On the Algebraic Limit Cycles of Quadratic Systems

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CERTIFICO que aquesta memòria ha estat realitzada per Jordi Sorolla Bardají sota la meva direcció.

Dr. Javier Chavarriga Soriano

To my parents and Carmen

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Preface

The resolution of differential equations began, in certain sense, as soon as the relation among the procedures of differentiation and integration was known in the XVII century by Newton and Leibnitz. Before Newton, the infinity was only thought in potential but he establishes its actualization writing functions as series expansions. In *Philosophiæ naturalis principia mathematica* (1687), he associated the infinite series with the change rate of two magnitudes flowing continuously in a geometric way. Ten years after Newton's discoveries, Leibnitz obtains the same results but using differential notation.

The procedures of differentiation were known as *calculus differentialis* and the integration ones as *calculus summatorius* and later *calculus integralis* and were used to compute the slope of a curve as a quotient of infinitesimal *differences* and the area under the curve as a *summation* of rectangles with infinitesimal base, respectively.

After Newton and Leibnitz, important mathematicians like Jacques Bernoulli and John Bernoulli followed the previous works and introduced the first differential equations, properly speaking. Other important differential equations were solved in the following years by Euler, D'Alambert, Clairaut, Riccati, Legendre, and so on.

Differential equations from a geometric and qualitative point of view were first studied by Poincaré centering his attention to the trajectories of a mobile point. In *"Sur les courbes définies par les équations différentielles"* (1886), Poincaré studies the differential equations, singular points and also closed trajectories:

> *"... au sujet des trajectoires qui s'approchent assez près d'une trajectoire fermée, une théorie tout à fait analogue à celle que nous avons faite pour les trajectoires qui s'approchent assez près d'un point singulier; de sorte que ces courbes fermées jouent dans une certaine mesure le même rôle que les points singuliers."*

The future idea and definition of a limit cycle was born. As he says, in certain cases

> *"... le point mobile se rapprocherait asymptotiquement de la trajectoire fermée."*

Poincaré's work is so important that other mathematicians have been unknown, like Darboux, who in his *"Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré"* (1878) studied algebraic solutions of differential equations in the complex projective plane. In the last years, his work has been studied again and has taken importance. For example, the mathematicians have kept in mind the way he considered projective differential equations and the relationship he established between the existence of algebraic solutions, and the existence of first integrals (*integrale generale*) and how to construct them:

> *"Si l'on a $m(m+1)/2 - q = p$ solutions particulières representant des courbes ne passant pas par q des points singuliers, $u_1, \ldots, u_p$ dèsignant ces solutions, l'inteégrale générale sera de la forme*
>
> $$u_1^{\alpha_1} u_2^{\alpha_2} \cdots u_p^{\alpha_p} = C."$$

With the XIX century disappear those great mathematicians who was able to concentrate all mathematic knowledge and its applications. In the next generation, a tendency to the specialization is manifested. This fact made Cantor the driver of the necessity of an *International Congress of Mathematicians* (ICM): The first was celebrated in Zurich (1897). The aim of these congresses was to establish meeting points for the communication and the group discussion. In the second ICM, celebrated in Paris (1900), Hilbert proposed a list of twenty-three problems to be solved along the XX century. One of the most difficult problems suggested by Hilbert is related with Poincaré limit cycles:

> *"Im Anshuß... die Frage nach der Maximalzahl und Lage der Poincaréschen Grenzzyklen für eine Differentialgleichung erster Ordnung und erster Grades von der Form:*
>
> $$\frac{dy}{dx} = \frac{Y}{X}$$
>
> *wo X, Y ganze rationale Funktionen n-ten Grades x, y sind."*

This problem is known in the literature as 16^{th} *Hilbert Problem* and involves two subproblems: relative position and number of limit cycles. Smale, in *Mathematical problems for the next century* (1998), reformulates the second part of Hilbert problem as follows:

> *Consider the differential equation in $\mathbb{R}^2$*
>
> $$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dy} = Q(x, y),$$
>
> *where P and Q are polynomials. Is there a bound k on the number of limit cycles of the form $K \leq d^q$ where d is the maximum of the degrees of P and Q, and q is a universal constant?*

It is well known that there are not limit cycles for linear systems but Hilbert Problem is unsolved even for *quadratic systems*. This fact, brings to a systematic study of quadratic systems which can have limit cycles and there can be found a *Russian* *classification* and a *Chinese classification*. According to the last one, quadratic systems that can have limit cycles are classified in the following three families

$$\dot{x} = \delta x - y + \ell x^2 + mxy + ny^2 \ , \quad \dot{y} = x(1 + ax + by) \ ,$$

according to: family (I) if $a = b = 0$; family (II) if $a \neq 0$ and $b = 0$; family (III) if $b \neq 0$.

The difficulty of such problem made necessary a weakness of the hypothesis and the most considered limit cycles are those that are included in algebraic curves. For this reason, Darboux theory emerges. Of course, the study of the degree of *algebraic limit cycles* is directly related to the study of the degree of *invariant algebraic curves*. To find an upper bound of the last one is known as *Poincaré Problem*.

This work is essentially dedicated to the existence, and therefore non-existence, of algebraic limit cycles. Thus, in the first chapter are given the definitions and some preliminary results that we need along the work. There are also some new results and its proof. In the following chapter we consider invariant algebraic curves of degree 4, and we study when these curves have an oval which is a limit cycle of a quadratic system. We find all the algebraic limit cycles of degree 4 for quadratic systems; the results of this chapter belong to

J. Chavarriga, J. Llibre and J. Sorolla, *Algebraic limit cycles for quadratic systems*, J. of Differential Equations **200** (2004), 206-244.

In the next two chapters, we study the existence of algebraic limit cycles from a different point of view: our starting object is a given quadratic system. Thus, we consider first the systems in the three families of the Chinese classification and using non-algebraic techniques we obtain results on existence which belong to

I.A. García, J. Giné and J. Sorolla, *On the existence of polynomial inverse integrating factors in quadratic systems with limit cycles*, to appear in Dynamics of Continuous, Discrete and Impulsive Systems.

Finally, we conclude algebraically that there are not algebraic limit cycles in the first family; as is also showed in

J. Chavarriga, I.A. García and J. Sorolla, *Resolution of the Poincaré Problem and Nonexistence of Algebraic Limit Cycles in Family (I) of Chinese Classification*, Chaos, Solitons & Fractals **24**, 2 (2005), 491–499.

In the last chapter, we prove among other facts that if two algebraic limit cycles belonging to different invariant algebraic curves coexist in the phase portrait, then they must be nested. The main body of this chapter belongs to

J. Chavarriga, I.A. García and J. Sorolla, *Non-nested configuration of algebraic limit cycles in quadratic systems*, Submitted to J. of Differential Equations.

Some other results which are not included in this memory can be found in the paper

J. Chavarriga, J. Giné and J. Sorolla, *Analytical integrability of a class of nilpotent cubic systems*, To appear in Mathematics and computers in simulation.

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I want to thank also Carmen and my family, as always *conditio sine qua non.*

Chapter 1

Introduction to planar differential systems

In this chapter we give a brief summary on differential equations and algebraic curves. We make a description of the most important questions related to algebraic curves in $\mathbb{C}P^2$, such as multiple points, intersection index, genus and so on. We also introduce formal differential equations, formal solutions and how to involve them in the study of differential equations.

1.1 Planar differential systems and solution curves

We consider planar polynomial differential systems of the form

$$\dot{x} = P(x,y) = \sum_{i=0}^{m} P_i(x,y) \ , \quad \dot{y} = Q(x,y) = \sum_{i=0}^{m} Q_i(x,y) \ , \tag{1.1}$$

where the dot stands for the derivative with respect to the independent variable t. Here P, $Q \in \mathbb{R}[x,y]$ are coprime polynomials such that $\max\{\deg P, \deg Q\} = m$, and P_i and Q_i are the homogeneous components of degree i. As usual, $\mathbb{R}[x,y]$ denotes the ring of the real polynomials in two variables.

In the literature equivalent mathematical objects to refer to this planar differential systems appear: as a *vector field*

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y},$$

as a *differential form*

$$\omega = Qdx - Pdy,$$

and also, some authors use a more geometric notation and they think a planar differential systems as a *foliation* $\mathcal{F}$ of *codimension* 1, because a phase portrait of a planar differential system consists on the plane formed (foliated) by 1-dimensional differential varieties.

Definition 1.1. A *flow* in $\mathbb{C}^2$ along a time $t \in \mathbb{R}$ is defined as

$$\begin{array}{rccc} \phi: & \mathbb{R} \times \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ & (t, \Omega) & \longmapsto & \phi^t(\Omega) \end{array}$$

such that

(i) $\phi^0(\Omega) = \Omega$,

(ii) $\phi^t\left(\phi^s(\Omega)\right) = \phi^{t+s}(\Omega)$,

for all Ω in $\mathbb{C}^2$ and t, s in $\mathbb{R}$.

System (1.1) defines a flow in $\mathbb{C}^2$, $\phi(x, y)$. It is known that this flow is a smooth function defined for all (x, y) in some neighborhood of the initial position and initial time. Also, it satisfies (1.1) in the sense that

$$\frac{d}{dt}\Big(\phi^t(x, y)\Big)_{t=\tau} = \mathcal{X}\Big(\phi^\tau(x, y)\Big).$$

Definition 1.2. A *solution* of (1.1) through a point $(x_0, y_0) \in \mathbb{C}^2$ is defined as $(x(y), y(t)) = \{\phi^t(x_0, y_0), t \in \mathbb{R}\}$.

The set of all the solutions is called *phase portrait.*

To found the solution of (**??**), when an initial condition $y(x_0) = y_0$ is given, is known as *Cauchy Problem.* The following theorem is well known:

Theorem 1.3. (Existence and uniqueness) *Consider equation $dy/dx = F(x, y)$ with initial condition $y(x_0) = y_0$. Then,*

(i) *If F is a continuous function in a neighborhood of (x_0, y_0) there exist a solution $y(x)$ of (**??**) through this point that is defined in $(x_0 - \Delta, x_0 + \Delta)$ for some $\Delta > 0$.*

(ii) *If F is also Lipschitz with respect to the second variable, then the solution is unique.*

Definition 1.4. The $\alpha - limit$ set (resp. $\omega - limit$) of a point (x_0, y_0) through ϕ^t is defined as $\alpha_\infty = \{(x, y) \in \mathbb{C}^2 | \phi^{t_n}(x_0, y_0) \to (x, y)$ for some $t_n \to -\infty\}$ (resp. $\omega_\infty = \{(x, y) \in \mathbb{C}^2 | \phi^{t_n}(x_0, y_0) \to (x, y)$ for some $t_n \to \infty\}$).

For flows in the plane, Bendixon-Poincaré Theorem is an important result from a topological point of view, that can not be generalized to higher dimensions. According to this theorem, there are three types of limit sets: singular points, closed periodic orbits, and the union of singular points and trajectories connecting them. The second ones are limit cycles, and the latter ones are referred as *heteroclinic orbits* when they connect distinct points and *homoclinic orbits* when they connect a point to itself. The next subsections are dedicated to the those objects which are invariant for the flow of a differential system; we are specially interested in its relation with limit cycles. Singular points are defined in the next section.

1.1.1 Singular points

Definition 1.5. A *singular point* or (*critical point*) *for system (1.1) is a point* (x_0, y_0) *such that* $P(x_0, y_0) = Q(x_0, y_0) = 0$.

A singular point is a particular case of solution, where $\phi^t(x_0, y_0) = (x_0, y_0)$ for all $t \in \mathbb{R}$.

We will denote by $D\mathcal{X}$ the jacobian matrix associated to vector field $\mathcal{X}$. The flow of (1.1) in a neighborhood of the singular point (x_0, y_0) is classified according to the eigenvalues of the matrix $D\mathcal{X}(x_0, y_0)$. Observe that, as system (1.1) is real, if (x_0, y_0) is a complex singular point, then its conjugated $(\bar{x}_0, \bar{y}_0)$ is also a singular point. Moreover, if (x_0, y_0) is a real singular point of a real system with non-real associated eigenvalues λ and μ, then $\mu = \bar{\lambda}$.

Definition 1.6. Let $p = (x_0, y_0)$ be a singular point of system (1.1). Let λ and μ be the eigenvalues of $D\mathcal{X}(p)$.

(i) If $\lambda = \mu = 0$, then p is called *degenerate*. If moreover $D\mathcal{X}(p) \not\equiv 0$, we say that p is a *nilpotent* point.

(ii) If $\lambda\mu = 0$ but $\lambda^2 + \mu^2 \neq 0$, then p is called *elementary degenerate*.

(iii) Otherwise, p is termed *non-degenerate*.

(a) When $D\mathcal{X}(p)$ can be diagonalized, p is
- of *focus type* ($\lambda = \bar{\mu} \in \mathbb{C} \setminus \mathbb{R}$), a *saddle* ($\lambda\mu < 0$ for $\lambda, \mu \in \mathbb{R}$) or a *node* ($\lambda\mu > 0$ for $\lambda, \mu \in \mathbb{R}$), if p is a real point.
- a *resonant node* ($\lambda/\mu \in \mathbb{Q}$) or a *non-resonant node* ($\lambda/\mu \notin \mathbb{Q}$), if p is a complex point.

(b) When $D\mathcal{X}(p)$ can not be diagonalized, p is a *logarithmic* singular point.

In fact, the definition of a center was first given by Poincaré [42]:

Definition 1.7. A singular point O of (1.1) is a *center* if it possesses a neighborhood $\mathcal{U}$ such that for all $p \in \mathcal{U} \setminus \{O\}$ verifies $P^2(p) + Q^2(p) \neq 0$, and the solution passing through p is closed, surrounding O.

1.1.2 Invariant curves

Once the flow is defined, take sense the fact that every set in the phase portrait is transformed into another along time. The more interesting sets to understand differential equations are those which are transformed into itself for all time.

Definition 1.8. A set Ω is said to be *invariant* for (1.1) if $\phi^t(\Omega) \subseteq \Omega$ for all $t \in \mathbb{R}$, where ϕ is the flow defined by (1.1).

Obviously, when the invariant set is a curve we talk about *invariant curves*. Since the solutions of planar differential equations are points or 1-dimensional components, the invariant curves play a very important role in the study of them. Every singular point and solution of a differential equation are invariant for the flow but the reciprocal is not true. An invariant curve may not be a solution of a differential equation but it is formed by solutions.

The tangents to the trajectories of a planar polynomial differential system are defined almost everywhere. So, if $f(x,y)=0$ is the equation of an invariant curve, its tangent must coincide with the tangents of the trajectories. In other words, the gradient to f, $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ and (P,Q) must be orthogonal over the curve $f=0$, that is,

$$\dot{f} = \left(P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} \right)_{f=0} = 0. \tag{1.2}$$

An invariant curve $f(x,y)=0$ is said to be *algebraic* and of degree n when $f(x,y)$ is a polynomial of degree n. Said this,

Definition 1.9. A curve $f(x,y)=0$ of degree n is an *invariant algebraic curve* if there exists a polynomial $k(x,y)$ of degree at most $m-1$ called *cofactor* such that

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = kf. \tag{1.3}$$

In fact, this last definition is a consequence of (1.2) when $f=0$ is algebraic.

1.1.3 Exponential factors

Definition 1.10. A function $F=\exp[g/h]$ where g and h are polynomials is said to be an *exponential factor* if there exists a polynomial $k(x,y)$ of degree at most $m-1$ called *cofactor* such that (1.3) is satisfied for F.

For any exponential factor $F=\exp[g/h]$ It is easy to check that $h=0$ is an invariant algebraic curve.

1.1.4 First integrals

Since the differential system is considered with real coefficients, we consider real first integrals.

Definition 1.11. A function $H(x,y)$ is said to be a *strong first integral* of system (1.1) in an open subset $\mathcal{U}$ of $\mathbb{R}^2$ if $H(x,y)$ is a nonconstant function in $\mathcal{U}$ which is constant on each solution curve $\big(x(t),y(t)\big) \in \mathcal{U}$ of (1.1). We say that $H(x,y)$ is a *weak first integral* in an open subset $\mathcal{U}$ of $\mathbb{R}^2 \setminus \Sigma$, if it is a nonconstant function which is constant over each solution curve in $\mathcal{U}$ of $\mathbb{R}^2 \setminus \Sigma$.

A strong first integral is the classical first integral. Notice that system $\dot{x}=x$, $\dot{y}=y$ in $\mathbb{R}^2$ does not have a strong first integral because it would be constant over all the plane. The function $H=xy/(x^2+y^2)$ is a weak first integral where $\Sigma=\{(0,0)\}$, see [8]. When H exists in $\mathcal{U}$, all the solutions of the differential system in $\mathcal{U}$ are known since every solution is given by $H(x, y) = c$, for some $c \in \mathbb{R}$. Clearly, if $H \in \mathcal{C}^1(\mathcal{U})$ verifies

$$\dot{H} = \frac{\partial H}{\partial x} P + \frac{\partial H}{\partial y} Q \equiv 0.$$

It has been seen that the existence of invariant algebraic curves (real or complex) forces the real integrability of a real differential system (1.1). This theory is due to Darboux [21], who studied differential equations in the projective complex plane. In particular, he studied singular points, algebraic curves and looked for first integrals in the form $H = f_1^{\lambda_1} \cdots f_n^{\lambda_n}$ with $\lambda_i \in \mathbb{C}$ and $f_i = 0$ real or complex invariant algebraic curves.

Some improvements to Darboux's theory are known: Jouanoulou [34] in 1979 studies the existence of rational first integrals. A rational first integral is more useful than a darbouxian one because taking into account it and its inverse, there is a first integral defined in any place of the plane. In particular, the existence of a rational first integral excludes the existence of limit cycles. When a differential system possesses a rational first integral $H = h/g$, then all the invariant curves can be defined by $f_c = 0$ where $f_c = h - cg$ for some constant $c \in \mathbb{R}$, and thus they are algebraic.

Prelle and Singer [45] prove that when a polynomial system possesses an elementary first integral it can be computed using the algebraic invariant curves. Chavarriga, Llibre and Sotomayor [12] introduce *independent points*: (x_h, y_h), $h = 1, \ldots, r$ are independent points with respect to $\mathbb{R}_{m-1}[x, y]$ if the intersection of the hyperplans $\{(a_{ij}) | \sum_{i+j=0}^{m-1} x_h^i y_h^j a_{ij} = 0\}$, $h = 1, \ldots, r$ is a vectorial subset of dimension $\frac{m(m+1)}{2} - r > 0$. Christopher [13] considers exponential factors $F = \exp[g/h]$, which play an important role in the construction of first integrals. First integrals with exponential factors are called *generalized Darboux first integrals.*

We summarize the most important results on first integrals of this theory in the following theorem. We emphasize again the fact that the curves and the exponential factors are in general complex but the first integral is real if the differential system is real.

Theorem 1.12. *Suppose that (1.1) has degree m and possesses*

(a) p invariant algebraic curves $f_i = 0$ with cofactors k_i for $i = 1, \ldots, p$.

(b) q exponential factors $F_j = \exp[g_j/h_j]$ with cofactors L_j for $l = 1, \ldots, q$.

(c) r independent singular points $(x_h, y_h) \in \mathbb{R}^2$ such that $f_i(x_h, y_h) \neq 0$ for $i = 1, \ldots, p$ and $h_j(x_h, y_h) \neq 0$ for $j = 1, \ldots, q$ for any $h = 1, \ldots, r$.

Then,

(i) *If there exists $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i k_i + \sum_{j=1}^{q} \mu_j L_j = 0,$$

the function $H = f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}$ is a first integral of system (1.1).

(ii) *If $p+q+r \geq \frac{m(m+1)}{2}+1$, there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i k_i + \sum_{j=1}^{q} \mu_j L_j = 0,$$

(iii) *If $p+q+r \geq \frac{m(m+1)}{2}+2$, then (1.1) has a rational first integral. In this case, a rational first integral can be constructed using $\frac{m(m+1)}{2}+1$ invariant algebraic curves.*

Recently, Llibre and Pereira [36] have introduced different notions of multiplicity for an invariant algebraic curve of a differential system which brings the authors to improve Darboux theory of integrability summarized in the last theorem.

Definition 1.13. A *Liouvillian first integral* is a first integral constructed from a rational function by a finite number of algebraic operations, compositions, exponentials and integrations.

For a more precise definition see [45].

1.1.5 Integrating factors

Definition 1.14. A function $R(x,y)$ is an *integrating factor* of system (1.1) in an open subset $\mathcal{U} \subseteq \mathbb{R}^2$ if $R \in \mathcal{C}^1(\mathcal{U})$, $R \not\equiv 0$ in $\mathcal{U}$ and

$$\frac{\partial (RP)}{\partial x} = -\frac{\partial (RQ)}{\partial y}, \quad div(RP, RQ) = 0, \quad \text{or} \quad \frac{\partial R}{\partial x} + \frac{\partial R}{\partial y} = -R div(P,Q),$$

where as usual the *divergence* of a vector field $\mathcal{X} = (A,B)$ is defined as

$$div(\mathcal{X}) = div(A,B) = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}.$$

The *first integral H associated to the integrating factor R* is given by

$$H(x,y) = \int R(x,y)P(x,y)dy + h(x), \tag{1.4}$$

satisfying $\frac{\partial H}{\partial x} = -RQ$.

When a polynomial differential system has an integrating factor R we can make a time rescaling and the associated 1-form $\omega = RQdx + RPdy$ becomes closed.

Following Darboux theory of integrability and improvements, we summarize the results on integrating factors.

Theorem 1.15. *Suppose that (1.1) has degree m and possesses*

(a) p invariant algebraic curves $f_i = 0$ with cofactors k_i for $i = 1, \ldots, p$.

(b) q exponential factors $F_j = \exp g_j/h_j$ with cofactors L_j for $j = 1, \ldots, q$.

(c) r independent singular points $(x_h y_h) \in \mathbb{R}^2$ such that $f_i(x_h y_h) \neq 0$ for $i = 1, \ldots, p$ and $h_j(x_h y_h) \neq 0$ for $j = 1, \ldots, q$ for any $h = 1, \ldots, r$.

Then,

(i) *If there exit $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i k_i + \sum_{j=1}^{q} \mu_j L_j = -div(P, Q),$$

then the function $R = f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}$ is an integrating factor of (1.1).

(ii) *If $p + q + r \geq \frac{m(m+1)}{2}$ and the independent singular points are weak (that is $div(P, Q)(x_h, y_h) = 0$), then exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i k_i + \sum_{j=1}^{q} \mu_j L_j = -div(P, Q).$$

1.1.6 Inverse integrating factors

Definition 1.16. A function $V(x, y)$ is an *inverse of integrating factor* of system (1.1) in an open subset $\mathcal{U} \subseteq \mathbb{R}^2$ if $V \in \mathcal{C}^1(\mathcal{U})$, $V \not\equiv 0$ in $\mathcal{U}$ and

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \Big(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\Big)V.$$

Clearly, from the definition, $V = 0$ is an invariant curve of system (1.1), not algebraic at first. Moreover, it is easy to check that the function $R = 1/V$ defines an integrating factor in $\mathcal{U} \setminus \{V = 0\}$ of system (1.1).

The following result on closed rational 1-forms is proved in page 205 of [47].

Lemma 1.17. *If ω is a closed complex rational differential 1-form, then there exist polynomials $f_i, f, g \in \mathbb{C}[x, y]$ and constants $\lambda_i \in \mathbb{C}$ for $i = 1, \ldots, m$, such that*

$$\omega = \sum_{i=1}^{m} \lambda_i \frac{df_i}{f_i} + d\left(\frac{g}{f}\right) \ . \tag{1.5}$$

The next corollary, works even for complex polynomial differential systems.

Corollary 1.18. *Assume that a polynomial system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ with $P, Q \in \mathbb{C}[x, y]$ possesses a rational inverse integrating factor V. Then it has a generalized Darboux first integral.*

Proof. We associate to the polynomial differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ the rational 1-form $\omega = A(x, y)dx + B(x, y)dy$ with $A = Q/V$ and $B = -P/V$. Since V is an inverse integrating factor of the system it is clear that ω is closed. Therefore, using Lemma 1.17, we can write ω as in (1.5). Hence, integrating we have that

$$\bar{H} = \sum_{i=1}^{m} \lambda_i \log f_i + \frac{g}{f} \ ,$$

verifies $\partial \bar{H}/\partial x = A$ and $\partial \bar{H}/\partial y = B$, i.e. $\bar{H}$ is a first integral of the system. Finally, taking exponentials, we have that $H = \exp(\bar{H})$ is a generalized Darboux first integral of the form

$$H = \exp\left(\frac{g}{f}\right) \prod_{i=1}^{m} f_i^{\lambda_i} \ ,$$

as stated in the corollary. ∎

We emphasize that the proof of Lemma 1.17 (and therefore the proof of Corollary 1.18) is constructive. Moreover, these same ideas with almost identical proof are used in the main result of Christopher [14]. In fact, Theorem 2 of that paper looks different, but works also for first integrals and its proof can be used to prove our Corollary 1.18.

1.1.7 Limit cycles

Definition 1.19. A *limit cycle* of system (1.1) is an isolated periodic solution in the set of all the periodic solutions.

Definition 1.20. An *algebraic limit cycle* is a limit cycle which is contained in the zeroes set of an invariant algebraic curve.

The existence of limit cycles was first detected by Poincaré [43], but one of the most interesting questions was proposed by Hilbert [32] in 1900 in the part (b) of 16^{th} Hilbert Problem: Compute $H(m)$ such that the number of limit cycles of any polynomial vector field of degree m is less or equal than $H(m)$.

Up to now, the more general result related with 16^{th} Hilbert Problem, due to Dulac [22] and corrected separately by Il'yashenko [33] and Ecalle, Martinet, Moussu and Ramis [23], is the fact that there are finitely many limit cycles for every polynomial vector field of degree m, but an upper bound for $H(2)$ is unknown. On the other hand it is well known that $H(2) \geq 4$, Żołądek [54] showed that $H(3) \geq 11$ perturbing a center, and in general it is proved by Christopher and Lloyd [17] that $H(m) \geq m^2 \log m$. It is known that a quadratic system with an invariant stright line has at most one limit cycle, see Coppel [18], or Coll and Llibre [20].

In Ye Yian-Qian [53] can be found a resum of the most important results on limit cycles but Hilbert Problem remains unsolved even for $m = 2$. So in Smale [49], the author includes Hilbert Problem in the list of unsolved problems.

It is known that the existence of a rational first integral excludes the existence of limit cycles because any region of the plane belongs to the definition domain of the first integral or his inverse. Also, when a rational first integral exists, there is not any focus.

In a paper of Giacomini, Llibre and Viano [31] a method has been introduced to study the existence and nonexistence of limit cycles of planar vector fields. This method is based on the following result:

Theorem 1.21. (Giacomini, Llibre & Viano) *Let (P, Q) be a C^1 vector field defined in the open subset U of $\mathbb{R}^2$. Let $V(x, y)$ be an inverse integrating factor If γ is a limit cycle of the vector field (P, Q) in the domain of definition of V, then γ is contained in $\Sigma = \{(x, y) \in U : V(x, y) = 0\}$.*

1.1.8 Quadratic systems

When $m = 2$, the differential system (1.1) is called *quadratic.*

$$\begin{array}{rclcl} \dot{x} & = & P_0 + P_1 + P_2 & = & a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \ , \\ \dot{y} & = & Q_0 + Q_1 + Q_2 & = & b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 \ . \end{array} \tag{1.6}$$

Of course, all the definitions and ideas on differential systems of arbitrary degree work also for systems of degree 2. We would like to remark some results only valid for quadratic systems.

The following Theorem establishes the coexistence of different type of real singular points in a quadratic system. A simple proof can be found in Kukles and Casanova [35] or Coppel [19] but the property was previously stated by Berlinskiĭ [1].

Theorem 1.22. (Berlinskiĭ) *Suppose that there are four real different critical points of a quadratic system. If the quadrilateral with vertices these points is convex then the opposite critical points are saddles and the other two are antisaddles (nodes, foci or centers). But if the quadrilateral is not convex then either the three exterior vertices are saddles and the interior vertex an antisaddle or the exterior vertices are antisaddles and the interior vertex is a saddle.*

The next result is well known, see Ye Yian-Qian [53].

Theorem 1.23. *Let C be a limit cycle of a quadratic system (1.6). There exists one only singular point inside the bounded region defined by the limit cycle and it is also a focus.*

1.2 The complex projective plane

Consider a real affine algebraic curve $f(x, y) = 0$ of degree n. If we want to compute the intersection points with a parametric straight line $\big(x(t), y(t)\big) = (t, at+b)$, we must solve the equation of degree n, $f(t, at + b) = 0$. This equation may be solved over the complex field, that is, there can be complex points that play an important role even when the curve is real. Moreover, it has been shown that the complex behavior plays a very important role even for real affine differential systems of equations. Darboux's theory of integrability is a good example of this fact.

Moreover, the behavior at infinity is as important as the affine behavior and algebraic projective curves are more useful than algebraic affine curves when we proceed to study their properties. In fact, this is in this way due to the compactness of projective spaces. Thus, the projective plane allows to work with the infinite line and provides a global vision of the curves necessary from now on. So, we must imagine the differential equations over the complex projective plane. Poincaré and Darboux, considered differential equations in this way, yet.

The complex projective plane is constructed as $\left(\mathbb{C}^3-\{0\}\right)/\sim$, where $(X_0,Y_0,Z_0)\sim(X_1,Y_1,Z_1)$ if $(X_0,Y_0,Z_0)=(\lambda X_1,\lambda Y_1,\lambda Z_1)$ for $(X_0,Y_0,Z_0),(X_1,Y_1,Z_1)\in\mathbb{C}^3-\{0\}$ and $\lambda\neq 0$. Thus, the points in $\mathbb{C}P^2$ are ratios $(X_0:Y_0:Z_0)$. The sets

$$\mathcal{U}_X=\{(X:Y:Z)\in\mathbb{C}P^2 \mid X\neq 0\},$$
$$\mathcal{U}_Y=\{(X:Y:Z)\in\mathbb{C}P^2 \mid Y\neq 0\},$$
$$\mathcal{U}_Z=\{(X:Y:Z)\in\mathbb{C}P^2 \mid Z\neq 0\},$$

with the difeoeomorphisms

$$\begin{array}{cccc} \phi_X: & \mathcal{U}_X & \longrightarrow & \mathbb{C}^2 \\ & (X:Y:Z) & \mapsto & \left(\frac{Y}{X},\frac{Z}{X}\right) \\ \\ \phi_Y: & \mathcal{U}_Y & \longrightarrow & \mathbb{C}^2 \\ & (X:Y:Z) & \mapsto & \left(\frac{X}{Y},\frac{Z}{Y}\right) \\ \\ \phi_Z: & \mathcal{U}_Z & \longrightarrow & \mathbb{C}^2 \\ & (X:Y:Z) & \mapsto & \left(\frac{X}{Z},\frac{Y}{Z}\right) \end{array}$$

define a differenciable atlas and give to $\mathbb{C}P^2$ a differenciable manifold structure.

To consider local coordinates of a projective curve at a point is to apply ϕ_X, ϕ_Y or ϕ_Z depending on the local chart where the point lives. By ϕ_Z^{-1}, every affine object can be extended to the projective plane.

1.2.1 Projective algebraic curves

A projective algebraic curve of degree n is the set of projective points where a homogeneous polynomial of degree n vanishes. The real affine curve $f(x,y)=0$ in the projective coordinates (X,Y,Z) is given by $F(X,Y,Z):=Z^nf(X/Z,Y/Z)=0$, a homogeneous polynomial on X, Y, Z.

From Euler's formula one has $X\frac{\partial F}{\partial X}+Y\frac{\partial F}{\partial Y}+Z\frac{\partial F}{\partial Z}=nF$.

Multiple points

Let $f(x,y)=0$ be an affine curve. By virtue of the implicit function derivative theorem

$$\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}},$$

what determines the slope of the tangent to the curve. Clearly, this angular coefficient is well defined at a point if $\frac{\partial f}{\partial y}\neq 0$ or $\frac{\partial f}{\partial x}\neq 0$. When such partial derivatives are both zero over a point of the curve, it is said to be a *multiple point* or singular point.

We will use the expression multiple point to distinguish between these points and the singular points of a differential equation. Intuitively, the extension of this concept to the projective plane is clear, but we will do it in a precise way.

Let $p = (X_0 : Y_0 : Z_0)$ be a point on the projective curve $F(X,Y,Z) = 0$. Since not all the coordinates of the point can be zero, we suppose that $Z_0 \neq 0$ and that $p = (0 : 0 : 1)$. If we consider the expression of the curve for $Z = 1$, we have

$$F(X,Y,1) = F_s(X,Y) + F_{s+1}(X,Y) + ... + F_n(X,Y), \tag{1.7}$$

where $F_i(X,Y)$ are homogeneous polynomials of degree i, with $F_s(X,Y) \not\equiv 0$. If $s = 0$ the curve does not contain the point.

