(t-s)C(s)(e^{-s\lambda}\varphi_{m-1}(s) - \mathbf{v})ds + (e^{-t\lambda}\varphi_2(t) - \mathbf{v}).\end{aligned}$$

Therefore

$$\begin{aligned}\|e^{-t\lambda}\varphi_m(t)-\mathbf{v}\| &\le \int_0^t e^{-a(t-s)}\|C(s)\|\|e^{-t\lambda}\varphi_{m-1}(t)-\mathbf{v}\|ds\\ &+\int_t^{\hat t} e^{a(t-s)}\|C(s)\|\|e^{-t\lambda}\varphi_{m-1}(t)-\mathbf{v}\|ds\\ &+\int_t^{\hat t}\|C(s)\|\|e^{-t\lambda}\varphi_{m-1}(t)-\mathbf{v}\|ds+\|e^{-t\lambda}\varphi_2(t)-\mathbf{v}\|\\ &\le \|\mathbf{v}\|\left(\sum_{k=1}^{m-2}(3\hat\epsilon)^k\left[3\int_0^{\hat t}\|C(s)\|ds\right]+3\hat\epsilon\right)\le\|\mathbf{v}\|\sum_{k=1}^{m-1}(3\hat\epsilon)^k\end{aligned}$$

and (3.40) follows.

From (3.39) we have that $\varphi_m$ tends to $\varphi$ uniformly on compacts.

Then, using (3.40) we have that

$$\|e^{-\lambda t}\varphi(t)-\mathbf{v}\|\le\|\mathbf{v}\|\textstyle\sum_{k=1}^{\infty}(3\hat\varepsilon)^k=\|\mathbf{v}\|\dfrac{3\hat\varepsilon}{1-3\hat\varepsilon}.$$

This ends the proof of the lemma. □

To prove lemma 3.3.4 we shall apply the lemma 3.4.1 to the system (3.27).

For a fixed value of $\varepsilon$ we consider $q_0>0$ small enough (we have that $q_0=\left(\dfrac{\delta}{|\gamma|}\right)^{\frac{2-\alpha}{2\alpha}}(1+O(\delta^{\frac{s}{\alpha}}))$) and $\tau\in[\tau_1,\tau_2]$ where we recall that $\tau_1,\tau_2$ depend on $q_0$. After a translation of time defined by $s=\tau-\tau_1$ we can restrict to the system

$$\frac{dw}{ds} = (\overline{D}+P^{-1}B_1(s+\tau_1)P)w, \tag{3.41}$$

for $s\in[0,\hat s]$, where $\hat s=\hat s(q_0)=\tau_2-\tau_1$. First we prove that the hypothesis (3.38) is satisfied for $C(s):=P^{-1}B(s+\tau_1)P$. We recall that $B_1(\tau)$ is defined in (3.22) and it depends on $\delta$. We get

$$\|C(s)\| \le \|P\|\|P^{-1}\|\|B_{11}(s+\tau_1)\|,$$

For any $\delta$ small enough, using lemma 3.2.2 we get

$$\textstyle\int_0^{\hat s}\|C(s)\|ds\le\|P\|\|P^{-1}\|\left(\dfrac{4\varepsilon}{Q_p-\varepsilon}+\dfrac{c_0\varepsilon}{2}\right).$$

Then (3.38) is satisfied with $\hat\varepsilon=\varepsilon c_1$ being $c_1=\|P\|\|P^{-1}\|\left(\dfrac{4}{Q_p-\varepsilon}+\dfrac{c_0}{2}\right)$. If $\varepsilon$ is small enough we have $\hat\varepsilon<\dfrac{1}{4}$. For technical reasons we shall assume that $\varepsilon<\dfrac{1}{12c_1}$.

