On Peer Networks and Group Formation

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Contents

Chapter 1
Introduction

The aim of this thesis work is to contribute to the analysis of the interaction of agents in social networks and groups.

In the chapter **"NP-completeness in Hedonic Games"**, we identify some significant limitations in standard models of cooperation in games: It is often impossible to achieve a stable organization of a society in a reasonable amount of time. The main implications of these results are the following. First, from a positive point of view, societies are bound to evolve permanently, rather than reach a steady state configuration rapidly. Second, from a normative perspective, a planner should take into account practical time limitations in order to implement a stable social order.

In order to obtain our results, we use the notion of NP-completeness, a well-established model of time complexity in Computer Science. In particular, we concentrate on group stability and individual stability in hedonic games. Hedonic games are a simple class of cooperative games in which each individual's utility is entirely determined by her group. Our complexity results, phrased in terms of NP-completeness, cover a wide spectrum of preference domains, including strict preferences, indifference in preferences or undemanding preferences over sizes of groups. They also hold if we restrict the maximum size of groups to be very small (two or three players).

The last two chapters deal with the interaction of agents in the social setting. It focuses on games played by agents who interact among them. The actions of each player generate consequences that spread to all other players throughout a complex pattern of bilateral influences.

In **"Who is Who in Networks. Wanted: The Key Player"** (joint with Antoni Calvó-Armengol and Yves Zenou), we analyze a model peer effects where interact in a game of bilateral influences. Finite population non-cooperative games with linear-quadratic utilities, where each player decides how much action she exerts, can be interpreted as a network game with local payoff complementarities, together with a globally uniform payoff substitutability component and an own-concavity effect.

For these games, the Nash equilibrium action of each player is proportional to her Bonacich centrality in the network of local complementarities, thus establishing a bridge with the sociology literature on social networks. This Bonacich-

Nash linkage implies that aggregate equilibrium increases with network size and density. We then analyze a policy that consists in targeting the key player, that is, the player who, once removed, leads to the optimal change in aggregate activity. We provide a geometric characterization of the key player identified with an intercentrality measure, which takes into account both a player's centrality and her contribution to the centrality of the others.

Finally, in the last chapter, **"Optimal Targets in Peer Networks"** (joint with Antoni Calvó-Armengol and Yves Zenou), we analyze the previous model in depth and study the properties and the applicability of network design policies.

In particular, the key group is the optimal choice for a planner who wishes to maximally reduce aggregate activity. We show that this problem is computationally hard and that a simple greedy algorithm used for maximizing submodular set functions can be used to find an approximation. We also endogeneize the participation in the game and describe some of the properties of the key group. The use of greedy heuristics can be extended to other related problems, like the removal or addition of new links in the network.

Chapter 2
NP-Completeness in Hedonic Games

2.1 Introduction

The significance of *hedonic coalition formation* is present in many actual socioeconomic situations like the creation of social clubs, groups, societies... An introductory work to hedonic games was Dreze and Greenberg (1980), where they studied situations of production of local (coalitional) public goods, in which agents' preferences are based upon both their individual consumption of the public good produced by their coalition and the coalition itself[1]. This concept was framed into more general studies by Banerjee *et al.* (2001), and by Bogomolnaia and Jackson (2002). Some stability concepts are studied by these authors, like the *core,* the *Nash stable set,* the *individually stable set* and the *contractually individually stable set*. Several interesting sufficient conditions for the nonemptiness of these sets are given in detail in these papers. Diamantoudi and Xue (2002) show that some of these stability concepts have deficiencies related to the myopic behavior of players. They analyze *farsighted stability*, where the decisions of agents are no longer myopic and coalition formation is based on a dynamic process. With a different approach, Konishi and Ray (2003) describe some aspects of the dynamic formation of coalitions.

In the context of computational complexity applied to non-cooperative game theory, Gilboa (1988) and Ben-Porath (1990) study the complexity of finding a best response. Hence, they do not study how difficult to implement a strategy is, but how hard to find it is. For computational purposes, the strategies in these papers are usually represented in the form of finite automata and NP-hardness results are used to explain why a best response is difficult to find.