Definition 1.24. Under the above assumptions, we say that p is a point of *multiplicity* s.

(i) If $s = 1$ we will say that p is a *simple point.*

(ii) If $s > 1$, then we will say that p is a *multiple point* with multiplicity $m_p = s$.

In particular, p is a *multiple point* of $F(X,Y,Z) = 0$ if and only if $\frac{\partial F}{\partial X}(p) = \frac{\partial F}{\partial Y}(p) = \frac{\partial F}{\partial Z}(p) = 0$. This is clear assuming $p = (0 : 0 : 1)$ and considering the partial derivatives of F as power expansions of Z and taking into account (1.7) with $s > 1$.

If p is a multiple point of multiplicity $s > 0$ we have $F_s = \prod_{i=1}^{k} L_i^{r_i}$ where L_i are different straight lines.

Definition 1.25. The lines L_i are called *tangent straight lines to* $F = 0$ at p; r_i is the multiplicity of the tangent.

Definition 1.26. We say that p is an *ordinary multiple point* if $r_i = 1$ for $i = 1, ...k$, otherwise we say that p is a *non ordinary multiple point.*

Relative to the multiplicity of the points of a curve we have the following theorem, whose proof can be seen in Fulton [28].

Theorem 1.27. *If $F(X,Y,Z) = 0$ is an irreducible curve in $\mathbb{C}P^2$ of degree n, then $\sum_p \frac{m_p(m_p-1)}{2} \leq \frac{(n-1)(n-2)}{2}$, where p runs over the multiple points of the curve.*

Genus of a projective algebraic curve

Let $p_0 = (X_0 : Y_0 : Z_0)$ be a multiple point on a given algebraic projective curve. By means of a birrational quadratic transformation, the curve is applied into another, and p_0 into the set of r_1 points $p_{1,1}$,$p_{1,2}$, $\dots$ p_{1,r_1}. we say that the given curve has r_1 points in the *first neighborhood* of p_0. By means of at most r_1 birrational quadratic transformations we obtain the r_2 points of the second neighborhood of p_0: $p_{2,1}$, $p_{2,2}$,$\dots$,p_{2,r_2}. Successively, we proceed until the points of some neighborhood are all simple, and we say that the singularity of the given curve at p_0 is *resolved.* The multiple point p_0 is said to be *explicit* and p_{k,r_k} $(k > 0)$ are said to be *implicit.* Now we are ready for the following definition:

Definition 1.28. We define the *genus* of a curve of degree n as

$$g = \frac{(n-1)(n-2)}{2} - \sum_p \frac{m_p(m_p-1)}{2}, \tag{1.8}$$

where the sum runs over all the multiple points, explicit and implicit, and m_p is their multiplicity.

Since the genus of a curve is a non negative integer, this is an improvement of Theorem 1.27. For more details see Primrose [46].

Relative to the genus of a curve we have the following theorem:

Theorem 1.29. (Harnack-Klein) *Let C be an algebraic curve in $\mathbb{R}P^2$. The number of real 1-dimensional connected components of C is at most $g+1$, where g is the genus of C.*

Intersection index

Here we present a brief introduction to the intersection index. For more detail see Foulton [28].

Definition 1.30. Let $p \in \mathbb{C}^2$. The *local ring* of $\mathbb{C}^2$ in p, $O_p(\mathbb{C}^2)$, is defined as the set of all the rational functions with complex coefficients such that the denominator does not vanish in p.

Let $p = (X_0 : Y_0 : Z_0) \in \mathbb{C}P^2$. Since not all the coordinates of p can be zero we can consider $Z_0 \neq 0$ and making the change $x_0 = X_0/Z_0$ and $y_0 = Y_0/Z_0$, p is given by its local coordinates (x_0, y_0) and one can define the local ring at p, O_p.

Definition 1.31. Let $C_1, \ldots, C_n$ be algebraic curves in $\mathbb{C}P^2$ defined in the local coordinates at p by $f_1(x,y) = 0, \ldots, f_n(x,y) = 0$. The *intersection index* of the curves at p is defined as

(i) $I_p(C_1, \ldots, C_n) = 0$ if $p \notin C_1 \cap \cdots \cap C_n$,

(ii) $I_p(C_1, \ldots, C_n) = \infty$ if $f_i = hg_i$ for $i = 1, \ldots, n$, where h is a polynomial that vanish on p,

(iii) $I_p(C_1, \ldots, C_n) = \dim_{\mathbb{C}} O_p/(f_1, \ldots f_n)$ otherwise, where $(f_1, \ldots f_n)$ is the ideal defined by the polynomials $f_1, \ldots, f_n$.

From the inclusion of ideals $(f_i, f_j) \subseteq (f_1, \ldots, f_n)$ for $i, j = 1, \ldots, n$ one have the following relation between the intersection index of n curves and the intersection index of each pair: $I_p(C_1, \ldots, C_n) \leq \min_{i,j}\{I_p(C_i, C_j)\}$.

Let $F(X,Y,Z) = 0$ and $G(X,Y,Z) = 0$ be two algebraic curves and let p be a point on them.

Definition 1.32. We say that $F = 0$ and $G = 0$ cut themselves *strictly* at p, if F and G does not have common factors that vanish on p. We say that $F = 0$ and $G = 0$ cut themselves *transversally* at p if p is a simple point of $F = 0$ and $G = 0$, and the tangent to $F = 0$ and to $G = 0$ at p are different.

On the intersection index of two curves we have the following theorem.

Theorem 1.33. *The intersection index of $F = 0$ and $G = 0$ at p, $I_p(F,G)$, is unique for all $p \in \mathbb{C}P^2$ and satisfies the following conditions:*

(i) *$I_p(F,G)$ is a non negative integer for all F, G and p when F and G cut themselves in strict sense. $I_p(F,G) = \infty$ if F and G does not cut themselves in strict sense.*

(ii) *$I_p(F,G) = 0$ if and only if p is not a common point of F and G. $I_p(F,G)$ only depends on the factors of F and G vanished on p.*

(iii) *If T is a coordinates change and $T(p) = q$, then $I_q(T(F),T(G)) = I_p(F,G)$.*

(iv) *$I_p(F,G) = I_p(G,F)$.*

(v) *$I_p(F,G) \geq m_p(F)m_p(G)$, verifying the equality if and only if F and G does not have common tangents at p, where $m_p(F)$ and $m_p(G)$ are the multiplicities of p with respect to F and G.*

(vi) *If $F = \prod_{i=1}^{r} F_i^{r_i}$ and $G = \prod_{j=1}^{s} G_j^{s_j}$, then the intersection index can be computed as $I_p(F,G) = \sum_{i=1}^{r}\sum_{j=1}^{s} r_i s_j I_p(F_i,G_j)$.*

(vii) *$I_p(F,G) = I_p(F,G+AF)$ for all homogeneous polynomial A on X, Y and Z.*

We will need a property like (vi) for the intersection index of three projective algebraic curves. It is given in the following Lemma.

Lemma 1.34. *Let A, B, C, C' be homogeneous polynomials in three variables Then*

$$I_p(A,B,CC') \leq I_p(A,B,C) + I_p(A,B,C').$$

Proof. Consider the following sequence of vector spaces:

$$0 \longrightarrow Ker(\psi) \xrightarrow{i} \frac{O_p}{(A,B,C)} \xrightarrow{\psi} \frac{O_p}{(A,B,CC')} \xrightarrow{\phi} \frac{O_p}{(A,B,C')} \longrightarrow 0,$$

where i is an inclusion, $\psi(z) = C'z$ and ϕ is the natural projection.

The sequence is exact because i is injective, ϕ is surjective, $Im(i) = Ker(\psi)$ and $Im(\psi) = Ker(\phi) = \{z \in \frac{O_p}{(A,B,CC')}$ such that $z = C'w$ for any $w\}$. So

$$\begin{aligned}
\dim_{\mathbb{C}} \frac{O_p}{(A,B,CC')} &= \dim_{\mathbb{C}} \frac{O_p}{(A,B,C)} + \dim_{\mathbb{C}} \frac{O_p}{(A,B,C')} - \dim_{\mathbb{C}} Ker(\psi) \\
&\leq \dim_{\mathbb{C}} \frac{O_p}{(A,B,C)} + \dim_{\mathbb{C}} \frac{O_p}{(A,B,C')},
\end{aligned}$$

and by the definition of the intersection index the lemma follows immediately. ∎

A very useful result for the developing of this work was the named Darboux Lemma that can be found in [21], but not correctly stated. See Chavarriga, Llibre and Moulin-Ollagnier [11] for a proof of the correct version.

Theorem 1.35. (Darboux Lemma) *Let A, A', B, B', C, C' be homogeneous polynomials in $\mathbb{C}P^2$ in the variables X, Y, Z of degrees l, l', m, m', n, n', respectively. Suppose that A, B, C and A', B', C' are two sets of coprime polynomials verifying $AA' + BB' + CC' \equiv 0$. Then,*

(i) $\sum_p I_p(A, B, C) + \sum_p I_p(A', B', C') \geq \frac{lmn+l'm'n'}{\lambda}$.

(ii) *If $A \cap B \cap C \cap A' \cap B' \cap C' = \emptyset$, then $\sum_p I_p(A, B, C) + \sum_p I_p(A', B', C') = \frac{lmn+l'm'n'}{\lambda}$, where $\lambda = l + l' = m + m' = n + n'$.*

Theorem 1.36. (Bézout) *Let $F = 0$ and $G = 0$ be two curves in $\mathbb{C}P^2$ of degrees r and s, respectively without common components. Then $\sum_p I_p(F, G) = rs$.*

1.2.2 Projective differential equations

Let $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ be homogeneous polynomials of degree $m + 1$ in the variables X, Y and Z. The homogeneous 1-form

$$\omega = \mathcal{P}dX + \mathcal{Q}dY + \mathcal{R}dZ$$

is said to be *projective* if $X\mathcal{P} + Y\mathcal{Q} + Z\mathcal{R} = 0$, that is,

$$\mathcal{P} = MZ - NY, \quad \mathcal{Q} = NX - LZ, \quad \mathcal{R} = LY - MX,$$

for some L, M and N homogeneous polynomials of degree m. Then

$$\omega = L(YdZ - ZdY) + M(ZdX - XdZ) + N(XdY - YdX),$$

and $\omega = 0$, that is,

$$L(YdZ - ZdY) + M(ZdX - XdZ) + N(XdY - YdX) = 0, \tag{1.9}$$

defines a differential equation. For more details, see Darboux [21].

The following result is well known.

Lemma 1.37. *If we take $\bar{L} = L + AX$, $\bar{M} = M + AY$, $\bar{N} = N + AZ$ being A a homogeneous polynomial of degree $m - 1$, then (1.9) remains invariant.*

Let F be a homogeneous polynomial of degree n in $\mathbb{C}P^2$. We say that $F = 0$ is an irreducible invariant algebraic curve of (1.9) if

$$\frac{\partial F}{\partial X}L + \frac{\partial F}{\partial Y}M + \frac{\partial F}{\partial Z}N = KF, \tag{1.10}$$

where K is a polynomial of degree $m - 1$. Using Euler's Formula we have

$$\frac{\partial F}{\partial X}\left(L - \frac{KX}{n}\right) + \frac{\partial F}{\partial Y}\left(M - \frac{KY}{n}\right) + \frac{\partial F}{\partial Z}\left(N - \frac{KZ}{n}\right) = 0. \tag{1.11}$$

Remark 1.38. Taking $\bar{L} = L - KX/n$, $\bar{M} = M - KY/n$ and $\bar{N} = N - KZ/n$ we can always consider that the cofactor of one invariant algebraic curve is zero.

1.2.3 Projective singular points

The *singular points* of (1.9) are those for which the tangent is not determined. These points verify the system

$$\mathcal{P} = MZ - NY = 0, \quad \mathcal{Q} = NX - LZ = 0, \quad \mathcal{R} = LY - MX = 0. \tag{1.12}$$

In order to determinate the number of singular points we use the following corollary of Theorem 1.35.

Corollary 1.39. *The number of singular points of the differential equation (1.9) where L, M, N are coprime polynomials of degree m, is $m^2 + m + 1$.*

1.2.4 Relationship among affine and projective objects

Now we show the behavior of a differential equation, and cofactors when we take local coordinates in the local chart determined by $Z = 1$. Of course, we can do the same for $X = 1$ and $Y = 1$, similarly.

Lemma 1.40. *Let (1.9) be a differential equation with L, M and N of degree m. Let $F = 0$ be an invariant algebraic curve of degree n of (1.9) with cofactor K. Then, the restriction of the projective differential equation to the affine plane is*

$$\left(L\left(X,Y,1\right) - XN\left(X,Y,1\right)\right)dY - \left(M\left(X,Y,1\right) - YN\left(X,Y,1\right)\right)dX = 0.$$

It has degree $m+1$ and $F(X,Y,1) = 0$ is an invariant algebraic curve with cofactor $\tilde{K}(X,Y,1) = K(X,Y,1) - nN(X,Y,1)$ of degree at most m, whenever $Z = 0$ is not an invariant straight line.

Proof. Since $F = 0$ is an invariant algebraic curve of (1.9) it follows (1.10). On the other hand, from Euler's Formula outside the infinite straight line we obtain

$$\frac{\partial F}{\partial Z} = \frac{1}{Z}\left(nF - X\frac{\partial F}{\partial X} - Y\frac{\partial F}{\partial Y}\right).$$

Replacing the right side of this expression in (1.10) and taking $Z = 1$ we see that $F(X,Y,1) = 0$ is invariant for the restricted differential equation and we obtain the expression of the cofactor. The line $Z = 0$ is invariant for (1.9) if and only if $N = ZA$ for some polynomial A of degree $m - 1$. When this does not happen, $N(X,Y,1)$ is a polynomial of degree m. ∎

System (1.1) defined on the affine plane can be extended to the projective plane. We write (1.1) as $Pdy - Qdx = 0$. Using projective coordinates $x = X/Z$, $y = Y/Z$ we can write the previous equation as

$$L(YdZ - ZdY) + M(ZdX - XdZ) = 0,$$

with

$$\begin{aligned} L &= Z^m P\left(X/Z, Y/Z\right), \\ M &= Z^m Q\left(X/Z, Y/Z\right). \end{aligned}$$

Notice that in this case we have $N \equiv 0$.

Of course, any singular point $p = (x_0, y_0)$ of the affine differential equation (1.1) becomes a singular point $p = (X_0 : Y_0 : 1)$ for the projective differential equation. The points satisfying $yP_m - xQ_m = 0$ are called *infinite singular points*. They are singular points of the projective differential equation which, from an affine point of view, live over the line at infinity, i.e., over $Z = 0$; are of the form $p = (X_0 : Y_0 : 0)$.

Definition 1.41. We say that system (1.1) has *degenerate infinity* if the line at infinity $Z = 0$ is fulfilled of singular points or equivalently $yP_m - xQ_m \equiv 0$.

Systems (1.1) of degree m with degenerate infinity can be reduced to differential equations of degree $m - 1$.

If $f = 0$ is an invariant algebraic curve of the affine differential equation with cofactor k, then the projectivized curve $F = 0$ defined by $F = Z^n f(X/Z, Y/Z)$ has cofactor $K = Z^{m-1} k(X/Z, Y/Z)$. As we have said in Remark 1.38, we can consider that this cofactor is identically null by making a change, but when the projective differential equation comes from an affine planar system then this change forces $N \not\equiv 0$.

1.3 Formal differential equations and formal solutions

In this section we summarize some definitions and results about formal differential equations and their solutions, that we shall use later on. For more details and proofs about these results see Seidenberg [48]. Walcher in [50] states also similar results with some precisions.

We consider the field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$). We denote by $\mathbb{K}[[x, y]]$ the ring of formal power series. A *unit* is an invertible element of this ring. In particular, if $U(x, y) = \sum_{i,j=0}^{\infty} u_{ij} x^i y^j$ is a unit then $u_{00} \neq 0$.

Let $F(x, y)$ be an irreducible non–unit of $\mathbb{K}[[x, y]]$ such that $F(x, y) \not\equiv 0$.

Definition 1.42. An *analytic branch centered at* $(0, 0)$ is the equivalence class in $\mathbb{K}[[x, y]]$ under the equivalence $F \sim G$ if $F = U \cdot G$ with U unit.

We note that here the adjective *analytic* does not mean the convergence of the power series. On the other hand $F(0, 0) = 0$ because $F(x, y)$ is non-unit.

Given a representative of an analytic branch $F(x, y)$ centered at the origin, there are power series $x(t) = \sum_{i=1}^{\infty} x_i t^i$ and $y(t) = \sum_{i=1}^{\infty} y_i t^i$, with $x_i, y_i \in \mathbb{K}$, not both identically null, such that $F(x(t), y(t)) = 0$.

Definition 1.43. Such a pair $(x(t), y(t))$ is called a *branch expansion* of the analytic branch.

Note that $x(0) = 0$ and $y(0) = 0$.

Given a branch expansion $x(t), y(t)$, there is an irreducible non–unit $F(x, y) \not\equiv 0$ in $\mathbb{K}[[x, y]]$, uniquely determined up to a unit factor, such that $F(x(t), y(t)) = 0$. $F(x, y) = 0$ is called the *equation of the branch*.

Consider the formal differential equation

$$P(x,y)dy - Q(x,y)dx = 0, \tag{1.13}$$

where $P(x,y), Q(x,y) \in \mathbb{K}[[x,y]]$. For a formal power series $F(x,y) = \sum_{i,j=0}^{\infty} f_{ij}x^i y^j$ we define $\partial F(x,y)/\partial x$ as the formal power series $\sum_{i=1,j=0}^{\infty} i f_{ij} x^{i-1} y^j$. Analogously, we define $\partial F(x,y)/\partial y$.

By a *solution* of the formal differential equation (1.13) we mean an analytic branch $(x(t), y(t))$, centered at the origin satisfying equation (1.13). More explicitly, if the equation of the solution branch $(x(t), y(t))$ is $F(x,y) = 0$ one has

$$P(x,y)\frac{\partial F}{\partial x} + Q(x,y)\frac{\partial F}{\partial y} = K(x,y)F(x,y), \tag{1.14}$$

for some $K \in \mathbb{K}[[x,y]]$. Conversely, every irreducible $F \in \mathbb{K}[[x,y]]$ with $F \not\equiv 0$ satisfying (1.14) for some $K \in \mathbb{K}[[x,y]]$, yields a solution of equation (1.13).

Definition 1.44. A branch $x(t) = \sum_{i=1}^{\infty} x_i t^i$ and $y(t) = \sum_{i=1}^{\infty} y_i t^i$, with $x_i, y_i \in \mathbb{K}$, centered at $(0,0)$, is called *linear* if x_1 or y_1 is not zero.

Using the following theorem, which summarizes the results from [48], we study the behavior of the solutions at a singular point according to the eigenvalues of the jacobian matrix $D\mathcal{X}$, where $\mathcal{X}$ is the vector field associated to the differential equation (1.13).

Theorem 1.45 (Seidenberg). *Let the origin $(0,0)$ be a critical point of the formal system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$, where $P, Q \in \mathbb{C}[[x,y]]$, with associated eigenvalues $\lambda, \mu \in \mathbb{C}$. In the following the dots denote higher order terms.*

1. *Let $(0,0)$ be a non-degenerate critical point. Then consider the formal differential system*

$$\dot{x} = \lambda x + \cdots \ , \ \dot{y} = \mu y + \cdots \ , \tag{1.15}$$

where $\lambda\mu \neq 0$. If $\lambda \neq \mu$ then every formal solution of (1.15) at the origin has a horizontal or vertical tangent. Moreover,

(i) *If $\lambda/\mu \notin \mathbb{Q}^+$ then (1.15) has exactly two formal solutions at the origin $F_i(x,y) = 0$ with $i = 1, 2$. They are linear branches with horizontal and vertical tangent respectively, i.e., $F_1(x,y) = x + \cdots$, $F_2(x,y) = y + \cdots$.*

(ii) *If $\lambda/\mu \in \mathbb{Q}^+$ then the following holds.*

(a) *If $\lambda = \mu$ then, for each direction there exists only one formal solution at the origin, a linear branch.*

(b) *If $\lambda/\mu \neq 1$ (with $\lambda/\mu > 1$) then there is one unique formal solution at the origin with horizontal tangent: a linear branch $F(x,y) = y + \cdots$. The other formal solutions at the origin, if they exists, have vertical tangent, i.e., are of the form $F(x,y) = x^s + \cdots$ with $s \in \mathbb{N}\backslash\{0\}$.*

(b.1) *If $\lambda/\mu \in \mathbb{N}$ then either there are no formal solutions at the origin with vertical tangent or there are infinitely many formal solution at the origin with vertical tangent, all linear.*

(b.2) *If $\lambda/\mu \notin \mathbb{N}$ then there is one unique linear branch formal solution at the origin with vertical tangent $F(x,y) = x + \cdots$. The other solutions are non–linear.*

2. *Let $(0,0)$ be a logarithmic critical point. Then, the formal differential system $\dot{x} = \lambda x + y + \cdots$, $\dot{y} = \lambda y + \cdots$, where $\lambda \neq 0$ has a unique formal solution at the origin, which is a linear branch with horizontal tangent $F(x,y) = y + \cdots$.*

3. *Let $(0,0)$ be a elementary degenerate critical point. Then, the formal differential system $\dot{x} = x + \cdots$, $\dot{y} = \cdots$, has exactly two formal solutions at the origin $F_i(x,y) = 0$ with $i = 1,2$. They are linear branches with horizontal and vertical tangent respectively, i.e., $F_1(x,y) = x + \cdots$, $F_2(x,y) = y + \cdots$.*

4. *Let $(0,0)$ be a nilpotent critical point. Then, the formal differential system $\dot{x} = y + \cdots$, $\dot{y} = \cdots$, can have either one formal solution at the origin or two linear branch formal solutions at the origin or infinity formal solutions at the origin.*

1.3.1 Relationship among formal solutions and invariant algebraic curves

Let us consider an irreducible algebraic curve $f(x,y) = 0$ with $f \in \mathbb{C}[x,y]$ such that $f(x_0,y_0) = 0$. We translate the point (x_0,y_0) to the origin. In particular $f \in \mathbb{C}[[x,y]]$ with $f(0,0) = 0$, hence f is not a unit element in $\mathbb{C}[[x,y]]$ and in this ring it is possible that f be a reducible element. By using the Newton-Poiseux algorithm, see [2] one can see that there are ℓ irreducible elements $\phi_i(x,y) \in \mathbb{C}[[x,y]]$, with $i = 1,\ldots,\ell$ such that f factorizes as

$$f(x,y) = x^r U(x,y) \prod_{i=1}^{\ell} \phi_i(x,y) \ , \tag{1.16}$$

being $r \in \mathbb{N} \cup \{0\}$ and $U \in \mathbb{C}[[x,y]]$ a unit element. Later on, in [6], it was proved that the above decomposition (1.16) is square free, that is, there is no repeated element ϕ_i neither $r \geq 2$.

Let the origin $(0,0)$ be a singular point of system (1.1) and let $f = 0$ be an irreducible invariant algebraic curve of that system such that $f(0,0) = 0$. The curve $f(x,y) = \sum_{i=s}^{n} f_i(x,y) = 0$ with f_i real homogeneous polynomials and $s \geq 1$, defines a finite number of branches at the origin corresponding to its irreducible nonunit factors in $\mathbb{C}[[x,y]]$. As f_s is homogeneous, it can be factorized as $f_s(x,y) = \prod_{i=1}^{s} L_i(x,y)$ where $L_i(x,y) = a_i x + b_i y$ are called the *tangents* of the curve $f = 0$ at the origin and $a_i, b_i \in \mathbb{C}$.

Finally, it is easy to see that each of the irreducible elements appearing in the above formal decomposition (1.16) of f is a formal solution of (1.1). Moreover, the tangents at the origin of these branches are given by $f_s = 0$ as defined above.

Let $(x_0,y_0) \in \mathbb{C}^2$ be a critical point with eigenvalues $\lambda, \mu \in \mathbb{C}$. Denoting by $v_\lambda, v_\mu \in \mathbb{C}^2$ the corresponding eigenvectors, we will call $L_\lambda(x,y)$ and $L_\mu(x,y)$ the non-null homogeneous polynomials of degree one belonging to $\mathbb{C}[x,y]$ such that $\nabla L_\lambda \perp v_\lambda$ and $\nabla L_\mu \perp v_\mu$ respectively. Here $\nabla := (\partial/\partial x, \partial/\partial y)$ is the gradient operator and $\perp$ means orthogonality with respect to the standard Euclidean scalar product in $\mathbb{C}^2$.

Taking into account all this background, in [6] the following results are proved, which describe the tangents and the value of the cofactor at some generic class of critical points.

Theorem 1.46. (Chavarriga, Giacomini & Grau) *Let $f(x,y)=0$ with $f \in \mathbb{C}[x,y]$ be an irreducible invariant algebraic curve with associated cofactor $k(x,y)$ of a real polynomial differential system. Let $(x_0,y_0) \in \mathbb{C}^2$ be a non-degenerate or elementary degenerate critical point of the system with different associated eigenvalues λ and μ verifying $f(x_0,y_0)=0$. Then, the equation of the tangents of the curve $f=0$ at (x_0,y_0) is $f_s(x,y)=L_\lambda^r(x,y)L_\mu^{s-r}(x,y)$ with $s,r \in \mathbb{N}$, $r \leq s$. Moreover $k(x_0,y_0)=r\mu+(s-r)\lambda$.*

Lemma 1.47. *Let $f(x,y)=0$ with $f \in \mathbb{R}[x,y]$ be an irreducible invariant algebraic curve in $\mathbb{R}[x,y]$ with associated cofactor $K(x,y)$ of a real polynomial differential system. Let $(x_0,y_0) \in \mathbb{R}^2$ be a real critical point of the system with complex eigenvalues $\lambda = a+\mathrm{i}b$ and $\mu = a-\mathrm{i}b$, where $b \neq 0$ and verifying $f(x_0,y_0)=0$. Then, the equation of the tangents of the curve $f=0$ at (x_0,y_0) is $f_2(x,y)=L_\lambda(x,y)L_\mu(x,y)$. Moreover $K(x_0,y_0)=\mu+\lambda$ and no other invariant algebraic curve $\tilde{f}(x,y)=0$ irreducible in $\mathbb{R}[x,y]$ with $\tilde{f}(x_0,y_0)=0$ can exist.*

Chapter 2

Algebraic Limit Cycles of Degree 4 for Quadratic systems

In this chapter we give a characterization of the irreducible invariant algebraic curves of fourth degree of a quadratic system containing an oval which is an algebraic limit cycle of the system, showing that there are exactly four families of algebraic limit cycles of degree 4 for quadratic systems.

2.1 Introduction

As we have said, 16^{th} Hilbert problem is unsolved even for quadratic systems. In this chapter we concentrate in algebraic limit cycles.

In 1958, Ch'in Yuan-shün summarizes in [3] the possible quadratic system having an algebraic limit cycle of degree 2 and he proves the uniqueness of this limit cycle:

If a quadratic system has an algebraic limit cycle of degree 2, then after an affine change of variables, the limit cycle becomes the circle $\Gamma := x^2 + y^2 - 1 = 0$. Moreover, Γ is the unique limit cycle of the quadratic system which can be written in the form

$$\begin{aligned} \dot{x} &= -y(ax + by + c) - (x^2 + y^2 - 1) \ , \\ \dot{y} &= x(ax + by + c) \ , \end{aligned}$$

with $a \neq 0$ and $c^2 > a^2 + b^2$.

The case of the limit cycles of degree 3 was studied later on. Using three papers Evdokimenco proves from 1970 to 1979 that there are no quadratic systems having limit cycles of degree 3, see [24, 25, 26]. An easier proof can be found in Chavarriga, Llibre and Moulin-Ollagnier [11].

The study of the algebraic limit cycles of degree 4 for quadratic systems began before the proof of Evdokimenco. Thus, Yablonskii [51] found one of them in 1966.

Seven years later, a new algebraic limit cycle of degree 4 was found by Filiptsov [27], and a third one was found in 1999, see Chavarriga [4]. The possible existence of other algebraic limit cycles of degree 4 and limit cycles of higher degree was unknown at the moment of composition of this work.

The study of invariant algebraic curves is closely related to the study of algebraic limit cycles. There is an essential open problem first stated by Poincaré [44]: find an upper bound for the degree of the invariant algebraic curves of quadratic systems without rational first integral. Lins Neto [38] conjectured that if a quadratic differential system possesses an invariant algebraic curve of degree greater than 4, it would be rationally integrable. At that moment, the study of all the algebraic limit cycles of degree 4 was an important objective because if we know all them and we believe Lins Neto conjecture, then we know all the algebraic limit cycles for quadratic systems. Unfortunately, the conjecture is false. This is first showed by Christopher and Llibre [15] and Moulin-Ollagnier [39].

2.2 The main result

We characterize the quadratic systems which have an algebraic limit cycle of degree 4. The main result is summarized in the following theorem.

Theorem 2.1. *After an affine change of variables the only quadratic systems having an algebraic limit cycle of degree 4 are*

(a) *Yablonskii system*

$$\begin{array}{rcl} \dot{x} &=& -4abcx-(a+b)y+3(a+b)cx^2+4xy\ , \\ \dot{y} &=& (a+b)abx-4abcy+4(abc^2-\frac{3}{2}(a+b)^2+4ab)x^2+8(a+b)cxy+8y^2\ , \end{array}$$

with $abc \neq 0$, $a \neq b$, $ab > 0$ *and* $4c^2(a-b)^2+(3a-b)(a-3b) < 0$. *This system posseses the irreducible invariant algebraic curve*

$$(y+cx^2)^2+x^2(x-a)(x-b)=0.$$

of degree 4 having two components, an oval (the algebraic limit cycle) and an isolated point (a singular point).

(b) *Filipstov system*

$$\begin{array}{rcl} \dot{x} &=& 6(1+a)x+2y-6(2+a)x^2+12xy\ , \\ \dot{y} &=& 15(1+a)y+3a(1+a)x^2-2(12+5a)xy+16y^2\ , \end{array}$$

with $0 < a < \frac{3}{13}$. *This system posseses the irreducible invariant algebraic curve*

$$3(1+a)(ax^2+y)^2+2y^2(2y-3(1+a)x)=0,$$

of degree 4 having two components, one is an oval and the other is homeomorphic to a straight line. This last component contains three singular points of the system.

(c) *The system*

$$\begin{array}{rcl} \dot{x} &=& 5x+6x^2+4(1+a)xy+ay^2\ , \\ \dot{y} &=& x+2y+4xy+(2+3a)y^2\ , \end{array}$$

with $\frac{-71+17\sqrt{17}}{32} < a < 0$ *posseses the irreducible invariant algebraic curve*

$$x^2 + x^3 + x^2y + 2axy^2 + 2axy^3 + a^2y^4 = 0,$$

of degree 4 having three components, one is an oval and each of the others two is homeomorphic to a straight line. Each one of these last two components contains one singular point of the system.

(d) *The system*

$$\begin{array}{rcl} \dot{x} & = & 2(1 + 2x - 2kx^2 + 6xy), \\ \dot{y} & = & (8 - 3k - 14kx - 2kxy - 8y^2), \end{array}$$

with $0 < k < \frac{1}{4}$ *posseses the irreducible invariant algebraic curve*

$$\frac{1}{4} + x - x^2 + kx^3 + xy + x^2y^2 = 0,$$

of degree 4 having three components, one is an oval and each of the other two is homeomorphic to a straight line. Each of these last two components contains one singular point of the system. it has an oval, which is a limit cycle and two real branches.

This result is obtained by using projective techniques, in particular, the infinite straight line plays an important role. From a projective point of view, and in order to make an algebraic classification of the curves that contain limit cycles, we can say that in case (a) (figure 2.1) the curve has two double points: a node at $(0 : 0 : 1)$ and a tacnode at $(0 : 1 : 0)$ that has real tangent but it is isolated because the branches through it are complex conjugated. Consequently, the genus of the curve is $g = 0$.

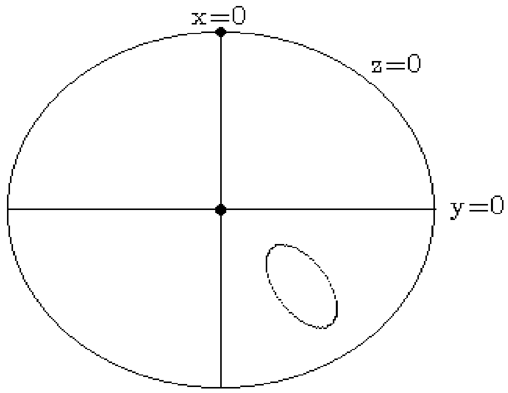

Figure 2.1: Case (a). Yablonskii.