We recall that $\bar{D} = \dfrac{Q_p}{4}\mathrm{diag}(\nu_1, \nu_2, \nu_3, \nu_4)$ where $\nu_j = \dfrac{Q_p}{4}\beta_j$, $j = 1, 2$, $\nu_3 = -\nu_1$, $\nu_4 = -\nu_2$. Using lemma 3.4.1, there exist $\hat{\varphi}_1(s), \hat{\varphi}_2(s), \hat{\varphi}_3(s), \hat{\varphi}_4(s)$ solutions of (3.41) such that for any $s \in [0, \hat{s}]$

$$\|e^{-\nu_i s}\hat{\varphi}_i(s) - \mathbf{e}_i\| \leq \|\mathbf{e}_i\| \tfrac{3\hat{\varepsilon}}{1-3\hat{\varepsilon}}, \quad i = 1, \ldots, 4,$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ is the canonical basis. Then, $\varphi_i(\tau) := \hat{\varphi}(\tau - \tau_1)$, $j = 1, \ldots, 4$, are solutions of (3.27) such that for any $\tau \in [\tau_1, \tau_2]$

$$\|e^{-\nu_i(\tau-\tau_1)}\varphi_i(\tau) - \mathbf{e}_i\| \leq \tfrac{3\hat{\varepsilon}}{1-3\hat{\varepsilon}}, \quad i = 1, \ldots, 4. \tag{3.42}$$

We denote as $Y(t)$ the matrix defined by $\varphi_1, \varphi_2, \varphi_3$ and $\varphi_4$ as column vectors. We write

$$\mathcal{D}^{-1}(\tau) = \mathrm{diag}(e^{-\nu_1(\tau-\tau_1)}, e^{-\nu_2(\tau-\tau_1)}, e^{-\nu_3(\tau-\tau_1)}, e^{-\nu_4(\tau-\tau_1)}),$$

and define $\Delta(\tau) = Y(\tau)\mathcal{D}^{-1}(\tau) - I_4$. Then, using (3.42)

$$\|\Delta(\tau)\| \leq \frac{3\hat{\varepsilon}}{1-3\hat{\varepsilon}} = \varepsilon\frac{3c_1}{1-3c_1\varepsilon},$$

for any $\tau \in [\tau_1, \tau_2]$. The last inequality above comes from the assumption that $\varepsilon < \dfrac{1}{12c_1}$. We remark that matrices $Y(\tau), \mathcal{D}(\tau)$, and then $\Delta(\tau)$, depend on $\delta$. Moreover, we can say that $\|\Delta(\tau)\| < \dfrac{1}{3}$ for any $\tau \in [\tau_1, \tau_2]$. In particular, $\|\Delta(\tau_1)\| < 1$. Then, $Y(\tau_1) = I_4 + \Delta(\tau_1)$ is a non singular matrix and $Y(\tau) = (I_4 + \Delta(\tau))\mathcal{D}(\tau)$ is a fundamental matrix for the system (3.27). Furthermore, $W(\tau)$ is the fundamental matrix of (3.27) such that $W(\tau_1) = I_4$. Then

$$W(\tau) = Y(\tau)Y(\tau_1)^{-1} = (I_4 + \Delta(\tau))\mathcal{D}(\tau)(I_4 + \Delta(\tau_1))^{-1}.$$

Furthermore, using standard results for natural matrix norms (see [I.K.],[Ch.]) if we define

$$R \ := \ I_4 - (I_4 + \Delta(\tau_1))^{-1},$$

we get $\|R\| \leq \dfrac{\hat{\hat{\varepsilon}}}{1-\hat{\hat{\varepsilon}}}$ where $\hat{\hat{\varepsilon}} = \dfrac{3\hat{\varepsilon}}{1-3\hat{\varepsilon}}$. Therefore,

$$\|R\| \ \leq \ \varepsilon\frac{3c_1}{1-6c_1\varepsilon} < 6c_1\varepsilon.$$

## 3.5 Proof of lemma 3.3.5

In this section we prove lemma 3.3.5. To this end we need some information about the matrices involved in $\mathcal{M}$. We recall that

$$\mathcal{M} \;=\; A_1\mathcal{D}(\tau_2)A_2\mathcal{D}(\tau_2)A_3,$$

where $A_1, A_2, A_3$ do not depend on $\delta$.