Faigle *et al.* (1997) apply computational complexity to cooperative games with transferable utility and conclude that computing some solution concepts in certain types of TU games is difficult. More recent works in complexity theory applied to cooperation in hedonic environments are Cechlárová and Hajduková (2004a, 2004b), where hedonic preferences on coalitions can be induced from a preference profile over players, and computational complexity is used to analyze core related concepts. This particular case of hedonic games is very reasonable in

1 An example of this situation is explained in detail by Bogomolnaia and Jackson (2002). If the decision for the production of the public good inside a coalition is done by the median voter of the coalition, then a hedonic environment appears.

many situations where we prefer a coalition S to a coalition S' if and only if the best (worst) player in S is preferred to the best (worst) player in S'. This specific type of induced preferences is called $\mathcal{B}$-preferences ($\mathcal{W}$-preferences) and it has the advantage of its concise representation: each player has a preference ordering over the rest of the players.

The aim of this paper is to show part of the *computational complexity* issues related to *hedonic games* in the general setting and in the anonymous preferences case. We prove that there cannot be efficient algorithms not only for deciding the nonemptiness of the core, but also for the case of the Nash stable set and the individually stable set. Furthermore, these results hold for both *preferences over coalitions* and *preferences over sizes*. Particularly, we find that the corresponding *decision problems* for these sets are *NP-complete* in a variety of versions.

The inherent complexity of these solution concepts constitutes, in part, a criticism to some processes of coalition formation. It is not a global objection to the proper stability definitions because situations which are not stable do not last, and hence they can be somehow discarded as the final outcome of such processes. Nevertheless, the difficulty of finding stable outcomes invalidates the possibility of reaching equilibria in reasonable time in processes of sequential coalition formation; and this difficulty also implies that the task of a social planner trying to implement a given stable solution, can be hard from the computational perspective.

Computational complexity can also provide some light into the goodness of theorems dealing with stability in hedonic games. As we explain in the conclusions, there cannot be good characterizations for any of these stability concepts in the hedonic setting. In fact, although we can easily verify the nonemptiness of the core (or the other stable sets) by analyzing a particular stable partition, there is no efficient way to check that a game has an empty core.

The following brief introductory subsection to the theory of NP-completeness can be skipped by those readers who are familiar with it. Section 2.2 is an introduction to *hedonic games* and to some *stability* concepts. In section 2.3, we start with three simple NP-completeness proofs for the general case of a hedonic game with *arbitrary preferences over coalitions*. We also consider those particular cases of requiring preferences to be *strict*. This particularization to strict preferences is supported by the fact that, for example, the *stable roommates problem* turns to be polynomial when preferences are required to be strict (Irving (1985) and Gusfield (1989)) but it is NP-complete when indifferences are permitted (Ronn (1986)). Finally, in section 2.4, a different structure for hedonic games, *anonymous* hedonic games, is analyzed showing that the corresponding decision problems remain NP-complete, even with strict preferences. In the conclusion section, we include some interpretations of our results.

2.1.1 A Short Description of NP-completeness

Throughout this article, we will deal with NP-completeness. Those readers who are familiar with this computational terminology may skip this section. Garey and Johnson (1979) and Cook (2000) are basic bibliographic material for this section.

A *computational problem* is a general question on some mathematical object, involving some *free parameters* which may take any value. Some examples:

Problem Π_1: *Given a graph $G = (V, E)$ where V is the set of vertices and E is the set of edges. Does it contain a cycle made up of at least K vertices?* In this case, the free parameters are G and K.

Problem Π_2: *Given a graph $G = (V, E)$ and two vertices $v, w \in V$, what are the shortest paths from v to w?* Here, the free parameters are G,v and w.

Problem Π_3: *Given a finite two-player game in normal form with action sets A_1* and *A_2 and with payoffs in $\mathbb{Q}$, what is the set of Nash equilibrium outcomes in pure strategies?* In this example, the free parameters are the rational pairs of a $|A_1| \times |A_2|$ payoff table.