In cases (b) (figure 2.2) and (c) (figure 2.3) the curve has one only double point, a ramphoid cusp, that is finite and can be put at $(0 : 0 : 1)$. The genus of the curve is $g = 1$.

In the new case (d) (figure 2.4) the curve has an infinite ramphoid cusp at $(0 : 1 : 0)$. So, the genus is $g = 1$, too.

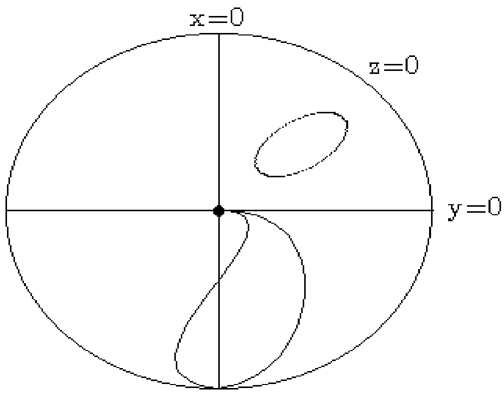

Figure 2.2: Case (b). Filipstov.

2.3 Some results on singular and multiple points

On the singular points on an straight line anyone we have the following Lemma.

Lemma 2.2. *Let $r = 0$ be a straight line. Then,*

$$\sum_p I_p(r, \mathcal{P}, \mathcal{Q}, \mathcal{R}) \leq m + 1.$$

Proof. We know that $I_p(r, \mathcal{P}, \mathcal{Q}, \mathcal{R}) \leq \min\{I_p(r, \mathcal{P}), I_p(r, \mathcal{Q}), I_p(r, \mathcal{R})\}$.

If $\sum_p I_p(r, \mathcal{P}, \mathcal{Q}, \mathcal{R}) > m + 1$, then

$$\sum_p I_p(r, \mathcal{P}) > m + 1, \quad \sum_p I_p(r, \mathcal{Q}) > m + 1 \text{ and } \sum_p I_p(r, \mathcal{R}) > m + 1,$$

from where r divides $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ against the hypothesis. ∎

When a line has less that $m+1$ singular points it can be invariant or not, depending on the singular points. Using the following result we give a characterization of invariant straight lines.

Theorem 2.3. *Let $r = 0$ be an straight line. It is invariant for equation (1.9) if and only if $\sum_p I_p(r, \mathcal{P}, \mathcal{Q}, \mathcal{R}) = m + 1$.*

Proof. By means of a projectivity we can consider that the straight line is $Z = 0$. Suppose that $\sum_p I_p(Z, \mathcal{P}, \mathcal{Q}, \mathcal{R}) = m + 1$. Then

$$\sum_p I_p(Z, \mathcal{P}) \geq m + 1 \text{ and } \sum_p I_p(Z, \mathcal{Q}) \geq m + 1.$$

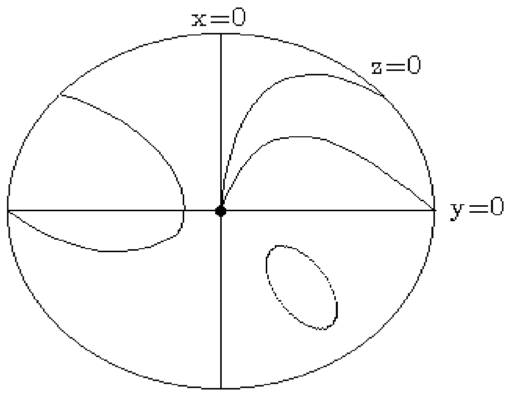

Figure 2.3: Case (c). Chavarriga.

On the other hand, since $\mathcal{P} = MZ - NY$, $\mathcal{Q} = NX - LZ$ and using Theorem 1.33(vii) follows that

$$\sum_p I_p(Z, \mathcal{P}) = \sum_p I_p(Z, NY) \text{ and } \sum_p I_p(Z, \mathcal{Q}) = \sum_p I_p(Z, NX).$$

Now, since X, Y and Z can not be zero simultaneously, follows that $\sum_p I_p(Z, N) \geq m + 1$ and from Bezout's Theorem Z divides N. Therefore, $Z = 0$ verifies (1.10), and is invariant for $\omega = 0$.

Reciprocally, if $Z = 0$ is an invariant straight line of $\omega = 0$ we have $N = ZA$ for some polynomial A of degree $m-1$. By taking $\bar{L} = L - AX$, $\bar{M} = M - AY$ and $\bar{N} = 0$ we have $\sum_p I_p(Z, \mathcal{P}, \mathcal{Q}, \mathcal{R}) = \sum_p I_p(Z, \bar{M}Z, -\bar{L}Z, LY - MX) = \sum_p I_p(Z, LY - MX) = m + 1$ by Bezout Theorem. ∎

The proof of the following result is due to Chavarriga and LLibre [10].

Proposition 2.4. *All the multiple points of an irreducible invariant algebraic curve of $\omega = 0$ (1.9) are singular points of the projective differential equation $\omega = 0$. The intersection points of two invariant algebraic curves of $\omega = 0$ are singular points of the differential projective equation $\omega = 0$.*

Let $F = 0$ be an invariant algebraic curve. From (1.10) and using Euler's Formula it follows that

$$\frac{\partial F}{\partial X}\Big(L - \frac{KX}{n}\Big) + \frac{\partial F}{\partial Y}\Big(M - \frac{KY}{n}\Big) + \frac{\partial F}{\partial Z}\Big(N - \frac{KZ}{n}\Big) = 0.$$

Then, following Darboux there are two types of singular points: those that are on the projective curve $F = 0$ and those that are not necessarily on this curve and for which one has

$$L - \frac{KX}{n} = 0, \quad M - \frac{KY}{n} = 0, \quad N - \frac{KZ}{n} = 0 \;, \tag{2.1}$$

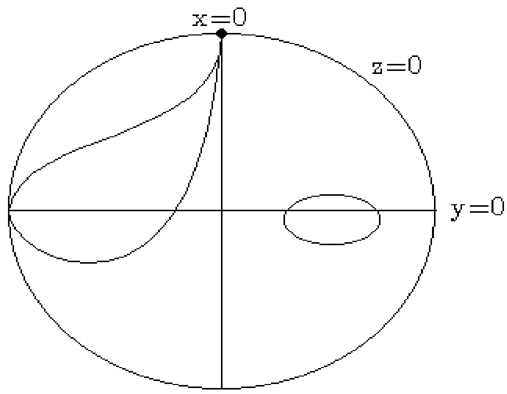

Figure 2.4: New case (d).

where n is the degree of the curve. Thus we define

$$h = \sum_p I_p \left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z} \right), \qquad h' = \sum_p I_p \left(L - \frac{KX}{n}, M - \frac{KY}{n}, N - \frac{KZ}{n} \right).$$

By applying Theorem 1.35(i) to equation (1.11) we have

$$h + h' \geq \frac{m^3 + (n-1)^3}{m+n-1}. \tag{2.2}$$

In order to simplify the notation, $I_p(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z})$ will be denoted by I_p.

When a projective differential equation is the extension of a differential equation defined in the affine plane we get $N \equiv 0$. Thus, in h' two types of points are counted: those given by $K = 0$, and those given by $Z = 0$. If we take

$$h'_1 = \sum_p I_p(L, M, K), \qquad h'_2 = \sum_p I_p \left(L - \frac{KX}{n}, M - \frac{KY}{n}, Z \right),$$

it follows from Lemma 1.34, that

$$h' \leq h'_1 + h'_2. \tag{2.3}$$

Notice that by Bezout's Theorem we have $h'_1 \leq m(m-1)$, otherwise the polynomials P, Q, k would not be coprime against the hypothesis.

If $h'_2 > m$, then $P_m \equiv k_{m-1}X/n$ and $Q_m \equiv k_{m-1}Y/n$. Replacing in (1.9) L for $L - k_{m-1}X/n$, M for $M - k_{m-1}Y/n$ and N, that is zero, for $-k_{m-1}Z/n$, we see that $Z = 0$ is a straight line of singular points, that is, has degenerate infinity. Affine quadratic systems with degenerate infinity can be reduced to a linear differential equation in $\mathbb{C}P^2$; in particular, they do not have limit cycles.

In what follows we suppose that system 1.1 has not degenerate infinity. In particular, we have $h_2 \le m$ and consequently $h' \le m^2$ which allows to obtain, using (2.2), an upper bound for h.

The next result is proved in [10].

Theorem 2.5. (Chavarriga & LLibre) *Let $f = 0$ be an invariant algebraic curve for system (1.1). Let h' be the number of points counted with their multiplicity in $\mathbb{C}P^2$ that verify (2.1). If $h' = m^2$, then system (1.1) has a rational first integral.*

A very useful result on invariant curves can be found in Christopher [13], but there are preliminary versions of this lemma in other authors, see for instance Theorem 1 of Yablonskii [52]. Here we present an improvement of such result.

Lemma 2.6. *Let $f := \sum_{i=0}^{n} f_i = 0$ be an affine invariant algebraic curve of system (1.1) of degree n. Let d be a real or complex linear divisor of f_n with multiplicity l. We denote by $k = \sum_{i=0}^{m-1} k_i$ the cofactor of $f = 0$. Then*

(i) *d is a divisor of $\Delta := yP_m - xQ_m$.*

(ii) *Let $\bar{l}$ be the multiplicity of d as a divisor of Δ. Then d is a divisor of $k_{m-1}x - nP_m$ and of $k_{m-1}y - nQ_m$ with multiplicity $\bar{l} - 1$.*

(iii) *$h_2' = m + 1 - r$ where r is the number of different factors of f_n.*

(iv) *d is a divisor of $f_{n-1}(k_{m-1}x - (n-1)P_m)$ and of $f_{n-1}(k_{m-1}y - (n-1)Q_m)$ with multiplicity $min\{l - 1, \bar{l}\}$.*

Proof. The curve $f = 0$ verifies (1.3) since it is invariant. Taking the terms of degree $m + n - 1$ and $m + n - 2$ of (1.3) we have

$$P_m \frac{\partial f_n}{\partial x} + Q_m \frac{\partial f_n}{\partial y} = k_{m-1} f_n, \tag{2.4}$$

$$P_m \frac{\partial f_{n-1}}{\partial x} + Q_m \frac{\partial f_{n-1}}{\partial y} + P_{m-1} \frac{\partial f_n}{\partial x} + Q_{m-1} \frac{\partial f_n}{\partial y} = k_{m-1} f_{n-1} + k_{m-2} f_n. \tag{2.5}$$

On the other hand, from Euler's Theorem $x\frac{\partial f_n}{\partial x} + y\frac{\partial f_n}{\partial y} = nf_n$. Consequently we obtain

$$\frac{\partial f_n}{\partial x} = \frac{f_n(k_{m-1}y - nQ_m)}{\Delta}, \qquad \frac{\partial f_n}{\partial y} = \frac{f_n(nP_m - k_{m-1}x)}{\Delta}. \tag{2.6}$$

Therefore, every divisor of f_n must be a divisor of $\frac{\partial f_n}{\partial x}\Delta$ and $\frac{\partial f_n}{\partial y}\Delta$. If d is a divisor of f_n with multiplicity l, then using Euler's formula it will be a divisor of $\frac{\partial f_n}{\partial x}$ and $\frac{\partial f_n}{\partial y}$ with multiplicity $l - 1$, and thus d must divide Δ, this proves (i). Since (2.6) must be verified it follows (ii).

Suppose that $f_n = \prod_{i=1}^{r} d_i^{l_i}$, $l_1 + \ldots + l_r = n$ and $\Delta = \left(\prod_{i=1}^{r} d_i^{\bar{l}_i}\right) B$ where B contains the divisors of Δ that do not divide f_n. Replacing the above expressions in

(2.6) we have

$$L_x\left(\prod_{i=1}^{r} d_i^{\bar{l}_i-1}\right)B = k_{m-1}y - nQ_m, \quad L_y\left(\prod_{i=1}^{r} d_i^{\bar{l}_i-1}\right)B = nP_m - k_{m-1}x, \tag{2.7}$$

where $L_x = d_1 d_2 ... d_r \sum_{i=1}^{r} \frac{1}{d_i}\frac{\partial d_i}{\partial x}$ and $L_y = d_1 d_2 ... d_r \sum_{i=1}^{r} \frac{1}{d_i}\frac{\partial d_i}{\partial y}$. Notice that L_x and L_y do not have common divisors.

Taking into account the degrees of the expressions that appear in (2.7) we have

$$r - 1 + \sum_{i=1}^{r}(\bar{l}_i - 1) + b = m,$$

where b is the degree of B. So

$$\sum_{i=1}^{r}(\bar{l}_i - 1) + b = m + 1 - r.$$

Therefore,

$$h_2' = \sum_p I_p\left(L - \frac{KX}{n}, M - \frac{KY}{n}, Z\right) = \sum_{i=1}^{r}(\bar{l}_i - 1) + b = m + 1 - r,$$

and proves (iii). This last equality becomes clear if we take into account that

$$L - \frac{KX}{n} = ZR + P_m - \frac{k_{m-1}X}{n} = ZR + L_x\left(\prod_{i=1}^{r} d_i^{\bar{l}_i-1}\right)B,$$

$$M - \frac{KY}{n} = ZS + Q_m - \frac{k_{m-1}Y}{n} = ZS + L_y\left(\prod_{i=1}^{r} d_i^{\bar{l}_i-1}\right)B,$$

and that their common points on $Z = 0$ come from the divisors of $\left(\prod_{i=1}^{r} d_i^{\bar{l}_i-1}\right)B$.

From (2.5) and Euler's formula for f_{n-1}, that is, $x\frac{\partial f_{n-1}}{\partial x} + y\frac{\partial f_{n-1}}{\partial y} = nf_{n-1}$, we obtain

$$\Delta\frac{\partial f_{n-1}}{\partial x} = yC + f_{n-1}(k_{m-1}y - (n-1)Q_m), \quad \Delta\frac{\partial f_{n-1}}{\partial y} = f_{n-1}((n-1)P_m - k_{m-1}x) - xC,$$

where $C = k_{m-2}f_n - P_{m-1}\frac{\partial f_n}{\partial x} - Q_{m-1}\frac{\partial f_n}{\partial y}$. Since d is a divisor of f_n, $\frac{\partial f_n}{\partial x}$ and $\frac{\partial f_n}{\partial y}$ with multiplicities l, $l-1$ and $l-1$, respectively, then d is a divisor of C with multiplicity greater or equal than $l-1$. Also, d divides Δ with multiplicity $\bar{l}$ and follows (iv). ∎

2.3.1 Structure of algebraic curves having double points

We will say that a double point p of $F = 0$ is a *node* (figure 2.5) if $I_p(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}) = 1$, in this case there are two branches and its tangents L_1 and L_2 are different and. We say that the node is *simple* if $I_p(L_i, F) = 3$ for $i = 1, 2$.

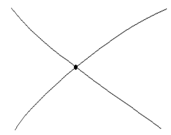

Figure 2.5: Node.

Let p be a non ordinary double point of $F = 0$, that is, it has a unique tangent with multiplicity two. Then we say that p is a *cusp* (figure 2.6) if $I_p(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}) = 2$. In this case, there exists a tangent line at the multiple point but the sense of the tangent is not continuous. The curve, formed by one only branch is at both sides of the tangent line. We say that p is a *tacnode* (figure 2.7) if $I_p(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}) = 3$. In this case, two branches cut themselves with the same tangent line. And p is a *ramphoid cusp* (figure 2.8) if $I_p(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}) = 4$. The only geometric difference with the cusp is that, locally, the curve is at only one side of the tangent.

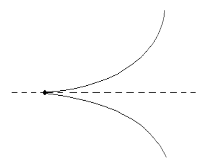

Figure 2.6: Cusp.

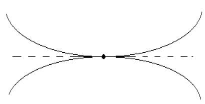

Figure 2.7: Tacnode.

Remark 2.7. A very important fact for us and very useful for the compute of the genus of a projective curve is that if a curve has a tacnode or a ramphoid cusp, then it has a node or a cusp in the first neighborhood, respectively. Nodes or cusps does not have implicit multiple points.

The following result gives us a technical characterization of the curves having a double point.

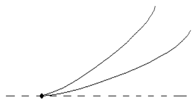

Figure 2.8: Ramphoid cusp.

Proposition 2.8. *Let $F = 0$ be a polynomial curve of degree 4 having a double point p. By making a projectivity and taking local coordinates at p, the curve can be written as $f := f_2 + f_3 + f_4 = 0$ with $f_2 = xy$ if the tangent are different or $f_2 = x^2$ if the tangents are the same. Then*

(i) *p is a node if $f_2 = xy$.*

(ii) *p is a cusp if $f_2 = x^2$ and $x \not| f_3$.*

(iii) *p is a tacnode if $f_2 = x^2$, $f_3 = xg_2$ and $x \not| f_4 - \frac{1}{4}g_2^2$.*

(iv) *p is a ramphoid cusp if $f_2 = x^2$, $f_3 = xg_2$, $x | f_4 - \frac{1}{4}g_2^2$ and $x^2 \not| f_4 - \frac{1}{4}g_2^2$.*

(iv) *$I_p \geq 5$ if $f_2 = x^2$, $f_3 = xg_2$, $x^2 | f_4 - \frac{1}{4}g_2^2$.*

Proof. (i) Since p is a node, $I_p = 1$. Therefore, from Theorem 1.33(v) the two tangents at p are different. Consequently $f_2 = xy$.

(ii) Since p is a cusp, $I_p = 2$. Therefore, from Theorem 1.33(v), the tangents at p are the same. Consequently, $f_2 = x^2$ and $f = x^2 + f_3 + ...$ By deriving f with respect to x and y it follows that $\frac{\partial f}{\partial x} = 2x + ...$, $\frac{\partial f}{\partial y} = \frac{\partial f_3}{\partial y} + ...$ The intersection index of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ is greater than two if and only if x divides $\frac{\partial f_3}{\partial y}$, from Theorem 1.33(v). Since $I_p = 2$, x does not divide $\frac{\partial f_3}{\partial y}$. So x does not divide f_3.

(iii) Since p is a tacnode, $I_p = 3$. By the arguments of the proof of (ii), x divides $\frac{\partial f_3}{\partial y}$. So x divides f_3 and we can write $f_3 = xg_2$. We have $f = x^2 + xg_2 + f_4$. Deriving with respect to x and y we have

$$\frac{\partial f}{\partial x} = 2x + x\frac{\partial g_2}{\partial x} + g_2 + \frac{\partial f_4}{\partial x}, \qquad \frac{\partial f}{\partial y} = x\frac{\partial g_2}{\partial y} + \frac{\partial f_4}{\partial y}.$$

Then

$$\begin{aligned} I_p\left(\tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}\right) &= I_p\left(2x + x\tfrac{\partial g_2}{\partial x} + g_2 + \tfrac{\partial f_4}{\partial x}, x\tfrac{\partial g_2}{\partial y} + \tfrac{\partial f_4}{\partial y}\right) \\ &= I_p\left(2x + x\tfrac{\partial g_2}{\partial x} + g_2 + \tfrac{\partial f_4}{\partial x}, x\tfrac{\partial g_2}{\partial y} + \tfrac{\partial f_4}{\partial y} - \tfrac{1}{2}\tfrac{\partial g_2}{\partial y}\left(2x + x\tfrac{\partial g_2}{\partial x} + g_2 + \tfrac{\partial f_4}{\partial x}\right)\right) \\ &= I_p\left(2x + x\tfrac{\partial g_2}{\partial x} + g_2 + \tfrac{\partial f_4}{\partial x}, \tfrac{\partial}{\partial y}(f_4 - \tfrac{1}{4}g_2^2) - \tfrac{1}{2}x\tfrac{\partial g_2}{\partial x}\tfrac{\partial g_2}{\partial y} - \tfrac{1}{2}\tfrac{\partial g_2}{\partial y}\tfrac{\partial f_4}{\partial x}\right), \end{aligned}$$

and from Theorem 1.33(v), since $I_p = 3$, x does not divide $\frac{\partial}{\partial y}(f_4 - \frac{1}{4}g_2^2)$, and therefore x does not divide $f_4 - \frac{1}{4}g_2^2$.

(iv) Since p is a ramphoid cusp, $I_p = 4$. By the arguments of the proof of (iii), $x | f_4 - \frac{1}{4}g_2^2$. Therefore $f_4 - \frac{1}{4}g_2^2 = xv_3$ for some homogeneous polynomial v_3 of degree 3, and then

$$\begin{aligned}
I_p\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) &= I_p\left(2x + x\frac{\partial g_2}{\partial x} + g_2 + \frac{\partial f_4}{\partial x}, x\left(\frac{\partial v_3}{\partial y} - \frac{1}{2}\frac{\partial g_2}{\partial x}\frac{\partial g_2}{\partial y}\right) - \frac{1}{2}\frac{\partial g_2}{\partial y}\frac{\partial f_4}{\partial x}\right) \\
&= I_p\Bigg(2x + x\frac{\partial g_2}{\partial x} + g_2 + \frac{\partial f_4}{\partial x}, x\left(\frac{\partial v_3}{\partial y} - \frac{1}{2}\frac{\partial g_2}{\partial x}\frac{\partial g_2}{\partial y}\right) - \frac{1}{2}\frac{\partial g_2}{\partial y}\frac{\partial f_4}{\partial x} - \\
&\qquad \frac{1}{2}\left(\frac{\partial v_3}{\partial y} - \frac{1}{2}\frac{\partial g_2}{\partial x}\frac{\partial g_2}{\partial y}\right)\left(2x + x\frac{\partial g_2}{\partial x} + g_2 + \frac{\partial f_4}{\partial x}\right)\Bigg) \\
&= I_p\Bigg(2x + x\frac{\partial g_2}{\partial x} + g_2 + \frac{\partial f_4}{\partial x}, -\frac{1}{2}\frac{\partial g_2}{\partial y}\frac{\partial f_4}{\partial x} - \frac{1}{2}\left(x\frac{\partial g_2}{\partial x} + g_2\right)\left(\frac{\partial v_3}{\partial y} - \frac{1}{2}\frac{\partial g_2}{\partial x}\frac{\partial g_2}{\partial y}\right) - \\
&\qquad \frac{1}{2}\left(\frac{\partial v_3}{\partial y} - \frac{1}{2}\frac{\partial g_2}{\partial x}\frac{\partial g_2}{\partial y}\right)\frac{\partial f_4}{\partial x}\Bigg).
\end{aligned}$$

Since $I_p = 4$, x does not divide

$$\begin{aligned}
&-\frac{1}{2}\frac{\partial g_2}{\partial y}\frac{\partial f_4}{\partial x} - \frac{1}{2}g_2\frac{\partial v_3}{\partial y} + \frac{1}{4}g_2\frac{\partial g_2}{\partial x}\frac{\partial g_2}{\partial y} = -\frac{1}{2}\frac{\partial g_2}{\partial y}\left(\frac{\partial f_4}{\partial x} - \frac{1}{2}g_2\frac{\partial g_2}{\partial x}\right) - \frac{1}{2}g_2\frac{\partial v_3}{\partial y} \\
&= -\frac{1}{2}\frac{\partial g_2}{\partial y}\left(v_3 + x\frac{\partial v_3}{\partial x}\right) - \frac{1}{2}g_2\frac{\partial v_3}{\partial y} = -\frac{1}{2}\frac{\partial(g_2v_3)}{\partial y} - \frac{1}{2}x\frac{\partial v_3}{\partial x}.
\end{aligned}$$

Thus, x does not divide $\frac{\partial(g_2v_3)}{\partial y}$ and therefore, x does not divide v_3g_2. In particular x does not divide v_3, and x^2 does not divide $f_4 - \frac{1}{4}g_2^2$.

(v) If $I_p \geq 5$, by the arguments used in the proof of (iv) we obtain that x^2 divides $f_4 - \frac{1}{4}g_2^2$. Hence the proposition is proved. ∎

Lemma 2.9. *Let p be a simple point of $G = 0$ and a double point of $F = 0$ with $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \geq 2$. Let $F = 0$ and $G = 0$ be tangent at p. In local coordinates the curves can be written as $f := x^2 + f_3... = 0$, $g := x + g_2 + ... = 0$.*

(i) *If $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \geq 2$, then $I_p(F, G) \geq 3$.*

Moreover, if $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \geq 3$, then $f_3 = xh_2$ and

(ii) *If $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \geq 3$, then $I_p(F, G) \geq 4$.*

(iii) *If $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \geq 4$ and x divides $h_2 - 2g_2$, then $I_p(F, G) \geq 5$.*

(iv) *If $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \geq 5$ and x divides $h_2 - 2g_2$, then $I_p(F, G) \geq 6$.*

Proof. Clearly, $I_p(f, g) \geq 3$ from Theorem 1.33(v).

When $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \geq 3$, in local coordinates

$$\begin{aligned}
I_p(f, g) = I_p(x^2 + xh_2 + ..., x + g_2 + ...) &= I_p\left(x^2 + xh_2 + ... - x(x + g_2 + ...), x + g_2 + ...\right) \\
&= I_p\left(x(h_2 - g_2) + ..., x + g_2 + ...\right),
\end{aligned}$$

and from Theorem 1.33(v) the intersection index must be ≥ 4.

If $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \geq 4$,

$$\begin{aligned}
I_p(f,g) &= I_p(x^2 + xh_2 + f_4 + \ldots, x + g_2 + \ldots) \\
&= I_p\left(x^2 + xh_2 + f_4 + \ldots - x(x+g_2) + \ldots, x + g_2 + \ldots\right) \\
&= I_p\left(x(h_2 - g_2) + f_4 + \ldots, x + g_2 + \ldots\right) \\
&= I_p\left(x(h_2 - g_2) + f_4 - (h_2 - g_2)(x + g_2) + \ldots, x + g_2 + \ldots\right) \\
&= I_p\left(f_4 + g_2(h_2 - g_2) + \ldots, x + g_2 + \ldots\right) \geq 5
\end{aligned}$$

if x divides $f_4 - g_2(h_2 - g_2) = (f_4 - \frac{1}{4}h_2^2) + (\frac{1}{2}h_2 - g_2)^2$.

The same argument can be used when $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \geq 5$, taking into account that x^2 divides $f_4 - g_2(h_2 - g_2) = (f_4 - \frac{1}{4}h_2^2) + (\frac{1}{2}h_2 - g_2)^2$, and so the last statement holds. ∎

2.4 Some results on quadratic systems

The following results are valid for quadratic systems (1.6), and they show some situations in which limit cycles can not appear.

Lemma 2.10. *Let $f := \sum_{i=0}^{n} f_i = 0$ be an invariant algebraic curve of (1.6) of degree n, that does not have multiple points in the infinite straight line. Let d_1 and d_2 be two linear divisors of f_n, real or complex, with multiplicity strictly greater than one. If $h' \geq 3$ or $h' = 2$ and $h_2' = 0$, then system (1.6) has a rational first integral.*

Proof. If $h' = 4$, then system (1.6) has a rational first integral by Theorem 2.5. Suppose that $h' = 3$; since f_n has at least two different divisors, from Lemma 2.6(iii) results $h_2' \leq 1$, and thus $h_1' \geq 2$. If $h' = 2$ and $h_2' = 0$, then $h_1' = 2$. In both cases $h_1' \geq 2$.

Let $k = k_0 + k_1$ be the cofactor of f with k_i homogeneous polynomials of degree i. Since $h_1' \geq 2$, the cofactor has either at least two singular points or one singular point with multiplicity greater or equal than 2.

We claim that if a linear divisor d_i, divides f_{n-1}, then there is a multiple point at $Z = 0$. To prove the claim, without loss of generality we can assume that $d_i = x$ and that $f = f_0 + f_1 + \ldots + f_{n-2} + xg_{n-2} + x^s g_{n-s}$ with $s \geq 2$. Then it is easy to check that the point $(0,1,0)$ is a multiple point. So the claim is proved.

Since the curve does not have multiple points in the infinite, from Lemma 2.6(iv) it follows that d_1 and d_2 are both divisors of $k_1 y - (n-1)Q_2$ and $k_1 x - (n-1)P_2$. Therefore,

$$P_2 = \lambda_1 d_1 d_2 + \frac{k_1 x}{n-1}, \qquad Q_2 = \lambda_2 d_1 d_2 + \frac{k_1 y}{n-1}.$$

Now system (1.6) takes the form

$$\dot{x} = P_0 + P_1 + \lambda_1 d_1 d_2 + \frac{k_1 x}{n-1}, \quad \dot{y} = Q_0 + Q_1 + \lambda_2 d_1 d_2 + \frac{k_1 y}{n-1},$$

and so, can be written as

$$\dot{x} = P_0 + P_1 - \frac{k_0 x}{n-1} + \lambda_1 d_1 d_2 + \frac{kx}{n-1}, \quad \dot{y} = Q_0 + Q_1 - \frac{k_0 y}{n-1} + \lambda_2 d_1 d_2 + \frac{ky}{n-1}.$$

By making the change $z = \lambda_2 x - \lambda_1 y$, we have $\dot{z} = B + \frac{kz}{n-1}$ where

$$B = \lambda_2 \left(P_0 + P_1 - \frac{k_0 x}{n-1} \right) - \lambda_1 \left(Q_0 + Q_1 - \frac{k_0 y}{n-1} \right).$$

The polynomials B and k have and B vanishes over the two points of $k = 0$ that are singular points. So we have $B = ak$. Therefore, $\dot{z} = k \left(a + \frac{z}{n-1} \right)$, that is, $a + \frac{\lambda_2 x - \lambda_1 y}{n-1} = 0$ is an invariant straight line with cofactor $\frac{k}{n-1}$. Thus, $H = f(a + \frac{\lambda_2 x - \lambda_1 y}{n-1})^{1-n}$ is a rational first integral of the system. ∎

Proposition 2.11. *Let $F = 0$ be an irreducible invariant algebraic curve of degree 4 of a quadratic system (1.6). Suppose that the curve has two multiple points over $Z = 0$, then*

(i) *If $h' \geq 3$, then the system has a rational first integral.*

(ii) *If the two multiple points are cusps, then either the system has a rational first integral, or the curve has three cusps.*

Proof. Let p_1 and p_2 be two multiple points on the infinite straight line. We can consider, without loss of generality, that $p_1 = (1 : 0 : 0)$ and $p_2 = (0 : 1 : 0)$ if they are real, or $p_1 = (1 : i : 0)$ and $p_2 = (1 : -i : 0)$ if they are complex. In both cases we can write

$$f = D^2 + D(m_{21}x + m_{12}y) + m_{20}x^2 + m_{11}xy + m_{02}y^2 + m_{10}x + m_{01}y + m_{00},$$

where $D = xy$ if the points are real and $D = x^2 + y^2$ if the points are complex. By means of a linear change we can write

$$f = D^2 + f_2 + f_1 + f_0,$$

where f_i are homogeneous polynomials of degree $i = 0, 1, 2$. Clearly, from Lemma 2.6(iii) we have $h'_2 = 1$.

Since the above curve is invariant for the flux defined by (1.6), developing (1.3) according to the different powers, we obtain for the terms of fifth and fourth degree after a simplification, the following

$$P_2 \frac{\partial D}{\partial x} + Q_2 \frac{\partial D}{\partial y} = \frac{k_1 D}{2}, \qquad P_1 \frac{\partial D}{\partial x} + Q_1 \frac{\partial D}{\partial y} = \frac{k_0 D}{2}.$$

Deriving D with respect to t and taking into account the above relations we have

$$\dot{D} = \dot{x} \frac{\partial D}{\partial x} + \dot{y} \frac{\partial D}{\partial y} = \frac{k}{2} D + l \tag{2.8}$$

where l is a linear function.

To prove (i). If $h' \geq 3$, then $h'_1 \geq 2$ because $h'_2 = 1$, and therefore there are at least two singular points of the system on the cofactor taking into account multiplicities, that is $l = ak$. Then equation (2.8) can be written as $\dot{D} = \frac{k}{2}(D + 2a)$ and $H = f(D + 2a)^{-2}$ is a rational first integral of the system.

To prove (ii). If the two multiple points are cusps, we have $f_2 = m_{11}D$ and

$$P_0\frac{\partial D}{\partial x} + Q_0\frac{\partial D}{\partial y} = \frac{m_{11}k_1}{4}.$$

Then (2.8) can be written as

$$\dot{D} = \dot{x}\frac{\partial D}{\partial x} + \dot{y}\frac{\partial D}{\partial y} = \frac{k}{2}\left(D + \frac{m_{11}}{2}\right) - m_{11}\frac{k_0}{4}. \tag{2.9}$$

- If $h'_1 > 0$, there must exist a singular point on the cofactor, and thus $D + \frac{m_{11}}{2} = 0$ is an invariant curve of system (1.6) with cofactor $\frac{k}{2}$, and therefore $H = f(D + \frac{m_{11}}{2})^{-2}$ is a rational first integral of the system.
- If $h'_1 = 0$, then $h' \leq h'_1 + h'_2 = 1$ and therefore $h \geq 6$. But for a curve with two cusps we have $h = 4$. Since the maximum number of multiple points on an irreducible quartic algebraic curve is three, there must be another multiple point and will be a cusp, too.