We begin with $A_3$. We recall that $A_3 = P^{-1}Z_1(\tau_{L_1})$ where $P$ is the constant matrix defined in (3.26) and $Z_1(\tau_{L_1})$ the fundamental matrix of the system (3.23), that is,

$$\mathbf{u}' \;=\; B_{L_1}(\tau)\mathbf{u}, \tag{3.43}$$

with $Z_1(0) = I_4$, where

$$B_{L_1}(\tau) \;=\; \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 & 0 & -\frac{Q_{L_1}(\tau)}{2} & 0 \\ 0 & \lambda_2 & 0 & -\frac{Q_{L_1}(\tau)}{2} \end{pmatrix}, \tag{3.44}$$

and $P = \begin{pmatrix} I_2 & I_2 \\ P_3 & P_4 \end{pmatrix}$, $P_3 = \mathrm{diag}(\rho_1^+, \rho_2^+)$, $P_4 = \mathrm{diag}(\rho_1^-, \rho_2^-)$.

Then, the system (3.43) splits in two uncoupled systems, one for $u_1, u_3$ and the second for $u_2, u_4$. Let us denote by $\mathbf{e}_j$, $j = 1, \ldots, 4$, the canonical basis of $\mathbb{R}^4$. If we take as initial condition $\mathbf{e}_1, \mathbf{e}_3$ ($\mathbf{e}_2, \mathbf{e}_4$) then $u_2(\tau) \equiv u_4(\tau) \equiv 0$ ($u_1(\tau) \equiv u_3(\tau) \equiv 0$). Then, $Z_1(\tau_{L_1})$ is a $4 \times 4$ block diagonal matrix, that is,

$$Z_1(\tau_{L_1}) \;=\; \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \quad \text{with} \quad C_j = \mathrm{diag}(c_{j1}, c_{j2}), j = 1, \ldots, 4. \tag{3.45}$$

In order to compute $A_3$ we need to know $P^{-1}$. A simple check shows that $P^{-1} = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}$, where

$$Q_1 = \mathrm{diag}\left(\frac{\rho_1^-}{\rho_1^- - \rho_1^+}, \frac{\rho_2^-}{\rho_2^- - \rho_2^+}\right), \qquad Q_2 = \mathrm{diag}\left(\frac{1}{\rho_1^+ - \rho_1^-}, \frac{1}{\rho_2^+ - \rho_2^-}\right),$$
$$Q_3 = \mathrm{diag}\left(\frac{\rho_1^+}{\rho_1^+ - \rho_1^-}, \frac{\rho_2^+}{\rho_2^+ - \rho_2^-}\right), \qquad Q_4 = \mathrm{diag}\left(\frac{1}{\rho_1^- - \rho_1^+}, \frac{1}{\rho_2^- - \rho_2^+}\right),$$

and $\rho_1^\pm$, $\rho_2^\pm$ are given in (3.25) Then,

$$A_3 \;=\; \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix} \quad \text{with} \quad E_j = \mathrm{diag}(e_{j1}, e_{j2}), j = 1, \ldots, 4. \tag{3.46}$$

Moreover,

$$E_1 = Q_1C_1 + Q_2C_3, \quad E_2 = Q_1C_2 + Q_2C_4,$$
$$E_3 = Q_3C_1 + Q_4C_3, \quad E_4 = Q_3C_2 + Q_4C_4. \tag{3.47}$$

As $A_1 = G_{00}A_3^T$ where $G_{00} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$, $C = \mathrm{diag}(-1, 1)$, we get that

$$A_1 \quad = \quad \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} \quad \text{with} \quad H_j = \mathrm{diag}(h_{j1}, h_{j2}), j = 1, \ldots, 4. \tag{3.48}$$

Moreover, $H_1 = CE_2$, $H_2 = CE_4$, $H_3 = CE_1$, $H_4 = CE_3$. Therefore, we get the following relations between $h_{ji}$ and $e_{ji}$,

$$h_{11} = -e_{21}, \quad h_{12} = e_{22}, \quad h_{21} = -e_{41}, \quad h_{22} = e_{42},$$
$$h_{31} = -e_{11}, \quad h_{32} = e_{12}, \quad h_{41} = -e_{31}, \quad h_{42} = e_{32}. \tag{3.49}$$

From (3.29) we know that $A_2 = P^T Z_2(0)^T G_M Z_2(0) P$, being $Z_2(\tau)$ the fundamental matrix of the system (3.24), that is $\mathbf{u}' = B_{L_2}(\tau)\mathbf{u}$, such that $Z_2(-\tau_{L_2}) = I_4$. We denote

$$A_2 \quad = \quad \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}, \tag{3.50}$$

for some $2 \times 2$ matrices $X_i$, $i = 1, \ldots, 4$. We write also $A_2 = (x_{ij})$.