A *decision problem* is a problem for which the only possible answers are *Yes* or *No*. Problem Π_1 is a decision problem. Decision problems are very important in computation because they are the simplest problems and many complex problems can be reduced to the solution of one or more decision problems. Given a decision problem Π, its *complementary decision problem* $\overline{\Pi}$ consists of the same object and parameters, but the opposite question. For instance, $\overline{\Pi}_1$ would be: *given a graph $G = (V, E)$, is it true that all its cycles (if any) have less than K vertices?*

An *instance* of a problem is just a particularization of the problem when the free parameters have been realized. For example, an instance of Problem Π_2 is $I_1 = (G_1 = (V_1, E_1), v_1, w_1)$ where $V_1 = \{1, 2, 3, 4\}$, $E_1 = \{(1, 2), (2, 3), (2, 4), (3, 4)\}$, $v_1 = 1$, $w_1 = 4$ and the answer for this particular instance is the path $\langle 1, 2, 4\rangle$. For decision problems, instances for which the answer is yes are called *yes-instances*.

The *length function of a problem* Π, $Length$, assigns a natural number (a *size*) to each instance of the problem Π. For instance, a reasonable length function for Problem Π_1 is $Length_1(I) = |V|+|E|+|K|$ where $|V|$ is the number of vertices, $|E|$ is the number of edges of the instance and $|K|$ denotes the number of digits needed in order to write the integer K in decimal notation. The study of the reasonability of different length functions is crucial but it is out of the scope of this introduction. More references can be found in Garey and Johnson (1979).

An *algorithm* is a set of step-by-step instructions that can be run in a computer and that, given some input data, it produces an output. An algorithm *solves a problem* Π if for any instance I of Π, the algorithm gives the correct answer for that instance. For example, an algorithm solves Π_3 if given any finite normal form game with two players, it produces the set of pure Nash equilibrium outcomes as the output. Given an algorithm A that is able to solve some problem Π, and given an instance I of Π, we can say that A takes $s(I)$ steps in order to solve instance I. We define the *time demand of* A or *cost function of* A as

$$time_A(n) = \max_{Length(I)=n} s(I)$$

For instance, consider some computational problem regarding a natural number (i.e, "Is k a prime number?") and an algorithm A that solves it. Then $time_A(n) = 2^n$ means that the most costly (from the point of view of the algorithm A) natural number with 3 digits can be solved by A in $2^3 = 8$ steps.

We are interested in providing upper bound functions for the cost function of an algorithm. It is common in computer science to work with the "*big Oh*" notation (O) to establish an upper bound for a function. Formally, we say that $f = O(g)$ when there exists a constant c such that $f(x) \leq cg(x)$ for sufficiently large x, that is, when f can be bounded above by g, up to a constant c. For example, $\ln(x) = O(x) = O(x^3) = O(2^x)$, but "$= O$" is not an equivalence relation: $O(x^2) \neq O(x)$.

An algorithm A solves Π in *polynomial time* if there is a polynomial function q such that for every instance I, the number of steps that A takes to solve Π is bounded above by $q(Length(I))$, that is, $time_A(n) = O(q(n))$. In other words, the cost of solving any instance is bounded above by a polynomial function of the size of the instance. For simplicity, we will often refer to these polynomial-time algorithms as "fast" algorithms.

A problem Π has *polynomial-time complexity* whenever there is an algorithm A that solves Π in polynomial time. For simplicity, in the following, these polynomial-time complexity problems will be referred to as "easy" or *tractable* problems. A problem is *intractable* when it does not have polynomial-time complexity, that is, when there cannot exist any algorithm that solves it rapidly. We do not mean that the problem cannot be solved algorithmically (here we are only dealing with problems that are computationally solvable) but that even the best algorithm will fail to solve it in polynomial time. The bad news about intractable problems are that even for moderately large instances, it may be impossible to solve them in reasonable time, even with the help of very powerful computers, because the rate of growth of their complexity is exponential.