∎

2.5 Proof of the main result

Let $f = 0$ be an irreducible invariant algebraic curve with real coefficients of a quadratic system (1.6). Suppose that it contains an oval that is a limit cycle of the system. Let $F = 0$ be the equation of the curve in the projective plane. Then, from (2.2) one has $h + h' \geq 7$. The existence of a rational first integral excludes the existence of a limit cycle. So, from (Theorem 2.5 se must have $h' < 4$ and it follows $h \geq 4$.

Taking into account Theorem 1.27, a quartic curve can have, at most, one triple point or three double points.

A. The curve $F = 0$ has a triple point p

In this case the curve can not have any oval. In case it exists, as p is real we can draw a straight line containing p and another point q in the bounded region defined by the oval. This straight line has five common points with the quartic curve, counted with their multiplicity. From Bézout Theorem, the curve is not irreducible.

B. The curve $F = 0$ has three double points p_1, p_2, p_3

At least one of the three multiple points must be real because when a curve has a complex point it has also the conjugated but only 3 points are allowed.

On the other hand $h \geq 4$, the genus (1.8) of a curve is never a negative integer, and taking into account Remark 2.7, the double points can be only cusps or nodes because the existence of more degenerated points implies the existence of implicit double points, which forces the genus to be negative. We have the following possibilities:

B.1. p_1 **is a cusp and** p_2 **and** p_3 **are nodes.**

In this case p_1 is a real point and since it is a cusp, its tangent must be real. So, the conic that contains the points p_1, p_2, p_3, a point q in the bounded region defined by the oval and is tangent to p_1, cuts the curve $F = 0$ in nine points, which is not possible from Bezout Theorem if the curve is irreducible.

B.2. p_1 **and** p_2 **are cusps and** p_3 **is a node.**

In this case p_1 and p_2 can not be real. If these were real, its tangents would be real too and the conic that contains the points p_1, p_2, p_3, a point q in the bounded region of the oval, and is tangent to p_1 cuts the curve $F = 0$ in nine points, which is not possible if the curve is irreducible.

The node p_3 can not have real tangents. It can be seen using the same conic now tangent to p_3. And using this conic not tangent to p_3 but containing a simple real point r which does not belong to the oval it follows that the only real points of the curve $F = 0$ are p_3 and the points of the oval.

From Proposition 2.11, neither p_1 nor p_2 can be infinite points. So, $F = 0$ does not cut the infinity and therefore $F_4 = D^2$, where D is a quadratic polinomial irreducible over the real field.

Without loss of generality we can assume that the local expression of the curve in the affine plane is

$$f = m_{00} + m_{10}x + m_{01}y + m_{20}x^2 + m_{11}xy + m_{02}y^2 + m_{30}x^3 +$$
$$m_{21}x^2y + m_{12}xy^2 + m_{03}y^3 + (x^2 + Bxy + Cy^2)^2,$$

where $B^2 - 4C < 0$ and $C \neq 0$.

We can consider $p_1 = (0 : i : 1)$, $p_2 = (0 : -i : 1)$ and $p_3 = (1 : 0 : 1)$. Since p_1 and p_2 are cusps and p_3 is a node, the following expressions must be identically zero. Notice that the last one means that the tangent of $f = 0$ at p_1 and p_2 is double.

$$f(p_{1,2}) = C^2 + m_{00} - m_{02} \pm (m_{01} - m_{03})i, \quad f(p_3) = 1 + m_{00} + m_{10} + m_{20} + m_{30},$$
$$\frac{\partial f}{\partial x}(p_{1,2}) = m_{10} - m_{12} \pm (m_{11} - 2BC)i, \quad \frac{\partial f}{\partial x}(p_3) = 4 + m_{10} + 2\,m_{20} + 3\,m_{30},$$
$$\frac{\partial f}{\partial y}(p_{1,2}) = m_{01} - 3m_{03} \pm 2(m_{02} - 2C^2)i, \quad \frac{\partial f}{\partial y}(p_3) = 2B + m_{01} + m_{11} + m_{21},$$
$$\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial xy}\right)^2\right)(p_{1,2}) = -12B^2C^2 + 48C^3 - 4B^2m_{02} - 8Cm_{02} + 12BCm_{11} - m_{11}^2$$
$$+4m_{12}^2 - 24C^2m_{20} + 4m_{02}m_{20} - 12m_{03}m_{21} \pm 4(3B^2m_{03} + 6Cm_{03} - 6BCm_{12} + m_{11}m_{12} -$$
$$3m_{03}m_{20} + 6C^2m_{21} - m_{02}m_{21})i.$$

We obtain $m_{10} = m_{12} = 2 - 2C^2 + m_{30}$, $m_{20} = -3 + C^2 - 2m_{30}$, $m_{11} = 2BC$, $m_{02} = 2m_{00} = 2C^2$, $m_{21} = -2B(1 + C)$, $m_{01} = m_{03} = 0$. Then the last expression can be written as

$$\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial xy}\right)^2\right)(p_{1,2}) = 4(2 + 2C + m_{30})(2 - 2C + 4C^2 + m_{30})$$

$$\pm 16BC(2+2C+m_{30})i.$$

If $2+2C+m_{30}=0$, then $f=(C-x-Cx+x^2+Bxy+Cy^2)^2$ and is not irreducible. So the only possibility is $2-2C+4C^2+m_{30}=0$ and we have $m_{30}=-2+2C-4C^2$. Also, since $C\neq 0$, it must be verified that $B=0$. In this case f depends only on even powers of y. So, if there is any oval for $f=0$ it is symmetrical with respect to the axis $y=0$ and there must be three intersection points of the curve $f=0$ with that axis: the points of the oval and p_3. In fact,

$$f(x,0)=(-1+x)^2(C^2+2Cx-4C^2x+x^2).$$

The first factor corresponds to p_3 and the second one must have two real roots. The roots are $x=-C+2C^2\pm 2\sqrt{(-1+C)C^3}$. Thus, a necessary condition for the existence of an oval is $C>1$.

Imposing to $f=0$ to be invariant for (1.6) with cofactor $mx+ny+p$, we define

$$M=\sum_{i+j=0}^{5} M_{ij}x^iy^j := \frac{\partial f}{\partial x}P+\frac{\partial f}{\partial y}Q-(mx+ny+p)f.$$

The coefficients M_{ij} must be zero.

In order five we have

$$\begin{aligned}
M_{50} &= 4a_{20}-m,\\
M_{41} &= 4a_{11}+4b_{20}C-n,\\
M_{32} &= 2(2a_{02}+2a_{20}C+2b_{11}C-Cm),\\
M_{23} &= 6a_{02}B+2a_{11}B^2+2B^2b_{02}+4a_{11}C+2a_{20}BC+4b_{02}C+6Bb_{11}C+4b_{20}C^2-\\
&\qquad 2BCm-B^2n-2Cn,\\
M_{14} &= C(4a_{02}+4b_{11}C-Cm),\\
M_{05} &= C^2(4b_{02}-n).
\end{aligned}$$

We obtain $m=4a_{20}$, $n=4b_{02}$, $a_{11}=b_{02}-b_{20}C$ and $a_{02}=(a_{20}-b_{11})C$.

The coefficients of M in order four are

$$\begin{aligned}
M_{40} &= 4a_{10}+2a_{20}-2a_{20}C+4a_{20}C^2-p,\\
M_{31} &= 2(2a_{01}+b_{02}-b_{02}C+2b_{10}C+5b_{20}C+2b_{02}C^2-9b_{20}C^2+6b_{20}C^3),\\
M_{22} &= 2C(2a_{10}-6a_{20}+2b_{01}+5b_{11}+12a_{20}C-9b_{11}C-6a_{20}C^2+6b_{11}C^2-p),\\
M_{13} &= 2C(2a_{01}-b_{02}+3b_{02}C+2b_{10}C-b_{20}C+3b_{20}C^2),\\
M_{04} &= C^2(2a_{20}+4b_{01}-2b_{11}-6a_{20}C+6b_{11}C-p).
\end{aligned}$$

We obtain $p=\frac{1}{2}(8b_{01}-b_{11}+3b_{11}C)$, $a_{10}=\frac{1}{4}(4b_{01}-2b_{11}+3b_{11}C-3b_{11}C^2)$, $a_{01}=C(-b_{10}-b_{20}+3b_{20}C)$, $b_{02}=-3b_{20}C$, $a_{20}=\frac{3}{4}b_{11}$.

In order three we have

$$\begin{aligned}
M_{30} &= \tfrac{1}{2}(8a_{00}+4b_{01}+b_{11}-4b_{01}C+5b_{11}C+8b_{01}C^2-7b_{11}C^2-15b_{11}C^3+18b_{11}C^4),\\
M_{21} &= 2C(2b_{00}+5b_{10}+5b_{20}-9b_{10}C-18b_{20}C+6b_{10}C^2+33b_{20}C^2-18b_{20}C^3),\\
M_{12} &= \tfrac{1}{2}C(8a_{00}-4b_{01}-b_{11}+12b_{01}C-3b_{11}C-3b_{11}C^2+9b_{11}C^3),\\
M_{03} &= 2C^2(2b_{00}-b_{10}-b_{20}+3b_{10}C+12b_{20}C-9b_{20}C^2).
\end{aligned}$$

We obtain $a_{00}=\frac{3}{4}b_{11}C^2(2+3C)$, $b_{00}=-3b_{20}C$, $b_{01}=\frac{-1}{4}b_{11}(1+3C)^2$, $b_{10}=b_{20}(-1+3C)$.

In order two,

$$M_{20} = \frac{3}{2}b_{11}(1-C)C(-1+2C)(1+3C)^2,$$

from where $b_{11} = 0$ since $C > 1$. Then, by making the time rescaling $\frac{dt}{d\tau} = \frac{1}{b_{20}}$ the system is

$$\begin{aligned}\dot{x} &= -4Cxy,\\ \dot{y} &= -3C - x + 3Cx + x^2 - 3Cy^2,\end{aligned}$$

and then the cofactor is $k = -12Cy$. Therefore, $H = \frac{f}{x^3}$ is a rational first integral and there is not any limit cycle.

B.3. p_1, p_2 and p_3 are cusps.

One of these cusps must be real and so we can use the argument used in **B.1.**

C. The curve $F = 0$ has two double points p_1, p_2

Taking into account that $h \geq 4$, this two points can not be nodes or cusps and someone must be more degenerated. On the other hand, from Remark 2.7 and the fact that the genus computed by (1.8) can not be a negative integer, follows that we have two explicit double points but we can not have more than three double points (explicit or implicit). So, p_1 or p_2 must be a node or a cusp. In this case, we have the following possibilities:

C.1. p_1 is a node and p_2 is a tacnode.

Notice that in this case $h = 4$ and $h' = 3$.

The points p_1 and p_2 must be real points because the curve has real coefficients, and can not be complex conjugated because its intersection index is different. Since p_2 is real a real tacnode of a curve with real coefficients , it has a double tangent with real coefficients. Hence, a real tangent.

The tangents to p_1 can not be real. If these was real, the conic that contains p_1, p_2, another point q in the bounded region of the oval and is tangent to p_1 and p_2 cuts the curve $F = 0$ in nine points from Lemma 2.9 (counting their multiplicities). Therefore the curve would not be irreducible from Bézout's Theorem.

Using the same argument with the conic not tangent to p_1 but containing a simple real point of $F = 0$ it follows that the only real points of $F = 0$ are p_1, p_2 and the points of the oval.

One of the points p_1 and p_2 must be in the infinity. If not, the points in the infinity would have to be simple and complex since the only real points of the curve are p_1, p_2 and the points of the oval. Then $f_4 = D^2$ where D is an homogeneous polynomial of degree two irreducible over the real field, and from Lemma 2.10 it follows, now, that the system is integrable.

But only p_2 can be in the infinity. p_1 can not be there because it has complex tangents and the infinite straight line is invariant. If p_2 is in the infinity and there are some other, these must be complex conjugated, so there are three points in the infinity. Then we have $h_2' = 0$ from Lemma 2.6 and since $h' = 3$ it follows $h_1' \geq 3$ from (2.3), which is not possible. On the other hand, since $h' = 3$, from Proposition 2.11 p_1 and p_2 can not be both over the infinite straight line.

Let us consider $p_1 = (0:0:1)$ with complex tangents and $p_2 = (0:1:0)$. Since p_2 is a tacnode, the affine equation of the curve is

$$f = ax^4 + x^2(bx + cy) + x^2 + y^2,$$

which corresponds to case (a) of Theorem 2.1.

C.2. p_1 is a node and p_2 is a ramphoid cusp.

In this case p_1 and p_2 must be real and then the tangent to p_2 is real too. The conic that contains these points, a point q in the bounded region defined by the oval and satisfies Lemma 2.9(iii) for p_2, cuts the curve $F = 0$ in nine points which is not possible if the curve is irreducible.

C.3. p_1 and p_2 are cusps.

By means of a projectivity we suppose that the cusps are $p_1 = (1:0:0)$ and $p_2 = (0:1:0)$ and the projective equation of the curve $F = 0$ is defined by

$$F = X^2Y^2 + \lambda_1 XYZ^2 + (\lambda_2 X + \lambda_3 Y)Z^3 + \lambda_4 Z^4.$$

If $\lambda_2 = 0$ or $\lambda_3 = 0$ then p_1 or p_2 are tacnodes, respectively. The projective differential equation is defined by

$$L = L_2 + L_1 Z + L_0 Z^2, \quad M = M_2 + M_1 Z + M_0 Z^2, \quad N = N_2 + N_1 Z + N_0 Z^2.$$

We can suppose that the cofactor of $F = 0$ is zero.

Since p_1 and p_2 are singular points of the projective differential equation, it is verified that

$$\begin{array}{ll} (LY - MX)(1,0,0) = -M_2(1,0,0) = 0, & (LY - MX)(0,1,0) = L_2(0,1,0) = 0, \\ (LZ - NX)(1,0,0) = -N_2(1,0,0) = 0, & (LZ - NX)(0,1,0) = 0, \\ (MZ - NY)(1,0,0) = 0, & (MZ - NY)(0,1,0) = -N_2(0,1,0) = 0. \end{array}$$

Therefore, $N_2 = a_0 XY$, $L_2 = (a_1 X + a_2 Y)X$ and $M_2 = (a_3 X + a_4 Y)Y$. We consider $L_1 = b_1 X + b_2 Y$, $M_1 = b_3 X + b_4 Y$ and $N_1 = b_5 X + b_6 Y$.

Since $F = 0$ is an invariant curve of the projective differential equation we obtain the following relations corresponding to the coefficients of the different powers of Z.

$$\begin{array}{l} 2XY^2L_2 + 2X^2YM_2 = 0, \\ 2XY^2L_1 + 2X^2YM_1 + 2\lambda_1 XYN_2 = 0, \\ 2XY^2L_0 + 2X^2YM_0 + \lambda_1 YN_2 + \lambda_1 XM_2 + 3(\lambda_2 X + \lambda_3 Y)N_2 + 2\lambda_1 XYN_2 = 0. \end{array}$$

That is

$$\begin{array}{l} 2YL_2 + 2XM_2 = 0, \\ 2YL_1 + 2XM_1 + 2\lambda_1 N_2 = 0, \\ 2YL_0 + 2XM_0 + 3(\lambda_2 X + \lambda_3 Y)a_0 + 2\lambda_1 N_2 = 0. \end{array} \tag{2.10}$$

In particular, from the first equation of (2.10) we obtain

$$a_1 + a_3 = 0, \quad a_2 + a_4 = 0. \tag{2.11}$$

From (2.10) it follows that

$$YL + XM + \lambda_1 ZN = \lambda_1 Z^3 N_0 - \frac{3}{2}(\lambda_2 X + \lambda_3 Y)a_0 Z^2.$$

In other words, if we define $G = XY + \frac{\lambda_1}{2}Z^2$, we have

$$\frac{\partial G}{\partial X}L + \frac{\partial G}{\partial Y}M + \frac{\partial G}{\partial Z}N = Z^2 r, \tag{2.12}$$

where $r = \lambda_1 Z N_0 - \frac{3}{2}(\lambda_2 X + \lambda_3 Y)a_0$.

When a curve has two cusps, we have $h = 4$, and thus $h' = \sum_p I_p(L, M, N) \geq 3$.

- If $\{L = 0\} \cap \{M = 0\} \cap \{N = 0\} \cap \{Z = 0\} = \emptyset$ then $I_p(r, L, M, N) = I_p(L, M, N)$ for all p from (2.12). Then

$$\sum_p I_p(r, \mathcal{P}, \mathbb{Q}, \mathcal{R}) = \sum_p I_p(r, LY - MX, LZ - NX, MZ - NY) \geq \sum_p I_p(r, L, M, N) =$$

$$\sum_p I_p(L, M, N) \geq 3$$

  Thus r is an invariant straight line from Theorem 2.3, that contains the singular points of the differential equation. There is never a foci over an invariant straight line. So, there is not any limit cycle in this case .

- If q belongs to $\{L = 0\} \cap \{M = 0\} \cap \{N = 0\} \cap \{Z = 0\}$ with $q \neq p_1$ and $q \neq p_2$, then the straight line $Z = 0$, that contains p_1, p_2, and q is invariant. In this case $a_0 = 0$ and

$$\frac{\partial G}{\partial X}L + \frac{\partial G}{\partial Y}M + \frac{\partial G}{\partial Z}N = \lambda_1 N_0 Z^3.$$

  Therefore, if $\lambda_1 N_0 \neq 0$ all the singular points that are not over the curve $F = 0$ are over the line $Z = 0$, which is invariant. In particular, the foci of a limit cycle belongs to $Z = 0$ which is not possible. If $\lambda_1 N_0 = 0$, we obtain $H = \frac{F}{G^2}$ as a zero degree homogeneous first integral.

- Suppose that q belongs to $\{L = 0\} \cap \{M = 0\} \cap \{N = 0\} \cap \{Z = 0\}$ with $q = p_1$ or $q = p_2$. We can consider, without loss of generality that $q = p_1$, and then from $L(1, 0, 0) = 0$ we obtain $a_1 = 0$, and from (2.11) $a_3 = 0$

  Taking local coordinates at p_1, the differential equation is $(N - ZL)dy - (M - YL)dz = 0$. Taking into account (2.11), it can be written as a differential system in the form

$$\begin{aligned} \dot{x} &= M - yL = b_3 z + \ldots, \\ \dot{y} &= N - zL = b_5 z + \ldots. \end{aligned} \tag{2.13}$$

  And the curve in local coordinates at p_1 is $f := y^2 + \lambda_1 y z^2 + \lambda_2 z^3 + \lambda_3 y z^3 + \lambda_4 z^4 = 0$. Imposing to it to be an invariant curve of (2.13) with cofactor $k = k_0 + k_1 y + k_2 z$, we obtain

$$(2y + \lambda_1 z^2 + \ldots)(b_3 z + \ldots) + (2\lambda_1 y z + 3\lambda_2 z^2 + \ldots)(b_5 z + \ldots) =$$

$$(k_0 + k_1 y + k_2 z)(y^2 + \lambda_1 y z^2 + \lambda_2 z^3 + \ldots).$$

Each one of the coefficients of the above expression must be zero. For the coefficients of y^2, yz and z^3 we obtain

$$k = 0, \quad 2b_3 = 0, \quad \lambda_1 b_3 + 3\lambda_2 b_5 - \lambda_2 k_0 = 0,$$

from where

$$b_3 = b_5 = 0, \tag{2.14}$$

since $\lambda_2 \neq 0$.

Taking local coordinates in p_2 for the differential equation and the curve, and using the same argument as used for p_1 we obtain

$$b_2 = 0, \quad b_6 = \frac{a_2}{3}, \tag{2.15}$$

Summarizing, from (2.11), (2.14) and (2.15) we obtain

$$\begin{aligned} L &= a_1 XY + b_1 XZ - L_0 Z^2, \\ M &= -a_2 Y^2 + b_4 YZ + M_0 Z^2, \\ N &= a_0 XY + \tfrac{a_2}{3} YZ + N_0 Z^2. \end{aligned}$$

Since $F = 0$ is an invariant algebraic projective curve of the above system with cofactor zero, the function

$$M = \sum_{i+j+k=5} M_{ijk} X^i Y^j Z^k := \frac{\partial F}{\partial X} L + \frac{\partial F}{\partial Y} M + \frac{\partial F}{\partial Z} N,$$

must be identically zero.

$$\begin{aligned} M_{005} &= -L_0\lambda_2 + \lambda_3 M_0 + 4\lambda_4 N_0, \\ M_{014} &= \tfrac{1}{3}(-3L_0\lambda_1 + 3b_4\lambda_3 + 4a_2\lambda_4 + 9\lambda_3 N_0), \\ M_{104} &= b_1\lambda_2 + \lambda_1 M_0 + 3\lambda_2 N_0, \\ M_{113} &= b_1\lambda_1 + b_4\lambda_1 + 2a_2\lambda_2 + 4a_0\lambda_4 + 2\lambda_1 N_0, \\ M_{122} &= \tfrac{1}{3}(-6L_0 + 2a_2\lambda_1 + 9a_0\lambda_3), \\ M_{212} &= 3a_0\lambda_2 + 2M_0, \\ M_{221} &= 2(b_1 + b_4 + a_0\lambda_1). \end{aligned}$$

The above expressions are zero non trivially if

$$\det\left(\frac{\partial[M_{005}, M_{014}, M_{113}, M_{122}, M_{212}, M_{221}]}{\partial[a_2, b_1, L_0, b_4, M_0, a_0, N_0]}\right) = \frac{32}{3}\lambda_2\Omega,$$

where $\Omega = \lambda_1^3\lambda_2\lambda_3 + 27\lambda_2^2\lambda_3^2 - \lambda_1^4\lambda_4 - 36\lambda_1\lambda_2\lambda_3\lambda_4 + 8\lambda_1^2\lambda_4^2 - 16\lambda_4^3$.

When $\lambda_2 = 0$ p_1 is a tacnode and when $\Omega = 0$ the curve $F = 0$ has another multiple point against the hypotesis

Therefore, there are not limit cycles in this case.

C.4. p_1 **is a cusp and** p_2 **is a tacnode.**

In this case the double points are real and their tangents are real too. The conic that contains p_1 p_2, a point q in the bounded region defined by the oval and is tangent to p_1 and p_2, cuts $F = 0$ in nine points which is not possible if the curve is irreducible.

C.5. p_1 **is a cusp and** p_2 **is a ramphoid cusp.**

The above argument can be used again and this case is not possible.

D. The curve $F = 0$ has one double point p_1

Taking into account that $h' \leq 3$ because otherwise there exists a rational first integral, follows that $h \geq 4$. We distinguish two cases:

D.1. p_1 **is ramphoid cusp.**

The point p_1 is a real point, and since $I_{p_1} = 4$, f_4 can not have three different divisors because otherwise $h_2' = 0$ from Lemma 2.6(iii). Then $h_1' \geq 3$ which is not possible because the system is quadratic. We will distinguish two important possibilities

D.1.1. Let the multiple point be finite.

Let us consider $p_1 = (0:0:1)$ a ramphoid cusp. If y is the tangent to the curve on p_1, then we have, since $I_{p_1} = 4$,

$$f = y^2 + yf_2 + f_4,$$

where y divides $f_4 - \frac{1}{4}f_2^2$ and f_4 can take one of the following forms: $f_4 = d_1^2 d_2^2$, $f_4 = d_1^3 d_2$ or $f_4 = kd_1^4$.

D.1.1.1. $f_4 = d_1^2 d_2^2$. Since p_1 is the unique multiple point of the curve, there are not multiple points of the curve in the infinity. On the other hand, $h' \geq 3$ and f_4 has two divisors with both multiplicities strictly greater than one. From Lemma 2.10, the system has a rational first integral.

D.1.1.2. $f_4 = d_1^3 d_2$. The general form of the curve is

$$f = y^2 + y(ax^2 + bxy + cy^2) + x^3(Ax + By).$$

Since y divides $f_4 - \frac{1}{4}f_2^2$ it follows $A = \frac{a^2}{4}$. We will consider $c \neq 0$ because if not there would be a double point in the infinity, $a \neq 0$ since the curve must be irreducible, and $B \neq 0$. The case $B = 0$ will be studied in D.1.1.3.

By making the change $x = X/(Bc)^{1/3}$, $y = Y/c$ we can consider

$$f = y^2 + y(ax^2 + bxy + y^2) + x^3(\frac{a^2}{4}x + y).$$

Imposing to $f = 0$ to be an invariant curve of (1.6) with cofactor $mx + ny + p$, then $a_{00} = b_{00} = 0$ since p_1 is a singular point, and we define

$$M = \sum_{i+j=0}^{5} M_{ij} x^i y^j := \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q - (mx + ny + p)f.$$

We will study the vanishing of the coefficients M_{ij}.

In order five,

$$\begin{aligned}
M_{50} &= \tfrac{1}{4}(4a^2 a_{20} + 4b_{20} - a^2 m), \\
M_{41} &= \tfrac{1}{4}(4a^2 a_{11} + 12a_{20} + 4b_{11} - 4m - a^2 n), \\
M_{32} &= a^2 a_{02} + 3a_{11} + b_{02} - n, \\
M_{23} &= 3a_{02}, \\
M_{14} &= M_{05} = 0,
\end{aligned}$$

from where we can write $a_{02}=0$, $n=3a_{11}+b_{02}$, $m=\frac{1}{4}(a^2a_{11}+12a_{20}-a^2b_{02}+4b_{11})$, $b_{20}=\frac{1}{16}a^2(a^2a_{11}-4a_{20}-a^2b_{02}+4b_{11})$.

In order four we have

$$\begin{aligned}
M_{40} &= \tfrac{1}{16}(16a^2a_{10}+a^5a_{11}-4a^3a_{20}-a^5b_{02}+16b_{10}+4a^3b_{11}-4a^2p),\\
M_{31} &= \tfrac{1}{8}(8a^2a_{01}+24a_{10}-2a^3a_{11}-8aa_{20}+a^4a_{11}b-4a^2a_{20}b+8b_{01}+2a^3b_{02}-\\
&\quad a^4bb_{02}+4a^2bb_{11}-8p),\\
M_{22} &= \tfrac{1}{16}(48a_{01}-16aa_{11}+3a^4a_{11}-12a^2a_{20}-4a^2a_{11}b-32a_{20}b-3a^4b_{02}+4a^2bb_{02}\\
&\quad +12a^2b_{11}+16bb_{11}),\\
M_{13} &= \tfrac{1}{4}(-a^2a_{11}-12a_{20}-8a_{11}b+a^2b_{02}+4bb_{02}+8b_{11}),\\
M_{04} &= -3a_{11}+2b_{02}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
p &= \tfrac{1}{72}(216a_{10}+30a^3a_{11}+3a^6a_{11}+12aa_{11}b-10a^4a_{11}b-2a^2a_{11}b^2+72b_{01}-48ab_{11}\\
&\quad -6a^4b_{11}+20a^2bb_{11}),\\
b_{02} &= \tfrac{3}{2}a_{11},\\
a_{20} &= \tfrac{1}{24}(a^2a_{11}-4a_{11}b+16b_{11}),\\
a_{01} &= \tfrac{1}{72}(24aa_{11}+3a^4a_{11}-4a^2a_{11}b-8a_{11}b^2-6a^2b_{11}+8bb_{11}),\\
b_{10} &= \tfrac{1}{288}a^2(-72a_{10}+42a^3a_{11}+3a^6a_{11}-10a^4a_{11}b-2a^2a_{11}b^2+72b_{01}-72ab_{11}-\\
&\quad 6a^4b_{11}+20a^2bb_{11}),
\end{aligned}$$

In order three,

$$\begin{aligned}
M_{30} &= \tfrac{1}{288}a^3\,(-72\,a_{10}+42\,a^3\,a_{11}+3\,a^6\,a_{11}-10\,a^4\,a_{11}\,b-2\,a^2\,a_{11}\,b^2+72\,b_{01}-\\
&\quad 72\,a\,b_{11}-6\,a^4\,b_{11}+20\,a^2\,b\,b_{11},\\
M_{21} &= \tfrac{1}{144}a\,(-144\,a_{10}-72\,a^3\,a_{11}-6\,a^6\,a_{11}-72\,a\,a_{10}\,b-12\,a\,a_{11}\,b+62\,a^4\,a_{11}\,b+\\
&\quad 3\,a^7\,a_{11}\,b+4\,a^2\,a_{11}\,b^2-10\,a^5\,a_{11}\,b^2-2\,a^3\,a_{11}\,b^3+72\,a\,b\,b_{01}+120\,a\,b_{11}+\\
&\quad 12\,a^4\,b_{11}-112\,a^2\,b\,b_{11}-6\,a^5\,b\,b_{11}+20\,a^3\,b^2\,b_{11}),\\
M_{12} &= \tfrac{1}{288}(-216\,a^2\,a_{10}+192\,a^2\,a_{11}+150\,a^5\,a_{11}+9\,a^8\,a_{11}-576\,a_{10}\,b+144\,a_{11}\,b-\\
&\quad 152\,a^3\,a_{11}\,b-42\,a^6\,a_{11}\,b-112\,a\,a_{11}\,b^2+34\,a^4\,a_{11}\,b^2+8\,a^2\,a_{11}\,b^3+\\
&\quad 216\,a^2\,b_{01}+288\,b\,b_{01}-288\,b_{11}-264\,a^3\,b_{11}-18\,a^6\,b_{11}+256\,a\,b\,b_{11}+\\
&\quad 84\,a^4\,b\,b_{11}-80\,a^2\,b^2\,b_{11}),\\
M_{03} &= \tfrac{1}{72}(-216\,a_{10}-108\,a_{11}-30\,a^3\,a_{11}-3\,a^6\,a_{11}+12\,a\,a_{11}\,b+13\,a^4\,a_{11}\,b-\\
&\quad 2\,a^2\,a_{11}\,b^2-8\,a_{11}\,b^3+144\,b_{01}+48\,a\,b_{11}+6\,a^4\,b_{11}-26\,a^2\,b\,b_{11}+8\,b^2\,b_{11}).
\end{aligned}$$

Considering the system of equations given by $M_{30}=M_{21}=M_{12}=M_{03}=0$, with respect to a_{10}, a_{11}, b_{01} and b_{11}, we will look for a non trivial solution because for the trivial one we have $P=Q=0$.

$$\det\left(\frac{\partial[M_{30},M_{21},M_{12},M_{03}]}{\partial[a_{10},a_{11},b_{01},b_{11}]}\right)=\frac{1}{576}a^4(-2+ab)(-1+ab)(108+8a^3-36ab-a^2b^2+4b^3)$$

and this expression must be zero to get a non trivial solution.

If $-2+ab=0$ or $108+8a^3-36ab-a^2b^2+4b^3=0$ there exists another multiple point, and so, since $a\neq 0$ it must happen that $-1+ab=0$. In this case if we take $b=\frac{1}{a}$, then we obtain the system of Theorem 2.1(c) by making the change $x=aX$, and the parameter is $\frac{a^3}{2}$.

D.1.1.3. $f_4=kd_1^4$. By making a linear change we can consider

$$f=y^2+y(ax^2+bxy+cy^2)+kx^4.$$

Since $I_{p_1} = 4$, y divides $f_4 - \frac{1}{4}f_2^2$, and then $k = \frac{1}{4}a^2$.

By making the rescaling $x = X/A$, $y = Y/B$ with $A = \frac{-b}{2}$ and $B = \frac{3b^2}{2(2b^2-3ac)}$ we obtain case (b) of Theorem 2.1 with a factor of proportionality $\frac{27b^6c^2}{2(2b^2-3ac)}$ and where the parameter is $\frac{3ac}{2b^2-3ac}$.

If $2b^2 - 3ac = 0$ we obtain $f = 24x^4 + 24x^2y + 6y^2 + 6xy^2 + y^3$, which does not have any oval.

D.1.2. Let the multiple point be infinite.

We can consider $p_1 = (0 : 1 : 0)$ the multiple point. Then $f = f_4 + xg_2 + f_2 + f_1 + f_0$, where f_4 can not have three different divisors and after linear changes of variables can be written in one of the following forms: $f_4 = x^2y^2$, $f_4 = x^3y$ or $f_4 = \alpha x^4$.