We note that using remark (3.3.3), the matrices $A_1$ and $A_3$ depend on $\lambda_1$, $\lambda_2$, $\alpha$ and $\gamma$ but they do not depend on function $V_1$. However, $A_2$ depends on $V_1$.

We recall that the matrix $\mathcal{D}(\tau)$ is defined in lemma 3.3.4. We can write $\mathcal{D}(\tau_2) = \mathrm{diag}(D_1, D_1^{-1})$, where $D_1 = \mathrm{diag}(\sigma^{\beta_1}, \sigma^{\beta_2})$ and $\sigma = \exp\left(\frac{Q_p}{4}(\tau_2 - \tau_1)\right)$. It is not difficult to check that $\mathcal{M} = \tilde{D}_1^{-1}\tilde{\mathcal{M}}\tilde{D}_1$ where $\tilde{D}_1 = \mathrm{diag}(D_1, D_1)$ and

$$\tilde{\mathcal{M}} \quad = \quad \tilde{D}_1^2\tilde{\mathcal{M}}_1 + \tilde{\mathcal{M}}_2 + \tilde{D}_1^2\tilde{\mathcal{M}}_3(\tilde{D}_1^{-1})^2 + \tilde{\mathcal{M}}_2(\tilde{D}_1^{-1})^2, \tag{3.51}$$

$$\tilde{\mathcal{M}}_1 = \begin{pmatrix} H_1X_1E_1 & H_1X_1E_2 \\ H_3X_1E_1 & H_3X_1E_2 \end{pmatrix}, \quad \tilde{\mathcal{M}}_2 = \begin{pmatrix} H_2X_3E_1 & H_2X_3E_2 \\ H_4X_3E_1 & H_4X_3E_2 \end{pmatrix},$$
$$\tilde{\mathcal{M}}_3 = \begin{pmatrix} H_1X_2E_3 & H_1X_2E_4 \\ H_3X_2E_3 & H_3X_2E_4 \end{pmatrix}, \quad \tilde{\mathcal{M}}_4 = \begin{pmatrix} H_2X_4E_3 & H_2X_4E_4 \\ H_4X_4E_3 & H_4X_4E_4 \end{pmatrix}. \tag{3.52}$$

We note that $\tilde{\mathcal{M}}_j$, $j = 1, \ldots, 4$, only depend on $A_1, A_2$ and $A_3$ and so, they do not depend on $\delta$.

So, we can reduce to consider the characteristic polynomial of $\tilde{\mathcal{M}}$. For the elements of these matrices we shall use the following notation $\tilde{\mathcal{M}}_1 = (u_{ij})$, $\tilde{\mathcal{M}}_2 = (v_{ij})$, $\tilde{\mathcal{M}}_3 = (p_{ij})$, $\tilde{\mathcal{M}}_4 = (w_{ij})$.

From (3.51) we get

$$\begin{aligned}\mathrm{trace}(\tilde{\mathcal{M}}) = \sigma^{2\beta_1}(u_{11}+u_{33}) + \sigma^{2\beta_2}(u_{22}+u_{44}) + \sigma^{-2\beta_1}(w_{11}+w_{33}) + \\ \sigma^{-2\beta_2}(w_{22}+w_{44}) + \mathrm{trace}(\tilde{\mathcal{M}}_2) + \mathrm{trace}(\tilde{\mathcal{M}}_3),\end{aligned}$$

where $\mathrm{trace}(\tilde{\mathcal{M}}_2)$ and $\mathrm{trace}(\tilde{\mathcal{M}}_3)$ do not depend on $\sigma$.