D.1.2.1. $f_4 = x^2y^2$. In this case,

$$f = x^2y^2 + x(ax^2 + bxy + cy^2) + m_{20}x^2 + m_{11}xy + m_{02}y^2 + m_{10}x + m_{01}y + m_{00}.$$

Notice that we can consider $b = c = 0$ by making the translation $x = X - \frac{c}{2}$, $y = Y - \frac{b}{2}$. Since $I_{p_1} = 4$ and the tangent of $f = 0$ on p_1 is x, we obtain $m_{02} = m_{01} = 0$ and $m_{00} = \frac{m_{11}^2}{4}$ and then

$$f = (\frac{m_{11}}{2} + xy)^2 + m_{10}x + m_{20}x^2 + ax^3.$$

Doing the change $x = m_{11}X$ we can take $m_{11} = 1$ since m_{11} must not be zero because if not the curve would be reducible. Since the branches of $f = 0$ are defined by $\frac{-1}{2x} \pm \frac{1}{x}\sqrt{-x(m_{10} + m_{20}x + ax^2)}$, a must not be zero because if not the curve would not have ovals.

Imposing to $f = 0$ to be invariant along the flow defined by (1.6), with cofactor $mx + ny + p$, we define

$$M = \sum_{i+j=0}^{5} M_{ij}x^iy^j := \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q - (mx + ny + p)f,$$

and we will study the vanishing of the coefficients M_{ij}.

The coefficients of the terms of order five are:

$$\begin{aligned}
M_{50} &= 0,\\
M_{41} &= 2b_{20},\\
M_{32} &= 2a_{20} + 2b_{11} - m,\\
M_{23} &= 2a_{11} + 2b_{02} - n,\\
M_{14} &= 2a_{02},\\
M_{05} &= 0,
\end{aligned}$$

from where $b_{20} = a_{02} = 0$, $m = 2(a_{20} + b_{11})$, $n = 2(a_{11} + b_{02})$.

For the coefficients of the terms of fourth order of M we have

$$\begin{aligned}
M_{40} &= a(a_{20} - 2b_{11}),\\
M_{31} &= aa_{11} - 2ab_{02} + 2b_{10},\\
M_{22} &= 2a_{10} + 2b_{01} - p,\\
M_{13} &= 2a_{01},\\
M_{04} &= 0,
\end{aligned}$$

and then $a_{20} = 2b_{11}$, $b_{10} = \frac{1}{2}(2ab_{02} - aa_{11})$, $p = 2(a_{10} + b_{01})$, $a_{01} = 0$.

In order three we have

$$\begin{aligned}
M_{30} &= aa_{10} - 2ab_{01} - b_{11}m_{20}, \\
M_{21} &= 2b_{00} - 3b_{11} - 2b_{02}m_{20}, \\
M_{12} &= 2a_{00} - a_{11} - b_{02}, \\
M_{03} &= 0,
\end{aligned}$$

and from the vanishing of these coefficients we obtain $a_{10} = \frac{2}{a}(ab_{01} + b_{11}m_{20})$, $b_{00} = \frac{1}{2}(3b_{11} + 2b_{02}m_{20})$, $a_{00} = \frac{1}{2}(a_{11} + b_{02})$.

The rest of coefficients of M are

$$\begin{aligned}
M_{20} &= \tfrac{1}{2}(-8b_{11}m_{10} + 2aa_{11} + 5ab_{02} - 4b_{01}m_{20}), \\
M_{11} &= \tfrac{-1}{a}(aa_{11}m_{10} + 2b_{02}m_{10} + 3ab_{01} + 2b_{11}m_{20}), \\
M_{02} &= 0, \\
M_{10} &= \tfrac{1}{a}(-4ab_{01}m_{10} - 2b_{11}m_{10}m_{20} + aa_{11}m_{20} + 2ab_{02}m_{20}), \\
M_{01} &= 0, \\
M_{00} &= \tfrac{1}{2a}(aa_{11}m_{10} + ab_{02}m_{10} - 3ab_{01} - 2b_{11}m_{20}).
\end{aligned}$$

Let us consider

$$\det\Big(\frac{\partial[M_{20}, M_{11}, M_{10}, M_{00}]}{\partial[a_{11}, b_{01}, b_{11}, b_{02}]}\Big) = \frac{2}{a}m_{10}(m_{10}^2 +^2 m_{20})(4am_{10} - m_{20}^2).$$

- If $m_{10} = 0$ the branches of $f = 0$ are defined by $y = \frac{-1}{2x} \pm \sqrt{-m_{20} - ax}$ which does not define an oval.
- If $4am_{10} - m_{20}^2 = 0$ the branches of $f = 0$ are defined by $y = \frac{-1}{2x} \pm \frac{m_{20}+2ax}{2\sqrt{-ax}}$, which does not define any oval.
- If $m_{10}^2 + m_{20} = 0$ and $m_{10} \neq 0$, we obtain $m_{20} = -m_{10}^2$. Then from $M_{20} = 0$ we obtain $b_{02} = \frac{2}{5a}(-2b_{01}m_{10}^2 + 4b_{11}m_{10} - aa_{11})$, and then from $M_{11} = 0$ we have $b_{11} = \frac{1}{6m_{10}^2}8b_{01}m_{10}^3 - aa_{11}m_{10} - 15ab_{01}$. Now, since $M_{10} = -b_{01}m_{10}$ and $m_{10} \neq 0$ it follows $b_{01} = 0$.

  Taking $a = km_{10}^3$, and by making the change of variables $x = X/m_{10}$, $y = m_{10}Y$, (1.6) takes the form

  $$\dot{x} = \frac{m_{10}}{6}(1 + 2x - 2kx^2 + 6xy), \quad \dot{y} = \frac{m_{10}}{12}(8 - 3k - 14kx - 2kxy - 8y^2),$$

  the curve is $f = \frac{1}{4} + x - x^2 + kx^3 + xy + x^2y^2$, with cofactor $\frac{m_{10}}{3}(2 - 3kx + 2y)$. The branches of $f = 0$ are defined by $y = \frac{-1}{2x} \pm \frac{\sqrt{-1+x-kx^2}}{\sqrt{x}}$ and there exists an oval if $1 - 4k > 0$, that is $k < \frac{1}{4}$. This case corresponds to the new system, case (d) of Theorem 2.1.

D.1.2.2. $f_4 = x^3y$. In this case

$$f = x^3y + x(ax^2 + bxy + cy^2) + m_{20}x^2 + m_{11}xy + m_{02}y^2 + m_{10}x + m_{01}y + m_{00}.$$

If we consider the projectivization of the curve we see that the tangent of the curve on p_1 is Z and the terms of order three are $X^3 + bX^2Z + m_{20}X^2Z + m_{01}Z^3$. Since p_1 is a ramphoid cusp, this case is not allowed because Z does not divide the terms of order three.

D.1.2.3. $f_4 = \alpha x^4$. In this case

$$f = \alpha x^4 + x(ax^2 + bxy + cy^2) + m_{20}x^2 + m_{11}xy + m_{02}y^2 + m_{10}x + m_{01}y + m_{00}.$$

Since p_1 is a ramphoid cusp and the tangent of the curve on this point is Z we obtain $c = 0$, $m_{02} = 1$ and $\alpha = \frac{b^2}{4} \neq 0$. The change of variables $x = X + A$, $y = Y + kX + B$, with $A = \frac{-1}{b^2}(a + bk)$, $B = \frac{-1}{2b^3}(a^2 + 2abk + b^2k^2 + b^3m_{01} - abm_{11} - b^2km_{11})$, and $k = \frac{-\left(-4\,a^2 - b^3 m_{01} + a\,b\,m_{11} + 2\,b^2\,m_{20}\right)}{3\,b\,(-2\,a + b\,m_{11})}$ vanishes the coefficients of x^3, y and x^2. Notice that k is well defined since the denominator never vanishes, because if $-2\,a + b\,m_{11} = 0$ then $I_{p_1} = 5$ which is studied later. So we can consider, making a rescaling of the variable x that $m_{11} = 1$, and then $f = \frac{b^2}{4}x^4 + bx^2y + y^2 + xy + m_{10}x + m_{00}$. Let us propose f as a particular solution of (1.6) with cofactor $cf = mx + ny + p$ and define

$$M = \sum_{i+j=0}^{5} M_{ij}x^iy^j := \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q - (mx + ny + p)f,$$

which must be identically zero and so we will study the vanishing of the coefficients M_{ij}.

In order five,

$$\begin{aligned}
M_{50} &= \tfrac{b^2}{4}(4a_{20} - m),\\
M_{41} &= \tfrac{b^2}{4}(4a_{11} - n),\ M_{32} = a_{02}b^2,\\
M_{23} &= M_{14} = M_{05} = 0,
\end{aligned}$$

from where $m = 4a_{20}$, $n = 4a_{11}$ and $a_{02} = 0$ since $b \neq 0$.

In order four, the coefficients are

$$\begin{aligned}
M_{40} &= \tfrac{b}{4}(4\,a_{10}\,b + 4\,b_{20} - b\,p),\\
M_{31} &= b\,(-2\,a_{20} + a_{01}\,b + b_{11}),\\
M_{22} &= b\,\left(-2\,a_{11} + b_{02}\right),\\
M_{13} &= M_{04} = 0,
\end{aligned}$$

from where $b_{20} = \frac{b}{4}(p - 4a_{10})$, $b_{11} = 2a_{20} - a_{01}b$ and $b_{02} = 2a_{11}$.

For the coefficients of the terms of third order, we have

$$\begin{aligned}
M_{30} &= \tfrac{b}{4}(-4\,a_{10} + 4\,a_{00}\,b + 4\,b_{10} + p),\\
M_{21} &= \tfrac{1}{2}(-2a_{20} - 2\,a_{01}\,b + 2\,b\,b_{01} - b\,p),\\
M_{12} &= -a_{11},\\
M_{03} &= 0,
\end{aligned}$$

from where $p = 4(a_{10} - a_{00}b - b_{10})$, $a_{11} = 0$ and $a_{20} = b(-a_{01} - 2a_{10} + 2a_{00}b + b_{01} + 2b_{10})$.

In order two,

$$\begin{aligned}
M_{20} &= b\,b_{00} + b_{10} + 3\,a_{01}\,b\,m_{10} + 6\,a_{10}\,b\,m_{10} - 6\,a_{00}\,b^2\,m_{10} - 3\,b\,b_{01}\,m_{10} - 6\,b\,b_{10}\,m_{10},\\
M_{11} &= -3\,a_{10} + 6\,a_{00}\,b + b_{01} + 6\,b_{10},\\
M_{02} &= a_{01} - 4\,a_{10} + 4\,a_{00}\,b + 2\,b_{01} + 4\,b_{10},
\end{aligned}$$

from where $a_{01} = 2\,(-a_{10} + 4\,a_{00}\,b + 4\,b_{10})$, $b_{01} = 3\,(a_{10} - 2\,a_{00}\,b - 2\,b_{10})$, and $b_{00} = \frac{1}{b}(-b_{10} + 9\,a_{10}\,b\,m_{10} - 36\,a_{00}\,b^2\,m_{10} - 36\,b\,b_{10}\,m_{10})$.

For the lower order terms, the coefficients to vanish are the following ones:

$$\begin{aligned}
M_{10} &= \tfrac{1}{b}(-b_{10} - 12\,a_{10}\,b^2\,m_{00} + 48\,a_{00}\,b^3\,m_{00} + 48\,b^2\,b_{10}\,m_{00} + 6\,a_{10}\,b\,m_{10} - \\
&\qquad 32\,a_{00}\,b^2\,m_{10} - 32\,b\,b_{10}\,m_{10}),\\
M_{01} &= \tfrac{1}{b}(a_{00}\,b - 2\,b_{10} + 16\,a_{10}\,b\,m_{10} - 64\,a_{00}\,b^2\,m_{10} - 64\,b\,b_{10}\,m_{10}),\\
M_{00} &= -4\,a_{10}\,m_{00} + 4\,a_{00}\,b\,m_{00} + 4\,b_{10}\,m_{00} + a_{00}\,m_{10}).
\end{aligned}$$

We can solve non trivially a_{00}, b_{10} and a_{10} from $M_{10} = M_{01} = M_{00} = 0$n if

$$\det\Big(\frac{\partial[M_{10}, M_{01}, M_{00}]}{\partial[a_{00}, b_{10}, a_{10}]}\Big) = \frac{4}{b}(-m_{00} + 108b^2m_{00}^2 - 36bm_{00}m_{10} - m_{10}^2 - 32bm_{10}^3) = 0.$$

The vanishing of this determinant is a condition for the existence of another multiple point, too. By hypothesis, this is not allowed.

D.2. p_1 is a double point with $I_{p_1} \geq 5$.

The only real points are p_1 and the points that are on the oval. If r is another real point, the conic that contains r, a point q in the bounded region of the oval and satisfies Lemma 2.9(iv) on p_1, cuts the quartic with index greater or equal than six in p_1, index one in r, and cuts the oval in two points, which is not possible from Bézout's Theorem if the curve is irreducible.

D.2.1. Let p_1 be a finite point. We can consider $p_1 = (0 : 0 : 1)$ and that the tangent to the curve in p_1 is x. Then from Lemma 2.8, in local coordinates $f = x^2 + xf_2 + f_4$ and x^2 divides $f_4 - \frac{1}{4}f_2^2$.

Since there are not real points at the infinity we have $f_4 = \lambda D^2$ where D is a quadratic polynomial irreducible over the real field. By means of linear changes we can take $D = x^2 + y^2$ and $\lambda = 1$ or $\lambda = -1$.

Moreover, since x^2 divides $f_4 - \frac{1}{4}f_2^2$, it follows that $\lambda D^2 - \frac{1}{4}f_2^2 = Ax^2$ for some polynomial A of degree 2. In particular, $\lambda \neq -1$ because otherwise the decomposition is not possible.

Since $I_p \geq 5$ it follows that x^2 divides $D^2 - \frac{1}{4}f_2^2 = (D - \frac{1}{2}f_2)(D + \frac{1}{2}f_2)$. If x divides both factors of the last expression, then x divides D, which is not allowed. Thus, the possibilities are $f_2 = \pm 2D + 2cx^2$ for some constant c. Then the curve can be written as $f = (x + \pm D)^2 + 2cx^3$. When the minus appears, we change the sign of x and c, and therefore, the curve is

$$f = (x + x^2 + y^2)^2 + cx^3.$$

If c is positive, any circle $x^2 + y^2 = \epsilon^2$ cuts the curve in $p(x) = (x + \epsilon^2)^2 + 2cx^3$. Notice that $p(0) = \epsilon^4$ and $p(-\epsilon^2) = -2c\epsilon^6$, from where there exists a real root of $p(x) = 0$ in $(-\epsilon^2, 0)$. Therefore, there are points of $f = 0$ in any neighborhood of the origin, that is, the origin is not isolated. Since the only real points of $f = 0$ are p_1 and the points of the oval, then p_1 must belong to the oval that will not be a limit cycle because the point is singular.

Suppose that c is negative. In this case

$$f = (x + x^2 + y^2)^2 - a^2x^3.$$

Since $f = 0$ is an invariant algebraic curve for (1.6) with cofactor $k = mx + ny + p$.

$$M = \sum_{i+j=0}^{5} M_{ij} x^i y^j := \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q - (mx + ny + p) f.$$

And the coefficients M_{ij} must be zero.

In order five we have

$$\begin{aligned}
M_{50} &= 4a_{20} - m, \\
M_{41} &= 4a_{11} + 4b_{20} - n, \\
M_{32} &= 2(2a_{02} + 2a_{20} + 2b_{11} - m), \\
M_{23} &= 2(2a_{11} + 2b_{02} + 2b_{20} - n), \\
M_{14} &= 4a_{02} + 4b_{11} - m, \\
M_{05} &= 4b_{02} - n.
\end{aligned}$$

We obtain $m = 4(a_{02} + b_{11})$, $n = b_{02} = 0$, $a_{20} = a_{02} + b_{11}$, $b_{02} = a_{11}$.

In order four,

$$\begin{aligned}
M_{40} &= -2a_{02} + a^2 a02 + 4a_{10} - 2b_{11} + a^2 b_{11} - p, \\
M_{31} &= 4a_{01} - 2a_{11} + a^2 a_{11} + 4b_{10}, \\
M_{22} &= -3a^2 a_{02} + 4a_{10} + 4b_{01} - 2b_{11} - 2p, \\
M_{13} &= 2(2a_{01} - a_{11} + 2b_{10}), \\
M_{04} &= 2a_{02} + 4b_{01} - p.
\end{aligned}$$

We obtain $a_{11} = 0$, $a_{01} = -b_{10}$, $p = 2(a_{02} + 2b_{01})$, $a_{10} = \frac{1}{4}(-4a_{02} + 3a^2 a_{02} + 4b_{01})$, $b_{11} = -4a_{02}$.

In order three we have,

$$\begin{aligned}
M_{30} &= \tfrac{1}{4}(-16a_{02} + 38a^2 a_{02} - 9a^4 a_{02} - 8b_{01} + 4a^2 b_{01}), \\
M_{21} &= (-2 + 3a^2) b_{10}, \\
M_{12} &= \tfrac{1}{2}(-8a_{02} + 3a^2 a_{02} - 4b_{01}), \\
M_{03} &= -2b_{10}.
\end{aligned}$$

We obtain $b_{10} = 0$, $b_{01} = \frac{1}{4}(-8 + 3a^2) a_{02}$ and now we have $M_{30} = \frac{3}{2}(2 - a) a^2 (2 + a) a_{02}$.

- If $a_{02} = 0$ we have $P = Q = 0$.
- If $a = 2$ or $a = -2$, after a time rescaling, the system is

$$\begin{aligned}
\dot{x} &= 3x - 3x^2 + y^2, \\
\dot{y} &= (1 - 4x) y.
\end{aligned}$$

It is easy to check that the point $(1, 0)$ is a finite singular point that is also over the curve $f = 0$ and must be over the oval because the only real points are the points on the oval and p_1. Therefore, there are not limit cycles.

Therefore, there are not limit cycles.

D.2.2 Let p_1 be an infinite point. We can take $p_1 = (1:0:0)$.

We can consider that the tangent to the curve at p_1 is z . Using the argument of **D.2.1.** follows that locally the curve is given by $g = (z + z^2 + y^2)^2 + cz^3$. Thus, the global projective curve is $F = (XZ + Z^2 + Y^2)^2 + cXZ^3$, and in the affine plane the curve is

$$f = (1 + x + y^2)^2 + cx.$$

If c is positive, using the above argument follows again that the multiple point can not be isolated. Thus, p_1 must belong to the oval because the only real points are p_1 and the points of the oval. We conclude that the oval can not be a limit cycle in this case.

Suppose that c is negative. In this case we can write

$$f = (1 + x + y^2)^2 - a^2 x.$$

The branches of $f = 0$ are given by

$$x_{1,2} = \frac{1}{2}(-2 + a^2 - 2\,y^2 \pm a\,\sqrt{-4 + a^2 - 4\,y^2}).$$

So, in order to exist a real oval, a must not be zero and the polynomial inside the root must have two different real roots. In particular, $a \neq 2$ and $a \neq -2$.

Impose to $f = 0$ to be invariant with cofactor $k = mx + ny + p$ and define

$$M = \sum_{i+j=0}^{5} M_{ij} x^i y^j := \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q - kf,$$

that must be identically zero.

The coefficients in order five are

$$\begin{aligned} &M_{50} = M_{41} = M_{32} = 0, \\ &M_{23} = 4b_{20}, \\ &M_{14} = 4b_{11} - m, \\ &M_{05} = 4b_{02} - n. \end{aligned}$$

We obtain $b_{20} = 0$, $m = 4b_{11}$ and $n = 4b_{02}$.

In order four,

$$\begin{aligned} &M_{40} = M_{31} = 0, \\ &M_{22} = 2(a_{20} - 2b_{11}), \\ &M_{13} = 2(a_{11} - 2b_{02} + 2b_{10}), \\ &M_{04} = 2a_{02} + 4b_{01} - p. \end{aligned}$$

We obtain $a_{20} = 2b_{11}$, $a_{11} = 2b_{02} - 2b_{10}$ and $p = 2a_{02} + 4b_{01}$.

In order three,

$$\begin{aligned} &M_{30} = M_{21} = 0, \\ &M_{12} = -2(a_{02} - a_{10} + 2b_{01} + 2b_{11}), \\ &M_{03} = 2(a_{01} + 2b_{00} - 2b_{02}), \end{aligned}$$

from where $a_{02} = a_{10} - 2b_{01} - 2b_{11}$ and $a_{01} = -2b_{00} + 2b_{02}$.

Then, in order two the coefficients are

$$\begin{aligned} M_{20} &= 2a^2 b_{11}, \\ M_{11} &= 2a^2(b_{02} + b_{10}), \\ M_{02} &= 2a_{00} - 2a_{10} - a^2 a_{10} + 2a^2 b_{01} + 4b_{11} + 2a^2 b_{11}. \end{aligned}$$

Thus, $b_{11} = 0$, $b_{02} = -b_{10}$, and $a_{00} = \frac{1}{2}(2a10 + a^2 a_{10} - 2a^2 b_{01})$.

In order one,

$$\begin{aligned} M_{10} &= 2a^2(a_{10} - b_{01}), \\ M_{01} &= 2a^2(b_{00} + b_{10}) \end{aligned}$$

We obtain $a_{10} = b_{01}$ and $b_{00} = -b_{10}$.

Finally, the coefficient in order one is $M_{00} = \frac{1}{2}(-2 + a)a^2(2 + a)b_{01}$, from where $b_{01} = 0$. In this case we have $P = Q = 0$. So, we conclude that the curve is not invariant for the flow defined by a quadratic system. Therefore, there are not limit cycles in this case.

Chapter 3

Polynomial inverse integrating factors in some quadratic systems

In this chapter we consider planar quadratic polynomial vector fields that can have limit cycles in families (I), (II) and (III) according to the Chinese classification. We study the existence polynomial inverse integrating factors and algebraic limit cycles of arbitrary degree for some of these systems in order to determinate the existence of limit cycles, algebraic or not.

3.1 Introduction

In Ye Yian-Qian [53] are classified quadratic systems that can have limit cycles in the following three families

$$\dot{x} = \delta x - y + \ell x^2 + mxy + ny^2 \ , \quad \dot{y} = x(1 + ax + by) \ ,$$

according to: family (I) if $a = b = 0$; family (II) if $a \neq 0$ and $b = 0$; family (III) if $b \neq 0$.

In the next sections, we study the inverse integrating factors for systems (I), $(II)_{n=0}$, $(III)_{a=0}$ and $(III)_{n=0}$.

Remark 3.1. Define $\Delta := xQ_2 - yP_2$ for these families, where P_2 and Q_2 are the homogeneous parts of degree 2 in $\dot{x}$ and $\dot{y}$, respectively. It is known that $\Delta = 0$ represents the singular points of the system that belong to the infinite straight line once the phase portrait has been compacted. In Family (I), $\Delta = yT_1$; in Family $(II)_{n=0}$, $\Delta = xT_2$; in Family $(III)_{a=0}$, $\Delta = yT_3$ and in Family $(III)_{n=0}$, $\Delta = xT_4$, where T_i are polynomials of degree 2. Each Δ_i appeared in the following theorems is the discriminant of the equation $T_i = 0$, for $i = 1, \ldots, 4$. In order to make possible the integration process developed in Theorem 3.2 we will consider, in certain cases, $\Delta_i < 0$. Therefore, the cases more widely studied are those in which the line at infinity contains just one or two real infinite singular points.

The preliminary results we use in this chapter work for polynomial differential systems of arbitrary degree. Let us consider a planar polynomial differential system of the form

$$\dot{x} = P(x,y) = \sum_{k=0}^{s} P_k(x,y) \ , \quad \dot{y} = Q(x,y) = \sum_{k=0}^{s} Q_k(x,y) \ , \tag{3.1}$$

in which P, $Q \in \mathbb{R}[x,y]$ are relative prime polynomials in the variables x and y and P_k and Q_k are homogeneous polynomials of degree k. Throughout this chapter we will denote by $s = \max\{\deg P, \deg Q\}$ the degree of system (3.1) in order to keep the parameter m for the coefficient of xy in the Chinese classification, as it is habitual. The degree of invariant algebraic curves will be denoted by d.

One interesting question to ask is whether some invariant curve of system (3.1) is algebraic, i.e. can be described implicitly by $f(x,y) = 0$ where f is a polynomial. In general, the answer is not easy but it is very interesting because it is known that the existence of invariant algebraic curves can be used to prove the existence or nonexistence of limit cycles of system (3.1). In short, invariant algebraic curves, first integrals and inverse integrating factors have a narrow relationship for planar polynomial systems like it is clearly shown in the Darboux theory (Theorems 1.12 and 1.15), but also the limit cycles with inverse of integrating factors as we have seen in Theorem 1.21.

Only a few mathematicians have worked with non-algebraicity. In this sense it is interesting to note the proof due to Odani [40] about the non-algebraicity of the famous van der Pol limit cycle, see [29] for a short proof, and the generalization into a family of polynomial Liénard systems. After this work, Żołądek in [55] almost completely solve the problem of algebraic invariant curves and algebraic limit cycles for polynomial Liénard systems of arbitrary degree. In general, to show the non-algebraicity of all solutions of some system (3.1) is a very hard problem. For instance Jouanolou in [34] devotes a large section to showing that one particular system has no invariant algebraic curves. Other explicit examples of polynomial systems (3.1) without invariant algebraic curves are presented by Żołądek in [56].

3.2 Some Preliminary Results

Now we give an algorithm, developed in [29], which gives, recursively from the higher homogeneous term to the other terms in descending form, all the invariant algebraic curves of arbitrary degree.

Theorem 3.2. (García) *Let $P(x,y) = \sum_{k=0}^{s} P_k(x,y)$ and $Q(x,y) = \sum_{k=0}^{s} Q_k(x,y)$ be the development in homogeneous components of the polynomials P and Q. Assume that polynomial system* (3.1) *without degenerate infinity possesses an invariant algebraic curve $f(x,y) = 0$ of degree d with associated cofactor $K(x,y)$ such that their developments in homogeneous components are given by $f(x,y) = \sum_{k=0}^{d} f_k(x,y)$ and $K(x,y) = \sum_{k=0}^{s-1} K_k(x,y)$. Then the polynomial sequence $\{\tilde{f}_i(u)\}$ where $\tilde{f}_i(u) := f_i(1,u)$ with $i = d, d-1, \ldots, 0$ is recursively obtained from*

$$\tilde{f}_i(u) = \frac{\int \frac{\Lambda_{s-1+i}(u)}{\Gamma(u)} \exp\left[\int \frac{\Gamma_i(u)}{\Gamma(u)}\ du\right]\ du + C_i}{\exp\left[\int \frac{\Gamma_i(u)}{\Gamma(u)}\ du\right]} \ , \tag{3.2}$$

where C_i are arbitrary real constants with $C_d \neq 0$ and

$$\Gamma(u) := Q_s(1,u) - uP_s(1,u) \ , \quad \Gamma_i(u) := iP_s(1,u) - K_{s-1}(1,u) \ , \tag{3.3}$$

and

$$\Lambda_{s-1+d}(u) \equiv 0 \ , \tag{3.4}$$

$$\begin{aligned}\Lambda_{s-1+i}(u) \;:=\; & \sum_{k=0}^{d-1-i}{}' \Big([uP_{s+i-d+k}(1,u) - Q_{s+i-d+k}(1,u)] \frac{d\tilde{f}_{n-k}(u)}{du} \\ & + [K_{s-1+i-d+k}(1,u) - (d-k)P_{s+i-d+k}(1,u)]\tilde{f}_{d-k}(u) \Big) \ ,\end{aligned} \tag{3.5}$$

where the dash in the previous sum should be understood in the following way: if the index of some term does not make sense then we take null that term.

In the proof of Theorem 3.2 it is used the blow–up $(x,y) \to (x,u)$ where $u = y/x$. Once we have determined the sequence $\{\tilde{f}_i(u)\}_{i=0}^d$ then $f(x,y) = \sum_{i=0}^d x^i \tilde{f}_i(y/x)$.

Remark 3.3. Let us notice that Theorem 3.2 can also be used for the study of the existence of inverse integrating factors $V(x,y)$ and first integrals $H(x,y)$ of polynomial class. The only changes in the algorithm consist on replacing K_i by either the homogeneous part of i-th degree of the divergence of (P,Q) or to take $K_i \equiv 0$ respectively for $i = 0, 1, \ldots, s-1$.

3.3 The main results

3.3.1 Invariant algebraic curves and inverse integrating factors in Family (I)

It is well known, see [53] for instance, that a quadratic system of the family (I) has at most one limit cycle. But when in this family we add the condition $\delta = 0$ then it does not have any limit cycle. In fact, Theorem 12.4, pag. 268, of [53] shows that the system of type $(I)_{\delta=0}$ has a center at the origin when $m(\ell+m) = 0$ and does not have any closed or singular closed orbit when $m(\ell+m) \neq 0$.

On the other hand, if $\ell = 0$, then family (I) is a quadratic Liénard system with constant damping whose invariant algebraic curves problem is completely solved by Żołądek in [55]. Any invariant algebraic curve in this case must be rational or hyperelliptic. The author also proves that there are not algebraic limit cycles in for such systems. The next theorem extends the above results for family (I) with $\ell \neq 0$.

Theorem 3.4. *Consider family (I) and define $\Delta_1 = m^2 - 4\ell n$. The following statements hold:*

1. *Consider $\ell n \neq 0$.*
   - **(a)** *Suppose that $\Delta_1 < 0$.*
     - **(i)** *If $\delta = 0$, then there is not any limit cycle. Moreover, the only irreducible invariant algebraic curve is $f(x,y) = -\ell + n + 2\ell(n-\ell)y + 2\ell^2 ny^2 + 2\ell^3 x^2 = 0$ and appears when $m = 0$. f is also an inverse of integrating factor.*

(ii) *If $\delta \neq 0$, then the unique irreducible invariant algebraic curve is $f(x,y) = nx^2 + mxy + ny^2 = 0$ which is also an inverse of integrating factor and appears when $\ell = n$ and $\delta = -\frac{m}{n}$. There are not algebraic limit cycles in this case, nor limit cycles when f exists.*

(b) *Suppose that $\Delta_1 > 0$. If $\delta = 0$, there is not any limit cycle. Moreover, the only polynomial inverse integrating factor is $f(x,y) = 1 - mx - 2ny - n^2x^2 + mnxy + n^2y^2 = 0$, and appears when $\ell = -n$ and $\delta = 0$.*

(c) *Suppose that $\Delta_1 = 0$. If $\delta = 0$, there is not any limit cycle. Moreover, the only polynomial inverse integrating factors are of the form $f(x,y) = (x \pm y)^2$ and exist when $\delta = 0$ and $m = \pm 2\ell$.*

2. Consider $\ell = 0$. In this case there is not any algebraic limit cycle. Moreover,

(a) *Suppose that $\Delta_1 \neq 0$.*

(i) *If $n \neq 0$, then there is not any polynomial inverse integrating factor.*

(ii) *If $n = 0$, then the only irreducible invariant algebraic curve is $f(x,y) = mx - 1 = 0$ and appears when $\delta = 0$. Moreover, this function f is a polynomial inverse integrating factor.*

(b) *Suppose that $\Delta_1 = 0$. If $n = 0$, then $f(x,y) = x^2 - \delta xy + y^2$ is a polynomial inverse integrating factor.*

3. Consider $n = 0$ and $\ell \neq 0$.

(a) *Suppose that $\Delta_1 \neq 0$.*

(i) *If $\delta = 0$, then there are not invariant algebraic curves nor limit cycles.*

(ii) *If $\delta \neq 0$, then the only invariant algebraic curves are $f(x,y) = \ell x + my = 0$ for $\delta = -(\ell^2 + m^2)/(\ell m)$, or $f(x,y) = mx - 1 = 0$ for $\delta = -\ell/m$ which are not inverse integrating factors. There are not algebraic limit cycles.*

(b) *Suppose that $\Delta_1 = 0$.*

(i) *If $\delta = 0$, then there is not any limit cycle. Moreover, $f(x,y) = 1 + 2\ell y - 2\ell^2 x^2 = 0$ is the only irreducible invariant algebraic curve, which is also a polynomial inverse integrating factor.*

(ii) *If $\delta \neq 0$, then there are not invariant algebraic curves, and therefore, there are not algebraic limit cycles.*

Proof. As we have said, in the Statements where $\delta = 0$ or $\ell = 0$ appears, we know the nonexistence of limit cycles and algebraic limit cycles, respectively. We have included these results in the theorem for completeness. Now we prove the rest of results.

Proof of Statement (1.a). Assume that system (I) has an invariant algebraic curve $f(x,y) = 0$ of degree d and set the homogeneous part of degree one of their cofactor $K_1(x,y) = \alpha x + \beta y$. In the generic case $\ell n \Delta_1 \neq 0$, doing the quadrature (3.2) with $i = d$ we have

$$\tilde{f}_d(u) = C_d(-4n)^{-\frac{\alpha}{2\ell}} u^{d-\frac{\alpha}{\ell}} (\sqrt{\Delta_1} + m + 2nu)^{\frac{2\beta\ell - \alpha m}{2\ell\sqrt{\Delta_1}} + \frac{\alpha}{2\ell}} (\sqrt{\Delta_1} - m - 2nu)^{-\frac{2\beta\ell - \alpha m}{2\ell\sqrt{\Delta_1}} + \frac{\alpha}{2\ell}},$$

where $C_d \neq 0$. Since $\tilde{f}_d$ is a polynomial, its exponents must be nonnegative integers.

Assume that $\Delta_1 < 0$. Then, we must consider

$$2\beta\ell - \alpha m = 0, \tag{3.6}$$

and we take $\beta = \alpha m/(2\ell)$.

In the next step, denoting by K_0 the independent term of the cofactor and carrying out integration (3.2) with $i = d-1$, we obtain

$$\begin{aligned}\tilde{f}_{d-1}(u) &= \frac{1}{2\ell^2 n\sqrt{\Delta_1}} u^{d-1-\frac{\alpha}{\ell}} (\ell + mu + nu^2)^{\frac{\alpha}{2\ell}-1} \Big[b_0^{(d-1)}(u) \\ &\quad + b_1^{(d-1)}(u) \operatorname{arctanh}\left(\frac{m+2nu}{\sqrt{\Delta_1}}\right) + b_2^{(d-1)}(u) \log\left(\frac{\ell+mu+nu^2}{u^2}\right)\Big] ,\end{aligned}$$

where $b_0^{(d-1)}$ is a polynomial of degree 2 and

$$\begin{aligned} b_1^{(d-1)}(u) &= 2nC_d(-\alpha\delta\ell + 2K_0\ell^2 - \alpha m + \ell m d)(\ell + mu + nu^2) , \\ b_2^{(d-1)}(u) &= n\sqrt{\Delta_1} C_d(\alpha - \ell d)(\ell + mu + nu^2) .\end{aligned}$$

Since $\tilde{f}_{d-1}$ must be polynomial we impose $b_1^{(d-1)}(u) = b_2^{(d-1)}(u) \equiv 0$. Hence we take $\alpha = d\ell$ and

$$K_0 = (\alpha\delta\ell + \alpha m - \ell m d)/(2\ell^2). \tag{3.7}$$

With these assignments $\tilde{f}_{n-1}$ becomes

$$\tilde{f}_{d-1}(u) = \frac{(\ell + mu + nu^2)^{-1+d/2}}{2nu} \sum_{i=0}^{2} a_i^{(d-1)} u^i ,$$

where $a_i^{(d-1)}$ are real constants. More concretely $a_0^{(d-1)} = \ell d C_d + 2\ell n C_{d-1} - dnC_d$ where C_{d-1} is an arbitrary constant. Obviously, from the above expression of $\tilde{f}_{d-1}$ we conclude that d must be even. Moreover, u must divide the polynomial $\sum_{i=0}^{2} a_i^{(d-1)} u^i$, i.e. $a_0^{(d-1)} = 0$ and therefore $C_{d-1} = d(n-\ell)C_d/(2\ell n)$.

The next step, that is, quadrature (3.2) with $i = d-2$ allows us to calculate

$$\tilde{f}_{d-2}(u) = -\frac{(\ell + mu + nu^2)^{-2+d/2}}{8\ell n^2 \Delta_1^{3/2} u^2} \left[b_0^{(d-2)}(u) + b_1^{(d-2)}(u) \operatorname{arctanh}\left(\frac{m+2nu}{\sqrt{\Delta_1}}\right)\right] ,$$

where $b_0^{(d-2)}$ is a polynomial and

$$b_1^{(d-2)}(u) = 8dn^2 C_d(\ell m + 2\delta\ell n + mn)(\ell + mu + nu^2)^2 .$$

Now, taking into account that $\tilde{f}_{d-2}$ is a polynomial we must take $b_1^{(d-2)} \equiv 0$ or equivalently $\delta = -m(\ell+n)/(2\ell n)$. In this situation $\tilde{f}_{d-2}$ takes the form

$$\tilde{f}_{d-2}(u) = \frac{(\ell + mu + nu^2)^{-2+d/2}}{32\ell^2 n^2 u^2} \sum_{i=0}^{4} a_i^{(d-2)} u^i ,$$

where $a_i^{(d-2)}$ are real constants. Here $a_0^{(d-2)} = 4\ell^2\Omega$ and $a_1^{(d-2)} = 8\ell m\Omega$ with $\Omega := dC_d(n-\ell)(2\ell-\ell d+dn) - 8\ell^2 n^2 C_{d-2}$. Imposing that u^2 divides to $\sum_{i=0}^4 a_i^{(d-2)} u^i$ we must take $a_0^{(d-2)} = a_1^{(d-2)} = 0$. This condition implies $\Omega = 0$ or equivalently $C_{d-2} = dC_d(n-\ell)(2\ell-\ell d+dn)/(8\ell^2 n^2)$.

The next step consists on quadrature (3.2) with $i = d-3$. So we compute

$$\tilde{f}_{d-3}(u) \quad = \quad \frac{(\ell+mu+nu^2)^{-3+d/2}}{384\ell^3 n^3(-\Delta_1)^{5/2}u^3}\left[b_0^{(d-3)}(u) + b_1^{(d-3)}(u)\,\mathrm{arctanh}\left(\frac{m+2nu}{-\sqrt{\Delta_1}}\right)\right] ,$$

where $b_0^{(d-3)}$ is a polynomial of degree 6 and

$$b_1^{(d-3)}(u) = 192C_d\ell m d n^2(\ell-n)(n+\ell)\Delta_1(\ell+mu+nu^2)^3. \tag{3.8}$$

Of course in the expression of $\tilde{f}_{d-3}$ there is implicitly an arbitrary constant C_{d-3} due to the made quadrature. In order to have $\tilde{f}_{d-3}$ polynomial we impose $b_1^{(d-3)} \equiv 0$. In short from the vanish of $b_1^{(d-3)}$ we obtain $\ell = n$ or $m = 0$ (we recall that $\Delta_1 < 0$).

Suppose that $\ell = n$. In this case $\tilde{f}_{d-3}$ becomes

$$\tilde{f}_{d-3}(u) = -\frac{C_{d-3}(n+mu+nu^2)^{d/2}}{u^3} .$$

Since $\tilde{f}_{d-3}$ is a polynomial, the only possibility is given by $C_{d-3} = 0$. Therefore $\tilde{f}_{d-3} \equiv 0$. From Theorem 3.2 we have $\Lambda_{1+i}(u) \equiv 0$ and $\tilde{f}_i(u) = C_i \exp\left[-\int \frac{\Gamma_i(u)}{\Gamma(u)}du\right] = C_i u^{i-d}(n+mu+nu^2)^{d/2}$ for $i \le d-4$. Here C_i is an arbitrary constant and from the previous expression of $\tilde{f}_i$ it follows that $C_i = 0$ and therefore $\tilde{f}_i \equiv 0$ for $i \le d-4$. Hence the invariant algebraic curve $f(x,y) = 0$ of system (I) is obtained from $\tilde{f}(u) = \sum_{i=d}^{d-2} \tilde{f}_i(u)$ going back through the blow–up. But it is easy to see that after the last condition $\ell = n$ we have $\tilde{f}_{d-1} = \tilde{f}_{d-2} \equiv 0$ and so $\tilde{f}(u) = \tilde{f}_d(u) = C_d(n+mu+nu^2)^{d/2}$. Therefore the irreducible invariant algebraic curve is the conic $f(x,y) = nx^2+mxy+ny^2 = 0$. Consequently, since f is homogeneous, under the conditions of the theorem, family (I) has not algebraic limit cycles.

Suppose that $m = 0$ and therefore $\delta = 0$. In this case there is not any limit cycle and moreover every invariant algebraic curve is a multiple of $f(x,y) = -\ell+n-2\ell^2x+2\ell nx+2\ell^2nx^2+2\ell^3y^2 = 0$ since the function $H(x,y) = e^{-2\ell y}f(x,y)$ is a first integral. Moreover, f is also an inverse integrating factor.

Proof of Statement (1.b). Suppose that $\Delta_1 > 0$. The divergence of the system is $\delta+2\ell x+my$. Therefore, looking for an inverse integrating factor is looking for an invariant algebraic curve with cofactor $K = K_0+\alpha x+\beta y$ such that $K_0 = \delta$, $\alpha = 2\ell$ and $\beta = m$. Of course, (3.6) and (3.7) of the proof of Statement (1.a) are satisfied for $d = 2$. Following the proof we arrive to (3.8) and since $\Delta_1 > 0$, we consider $\ell = -n$. Hence

$$\tilde{f}_{d-3}(u) = C_{d-3}(-n+mu+nu^2)/u^3,$$

where C_{d-3} is an arbitrary constant. Therefore we must take $C_{d-3} = 0$. In an analogous way, it is easy to see that we must choose $C_i = 0$ in order to have $\tilde{f}_i \equiv 0$ for $i \le d-4$.

In summary, $\tilde{f}(u) = \sum_{i=d}^{d-2} \tilde{f}_i(u) = C_d[-n+mu+nu^2-(m+2nu)/n+1/n]$ and going back through the blow–up $u = y/x$ we obtain that $f(x, y) = 1-mx-2ny-n^2x^2+mnxy+n^2y^2$ is the only inverse integrating factor and appears when $\ell = -n$ and $\delta = 0$.

Proof of Statement (1.c). From $\Delta_1 = m^2 - 4\ell n = 0$ we take $n = m^2/(4\ell)$. Doing the quadrature (3.2) with $i = d$ and with a cofactor K equal to the divergence of the system, we obtain

$$\tilde{f}_d(u) = C_d u^{d-2}(2\ell + mu)^2,$$

where $C_d \neq 0$. In the next step, denoting by K_0 the independent term of the divergence of (P, Q), that is $K_0 = \delta$, and carrying out integration (3.2) with $i = d-1$, we obtain

$$\tilde{f}_{d-1}(u) = \frac{u^{d-3}}{\ell m^2}\left[B_0^{(d-1)}(u) + B_1^{(d-1)}(u)\log\left(\frac{2\ell+mu}{u}\right)\right] ,$$

where $B_0^{(d-1)}$ is a polynomial of degree 2 and $B_1^{(d-1)}(u) = C_d(2-d)m^2(2\ell+mu)^2$. Clearly, in the generic case $\ell m \neq 0$, the logarithmic term of $\tilde{f}_{d-1}$ does not vanish except for $d = 2$. So, in this case, family (I) does not have a polynomial inverse integrating factor of degree different from 2.

Now, $\tilde{f}_{d-1}$ becomes $\tilde{f}_{d-1}(u) = (\sum_{i=0}^{2} a_i^{(d-1)}u^i)/(m^2u)$ where $a_i^{(d-1)}$ are real constants and $a_0^{(d-1)} = 4\ell(4C_d\ell^2 - C_dm^2 + C_{d-1}\ell m^2)$. In order to have $\tilde{f}_{d-1}(u)$ a polynomial we need $a_0^{(d-1)} = 0$, that is $C_{d-1} = C_d(m^2-4\ell^2)/(\ell m^2)$. The next step, that is, quadrature (3.2) with $i = d-2$ allows us to calculate

$$\tilde{f}_{d-2}(u) = \frac{1}{3m^2u^2(2\ell+mu)}\sum_{i=0}^{3} a_i^{(d-2)}u^i ,$$

where $a_i^{(d-2)}$ are real constants. Here $a_0^{(d-2)} = 4\ell\Omega$ and $a_1^{(d-2)} = 6m\Omega$ where $\Omega = C_d(20\ell^2+4\delta\ell m-m^2)+6C_{d-2}\ell^2m^2$. Now, taking into account that $\tilde{f}_{d-2}$ is a polynomial we must take $a_0^{(d-2)} = a_1^{(d-2)} = 0$, i.e., $\Omega = 0$. From this we have $C_{d-2} = (m^2-4\delta\ell m-20\ell^2)/(6\ell^2m^2)$. In this situation $\tilde{f}_{d-2}$ takes the form

$$\tilde{f}_{d-2}(u) = \frac{C_d[6\ell(m^2-4\ell^2) + m(m^2-4\delta m-20\ell^2)u]}{6\ell^2(2\ell+mu)} .$$

It is easy to see that the above expression is polynomial if and only if $4\ell^2+2\delta\ell m+m^2 = 0$. From such condition we have $\delta = -(m^2+4\ell^2)/(2\ell m)$.

Finally, since $d = 2$ we must impose $\tilde{f}_i \equiv 0$ for $i \leq d-3$. Hence we do a new step which consists on quadrature (3.2) with $i = d-3$ and compute

$$\tilde{f}_{d-3}(u) = \frac{1}{3\ell^2m^4u^3(2\ell+mu)}\sum_{i=0}^{3} a_i^{(d-3)}u^i ,$$

where $a_i^{(d-3)}$ are real constants. Here $a_0^{(d-3)} = 4\ell^2\Psi$, $a_1^{(d-3)} = 6\ell m\Psi$ and $a_2^{(d-3)} = 3m^2\Psi$ where $\Psi = C_d(16\ell^4 - m^4) + 6C_{d-3}\ell^3m^4$. From the vanishing of $\tilde{f}_{d-3}$ we must take $C_{d-3} = C_d(m^4-16\ell^4)/(6\ell^3m^4)$. In this situation $\tilde{f}_{d-3}$ becomes

$$\tilde{f}_{d-3}(u) = \frac{C_d(m-2\ell)(m+2\ell)(m^2+4\ell^2)}{6\ell^2m(2\ell+mu)} ,$$

which vanish if and only if $m = \pm 2\ell$. Again, it is easy to see that we must choose $C_i = 0$ in order to have $\tilde{f}_i \equiv 0$ for $i \leq d-4$.

With all these conditions $\tilde{f}_{d-1} \equiv \tilde{f}_{d-2} \equiv 0$ and therefore $\tilde{f}(u) = \tilde{f}_d(u) = 2\ell C_d(1 \pm u)^2$ according to $m = \pm 2\ell$. The statement is proved going back through the blow–up $u = y/x$.

Proof of Statement (2.a). If $n \neq 0$, doing the quadrature (3.2) with $i = d$ we have $\tilde{f}_d(u) = C_d u^{d-1}(m + nu)$, where $C_d \neq 0$. Hence in the next step, carrying out integration (3.2) with $i = d-1$, we obtain

$$\tilde{f}_{d-1}(u) = \frac{u^{d-3}}{nm^2}\left[B_0^{(d-1)}(u) + B_1^{(d-1)}(u)\log\left(\frac{m+nu}{u}\right)\right],$$

where $B_0^{(d-1)}$ is a polynomial of degree 2 and $B_1^{(d-1)}(u) = C_d(d-2)n^2u(m+nu)$. Clearly, the logarithmic term of $\tilde{f}_{d-1}$ does not vanish except for $d = 2$. So, in this case, family (I) does not have a polynomial inverse integrating factor of degree different from 2. Now we have $\tilde{f}_{d-1}(u) = [-C_d n + (C_d m + C_d\delta n + C_{d-1}mn)u + C_{d-1}n^2u^2]/(nu)$ which never is polynomial because $n \neq 0$.

When $n = 0$, and doing the quadrature (3.2) with $i = d$, we obtain

$$\tilde{f}_d(u) = C_d e^{\frac{\alpha}{mu}} u^{-\frac{\beta}{m}+d},$$

from where $\alpha = 0$ and $\beta = km$ for some nonnegative integer k. That is, $\tilde{f}_d(u) = C_d u^{-k+d}$.

In the next step,

$$\tilde{f}_{d-1}(u) = -\frac{1}{m}u^{d-2-k}(C_{d-1}mu - C_d(d + k(u^2-1)) + C_d(\delta k - K_0)u\log(u)),$$

which is a polynomial if $K_0 = \delta k$.

Following the quadrature with $i = d-2$ follows

$$\tilde{f}_{d-2}(u) = \frac{1}{2m^2}u^{-4-k+d}(b_0^{(d-2)} + 2C_d(d-k)u^2\log(u)),$$

where $b_0^{(d-2)}$ is a polynomial of degree 4. Therefore, we must take $k = d$ and we have

$$\begin{aligned}\tilde{f}_{d-1}(u) &= \frac{C_{d-1}mu - C_d du^2}{mu^2},\\ \tilde{f}_{d-2}(u) &= \frac{2C_{d-2}m^2 + 2C_d\delta du - C_d du^2 + C_d d^2u^2}{2m^2u^2}.\end{aligned}$$

Clearly, in order to get algebraic curves we must have $C_{d-1} = C_{d-2} = \delta = 0$.

Now the system is $\dot{x} = y(mx-1)$, $\dot{y} = x$ and it is easy to check that $f(x,y) = mx-1$ is an inverse integrating factor. Moreover, any invariant algebraic curve is a power of f because $H(x,y) = e^{m(2x-my^2)/2}(mx-1)$ is a first integral of the system.

Proof of Statement (2.b). In this case the system becomes linear. In particular it does not have limit cycles and it is easy to check that $f(x,y) = x^2 - \delta xy + y^2$ is a polynomial inverse integrating factor.

Proof of Statement (3.a). Consider $n = 0$, $\ell \neq 0$ and $\Delta_1 \neq 0$. In this case, $m \neq 0$.

Assume that system (I) has an invariant algebraic curve $f(x, y) = 0$ of degree d and set the homogeneous part of degree one of their cofactor $K_1(x, y) = \alpha x + \beta y$. Quadrature (3.2) with $i = d$ leads to

$$\tilde{f}_d(u) = C_d u^{d-s}(\ell + mu)^{s-k},$$

where $s := \alpha/\ell$ and $k := \beta/m$ are nonnegative integers.

In the next step,

$$\tilde{f}_{d-1}(u) = b_0^{(d-1)}(u) + b_1^{(d-1)}(u)\log(u) + b_2^{(d-1)}(u)\log(\ell + mu),$$

where $b_0^{(d-1)}$ is a rational function and $b_i^{(d-1)}$ with $i = 1, 2$ are polynomials that must be identically zero in order to $\tilde{f}_{d-1}$ be a polynomial. More concretely,

$$b_1^{(d-1)}(u) = \frac{C_d}{\ell}(d - s)u^{-1+d-s}(\ell + mu)^{-k+s}.$$

Therefore, we obtain $s = d$. In this case,

$$b_2^{(d-1)}(u) = \frac{C_d}{m^2 u}(2\ell k + \delta km - K_0 m - \ell d)(\ell + mu)^{-k+d},$$

from where $K_0 = (2\ell k + \delta km - \ell d)/m$. Therefore, $\tilde{f}_{d-1}$ is a rational function and taking $C_{d-1} = C_d(\ell^2 + \delta \ell m + m^2)(d - k)/(\ell m^2)$ becomes the polynomial

$$\tilde{f}_{d-1}(u) = -\frac{C_d(\ell + mu)^{-1-k+d}}{\ell m}(a_0^{(d-1)} + a_1^{(d-1)}u);$$

for some real numbers $a_i^{(d-1)}$, $i = 0, 1$.

In the next step,

$$\tilde{f}_{d-2}(u) = b_0^{(d-2)}(u) + b_1^{(d-2)}(u)\log(\ell + mu),$$

where $b_0^{(d-2)}$ and $b_1^{(d-2)}$ are rational functions. The second one is

$$b_1^{(d-2)}(u) = -\frac{C_d}{m^4 u^2}\left(m^2(k - d) + \ell^2(2k - d) + \delta \ell m\,(2\,k - d)\right)(\ell + mu)^{-k+d}.$$

If $d = 2k$, the above expression vanishes if $k = 0$ and therefore, $d = 0$. So, there are not invariant algebraic curves in this case. Suppose that $d \neq 2k$ and we obtain, from $b_1^{(d-2)} \equiv 0$,

$$\delta = -\frac{2\ell^2 k + km^2 - \ell^2 d - m^2 d}{\ell m\,(2k - d)}.$$

Moreover, $\tilde{f}_{d-2}$ must be a polynomial. So we write C_{d-2} in function of the rest of parameters.

In the next step,

$$\tilde{f}_{d-3}(u) = b_0^{(d-3)}(u) + b_1^{(d-3)}(u)\log(\ell + mu),$$

where $b_0^{(d-3)}$ is a rational function and

$$b_1^{(d-3)}(u) = \frac{2C_d k\,(k-d)\,(\ell+mu)^{-k+d}}{\ell m^2\,(2k-d)\,u^3}.$$

We have two possibilities: either $k=0$ or $k=d$.

If $k=0$, we obtain

$$\tilde{f}_{d-3}(u) = \frac{C_{d-3}(\ell+mu)^d}{u^3},$$

from where $C_{d-3}=0$ and consequently $\tilde{f}_{d-i}\equiv 0$ for $i\geq 3$. Then we obtain $\tilde{f}(u) = \tilde{f}_d(u)+\tilde{f}_{d-1}(u)+\tilde{f}_{d-2}(u) = C_d(\ell+mu)^d$ and going back to the variables (x,y) follows that the only irreducible invariant algebraic curve is $f(x,y)=\ell x+my=0$. It can be also seen that f is not an inverse integrating factor. We remark that $\delta=-(\ell^2+m^2)/(\ell m)\neq 0$.

On the other hand, if $k=d$, we must take $C_i=0$ in order to $\tilde{f}_i$ be polynomials for $i\leq d-3$ and moreover, this $\tilde{f}_i$ become constants. Following the quadrature for $i=d-j$, and taking $C_{d-j}=0$ in order to have a polynomial, we obtain

$$\tilde{f}_{d-j}(u) = \frac{u^{-j}}{m}\int \frac{u^{-1+j}}{\ell+mu}\omega_j(u)du.$$

where $\omega_j(u) = (-(1+d-j)(\ell+mu)\tilde{f}_{d-(j-1)}(u) + (m+\ell u+m\,u^2)\tilde{f}'_{d-(j-1)}(u)$. Inductively, it can be seen that $\tilde{f}_{d-j}$ is a constant for all j and going back to the variables (x,y) we obtain a function in one only variable. Obviously, since the system is $\dot{x}=(mx-1)(\ell x+my)/m$, $\dot{y}=x$, the only invariant algebraic curve is $f(x,y)=mx-1=0$ which is not an inverse integrating factor. In particular, there are not algebraic limit cycles. We remark that $\delta=-\ell/m\neq 0$.

Thus, for $\delta=0$ there are not limit cycles nor invariant algebraic curves.

Proof of statement (3.b). Consider $n=0$, $\ell\neq 0$ and $\Delta_1=0$. In this case, $m=0$.

Assume that system (I) has an invariant algebraic curve $f(x,y)=0$ of degree d and set the homogeneous part of degree one of its cofactor $K_1(x,y)=\alpha x+\beta y$. Doing the quadrature (3.2) with $i=d$ we have

$$\tilde{f}_d(u) = C_d e^{-\frac{\beta u}{\ell}} u^{-\frac{\alpha}{\ell}+d},$$

from where $\beta=0$ and $\alpha=k\ell$ for some nonnegative integer k and we obtain $\tilde{f}_d(u) = C_d u^{(-k+d)}$. In the next step,

$$\tilde{f}_{d-1}(u) = \frac{u^{-1-k+d}}{2\ell}(2C_{d-1}\ell - C_d u(-2\delta k+2K_0+ku) + 2C_d(k-d)\log(u)),$$

from where $k=d$ to vanish the logarithmic term. In this case $\tilde{f}_{d-1}$ is a rational function and we take $C_{d-1}=0$ in order to be a polynomial. It follows $\tilde{f}_{d-1}(u) = -C_d(2K_0-2\delta d+du)/(2\ell)$.

Following the quadrature (3.2) for $i=d-2$ we obtain

$$\tilde{f}_{d-2}(u) = \frac{R(u)}{24\ell^2 u^2},$$

where $R(u)$ is a polynomial without linear term and its independent term is $24C_{d-2}\ell^2$ from where $C_{d-2} = 0$. Again, in order to $\tilde{f}_i$ be polynomials, we impose $C_i = 0$ for $i < d-2$. It is also easy to see that when $\tilde{f}_i \equiv 0$ for some i the functions $\tilde{f}_j$, with $j < i$, that we obtain in the next steps are identically zero. Moreover, doing the quadrature (3.2) for $i = d - j$ and taking $C_{d-j} = 0$ we obtain

$$\tilde{f}_{d-j}(u) = \frac{u^{-j}}{\ell} \int \omega_j(u) du,$$

where $\omega_j(u) = -(K_0+\delta(-1-d+j)+u+du-ju)\tilde{f}_{d-(j-1)}(u)+(1-\delta u+u^2)\tilde{f}'_{d-(j-1)}(u)$.

It can be seen that $\tilde{f}_{d-3} \equiv 0$ if and only if $d = 2$ and $\delta = K_0 = 0$. In this situation, we consider $\tilde{f} = \tilde{f}_2 + \tilde{f}_1 + \tilde{f}_0$ and going back through the blow-up we obtain $f(x, y) = -1 + 2\ell^2x^2 - 2\ell y = 0$ as invariant algebraic curve. Moreover, f is also an inverse integrating factor.

We claim that the only invariant algebraic curves have even degree and appear only when $\delta = K_0 = 0$. The prove of the claim follows by induction. We give here an sketch of the steps of the proof.

It can be seen that $\tilde{f}_{d-3} \equiv 0 \Leftrightarrow \tilde{f}_{d-4} \equiv 0 \Leftrightarrow \tilde{f}'_{d-3} \equiv 0 \Leftrightarrow \omega_4 \equiv 0 \Leftrightarrow d = 2$ and $\delta = K_0 = 0$ and that $\tilde{f}'_{d-4} \equiv 0 \Leftrightarrow \omega_5 \equiv 0 \Leftrightarrow d = 2, 4$ and $\delta = K_0 = 0$.

Suppose that for some j even one has $\tilde{f}_{d-(j-1)} \equiv 0 \Leftrightarrow \tilde{f}_{d-j} \equiv 0 \Leftrightarrow \tilde{f}'_{d-(j-1)} \equiv 0 \Leftrightarrow \omega_j \equiv 0 \Leftrightarrow d = 2, \ldots, j-2$ and $\delta = K_0 = 0$ and that $\tilde{f}'_{d-j} \equiv 0 \Leftrightarrow \omega_{j+1} \equiv 0 \Leftrightarrow d = 2, \ldots, j$ and $\delta = K_0 = 0$.

It is easy to check that $\tilde{f}_{d-(j+1)} \equiv 0 \Leftrightarrow \tilde{f}_{d-(j+2)} \equiv 0 \Leftrightarrow \tilde{f}'_{d-(j+1)} \equiv 0 \Leftrightarrow \omega_{j+2} \equiv 0 \Leftrightarrow d = 2, \ldots, j$ and $\delta = K_0 = 0$ and also that $\tilde{f}'_{d-(j+2)} \equiv 0 \Leftrightarrow \omega_{j+3} \equiv 0 \Leftrightarrow d = 2, \ldots, j+2$ and $\delta = K_0 = 0$, because when $d = j+2$ we have $\omega_{j+3} = K_0\tilde{f}_0 + (1 - \delta u + u^2)\tilde{f}'_0$, which is zero when $K_0 = 0$, since $\tilde{f}_0$ is a constant. So the claim is proved.

It can be checked that $H(x, y) = e^{-2\ell y} f(x, y)$ is a first integral. Therefore, every invariant algebraic curve is a power of f. Moreover, there are not limit cycles because the set of points where f vanishes, does not contain any oval.

And finally we remark that we have also shown that when $\delta \neq 0$ there is not any invariant algebraic curve. ∎

3.3.2 Polynomial Inverse Integrating Factors in Family $(II)_{n=0}$

In the following theorem we study the existence of polynomial inverse integrating factors for the family $(II)_{n=0}$. In [53] can be found some results on non existence, existence and uniqueness of limit cycles for these systems depending on the value of the parameter δ.

Theorem 3.5. *Consider system $(II)_{n=0}$ and define $\Delta_2 := \ell^2 + 4am$. The following statements hold:*

1. *Suppose that $\Delta_2 \neq 0$.*

(a) *In the generic case $\ell m \neq 0$, then there exist a polynomial inverse integrating factor of degree d if* $\pm\dfrac{\ell(d-3)}{2\sqrt{\Delta_2}} + \dfrac{d-1}{2} \in \mathbb{N} \cup \{0\}$.

Moreover, if $\Delta_2 < 0$ the only polynomial inverse integrating factor is $f(x,y) = (1+ax)(x^2+\delta xy+y^2)$ and appears when $m=-a$ and $\ell = a\delta$.

(b) *If $\ell = 0$, then $f(x,y) = ax+1$ is the only polynomial inverse integrating factor, and appears when $m=-a$ and $\delta = 0$.*

(c) *If $m = 0$, then there is not any polynomial inverse integrating factor.*

2. Suppose that $\Delta_2 = 0$. In this case, the only polynomial inverse integrating factors are $f(x,y) = (1+ax)(x\pm y)^2$ and appear when $\ell = \mp 2a$, $\delta = \mp 2$ and $m=-a$.

Proof. Assume that the vector field associated to family $(II)_{n=0}$ is given by (P,Q) and has a polynomial inverse integrating factor $f(x,y)$ of degree n. Let K_1 be the homogeneous part of degree 1 of the divergence of (P,Q), i.e. $K_1(x,y) = 2\ell x + my$.

Proof of Statement (1.a). Consider $\Delta_2 \ell n \neq 0$. Doing the quadrature (3.2) with $i = d$ we have

$$\tilde{f}_d(u) = C_d(-4m)^{\frac{1-d}{2}}(-\ell+\sqrt{\Delta_2}-2mu)^{\frac{\ell(d-3)}{2\sqrt{\Delta_2}}+\frac{d-1}{2}}(\ell+\sqrt{\Delta_2}+2mu)^{-\frac{\ell(d-3)}{2\sqrt{\Delta_2}}+\frac{d-1}{2}}.$$

In order have a polynomial, the exponents must be nonnegative integers. Moreover, if $\Delta_2 < 0$, the only possibility is $d = 3$ and straight forward calculations show that the only polynomial inverse integrating factor is $f(x,y) = (1+ax)(x^2+\delta xy+y^2)$.

Proof of Statement (1.b). Suppose that $\ell = 0$. In this case, $f_d = C_d(a - mu^2)^{(d-1)/2}$, and d must be odd. In the next step, we obtain

$$\tilde{f}_{d-1}(u) = C_{d-1}(a-mu^2)^{\frac{d-2}{2}} + b^{(d-1)}(u)(a-mu^2)^{\frac{d-3}{2}},$$

where $b^{(d-1)}$ is a polynomial. Since d must be odd, follows $C_{d-1} = 0$.

Following the quadrature (3.2) with $i = d-2$ we obtain

$$\tilde{f}_{d-2}(u) = (a-mu^2)^{\frac{d-3}{2}}[b_0^{(d-2)}(u) + b_1^{(d-2)}\log(\sqrt{a}+\sqrt{m}u) + b_2^{(d-2)}\log(\sqrt{a}-\sqrt{m}u)],$$

where $b_0^{(d-2)}$ is a rational function which must be a polynomial and the constants

$$\begin{aligned} b_1^{(d-2)} &= \frac{1-d}{m}(a+\sqrt{a}\sqrt{m}\delta+m) \\ b_2^{(d-2)} &= \frac{1-d}{m}(a-\sqrt{a}\sqrt{m}\delta+m) \end{aligned}$$

must be zero. We have two possibilities: either $d = 1$ or $m = -a$ and $\delta = 0$. Straightforward calculations show that does not exist any polynomial inverse integrating factor of degree 1 except for the case $m = -a$ and $\delta = 0$. In this case the system is $\dot{x} = -y(1+ax)$, $\dot{y} = x(1+ax)$, which can be transformed into a linear system by doing a time-rescaling and $f(x,y) = 1+ax$ is a polynomial inverse integrating factor.

Proof of Statement (1.c). Suppose that $m = 0$. Doing the quadrature (3.2) for $i = d$ we obtain

$$\tilde{f}_d(u) = C_d(a - \ell u)^{-2+d}.$$

In the next step,

$$\tilde{f}_{d-1}(u) = -\frac{(a-\ell u)^{-3+d}}{\ell^2}(b_0^{(d-1)}(u) + C_d(a^2 - a\delta\ell + \ell^2)(-2+d)\log(a-\ell u)),$$

where $b_0^{(d-1)}(u) = \ell(-C_{d-1}\ell + C_d(a(-2+d) + \ell(d-u))u)$. Since f_{d-1} must be a polynomial we have two possibilities: either $\delta = (a^2 + \ell^2)/(a\ell)$ or $d = 2$.

In the first case, we obtain

$$\tilde{f}_{d-1}(u) = -\frac{(a-\ell u)^{-3+d}}{a\ell}(-aC_{d-1}\ell - a^2C_d u + C_d\ell^2 u + a^2C_d du - aC_d\ell u^2),$$

and in the next step we obtain for $\tilde{f}_{d-2}$ a polynomial of degree $d-1$, what is not possible. Thus, there are not polynomial inverse integrating factor in this case.