Taking into account that $b_3 = -\mathrm{trace}(\tilde{\mathcal{M}})$, we obtain (3.33) by taking $d_1 = u_{11}+u_{33}$, $d_2 = u_{22}+u_{44}$, $d_3 = w_{11}+w_{33}$, $d_4 = w_{22}+w_{44}$ and $d_5 = \mathrm{trace}(\tilde{\mathcal{M}}_2) + \mathrm{trace}(\tilde{\mathcal{M}}_3)$.

From (3.52) and (3.49) we get

$$u_{11} = h_{11}x_{11}e_{11}, \quad u_{33} = h_{31}x_{11}e_{21}, \quad u_{13} = h_{11}x_{11}e_{21}, \quad u_{31} = h_{31}x_{11}e_{11} \tag{3.53}$$

$d_1 = h_{11}x_{11}e_{11} + h_{31}x_{11}e_{21} = -2x_{11}e_{11}e_{21}$. Then, if we denote by

$$d_n = -2e_{11}e_{21} \quad \text{and} \quad d_g = x_{11},$$

$d_n$ depends on $\lambda_1, \lambda_2, \alpha$ and $\gamma$ but not on $V_1$ and $d_g$ depends also on $V_1$.

Now we study the dominant terms in $b_2$.

Given a $4 \times 4$ matrix $A = (a_{ij})$, the coefficient of $x^2$ in the characteristic polynomial of $A$ is

$$\begin{aligned}a_{11}(a_{22}+a_{33}+a_{44}) + a_{22}(a_{33}+a_{44}) + a_{33}a_{44} - a_{12}a_{21} - a_{13}a_{31} - a_{14}a_{41} - \\ a_{23}a_{32} - a_{34}a_{43} - a_{24}a_{42}.\end{aligned}$$

In our case, the terms on $\tilde{\mathcal{M}}$ are of the form $k_1\sigma^{2\beta_i} + k_2\sigma^{-2\beta_i} + k_3\sigma^{2(\beta_i-\beta_j)} + k_4$ for $i \neq j$, $i,j \in \{1,2\}$, and some constants $k_m$, $m = 1,\dots,4$. By taking into account this fact, it is easy to see that the dominant terms in $b_2$ is $(u_{11}u_{33} - u_{13}u_{31})\sigma^{4\beta_1}$. However using (3.53) one has $u_{11}u_{33} - u_{13}u_{31} = 0$.

An analogous computation shows that the coefficient of $\sigma^{4\beta_2}$ in $b_2$ is $u_{22}u_{44} - u_{24}u_{42}$, which is equal to zero. Then, $b_2$ does not contain terms in $\sigma^{4\beta_1}$ nor $\sigma^{4\beta_2}$.

Then, $b_2$ is written as (3.34) with

$$e_3 \;=\; (u_{11}+u_{33})(u_{22}+u_{44}) - u_{12}u_{21} - u_{14}u_{41} - u_{23}u_{32} - u_{34}u_{43}.$$

It is easy to check that $e_3 = -4e_{11}e_{12}e_{21}e_{22}\det(X_1)$. If we denote by

$$e_n = -4e_{11}e_{12}e_{21}e_{22} \quad \text{and by} \quad e_g = \det(X_1),$$

we have that $e_n$ depends on $\lambda_1, \lambda_2, \alpha$ and $\gamma$, but not on $V_1$, and $e_g$ depends also on $V_1$.

Now we assume that $\lambda_2 < \gamma\dfrac{(2-\alpha)^2}{8}$. In this case $\beta_2$ is pure imaginary. Due to the real character of $\mathcal{M}$ its characteristic polynomial is real and therefore

$$
\begin{aligned}
d_2\sigma^{2\beta_2} + d_4\sigma^{-2\beta_2} \in \mathbb{R} \Rightarrow d_4 = \overline{d}_2 \quad ,\\
e_3\sigma^{2\beta_2} + e_4\sigma^{-2\beta_2} \in \mathbb{R} \Rightarrow e_4 = \overline{e}_3.
\end{aligned}
$$

To finish the proof of lemma 3.3.5 we only need to prove that $d_n, e_n \neq 0$. To do that, we shall see that $e_{11}, e_{12}, e_{21}, e_{22} \neq 0$.