Otherwise, if $d = 2$, we obtain

$$\tilde{f}_1(u) = \frac{aC_1\ell - a^2C_2u - C_2\ell^2u + aC_2\ell u^2}{a\ell(a-\ell u)},$$

which is a polynomial if $C_1 = C_2$ and in this case,

$$\tilde{f}_1(u) = \frac{C_2(\ell - au)}{a\ell}.$$

In the next step, $\tilde{f}_0$ must be a constant but

$$\tilde{f}_0(u) = \frac{6C_0\ell^2 - 6aC_2\ell u + 3a^2C_2u^2 + 3C_2\ell^2u^2 - 2aC_2\ell u^3}{6\ell^2(a-\ell u)^2},$$

which is never a constant. Therefore, there are not polynomial inverse integrating factors in this case.

Proof of Statement (2). Consider $\Delta_2 = 0$. In this case we write $m = -\ell^2/(4a)$ and in the first step of the algorithm we obtain

$$\tilde{f}_d(u) = C_d e^{\frac{2a(d-3)}{\ell d - 2a}}(2a - \ell d)^{d-1},$$

from where d must be 3 in order to have a polynomial since $a \neq 0$. Straightforward calculations show that the only polynomial inverse integrating factors are the stated ones. ∎

3.3.3 Polynomial Inverse Integrating Factors in Family $(III)_{a=0}$

It is well known that family $(III)_{a=0}$ has at most one limit cycle, see [53] for instance. Moreover, if $\delta = 0$ the system has a center at the origin when $m(\ell + n) = 0$. Here we study some properties for the family $(III)_{a=0}$.

Theorem 3.6. *Consider system $(III)_{a=0}$ and define $\Delta_3 := m^2 + 4n(b-\ell)$. The following statements hold:*

1. *Consider the generic case $(b-\ell)n \neq 0$.*

   (a) *If $\Delta_3 \neq 0$, then there exist a polynomial inverse integrating factor of degree d if $\dfrac{\ell(d-2)-b}{\ell-b}$, $\pm\dfrac{bm(3-d)}{(b-\ell)\sqrt{\Delta_3}} - \dfrac{2\ell+b(1-d)}{2(b-\ell)} \in \mathbb{N}\cup\{0\}$. Moreover, when $\Delta_3 < 0$ and $m \neq 0$, then the only polynomial inverse integrating is $f(x,y) = (ny-1)(x^2-\delta xy+y^2)$ and appears when $m=-\delta n$, $b=-n$ and $\ell=0$. There is not any limit cycle when f exists.*

   (b) *If $\Delta_3 = 0$, then the only polynomial inverse integrating factors are $f(x,y) = (x-y)^2(1+by)$ which appears when $\ell=0$, $m=2b$ and $\delta=2$ and $f(x,y) = (x\pm y)^2(1+2\ell y)$ which appear when $b=2\ell$, $m=\mp 2\ell$ and $\delta=\mp 2$. There is not any limit cycle when someone exists.*

2. *Suppose that $b-\ell=0$ and $n\neq 0$. The only polynomial inverse integrating factors are $f(x,y) = (ny-1)^2(-1+mx+ny)$ which appears when $\delta=m=0$ and $b=-n$, and $f(x,y)=(1+by)^3$ which appears when $\delta=0$ and $m\neq 0$. There are not limit cycles when someone exist.*

3. *Suppose that $n=0$ and $b-\ell\neq 0$. In this case,*

   (a) *If $m\neq 0$ then there exist a polynomial inverse integrating factor of degree d if $\dfrac{\ell(d-2)-b}{\ell-b}$, $\dfrac{b(d-2)-\ell}{b-\ell} \in \mathbb{N}\cup\{0\}$.*

   (b) *If $m=0$ then there exist a polynomial inverse integrating factor of degree d if $\dfrac{\ell(d-2)-b}{\ell-b} \in \mathbb{N}\cup\{0\}$.*

4. *In the case $b-\ell=n=0$, the only polynomial inverse integrating factor appears when $m=0$ and is $f(x,y)=(1+by)(x^2-\delta xy+y^2)$. In this case, there is not any limit cycle.*

Proof. Assume that the vector field associated to family $(III)_{a=0}$ is given by (P,Q) and has a polynomial inverse integrating factor $f(x,y)$ of degree d. Let K_1 be the homogeneous part of degree 1 of the divergence of (P,Q), i.e. $K_1(x,y) = (b+2\ell)x+my$.

Proof of Statement (1.a). In the generic case $(b-\ell)n\Delta_3 \neq 0$, doing the quadrature (3.2) with $i=d$ we have

$$\begin{aligned}\tilde{f}_d(u) &= C_d(4n)^{\frac{2\ell+b(1-d)}{2(b-\ell)}} u^{\frac{\ell(d-2)-b}{\ell-b}} (\sqrt{\Delta_3}+m+2nu)^{\frac{bm(3-d)}{(b-\ell)\sqrt{\Delta_3}}-\frac{2\ell+b(1-d)}{2(b-\ell)}}\\ &\quad \times(\sqrt{\Delta_3}-m-2nu)^{-\frac{bm(3-d)}{(b-\ell)\sqrt{\Delta_3}}-\frac{2\ell+b(1-d)}{2(b-\ell)}},\end{aligned}$$

where $C_d \neq 0$. Since $\tilde{f}_d$ is a polynomial, its exponents must be nonnegative integers.

On the other hand, if we assume $\Delta_3 < 0$, we obtain the condition $d=3$ when $m\neq 0$. Therefore, system $(III)_{a=0}$ does not have any polynomial inverse integrating factor of degree different from 3. Straightforward calculations show that the only polynomial inverse integrating factor is $f(x,y) = (-1+ny)(x^2-\delta xy+y^2)$ and appears when $m=-\delta n$, $b=-n$ and $\ell=0$.

Proof of Statement (1.b). Let $f(x,y)$ be a polynomial inverse integrating factor of degree d for family $(III)_{a=0}$. From the condition $\Delta_3=0$ we take $n=m^2/[4(\ell-b)]$. In this case, the quadrature (3.2) with $i=d$ leads to

$$\tilde{f}_d(u)=C_d\exp\left(\frac{2b(d-3)}{mu+2(\ell-b)}\right)u^{\frac{\ell(d-2)-b}{\ell-b}}(2(b-\ell)-mu)^{\frac{b(1-d)+2\ell}{\ell-b}},$$

where $C_d\neq 0$. Since $\tilde{f}_d(u)$ is polynomial, we conclude that $d=3$ since $b\neq 0$. Straightforward calculations show that the only polynomial inverse integrating factors are the stated ones.

Proof of Statement (2). Assume that family $(III)_{a=0}$ has a polynomial inverse integrating factor $f(x,y)$ of degree d.

Let us suppose $m\neq 0$. In this case, doing the quadrature (3.2) with $i=d$ we have

$$\tilde{f}_d(u)=C_d\exp\left(\frac{(3-d)b}{mu}\right)u^{\frac{m^2(d-1)+bn(3-d)}{m^2}}(m+nu)^{\frac{m^2bn(d-3)}{m^2}},$$

where $C_d\neq 0$. Since $\tilde{f}_d(u)$ is polynomial, we conclude that $d=3$ because family (III) has always $b\neq 0$.
Otherwise, when $m=0$, we have $\tilde{f}_d(u)=C_d\exp\left(\frac{(3-d)b}{2nu^2}\right)u^d$, which is polynomial if and only if $d=3$.

Straightforward calculations with $d=3$ for $m\neq 0$ and $m=0$ show that the possible inverse integrating factors are the stated ones.

Proof of Statement (3). Let $f(x,y)$ be a polynomial inverse integrating factor of degree d for family $(III)_{a=0}$. Firstly, let us suppose $m\neq 0$. Hence the quadrature (3.2) with $i=d$ leads to

$$\tilde{f}_d(u)=C_d u^{\frac{\ell(2-d)+b}{b-\ell}}(b-\ell-mu)^{\frac{b(d-2)-\ell}{b-\ell}},$$

where $C_d\neq 0$. Since $\tilde{f}_d$ is polynomial, its exponents must be nonnegative integers. Then we conclude the Statement (3.a). Otherwise, when $m=0$, $\tilde{f}_d$ becomes

$$\tilde{f}_d(u)=C_d u^{\frac{\ell(d-2)-b}{\ell-b}},$$

and in analogous way we obtain Statement (3.b).

Proof of Statement (4). Assume that family $(III)_{a=0}$ with $b=\ell$ and $n=0$ has a polynomial inverse integrating factor $f(x,y)$ of degree d.

Let us suppose $m\neq 0$. In this case, doing the quadrature (3.2) with $i=d$ we have

$$\tilde{f}_d(u)=C_d u^2.$$

In the next step,

$$\tilde{f}_{d-1}(u) = \frac{1}{bm^2}\Big(C_d mu(2m-bu) + be^{\frac{b}{mu}}u(C_{d-1}m^2 + b\, C_d\int_{\frac{b}{mu}}^{\infty}\frac{e^{-z}}{z}dz)\Big),$$

which is a polynomial if $b=0$, but in this case the system belongs to Class (I) of Ye Yian-Qian.

In the case $m=0$, the system becomes $\dot{x} = \delta x - y + bx^2$, $\dot{y} = x(1+by)$, which is degenerate infinity and consequently we can not apply the algorithm of Theorem 3.2, but this system can not have limit cycles because can be transformed into a linear system into the projective plane. Moreover, the function $f(x,y) = (1+by)(x^2-\delta xy+y^2)$ is a polynomial inverse integrating factor, see for instance [5]. ∎

3.3.4 Polynomial Inverse Integrating Factors in Family $(III)_{n=0}$

It is well known that family $(III)_{n=0}$ can be transformed into a equation of Liénard type by a change of variables. But, in general, such equation is not polynomial. In this section we give some results for this family.

Theorem 3.7. *Consider system $(III)_{n=0,a\neq 0}$ and define $\Delta_4 := 4am + (b-\ell)^2$. The following statements hold:*

1. *If $m\Delta_4 \neq 0$, then there exist a polynomial inverse integrating factor of degree d if $\pm\dfrac{(d-3)(b+\ell)}{2\sqrt{\Delta_4}} + \dfrac{d-1}{2} \in \mathbb{N}\cup\{0\}$ and*
   (a) *when $b+\ell\neq 0$ and $\Delta_4<0$ there is not any polynomial inverse integrating factor.*
   (b) *when $b+\ell = 0$ there is not any polynomial inverse integrating factor.*
2. *If $m=0$, then*
   (a) *when $\ell - b\neq 0$ the system possesses a polynomial inverse integrating factor of degree d if $\dfrac{\ell(d-2)-b}{\ell-b}\in\mathbb{N}\cup\{0\}$.*
   (b) *when $\ell-b=0$ there is not any polynomial inverse integrating factor.*
3. *If $\Delta_4 = 0$ and $m\neq 0$, then there is not any polynomial inverse integrating factor.*

Proof. Let (P,Q) be the vector field associated to family $(III)_{n=0}$ and assume that it has a polynomial inverse integrating factor $f(x,y)$ of degree d. Let K_1 be the homogeneous part of degree 1 of the divergence of (P,Q), i.e. $K_1(x,y) = (b+2\ell)x + my$.

Proof of Statement (1.a). In the generic case $m\Delta_4\neq 0$, the quadrature (3.2) with $i=d$ leads to

$$\begin{aligned}\tilde{f}_d(u) &= C_d(-4m)^{\frac{1-d}{2}}(\sqrt{\Delta_4}+b-\ell-2mu)^{\frac{(d-3)(b+\ell)}{2\sqrt{\Delta_4}}+\frac{d-1}{2}}\\ &\quad\times(\sqrt{\Delta_4}-b+\ell+2mu)^{\frac{(d-3)(b+\ell)}{2\sqrt{\Delta_4}}-\frac{d-1}{2}},\end{aligned}$$

where $C_d \neq 0$. Since $\tilde{f}_d(u)$ is a polynomial, its exponents must be nonnegative integers and we obtain the first part of the statement.

On the other hand, if we assume $\Delta_4 < 0$, we obtain the condition $d = 3$ when $b + \ell \neq 0$. But straightforward calculations show that there is not any polynomial inverse integrating factor of such degree.

Proof of Statement (1.b). If we take $\ell = -b$ then

$$\tilde{f}_d(u) = C_d(-a - 2bu + mu^2)^{(d-1)/2}.$$

Notice that this expression implies that d must be an odd number under condition $\Delta_4 \neq 0$. Furthermore, a new step in the algorithm of Theorem 3.2 gives

$$\begin{aligned} \tilde{f}_{d-1}(u) \quad &= \quad \frac{mu^2 - 2bu - a}{m^{3/2}} \Big[a_0^{(d-1)}(u) + a_1^{(d-1)}(u)\sqrt{mu^2 - 2bu - a} \\ &\quad + a_2^{(d-1)}(u) \log[-b + mu + \sqrt{m(mu^2 - 2bu - a)}] \Big] \ , \end{aligned}$$

where $a_i^{(d-1)}$ are polynomials of degree 2 and more concretely $a_2^{(d-1)}(u) = C_d b(2 - d)(a + 2bu - mu^2)$. Clearly, there is implicitly an arbitrary constant C_{d-1} in the expression of $\tilde{f}_{d-1}(u)$ due to the made quadrature. Since $\tilde{f}_{d-1}$ must be a polynomial we conclude that $a_2^{(d-1)} \equiv 0$ and this implies $d = 2$ because in family (III) always $b \neq 0$. But this leads to a contradiction with the above condition d odd.

Proof of Statement (2). In the case $m = 0$ and $\ell - b \neq 0$ the quadrature (3.2) with $i = d$ takes the form

$$\tilde{f}_d(u) = C_d[a + (b - \ell)u]^{\frac{\ell(d-2)-b}{\ell - b}} \ ,$$

where $C_d \neq 0$. Since $\tilde{f}_d$ is a polynomial, its exponent must be a nonnegative integer.

Otherwise, that is, if $m = 0$ and $\ell - b = 0$ then $\tilde{f}_d(u) = C_d \exp[b(3 - d)u/a]$. This implies $d = 3$. But it is easy to see that this inverse integrating factor of degree 3 does not exist.

Proof of Statement (3). Assume that system $(III)_{n=0,a\neq 0}$ possesses a polynomial inverse integrating factor of degree d. From the condition $\Delta_4 = 0$ we take $m = -(b - \ell)^2/(4a) \neq 0$ and the quadrature (3.2) with $i = d$ leads to

$$\tilde{f}_d(u) = C_d \exp\left[\frac{2a(b + \ell)(d - 3)}{(\ell - b)((\ell - b)u - 2a)}\right] \ [2a + (b - \ell)u]^{d-1} \ ,$$

where $C_d \neq 0$. Since $\tilde{f}_d(u)$ is a polynomial, the exponential term must vanish an so, $b + \ell = 0$ or $d = 3$ and $b + \ell \neq 0$.

If we assume $b + \ell = 0$ then a new step in the algorithm of Theorem 3.2 gives

$$\tilde{f}_{d-1}(u) = \frac{(a + bu)^{d-3}}{b^2} \left[a_0^{(d-1)}(u) + a_1^{(d-1)}(u) \log[a + bu] \right] \ ,$$

where $a_0^{(d-1)}$ is a polynomial of degree 2 and $a_1^{(d-1)}(u) = C_d a^2(d - 2)(a + bu)$. Since $\tilde{f}_{d-1}$ must be a polynomial we conclude that $a_1^{(d-1)} \equiv 0$ which implies $d = 2$. But a straightforward calculation shows that this inverse integrating factor of degree 2 does not exist.

On the other hand, if we assume $d = 3$ and $b + \ell \neq 0$, straightforward calculations show that the inverse integrating factor does not exist. ∎

Chapter 4

Resolution of the Poincaré Problem in Family (I)

In this chapter, family (I) of the Chinese classification is widely studied. For this type of systems, the Poincaré problem is solved: any irreducible invariant algebraic curve has degree at most 3. As a corollary, we prove that these systems does not have algebraic limit cycles.

4.1 Introduction

In Ye Yian-Qian [53] are classified quadratic systems that can have limit cycles in three families as we have seen in the above chapter. Now, our target is family (I), i.e.,

$$\dot{x} = \delta x - y + \ell x^2 + mxy + ny^2 \ , \quad \dot{y} = x \ .$$

The authors expend pages to de discussion on the non existence of limit cycles and its uniqueness when exist. The limit cycle may appear in a neighborhood of the origin for $\delta m(\ell + n) < 0$ and $|\delta|$ sufficiently small. However, it is not known whether such limit cycle is an algebraic or transcendent curve. This question will be solved along this chapter.

Many results that we obtain in this chapter are already obtained in section 3.3.1 in the above chapter. Even so, this chapter is strictly algebraic, the proves we present are different and it makes this chapter self-contained.

4.2 The main results

Concerning the Poincaré problem for family (I) we have the following result.

Theorem 4.1. *Any irreducible invariant algebraic curve for family (I) has at most degree 3.*

Proof. Consider the planar polynomial differential system of family (I). When $\ell = 0$, family (I) becomes a quadratic Liénard system with linear damping. In this case, following Żołądek's results [55], the invariant algebraic curves associated can be only of two types: either rational curves of the form $x = \xi(y)$ or hyperelliptic curves like $(x - \xi(y))^2 = \eta(x)$ with ξ and η polynomials. It is easy to check that any invariant algebraic curve of the first type must have at most degree 2 and for the second type the degree is bounded by 3. On the other hand, there are not algebraic limit cycles for such systems.

So we continue the proof assuming $\ell \neq 0$. In order to control the behavior of the solutions of family (I) at infinity, we extend this family to a differential equation in the complex projective plane $\mathbb{C}P^2$. Thus, following the ideas of Darboux [21] one has

$$\mathcal{P}(X,Y,Z)dX + \mathcal{Q}(X,Y,Z)dY + \mathcal{R}(X,Y,Z)dZ = 0, \tag{4.1}$$

where $\mathcal{P} = MZ$, $\mathcal{Q} = -LZ$ and $\mathcal{R} = LY - MX$. Here L and M are the following homogeneous polynomials of second degree

$$\begin{array}{rcl} L = Z^2P(X/Z,Y/Z) & = & \ell X^2 + mXY + nY^2 + \delta XZ - YZ, \\ M = Z^2Q(X/Z,Y/Z) & = & XZ. \end{array}$$

Case 1: Consider $n \neq 0$.

Since the singular points in $\mathbb{C}P^2$ of the differential equation (4.1) are defined by $\mathcal{P} = \mathcal{Q} = \mathcal{R} = 0$, we obtain $p_1 = (0:0:1)$ and $p_2 = (0:1/n:1)$ which correspond to finite singular points of family (I), and $p_3 = (1:0:0)$, $p_4 = (1:\alpha:0)$ and $p_5 = (1:\beta:0)$, where α and β satisfy the equation $\ell + mY + nY^2 = 0$, which corresponds to the infinity ones. Notice that $p_4 = p_5$ if and only if $\alpha = \beta$, i.e., $m^2 - 4\ell n = 0$. Moreover $\alpha\beta \neq 0$ because $\ell \neq 0$.

The jacobian matrix of the associated vector of family (I) at p_1 is

$$D\mathcal{X}(p_1) = \begin{pmatrix} \delta & -1 \\ 1 & 0 \end{pmatrix},$$

and the characteristic polynomial is $p(t) = 1 - \delta t + t^2$.

Similarly, at p_2 one has

$$D\mathcal{X}(p_2) = \begin{pmatrix} \frac{m}{n} + \delta & 1 \\ 1 & 0 \end{pmatrix},$$

with eigenvalues $\lambda_2 = \frac{m+n\,\delta+\sqrt{4\,n^2+(m+n\,\delta)^2}}{2\,n}$ and $\mu_2 = \frac{m+n\,\delta-\sqrt{4\,n^2+(m+n\,\delta)^2}}{2\,n}$, satisfying $\lambda_2\mu_2 < 0$ and thus p_2 is a saddle.

Around p_3, p_4 and p_5, the differential equation (4.1) can be written in local coordinates, taking $X = 1$, as

$$\begin{array}{rcl} \dot{Y} & = & -\ell Y + Z - mY^2 - \delta YZ - nY^3 + Y^2Z, \\ \dot{Z} & = & -Z(\ell + mY + \delta Z + nY^2 - YZ). \end{array} \tag{4.2}$$

Now, denoting $D\mathcal{X}$ the jacobian matrix of the associated vector field to system (4.2) and, taking into account that α and β are solutions of the equation $\ell+mY+nY^2=0$, we can write

$$D\mathcal{X}(p_3)=\begin{pmatrix} -\ell & 1 \\ 0 & -\ell \end{pmatrix},$$

$$D\mathcal{X}(p_4)=\begin{pmatrix} -\ell-2m\alpha-3n\alpha^2 & 1-\delta\alpha+\alpha^2 \\ 0 & 0 \end{pmatrix},$$

$$D\mathcal{X}(p_5)=\begin{pmatrix} -\ell-2m\beta-3n\beta^2 & 1-\delta\beta+\beta^2 \\ 0 & 0 \end{pmatrix}.$$

At this point, we introduce the fact that family (I) has an invariant algebraic curve $f(x,y)=0$ of degree d with cofactor $k(x,y)=k_0+k_1x+k_2y$. We extend it to the projective plane taking $F(X,Y,Z)=Z^d f(X/Z,Y/Z)$ and cofactor $K(X,Y,Z)=Zk(X/Z,Y/Z)=k_0Z+k_1X+k_2Y$. It is known that in the local chart of the points p_3, p_4 and p_5, the curve is given by $F(1,Y,Z)=0$ and the cofactor by $\tilde{K}(1,Y,Z)=k_0Z+k_1+k_2Y-dL(1,Y,Z)$.

Since $\ell\neq 0$, p_3 is a logarithmic singular point and from statement 2 of Theorem 1.45 the differential equation has only one formal solution at it. Since $Z=0$ is solution of (4.2) through p_3, there are not more solutions at this point. In particular, it follows that $F(p_3)\neq 0$. Therefore, from (1.3), $\tilde{K}(p_3)=0$ and we obtain $k_1=\ell d$.

The eigenvalues of the linear part at p_4 and p_5 are $\lambda_4=-\ell-2m\alpha-3n\alpha^2$, $\mu_4=0$ and $\lambda_5=-\ell-2m\beta-3n\beta^2$, $\mu_5=0$, respectively.

We claim that $\lambda_4=0$ if and only if $\alpha=\beta$, and $\lambda_5=0$ if and only if $\alpha=\beta$.

We prove the claim for λ_4 and the claim for λ_5 holds by symmetry. Suppose that $\lambda_4=0$. We know that α is a solution of the equation $\ell+mY+nY^2=0$ but $\alpha\neq 0$ because $\ell\neq 0$. Of course, it is also a solution of $Y(\ell+mY+nY^2)=0$. Since $\lambda_4=0$, α is also a solution of the derivative of the above expression. Therefore, α must be a double solution of $Y(\ell+mY+nY^2)=0$ which has $Y=0$, $Y=\alpha$ and $Y=\beta$ as solutions. So, the only possibility is $\alpha=\beta$. It is easy to see that the reciprocal is true.

Therefore, we must distinguish two possibilities for α and β:

Case 1.a: Consider $\alpha\neq\beta$.

In this case, from statement 3 of Theorem 1.45 the differential equation (4.2) has only two different formal solutions on each point p_4 and p_5. Since we know that $Z=0$ is one of these solutions, there is only another one for each point.

Taking into account that the relation $\ell+mY+nY^2=0$ is satisfied by α and β one has $L(p_i)=0$, for $i=4,5$ and then $\tilde{K}(p_i)=K(p_i)$ for $i=4,5$. Moreover, from Theorem 1.46 and the fact that $\mu_i=0$ for $i=4,5$, the cofactor must satisfy $\tilde{K}(p_4)=0$ or $\tilde{K}(p_4)=\lambda_4$ and $\tilde{K}(p_5)=0$ or $\tilde{K}(p_5)=\lambda_5$.

Summarizing, we must study the following possibilities:

1. $\tilde{K}(p_4) = \tilde{K}(p_5) = 0$.

   We have
   $$\ell d + k_2\alpha = 0,$$
   $$\ell d + k_2\beta = 0,$$
   from where $k_2(\alpha - \beta) = 0$ and $k_2 = 0$ when $\alpha \neq \beta$. Hence, $\ell d = 0$ and then $d = 0$ because $\ell \neq 0$. Therefore, there are not invariant algebraic curves in this case.

2. $\tilde{K}(p_4) = 0$, $\tilde{K}(p_5) = \lambda_5$ or $\tilde{K}(p_4) = \lambda_4$, $\tilde{K}(p_5) = 0$.

   Suppose that $\tilde{K}(p_4) = 0$, $\tilde{K}(p_5) = \lambda_5$. We have that
   $$\begin{aligned} &\ell d + k_2\alpha = 0, \\ &\ell d + k_2\beta + \ell + 2m\beta + 3n\beta^2 = 0. \end{aligned} \tag{4.3}$$
   We know that α and β are solutions of $\ell + mY + nY^2 = 0$, in other words, $\ell = n\alpha\beta$ and $m = -n(\alpha + \beta)$.

   Then, from equations (4.3) we obtain $(k_2+n\beta)(\beta-\alpha) = 0$, from where $k_2 = -n\beta$, because $\alpha \neq \beta$.

   From this last condition and equations (4.3) we get $n\alpha\beta(d-1) = 0$ and so $d = 1$ because $\alpha\beta n \neq 0$.

   When $\tilde{K}(p_4) = \lambda_4$ and $\tilde{K}(p_5) = 0$, we obtain also $d = 1$ by symmetry.

3. $\tilde{K}(p_4) = \lambda_4$, $\tilde{K}(p_5) = \lambda_5$.

   In this case we have
   $$\ell d + k_2\alpha + \ell + 2m\alpha + 3n\alpha^2 = 0,$$
   $$\ell d + k_2\beta + \ell + 2m\beta + 3n\beta^2 = 0.$$
   As in the former case, we add the relations $\ell = n\alpha\beta$ and $m = -n(\alpha+\beta)$ to these equations obtaining $n(\alpha - \beta)(d-2) = 0$. Clearly, $d = 2$.

Case 1.b: Consider $\alpha = \beta$.

In this case, the singular points at infinity of family (I) are p_3 and p_4 being the last a double point $(1 : \alpha : 0)$ where α is the double root of $\ell + mY + nY^2 = 0$. In other words $m^2 - 4\ell n = 0$ from where $\ell = m^2/(4n)$ and $\alpha = -m/(2n)$. Recall that p_3 is the logarithmic singular point already analyzed.
Family (I) can be written now in the form
$$\dot{x} = \delta x - y + \frac{1}{4n}(mx + 2ny)^2, \;\; \dot{y} = x, \tag{4.4}$$
where we can always take $n = 1/2$ by using the scaling of variables $X = 2nx$, $Y = 2ny$.

Now, we have
$$D\mathcal{X}(p_4) = \begin{pmatrix} 0 & \Delta \\ 0 & 0 \end{pmatrix},$$
with $\Delta = 1 + m\delta + m^2$. In function of the value we distinguish two cases.

1. Suppose that $\Delta = 0$. In this case p_4 is a degenerated singular point. The characteristic polynomial $p(t) = 1 - \delta t + t^2$ associated to $D\mathcal{X}(p_1)$ has $t = -m$ as a real root and therefore p_1 is not a focus. There are not limit cycles in this case because there are not finite foci.

   Moreover, from $\Delta = 0$ we obtain $\delta = -(1+m^2)/m$. Making the change $u = x$, $w = y + mx$ and the time rescaling $dt = 2mdT$ we can write family (I) as

$$u' = -2u - 2mv + mv^2, \;\; v' = m^2(-2+v)v, \tag{4.5}$$

   where the prime denotes derivative with respect to T. We emphasize that the differential equation of the orbits of system (4.5), i.e., $du/dv = \xi(u, v)$ is linear. But its associated first integral $H(u, v)$ involves hypergeometric functions depending of the parameter m. Hence is not easy to analyze from H the existence of invariant algebraic curves for (4.5).

   However, notice that system (4.5) has $v = 2$ and $v = 0$ as affine invariant straight lines. We extend now system (4.5) to $\mathbb{C}P^2$ by using the homogeneous variables V W and Z. Due to the changes of variables, the singular points at infinity of (4.5) become $p_3 = (1 : m : 0)$ and $p_4 = (1 : 0 : 0)$. Therefore, the projective invariant straight lines $V = 0$, $V = 2$ and $Z = 0$ have the degenerated singular point p_4 as common point.

   Let $g(u, v) = 0$ be the irreducible invariant algebraic curve of degree d for system (4.5) which comes from the initial invariant curve $f(x, y) = 0$ through the made linear changes of variables. Denote by $c(u, v) = c_0 + c_1 u + c_2 v$ the cofactor associated to $g = 0$. Let $G(U, V, Z) = 0$ be the projectivization of $g = 0$. Since p_3 is a logarithmic point, we recall that the only solution of (4.5) at p_3 is $Z = 0$. Hence, $G = 0$ crosses the line at infinity only at p_4. Therefore, from Lemma 2.6(i), the highest order homogeneous degree term of $g = 0$ must be v^d. In addition, from Lemma 2.6(ii) we conclude that v must divide $(c_1 u + c_2 v)u - dmv^2$. Hence $c_1 = 0$ and we get that the cofactor c only depends on v. Since additionally v' also depends only on v, taking into account (1.3), i.e., $g' = cg$, it follows that

$$\int \frac{dg}{g} = \int c(v)dT = \int c(v)\frac{v}{m^2(-2+v)v}dv. \tag{4.6}$$

   In short, the irreducible real algebraic curve $g = 0$ only depends on v and therefore it is either a real straight line or the product of two complex conjugated straight lines.

2. Suppose that $\Delta \neq 0$. Doing the change of variables

$$x = \frac{1}{\Delta}\left(u + mv - \frac{m}{2}v^2\right) \;,\; y = \frac{1}{\Delta}\left(-mu + (1+m\delta)v + \frac{m^2}{2}v^2\right) \;,$$

   which is a global diffeomorphism on $\mathbb{R}^2$ (in fact is a bipolinomial transformation), system (4.4) becomes

$$\dot{u} = -v + \delta u + muv + \frac{1}{2}v^2 \;,\; \dot{v} = u \;.$$

   Therefore, the transformed system belongs to family (I) with $\ell = 0$, studied in [55]. As we have said before, the degree of any invariant algebraic curve is 2 or 3 and there are not algebraic limit cycles.

Case 2: Consider $n = 0$ and $m \neq 0$.

In this case, there is a unique finite singular point $p_1 = (0 : 0 : 1)$ and there are three singular points over the line at infinity $Z = 0$: $p_3 = (1 : 0 : 0)$, $p_4 = (1 : -\ell/m : 0)$ and $p_5 = (0 : 1 : 0)$. Around p_3 and p_4 the differential equation can be written in local coordinates, taking $X = 1$, as

$$\begin{array}{rcl} \dot{Y} & = & -\ell Y - mY^2 + Z + Y^2 Z - YZ\delta, \\ \dot{Z} & = & -Z(\ell + mY - YZ + Z\delta). \end{array}$$

Around p_5 and taking $Y = 1$ we have

$$\begin{array}{rcl} \dot{X} & = & -mX - \ell X^2 + Z + X^2 Z - XZ\delta, \\ \dot{Z} & = & XZ^2. \end{array}$$

If $D\mathcal{X}$ is the corresponding jacobian matrix, we have

$$D\mathcal{X}(p_3) = \begin{pmatrix} -\ell & 1 \\ 0 & -\ell \end{pmatrix},$$

$$D\mathcal{X}(p_4) = \begin{pmatrix} \ell & 1 + \ell^2/m^2 + \ell\delta/m \\ 0 & 0 \end{pmatrix},$$

with eigenvalues $\lambda_4 = 0$ and $\mu_4 = \ell$,

$$D\mathcal{X}(p_5) = \begin{pmatrix} -m & 1 \\ 0 & 0 \end{pmatrix},$$

with eigenvalues $\lambda_5 = 0$ and $\mu_5 = -m$.
By statement 2 of Theorem 1.45, it follows that there is a unique linear branch at the logarithmic point p_3, and by statement 3 of the same theorem, there are two linear branches at p_4 and p_5.

Let $f = 0$ be an invariant algebraic curve for family (I) of degree d with cofactor $k(x, y) = k_0 + k_1 x + k_2 y$. The cofactor of the projectivized curve at the local chart of p_3 and p_4 is given by $\tilde{K}_1(1, Y, Z) = k_0 Z + k_1 + k_2 Y - dL(1, Y, Z)$ and at the local chart of p_5, by $\tilde{K}_2(X, 1, Z) = k_0 Z + k_1 X + k_2 - dM(X, 1, Z)$. Since $Z = 0$ is the only invariant curve through p_3, from (1.3) it follows $\tilde{K}_1(p_3) = 0$ from where $k_1 = d\ell$. On the other hand, $Z = 0$ is one of the invariant curves through p_4 and p_5, and from Theorem 1.46 we have either $\tilde{K}_2(p_4) = 0$ or $\tilde{K}_2(p_4) = \ell$ and $\tilde{K}_2(p_5) = 0$ or $\tilde{K}_2(p_5) = -m$. Then, we obtain $k_2 = dm$ or $k_2 = (d-1)m$ and $k_2 = 0$ or $k_2 = -m$, respectively. Taking into account that $m \neq 0$, the possibilities for the degree of f are $d = 0$ or $d = 1$.

Case 3: Consider $n = m = 0$.

In this case, after changing $t \to -t$, family (I) reduces to $\dot{x} = y - F(x)$, $\dot{y} = -g(x)$ with $F(x) = \delta x + \ell x^2$ and $g(x) = x$. Hence it is a Liénard system written in the called Liénard plane. Moreover, taking $u = x$, $v = y - F(x)$ it can be written as a quadratic Liénard system with linear damping: $\dot{u} = v$, $\dot{v} = -f(u)v - g(u)$. Here $f(x) = dF(x)/dx$. Since $\deg f = \deg g = 1$, for $\delta \neq 0$, g is not a constant multiple of f we can apply Odani's results [40] concluding the non existence of any invariant algebraic curve.

If additionally $\delta = 0$, then family (I) writes as

$$\dot{x} = -y + \ell x^2, \quad \dot{y} = x, \tag{4.7}$$

and possesses the Darboux first integral $H(x,y) = h(x,y)\exp(-2\ell y)$ with $h(x,y) = -1 - 2\ell y + 2\ell^2 x^2$ as an invariant parabola.

We will show now, that this parabola is the only irreducible invariant algebraic curve of system 4.7. We recall that for any invariant algebraic curve $f = 0$ of degree d with cofactor $k = k_0 + k_1 x + k_2 y$ we get $k_1 = \ell d$ since $p_3 = (1:0:0)$ is a logarithmic singular point. Also, the origin $p_1 = (0:0:1)$ is a weak focus which can not belong to an algebraic curve. So, $k(p_1) = 0$. and then $k_0 = 0$. Finally, the projectivized system of 4.7 written in local coordinates around $p_2 = (0:1:0)$ is

$$\begin{aligned} \dot{X} &= Z - \ell X^2 Y + X^2 Z, \\ \dot{Z} &= Z(Z - \ell X^2). \end{aligned}$$

Clearly, p_2 is a nilpotent singular point. It can be checked that the cofactor in this coordinates $\tilde{K}(X,1,Z) = \ell dX + k_2 Y - dM(X,1,Z) = d(\ell - 1)X + k_2 Y$ must satisfy $\tilde{K}(p_2) = 0$, from where $k_2 = 0$.

Summarizing, the cofactor of any invariant algebraic curve of degree d $k(x,y) = \ell dx$. The existence of two different invariant algebraic curves of degrees d_1 and d_2 with their respective cofactors implies the existence of a rational first integral. But it is well known, see Poincaré [44] that such first integral can not coexist with a logarithmic singular point.

Therefore the only invariant algebraic curve of the system is the parabola $h(x,y) = 0$. ∎

Relative to the existence of algebraic limit cycles, the following result follows taking into account the proof of Theorem 4.1.

Corollary 4.2. *There are not algebraic limit cycles in family (I).*

Proof. As we have seen, the real invariant algebraic curves for family (I) have at most degree 3, and in some cases the existence of algebraic limit cycles has been discounted along the proof. Moreover, it is well known that a quadratic system cannot have cubic algebraic limit cycles. Hence, the only possibility for algebraic limit cycles are conics.

On the other hand, any invariant algebraic curve must intersect the line at infinity in at least one of the singular points. In all the non discounted cases described in the proof of Theorem 4.1 except perhaps in in Case 1.a, all the infinite singular points are real, then the invariant conic cannot have real ovals contained in the affine plane. Therefore, there are not algebraic limit cycles for family (I).

In Case 1.a, the singular points p_4 and p_5 may be complex conjugated. As we have seen, if there exists an invariant algebraic curve $f = 0$ it must have degree 2 and must not contain the singular points p_1 and p_2. Let it's cofactor be $k(x,y) = k_0 + k_1 x + k_2 y$; it follows $k(p_i) = 0$, $i = 1,2$ and therefore $k_0 = k_2 = 0$. Moreover, we recall that p_3 is a logarithmic singular point what brings us to an specific expression for the cofactor: $k(x,y) = 2\ell x$. Finally, imposing $\mathcal{X} f = kf$ it follows $\delta = 0$ which excludes de existence of limit cycles. ∎

Chapter 5

Nested Configuration of Algebraic Limit Cycles in Quadratic Systems

This chapter deals with algebraic limit cycles of planar polynomial differential systems of degree two. More concretely, we show among other facts that a quadratic vector field can not possess two non nested algebraic limit cycles contained in different irreducible invariant algebraic curves.

5.1 Introduction and statement of the results

We will concentrate our study in invariant algebraic curves satisfying (1.3), containing ovals which are limit cycles for a quadratic systems (1.6).

Differential systems and limit cycles of degree 4 presented in chapter 2 have been studied by many mathematicians later. When the algebraic limit cycles of degree 4 where known, the next question was the uniqueness, that is, the fact that when a differential equation has one of the known algebraic limit cycle there are not more limit cycles for the system; this question is solved by Chavarriga, Giacomini and Llibre [7]. In such paper is proved the following result involving projective notation, which provides sufficient conditions i order to have a quadratic system with all its limit cycles algebraic.

Theorem 5.1. (Chavarriga, Giacomini & Llibre) *Let $f(x, y) = 0$ be a real invariant algebraic curve of degree great or equal than two of a real quadratic system (1.6). Let k be the cofactor of $f = 0$. Suppose that there are two points $p_1, p_2 \in \mathbb{C}P^2$ such that $L(p_i) = M(p_i) = K(p_i) = 0$ for $i = 1, 2$, where $L = Z^2P(X/Z, Y/Z)$, $M = Z^2Q(X/Z, Y/Z)$ and $K = Zk(X/Z, Y/Z)$. Then all the limit cycles of (1.6) must be algebraic and contained into $f(x, y) = 0$.*

In a work due to Llibre and Rodríguez [37] is proved that any configuration of limit cycles is possible using algebraic limit cycles, what gives extra importance to the study of algebraic limit cycles.

In a recent work due to Christopher, Llibre and Świrszcs [16] two families of quadratic systems with an algebraic limit cycle of degrees 5 and 6, respectively, are given. These two families are constructed by means of a birrational transformation of system (d) given in Theorem 2.1. Moreover, they prove that there is also a birrational transformation which converts Yablonskii system into the system found by Ch'in Yuan-shün. More recently, Chavarriga, Giacomini and Grau [6] have proved that none of the quadratic systems with known algebraic limit cycles have a liouvillian first integral and that these systems have only one invariant algebraic curve when the limit cycle exists. Moreover, Giacomini and Grau [30] show the hyperbolicity of these limit cycles.

Summarizing, as far we know, it seems as if the uniqueness of the invariant algebraic curves containing algebraic limit cycles was unavoidable, and also perhaps the uniqueness of the algebraic limit cycles itself. Concretely, the open question that we think about is the following one: *Can a quadratic system possess more than one algebraic limit cycle?*

Of course, if system (1.6) has more than one limit cycle, then they can be distributed in many different ways. Assuming system (1.6) possesses two algebraic limit cycles γ_i with $i = 1, 2$, two algebraically differentiated situations are presented. Either the two limit cycles are contained in a unique irreducible invariant algebraic curve or there are two different irreducible invariant algebraic curves $f_i(x, y) = 0$ with $i = 1, 2$, such that each one of them contains only one limit cycle. In this work we will concentrate on the second case. But one still has two cases with different topology respect to the configuration of limit cycles: either the two algebraic limit cycles are nested or not. The main result of this chapter is the following one.

Theorem 5.2. *A quadratic system (1.6) can not possess two non-nested algebraic limit cycles contained in different irreducible invariant algebraic curves.*

It is well known that in a given quadratic system at most two singularities are surrounded by limit cycles and that these singularities necessarily are foci. We say that limit cycles of system (1.6) have (p, q)-distribution if it possesses p nested limit cycles surrounding one focus and q nested limit cycles surrounding another different focus. In [41], Z. Pingguang proves that limit cycles of a quadratic system with two foci must be $(1, i)$-distribution $(i = 0, 1, \ldots)$.

Corollary 5.3. *If a quadratic system (1.6) with two foci possesses r limit cycles $C_1, \ldots, C_r$ $(r > 1)$ surrounding the same focus and at least one of them is algebraic, i.e., $C_1, \ldots, C_s$ $(1 \leq s \leq r)$ are algebraic, then there exists another limit cycle $C^\star$ surrounding the other focus. Moreover, either*

(i) *$C^\star$ is a non-algebraic limit cycle or,*

(ii) *$C^\star$ is an algebraic limit cycle and the algebraic limit cycles C_i and $C^\star$ are contained in the same irreducible invariant algebraic curve for some $i = 1, \ldots, s$.*

In the next section we present the necessary concepts on quadratic systems, projective differential equations, formal differential equations and some known results. Next, we prove some technical results. We get the proof of Theorem 5.2 in the last section.

5.2 Preliminary results

We make some considerations on the arguments used along the proofs in order to to facilitate the understanding:

- It is clear from (1.3) that, given an invariant algebraic curve $f = 0$ with cofactor k, then all the finite critical points of a polynomial differential system (1.6) verify either $f(x_0, y_0) = 0$ or $k(x_0, y_0) = 0$ or both above conditions. Moreover, since $f, k \in \mathbb{R}[x, y]$, if (x_0, y_0) is a complex critical point of (1.6) with $f(x_0, y_0) \neq 0$ then $k(x_0, y_0) = k(\bar{x}_0, \bar{y}_0) = 0$.
- Said this, Theorem 1.46 and Lemma 1.47 give the possible values of the cofactor k of an invariant algebraic curve $f = 0$ of system (1.6) at a nondegenerate or degenerate elementary critical point $(x_0, y_0) \in \mathbb{C}^2$ whose ratio of eigenvalues does not equal one. Of course, we can extend system (1.6) to $\mathbb{C}P^2$. Hence, if $p_0 = (X_0 : Y_0 : Z_0)$ is a singular point of the associated projective equation we can take local coordinates at this point and Theorem 1.46 can be applied. We notice that, for an infinite critical point $p_0 = (X_0 : Y_0 : 0)$ we will obtain by the above procedure conditions on the degree n of the curve $f = 0$ because the coefficients of the cofactor also depend on n.
- From now on, we will write $k(p) = \text{div}\mathcal{X}(p)$ in case that $k(p) = \lambda + \mu$.

We will study the algebraic limit cycles of system (1.6) under the next assumption:

Hypothesis A: Let us suppose that system (1.6) has two non nested algebraic limit cycles γ_i with $i = 1, 2$. We will assume moreover that system (1.6) has two different irreducible real invariant algebraic curves $f_i(x, y) = 0$ with $i = 1, 2$, such that $\gamma_i \subset \{(x, y) \in \mathbb{R}^2 : f_i(x, y) = 0\}$.

Since system (1.6) is a quadratic system, a consequence of hypothesis A is the existence of two different critical points of nondegenerate focus type p_i with $i = 1, 2$, such that $p_i \in \text{Int}(\gamma_i)$, as is stated in Theorem 1.23.

Lemma 5.4. *Under hypothesis A the next holds.*

(i) $f_i(p_1) f_i(p_2) = 0$ *for* $i = 1, 2$.

(ii) $f_1^2(p_j) + f_2^2(p_j) \neq 0$ *for* $j = 1, 2$.

Proof. Let $k_i(x, y)$ be the cofactor of the invariant algebraic curve $f_i(x, y) = 0$. Let us assume the contrary of statement (i), that is, suppose that $f_i(p_1) f_i(p_2) \neq 0$ for some $i \in \{1, 2\}$. Then, it follows $k_i(p_j) = 0$ for $j = 1, 2$ and, since $\deg f_i > 1$, by Theorem 5.1 all the algebraic limit cycles of the quadratic system must be contained in either curve $f_1(x, y) = 0$ or $f_2(x, y) = 0$. Of course, this is in contradiction with hypothesis A and so either p_1 or p_2 must belong to the zero level set of f_i for $i = 1, 2$ proving thus statement (i).

In order to prove statement (ii) we suppose the contrary, i.e., $f_1(p_j) = f_2(p_j) = 0$ for some $j \in \{1, 2\}$. Since p_j is a nondegenerate focus, its associated eigenvalues λ and μ are complex numbers $\alpha \pm \mathrm{i}\beta$ verifying $\lambda/\mu \notin \mathbb{Q}^+$. We can translate the focus p_j to the origin and make a complex linear change of coordinates in order to bring system (1.6) to the form $\dot{x} = \lambda x + \cdots$, $\dot{y} = \mu y + \cdots$. After, applying Theorem 1.45(1.i), we conclude that there are exactly two formal solutions at the origin. Therefore, going back through the change of variables, there are exactly two formal solutions at p_j: $F_1(x, y) = T_1(x, y) + \cdots$, $F_2(x, y) = \bar{F}_1(x, y) = \bar{T}_1(x, y) + \cdots$, being T_1 the tangent of F_1 at p_j and where the over bar denotes complex conjugation operation. Finally, since $f_i(x, y) = 0$ are real invariant algebraic curves, we conclude that $f_1 = f_2 = F_1\bar{F}_1$ in contradiction with hypothesis A. ∎

Lemma 5.5. *Under hypothesis A, either*

$$k_i(p_j) = \begin{cases} \operatorname{div}\mathcal{X}(p_j), & i = j, \\ 0, & i \neq j, \end{cases} \qquad \text{for } \; i, j = 1, 2.$$

or

$$k_i(p_j) = \begin{cases} 0, & i = j, \\ \operatorname{div}\mathcal{X}(p_j), & i \neq j, \end{cases} \qquad \text{for } \; i, j = 1, 2.$$

Proof. From Lemma 5.4 it follows that one focus belongs to a curve and the other one belongs to the other curve of hypothesis A. So, the cofactor must be zero over at least one of the foci. On the other hand, if any cofactor vanishes at more that one foci, from Theorem 5.1 we get a contradiction with hypothesis A. In short, any cofactor is zero exactly at one focus. The value of the cofactor at the other focus is given by Lemma 1.47. ∎

Anyway, respect to the configuration of the finite critical points of system (1.6), the next possibilities are presented. Two foci p_1 and p_2 exist always and: (a) There are not more finite critical points; (b) There is exactly one more finite critical point p_3 which has multiplicity one; (c) The rest of finite critical points p_3 and p_4 are real. Here it is possible $p_3 = p_4$; (d) The rest of finite critical points p_3 and p_4 have complex conjugate coordinates.

We will see that the first two former cases (a) and (b) are in contradiction with hypothesis A. First we present this preliminary result.

Lemma 5.6. *Let us assume that quadratic system (1.6) has a common factor in their highest order terms, i.e., $P_2 = \Lambda L_1$ and $Q_2 = \Lambda L_2$ where Λ, L_1 and L_2 are linear polynomials. Then system (1.6) does not satisfy hypothesis A.*

Proof. By linear change of variables we consider the case $\Lambda = x$ without lost of generality. Then the point $q_1 = (0 : 1 : 0) \in \mathbb{C}P^2$ is an singular point of (1.6) at infinity.

Assume the contrary of the thesis, i.e., hypothesis A is verified. Let $F_i(X, Y, Z) = 0$ and $K_i(X, Y, Z)$ be the projectivizations of the invariant algebraic curves $f_i(x, y) = 0$ and its associated cofactors for $i = 1, 2$, respectively.

We take local coordinates in a neighborhood of the singular point q_1 and denote by $\tilde{F}_i(X, 1, Z) = 0$ and $\tilde{K}_i(X, 1, Z)$ the transformed invariant curves and cofactors in such coordinates respectively, see the preliminaries. Since by definition $\tilde{K}_i(X,1,Z) = K_i(X,1,Z) - \deg f_i M(X,1,Z)$ with $M(X,Y,Z) = Z^2 Q(X/Z, Y/Z)$ and $M(q_1) = 0$ it follows

$$\tilde{K}_i(q_1) = K_i(q_1) \ . \tag{5.1}$$

Additionally, it is easy to see that the coefficients of the linear part of the system in local coordinates at q_1 are given by

$$\begin{pmatrix} L_1(0,1) & P_1(0,1) \\ 0 & 0 \end{pmatrix} .$$

This means that q_1 has at least one associated eigenvalue different from zero. If both eigenvalues vanish then $\tilde{K}_i(q_1) = 0$ for $i = 1, 2$. Otherwise, if exactly one eigenvalue is zero then, from statement 3 of Seidenberg's Theorem 1.45, it follows that there are two formal solutions through q_1. Since one of them is the line at infinity $Z = 0$, it is clear that $\tilde{F}_1(q_1) \neq 0$ or $\tilde{F}_2(q_1) \neq 0$. This implies that $\tilde{K}_1(q_1) = 0$ or $\tilde{K}_2(q_1) = 0$ respectively. Hence, taking into account (5.1) we get $K_1(q_1) = 0$ (re–indexing if necessary).

We know that the affine cofactor $k_1(x,y)$ vanishes also at one of the two foci by Lemma 5.5. Hence $K_1(X,Y,Z)$ vanishes at such focus, too. Therefore, we are under hypothesis of Theorem 5.1 and we get a contradiction with hypothesis A. ∎

Proposition 5.7. *Let us assume that quadratic system (1.6) has two real finite different critical points p_1 and p_2 of nondegenerate focus type. If either there are not more finite critical points or there is exactly one more finite critical point p_3 with multiplicity one then system (1.6) does not satisfy hypothesis A.*

Proof. We consider the homogeneous polynomials $L(X,Y,Z) = Z^2 P(X/Z, Y/Z)$ and $M(X,Y,Z) = Z^2 Q(X/Z, Y/Z)$. We denote $I_p(L,M)$ the intersection index of $L = 0$ and $M = 0$ at the point $p \in \mathbb{C}P^2$, see the preliminaries of this work. From Bézout Theorem it follows $\sum_p I_p(L,M) = 4$. Since p_1 and p_2 are nondegenerate foci, its associated eigenvalues are different from zero and then p_1 and p_2 have multiplicity one as common roots of $P(x,y)$ and $Q(x,y)$. Hence $I_{p_i}(L,M) = 1$ for $i = 1, 2$. We split the study of each situation described in the proposition.

- If there are not more finite critical points of system (1.6) then there are points $q_j \in \{Z = 0\} \cap \{L = 0\} \cap \{M = 0\}$ such that $\sum_{q_j} I_{q_j}(L,M) = 2$. Therefore $Q_2(x,y) = \alpha P_2(x,y)$ with $\alpha \in \mathbb{R}$ and from Lemma 5.6 system (1.6) does not satisfy hypothesis A.
- If there is exactly one more finite critical point p_3 of system (1.6) with multiplicity one then $\sum_{i=1}^{3} I_{p_i}(L,M) = 3$. So there is exactly one point $q \in \{Z = 0\} \cap \{L = 0\} \cap \{M = 0\}$ such that $I_q(L,M) = 1$. Therefore P_2 and Q_2 have exactly one real common divisor of degree 1. Hence, applying Lemma 5.6, system (1.6) does not verify hypothesis A. ∎

The next two propositions explore the possibilities of the above cases (c) and (d). In such study we shall consider the real straight line $L(x,y) := pk_1(x,y) + qk_2(x,y) - \mathrm{div}\mathcal{X}(x,y) = 0$, with $p, q \in \mathbb{R}$. The main idea in what follows consists on to look for three finite critical points of system (1.6) such that L vanishes at them. Of course such critical points are not in any straight line because in this case $P(x,y)$ and $Q(x,y)$ are not coprime. So the only possibility is $L(x,y) \equiv 0$ and therefore, applying Darboux's integrability theory we conclude that $f_1^p f_2^q$ is an inverse integrating factor of the system.

Proposition 5.8. *Let us assume that quadratic system (1.6) verifies hypothesis A and moreover the other finite critical points p_3 and p_4 are real. Then $f_1(x,y)f_2(x,y)$ is an inverse integrating factor of the system.*

Proof. We will start with two different cases which are either $p_3 \neq p_4$ or $p_3 = p_4$.

If $p_3 \neq p_4$ then each one have multiplicity one. Since p_1 and p_2 are foci of the quadratic system, using Theorem 1.22, we can suppose that p_3 is a topological saddle. Hence the quotient of the eigenvalues associated to p_3 is negative. So following Seidenberg's results and more concretely Theorem 1.45(1.i), there are exactly two formal solutions (linear branch) with different tangent at p_3.

If $f_1(p_3) \neq 0$ or $f_2(p_3) \neq 0$, then $k_1(p_3) = 0$ or $k_2(p_3) = 0$. Then applying Theorem 5.1 and Lemma 5.5 we have that all the limit cycles are contained in $f_1 = 0$ or $f_2 = 0$, respectively. This is a contradiction with hypothesis A. Therefore, the only possibility consists in that the invariant algebraic curve $f_1 = 0$ contains exactly one branch at p_3 and $f_2 = 0$ the other one.

Hence, translating the critical point p_3 to the origin, and making a linear change of coordinates we will continue assuming $f_1(x,y) = x + \cdots$, $f_2(x,y) = y + \cdots$ and the system becomes $\dot{x} = \lambda x + \cdots$, $\dot{y} = \mu y + \cdots$, where λ and μ are the eigenvalues associated to p_3. Now, equating the same powers of x and y in both members of the equations $\mathcal{X} f_i = k_i f_i$ for $i = 1, 2$, we have that $k_1(p_3) = \lambda$ and $k_2(p_3) = \mu$. Since $\mathrm{div}\mathcal{X}(p_3) = \lambda + \mu$ we have in short $k_1(p_3) + k_2(p_3) - \mathrm{div}\mathcal{X}(p_3) = 0$. As we also knew that $k_1(p_i) + k_2(p_i) - \mathrm{div}\mathcal{X}(p_i) = 0$ for $i = 1, 2$, this implies $k_1(x,y) + k_2(x,y) \equiv \mathrm{div}\mathcal{X}(x,y)$ because k_1, k_2 and $\mathrm{div}\mathcal{X}$ are polynomials of degree at most one. Finally, by the Darboux's integrability theory we conclude that $f_1(x,y)f_2(x,y)$ is an inverse integrating factor of system (1.6).

In the second option, i.e. when $p_3 = p_4$, we have that p_3 is a critical point of system (1.6) with multiplicity two and therefore either p_3 is a nilpotent singular point or exactly one of the eigenvalues associated to p_3 is null. Now we put p_3 at the origin and in the first case the quadratic system can be written after a linear change of coordinates as $\dot{x} = y + \cdots$, $\dot{y} = \cdots$. From (1.3) at lower degree it follows $k_i(p_3) = 0$ for $i = 1, 2$. Taking into account Lemma 5.5 and Theorem 5.1 we get that $f_1 = 0$ and $f_2 = 0$ contain each one all the limit cycles. This is a contradiction because $f_1 \neq f_2$ and are irreducible.

We can assume that exactly one eigenvalue associated to p_3 (now at the origin) is equal zero. Then we can write the system as $\dot{x} = \lambda x + \cdots$, $\dot{y} = \cdots$. By statement 3 of Seidenberg Theorem 1.45, it follows that the above system has exactly two formal solutions at the origin $F_i(x,y) = 0$ with $i = 1, 2$ of the form $F_1(x,y) = x + \cdots$ and $F_2(x,y) = y + \cdots$. The following possibilities appear: either $f_i(p_3) \neq 0$ for some $i \in \{1, 2\}$ and so $k_i(p_3) = 0$ for such i or $f_1(p_3) = 0$ and $f_2(p_3) = 0$. The first case leads to a contradiction with hypothesis A because we have two critical points (p_3 and one focus) in the straight line $k_i(x,y) = 0$ for some i and we can apply Theorem 5.1. In the second option, when $f_i(p_3) = 0$ for $i = 1, 2$, it follows that $f_1 = 0$ contains exactly one branch and $f_2 = 0$ the other one. Moreover, from Theorem 1.46 we have either $k_1(p_3) = 0$ or $k_2(p_3) = 0$. Again, using Theorem 5.1 we get a contradiction with hypothesis A. ∎

Proposition 5.9. *Let us assume that quadratic system (1.6) verifies hypothesis A and moreover the other finite critical points p_3 and p_4 are not real. Then $f_1(x,y)f_2(x,y)$ is an inverse integrating factor of the system.*

Proof. Of course, since system (1.6) is real, if $p_3 = (x_3, y_3)$ and $p_4 = (x_4, y_4)$ are not real then its coordinates are complex conjugates, i.e., $x_4 = \bar{x}_3$ and $y_4 = \bar{y}_3$. This will be denoted by $p_4 = \bar{p}_3$. Moreover, the eigenvalues associated to each point verify the same property. So if λ and μ are the eigenvalues associated to p_3 then $\bar{\lambda}$ and $\bar{\mu}$ are the eigenvalues associated to p_4.

Let us suppose that p_3 (and therefore p_4) is not a resonant node. This means that $\lambda/\mu \notin \mathbb{Q}^+$. In this case we may simply repeat verbatim the first part in the proof of Proposition 5.8 when we apply Theorem 1.45(1.i) to conclude that $f_1(x,y)f_2(x,y)$ is an inverse integrating factor of system (1.6).

We continue supposing that p_3 and $p_4 = \bar{p}_3$ are resonant nodes. Hence the ratio of the eigenvalues λ and μ associated to p_3 is a positive rational number and are related by means of $\mu = \kappa\lambda$ with $\kappa \in \mathbb{Q}^+$. Of course the eigenvalues $\bar{\lambda}$ and $\bar{\mu}$ associated to p_4 verify $\bar{\mu} = \kappa\bar{\lambda}$. Moreover $\mathrm{div}\mathcal{X}(p_3) = (\kappa+1)\lambda$ and $\mathrm{div}\mathcal{X}(p_4) = (\kappa+1)\bar{\lambda}$.

If $f_i(p_3) = 0$ with $i = 1, 2$ then, applying Theorem 1.46 we have that $k_i(p_3) = r_i\mu + (s_i - r_i)\lambda$ for $i = 1, 2$ where $s_i, r_i \in \mathbb{N}$ and $r_i \leq s_i$. Clearly this implies $k_i(p_3) = \alpha_i\lambda$ where $\alpha_i := r_i\kappa + s_i - r_i \in \mathbb{Q}^+$. Furthermore since $k_i \in \mathbb{R}[x,y]$ and $p_4 = \bar{p}_3$ then $k_i(p_4) = \alpha_i\bar{\lambda}$ for $i = 1, 2$.

Now let us consider the real straight line $S(x,y) := pk_1(x,y) + qk_2(x,y) - \mathrm{div}\mathcal{X}(x,y) = 0$, with $p, q \in \mathbb{R}$. We have

$$S(p_3) = [p\alpha_1 + q\alpha_2 - (\kappa+1)]\lambda \ , \tag{5.2}$$

where $\lambda \neq 0$. We recall here that, since $p_4 = \bar{p}_3$ and $S \in \mathbb{R}[x,y]$, if $S(p_3) = 0$ then $S(p_4) = 0$.

If we are in Case 1 of Lemma 5.5, then $k_i(p_i) = \mathrm{div}\mathcal{X}(p_i)$ and $k_i(p_j) = 0$ for $i \neq j$ and $i, j \in \{1, 2\}$. This implies

$$S(p_1) = (p-1)\mathrm{div}\mathcal{X}(p_1) \ , \quad S(p_2) = (q-1)\mathrm{div}\mathcal{X}(p_2) \ . \tag{5.3}$$

First of all we claim that none of the foci p_1 and p_2 can be weak foci because in this case $\mathrm{div}\mathcal{X}(p_i) = 0$ for some $i \in \{1, 2\}$ and so either $k_1(p_i) = 0$ for $i = 1, 2$ or $k_2(p_i) = 0$ for $i = 1, 2$ in contradiction with hypothesis A by Theorem 5.1.

So we continue the proof assuming $\mathrm{div}\mathcal{X}(p_i) \neq 0$ for $i = 1, 2$. If we impose $S(p_1) = 0$ then $p = 1$ from the first equation (5.3). Moreover, from (5.2) we can take $q = (\kappa + 1 - \alpha_1)/\alpha_2$ so that $S(p_3) = S(p_4) = 0$. Hence $S(p_i) = 0$ for $i = 1, 3, 4$ and therefore $S(x,y) \equiv 0$. But now, from the second equation of (5.3) we have that, in fact, $q = 1$. So, quadratic system (1.6) admits the polynomial inverse integrating factor $f_1(x,y)f_2(x,y)$.

If Case 2 of Lemma 5.5 is verified then the proof is similar. ∎

Proposition 5.10. *Under hypothesis A, the curves $f_1 = 0$ and $f_2 = 0$ are the unique invariant algebraic curves of system (1.6).*

Proof. We suppose that another invariant algebraic curve $f_3 = 0$ irreducible in $\mathbb{R}[x, y]$ exists with $\mathcal{X} f_3 = k_3 f_3$ for some polynomial k_3. Assuming hypothesis A, f_3 must have degree greater than one because it is well known that quadratic systems with an invariant straight line have at most one limit cycle.

As we have proved in Lemma 5.4, the foci p_i, $i = 1, 2$ are contained in the curves $f_i = 0$, $i = 1, 2$ (each focus in one curve). Then, from Lemma 1.47, $f_3(p_i) \neq 0$ and so $k_3(p_i) = 0$, $i = 1, 2$. Now, applying Theorem 5.1, it follows that all the limit cycles of system (1.6) must be contained in $f_3 = 0$, against hypothesis A. ∎

5.3 Proof of the main result

We will see that hypothesis A can not be satisfied for system (1.6). Assuming the contrary, i.e. hypothesis A is fulfilled, we have shown that system (1.6) has the polynomial inverse integrating factor $V = f_1 f_2$. Hence, it must have a Darboux first integral, see Corollary 1.18.

Since $f_1 = 0$ and $f_2 = 0$ are real curves and, from Proposition 5.10, they are the unique invariant algebraic curves of system (1.6) it follows that

$$H = f_1^{\lambda_1} f_2^{\lambda_2} \left[\exp\left(\frac{h_1}{f_1^{n_1}} \right) \right]^{\mu_1} \left[\exp\left(\frac{h_2}{f_2^{n_2}} \right) \right]^{\mu_2},$$

for some $\lambda_i, \mu_i \in \mathbb{C}$, $n_i \in \mathbb{N} \setminus \{0\}$, $h_i \in \mathbb{C}[x, y]$, where h_i and f_i are coprime polynomials for $i = 1, 2$.

Following the ideas of [9], we compute

$$\log H = \lambda_1 \log f_1 + \lambda_2 \log f_2 + \mu_1 \frac{h_1}{f_1^{n_1}} + \mu_2 \frac{h_2}{f_2^{n_2}}$$

which is also a first integral for system (1.6) whose partial derivatives are rational functions.

The inverse integrating factor $\hat{V}$ related to the first integral $\log H$ is given by

$$\hat{V} = -\frac{P}{\frac{\partial}{\partial y} \log H} = \frac{Q}{\frac{\partial}{\partial x} \log H}.$$

It must be verified $\hat{V} = V$ (modulus a multiplicative constant). Otherwise, $\hat{H} = \frac{\hat{V}}{V}$ is a rational first integral and excludes the existence of limit cycles. In other words

$$f_1 f_2 \frac{\partial}{\partial x} \log H = Q\ , \tag{5.4}$$

must be verified. Moreover, it can be checked that $\frac{\partial}{\partial x} \log H = \frac{\Phi}{\Lambda}$, where $\Lambda = f_1^{n_1+2} f_2^{n_2+2}$ and

$$\begin{aligned}
\Phi \quad = \quad & \lambda_1 f_1^{n_1+1} f_2^{n_2+2} \frac{\partial f_1}{\partial x} + \lambda_2 f_1^{n_1+2} f_2^{n_2+1} \frac{\partial f_2}{\partial x} + \\
& \mu_1 f_1 f_2^{n_2+2} \left(f_1 \frac{\partial h_1}{\partial x} - n_1 h_1 \frac{\partial f_1}{\partial x} \right) + \mu_2 f_1^{n_1+2} f_2 \left(f_2 \frac{\partial h_2}{\partial x} - n_2 h_2 \frac{\partial f_2}{\partial x} \right).
\end{aligned}$$

Relation (5.4) becomes $\Phi = Q f_1^{n_1+1} f_2^{n_2+1}$, from where $f_1^{n_1+1} f_2^{n_2+1}$ divides Φ. Therefore, f_1 must divide $-n_1 h_1 \frac{\partial f_1}{\partial x}$ and then $h_1 = \Omega f_1$ for certain polynomial $\Omega \in \mathbb{R}[x,y]$. Thus, h_1 and f_1 are not coprime, which is a contradiction. ∎

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