From (3.47) and (3.45), a simple computation shows that

$$
\begin{aligned}
e_{11} &= \frac{1}{\rho_1^+ - \rho_1^-}(c_{31} - \rho_1^- c_{11}), \qquad & e_{21} &= \frac{1}{\rho_1^+ - \rho_1^-}(c_{41} - \rho_1^- c_{21}),\\
e_{12} &= \frac{1}{\rho_2^+ - \rho_2^-}(c_{32} - \rho_2^- c_{12}), \qquad & e_{22} &= \frac{1}{\rho_2^+ - \rho_2^-}(c_{42} - \rho_2^- c_{22}).
\end{aligned}
\tag{3.54}
$$

System (3.43) can be written as two uncoupled systems of the following type

$$
\begin{aligned}
v_1' &= v_2,\\
v_2' &= \lambda v_1 - \frac{Q_{L_1}(\tau)}{2} v_2,
\end{aligned}
\tag{3.55}
$$

where $\lambda = \lambda_1$ for the system corresponding to $u_1, u_3$ and $\lambda = \lambda_2$ for $u_2, u_4$.

We note that $\begin{pmatrix} c_{11} & c_{21} \\ c_{31} & c_{41} \end{pmatrix}$ and $\begin{pmatrix} c_{12} & c_{22} \\ c_{32} & c_{42} \end{pmatrix}$ are the fundamental matrices of (3.55) for $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively.

Therefore any $e_{ij}$ in (3.54) can be written in terms of $v_2(\tau_{L_1}) - \rho^- v_1(\tau_{L_1}))$ being $(v_1(\tau), v_2(\tau))$ a solution of (3.55) and

$$
\rho^- = \frac{Q_p}{4}(1-\beta), \qquad \text{with} \qquad \beta = \sqrt{1 - \frac{8\lambda}{\gamma(2-\alpha)^2}}.
\tag{3.56}
$$

We note that if $\lambda = 0$, $\rho^- = 0$ and (3.55) has solutions with $v_2(\tau) \equiv 0$. Then, if $\lambda_1 = 0$, $c_{31} = 0$ and $e_{11} = 0$. In a similar way, if $\lambda_2 = 0$, $c_{32} = 0$ and $e_{12} = 0$.

Next lemma applied to our systems will be very useful in order to finish the proof of lemma 3.3.5.

**Lemma 3.5.1.** *Assume $\lambda \neq 0$. Let $\mathbf{v}(\tau) = (v_1(\tau), v_2(\tau))^T$ be one of the solutions of (3.55) with initial conditions $\mathbf{v}(0) = (1,0)^T$ or $\mathbf{v}(0) = (0,1)^T$. Let be $\rho^-$ defined in (3.56). Then, for any $\tau > 0$ sufficiently large*

$$
v_2(\tau) - \rho^- v_1(\tau) \neq 0.
\tag{3.57}
$$

**Proof**

Assume that $\lambda < \gamma(2-\alpha)^2/8 < 0$. In this case $\rho^-$ is a complex number and as we consider real solutions of the real system (3.55) then the result of the lemma follows trivially.

We assume $\lambda > \dfrac{\gamma(2-\alpha)^2}{8}$. We introduce polar coordinates in (3.55) as $v_1 = r\cos\phi$, $v_2 = r\sin\phi$. Then

$$\begin{aligned} r' &= r\left[(1+\lambda)\cos\phi\sin\phi - \frac{Q_{L_1}(\tau)}{2}\sin^2\phi\right], \\ \phi' &= \lambda\cos^2\phi - \sin^2\phi - \frac{Q_{L_1}(\tau)}{2}\sin\phi\cos\phi. \end{aligned} \tag{3.58}$$

We are interested in the solutions of (3.55) with $\mathbf{v}(0) = (1,0)^T$ and $\mathbf{v}(0) = (0,1)^T$, that is, the solutions of (3.58) with $r(0) = 1$ and, $\phi(0) = 0$ and $\phi(0) = \dfrac{\pi}{2}$, respectively. Moreover we have $v_2(\tau) - \rho^- v_1(\tau) = r(\tau)[\sin\phi(\tau) - \rho^-\cos\phi(\tau)]$. We note that $r = 0$ is invariant for (3.58). Then, we have to prove that $\sin\phi(\tau) - \rho^-\cos\phi(\tau) \neq 0$, or equivalently, that

$$\tan\phi(\tau) \neq \rho^-, \tag{3.59}$$

for any $\tau > 0$ sufficiently large. To do that we only need to consider the non autonomous equation for $\phi$ in (3.58). If we consider time as a variable we are faced to a system defined on a cilinder. For convenience, we shall consider a 2–dimensional system on the plane taking into account the corresponding identifications.

So, we define new variables $u = \tanh\left(\dfrac{\alpha}{2(2-\alpha)}Q_p\tau\right)$ and $w = \tan(\phi(\tau))$. Condition (3.59) reduces to $w(\tau) \neq \rho^-$ for $\tau$ sufficiently large.

We get for $w$ and $u$ the following planar system

$$\begin{aligned} w' &= -w^2 + \frac{Q_p}{2}uw + \lambda, \\ u' &= \frac{\alpha}{2(2-\alpha)}Q_p(1-u^2), \end{aligned} \tag{3.60}$$

which is well defined for any $u, w$. However, for us, it only makes sense for $|u| \leq 1$. It is also clear that $u(\tau)$ is an increasing function for $|u| < 1$. Moreover, in order to recover the solutions of (3.58) from (3.60) we must identify the solutions of (3.60) with $w(\tau) \to -\infty$ for $\tau \to \tau_*^-$ with the corresponding ones with $w(\tau) \to +\infty$ for $\tau \to \tau_*^+$.

If $\lambda > \gamma(2-\alpha)^2/8$ the system (3.60) has two equilibrium points on the line $u = 1$ located at $(w,u) = (\rho^-, 1)$ and $(w,u) = (\rho^+, 1)$ respectively, where

$$\rho^{\pm} = \frac{Q_p}{4}(1 \pm \beta).$$

We study the stability of these equilibrium points. The Jacobian of the planar vector field that defines (3.60) is

$$\begin{pmatrix} -2w + \frac{Q_p}{2}u & \frac{Q_p}{2}w \\ 0 & -\frac{\alpha}{2-\alpha}Q_p u \end{pmatrix} . \tag{3.61}$$

Then, for $(\rho^-, 1)$ matrix (3.61) has eigenvalues $-\dfrac{\alpha}{2-\alpha}Q_p < 0$ and $\dfrac{Q_p}{2}\beta > 0$, and then $(\rho^-, 1)$ is a saddle point. The eigenvalues of (3.61) on $(\rho^+, 1)$ are $-\dfrac{\alpha}{2-\alpha}Q_p < 0$ and $-\dfrac{Q_p}{2}\beta < 0$. Therefore $\rho^+$ is an attractor. Moreover, (3.60) has a vertical isocline defined by $-w^2 + \dfrac{Q_p}{2}uw + \lambda = 0$.

First we assume that $\lambda > 0$. In this case, $\rho^- < 0$ and $\rho^+ > 0$. The region $\mathcal{R}_1 = \{(w,u)\,|\,w \geq 0, 0 \leq u \leq 1\}$ is positively invariant for the flow defined by (3.60). In this region all the orbits tend to the attractor. Then, the orbits we are interested in are contained for all positive time in $\mathcal{R}_1$. As $\rho^- \notin \mathcal{R}_1$, $w(\tau) \neq \rho^-$ for $\tau > 0$. (see Figure 3.5).

Now we consider values of $\lambda$ such that $\gamma(2-\alpha)^2/8 < \lambda < 0$. Then $0 < \rho^- < \rho^+$. Let $W^s$ be the branch of the unstable invariant manifold of the point $(\rho^-, 1)$ contained in the band $\{(w,u)\,|\,|u| \leq 1\}$, and $\mathcal{R}_2 \subset \{(w,u)\,|\,w \geq 0, \quad 0 \leq u \leq 1\}$ the unbounded region with boundaries $W^s$ and $\{(w,u)\,|\,w \geq \rho^-, u = 1\}$. Then $\mathcal{R}_2$ is positively invariant and all the orbits in $\mathcal{R}_2$ tend to $(\rho^+, 1)$ when $\tau \to \infty$. As the orbits we are interested in enter in $\mathcal{R}_2$ for some $\tau$ large enough, then $w(\tau) \neq \rho^-$ if $\tau$ is sufficiently large (see Figure 3.6).

$\square$

In order to apply this lemma to our case, first we take $\mathbf{v}(\tau) = (c_{11}(\tau), c_{31}(\tau))^T$ and $\lambda = \lambda_1$. We know that $\mathbf{v}(0) = (1,0)^T$. Then,

$$c_{31}(\tau) - \rho^- c_{11}(\tau) \quad \neq \quad 0,$$

for $\tau$ sufficiently large. We take $\delta$ small enough in order to obtain the relation above. Then, $e_{11} \neq 0$.

Analogously one can see that $e_{12}, e_{21}, e_{22} \neq 0$.

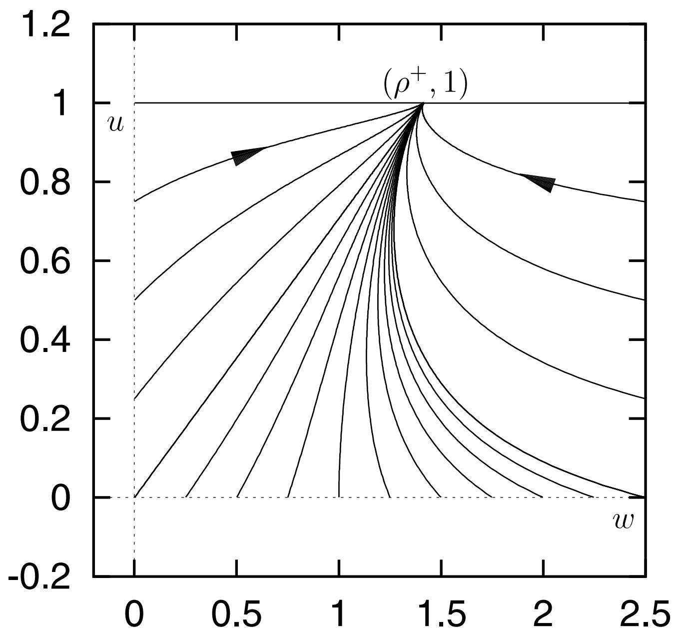



Figure 3.5: Phase portrait of system (3.60) for $\gamma = -1$ and $\lambda = 1$ on the region $\mathcal{R}_1$.

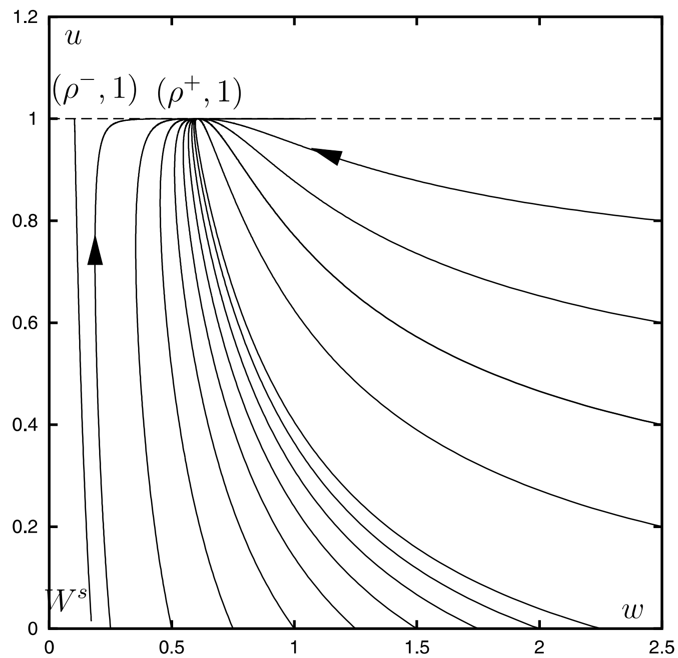



Figure 3.6: Phase portrait of system (3.60) for $\gamma = -1$ and $\lambda = -\frac{1}{16}$ on the region $\mathcal{R}_2$.