Going back to decision problems, a decision problem Π_1 can be *polynomially reduced* to the decision problem Π_2, or in notation $\Pi_1 \propto \Pi_2$, if we can find a function ψ that maps instances of Π_1 into instances of Π_2, such that ψ can be computed by some algorithm in polynomial time and ψ preserves the answer, that is, I is a yes-instance of problem Π_1 if and only if $\psi(I)$ is a yes-instance of problem Π_2. From another perspective, if we can write an algorithm that, given any instance of Π_1 as input, it produces an instance of Π_2 as output with the particularity that this algorithm is fast and that it preserves the answer, then we have constructed a polynomial reduction from Π_1 to Π_2, in the form of an algorithm that computes ψ. When $\Pi_1 \propto \Pi_2$, we say that problem Π_2 is at least as hard as Π_1, because instances of Π_1 can be easily reduced to instances of Π_2. It is not difficult to show that $\propto$ induces a transitive relation between problems. In the paper, we are presenting plenty of polynomial reductions in order to prove our NP-completeness results.

We end this section with the most important definitions. A decision problem belongs to the class P (P stands for polynomial time) whenever it has polynomial-time complexity. In other words, the *class* P is the family of easy decision problems. The *class* NP (NP stands for non-deterministic polynomial time) is a superset of P and it can be roughly defined as the class of problems that can be solved in polynomial time by a hypothetical computer having an unbounded number of processors. There are several equivalent definitions of the class NP. Here, we use a formal definition that can be easily understood. A problem Π is in the class NP whenever there is a polynomial time algorithm A such that for any instance I of the problem Π:

- if I is a yes-instance, there is a certificate $C(I)$ such that when the tuple $(I, C(I))$ is the input of A, the output of A is yes (in polynomial time).
- if I is a no-instance, there is no certificate $C(I)$ such that when $(I, C(I))$ is the input of A, the output of A is yes. That is, if I is a no-instance, any certificate leads to a negative answer by algorithm A (in polynomial time).

In plain words, a problem is in NP if any yes-instance can be quickly verified with a certificate on hand. For example, consider the following decision problem, also known as the *compositeness test: given an integer k, is it composite?* This problem is clearly in NP. To see this, consider an instance of the compositeness test $k_1 = 1465038781$. If someone told us that k_1 is composite, we may have the right not to believe her. But if she strengthened her assertion by saying '$1465038781 = 14867 \times 98543$', we would only need to use a calculator and multiply these two numbers in order to see that their product is exactly k_1, thus certifying that she was right. And verifying that this certificate is correct can be done in polynomial time since multiplication of two integers is an easy computational operation. On the other hand, when the instance is a prime number, like 23, there is no certificate (pair of non-trivial numbers other that 1) such that their product is 23. Hence, when dealing with the compositeness test, we can easily state that this problem is in NP: the structure of a certificate can be a pair of factors greater than 1. Therefore, when talking about the class NP we are not dealing with the complexity of finding a certificate, but with the complexity of checking the yes-answer with the help of that certificate.

Finally, the property

$$P \subseteq NP$$

holds because all decision problems in P are easy and hence, the answer for a particular instance can be computed in polynomial time with the help of an empty certificate.

A problem Π belongs to $co - NP$ if its complementary problem $\overline{\Pi}$ belongs to NP. It is obvious that $P \subseteq NP \cap co - NP$ because easy problems' instances can be efficiently (quickly) solved either if they are yes-instances or no-intances. Problems which are both in NP and in $co - NP$ are called *well-characterized problems*. Examples of well-characterized problems are linear programming[2] and the compositeness test[3]. It is a generally accepted fact that $NP \neq co - NP$ since there are many problems that have been proven to belong to one of these sets but, dispite of the continuous efforts of computer scientists, cannot be proven to be in or out of the other.

A decision problem Π is NP*-complete* (NP-c for short) whenever $\Pi \in NP$ and $\Pi' \propto \Pi$ for all $\Pi' \in NP$. That is, NP-complete problems are the hardest problems in NP. In fact, no one has found a polynomial-time algorithm for solving any NP-complete problem and there is the accepted conjecture that this will not ever happen, that is, $P \cap NP$-c $= \emptyset$. Moreover, by the transitivity of $\propto$, finding a fast algorithm for a single NP-complete problem would imply $P = NP$, which means that many decision problems that have been suspected to be hard for decades would be proven to be easy. For this reason, although no proof has been provided at this point, the accepted conjecture is that $P \neq NP$: all NP- *complete problems are intractable*, that is, none of them can be

[2] Existence of feasible solutions can be verified with a particular solution. By *strong duality*, linear programming is also in $co - NP$. In fact, linear programming is polynomial due to the *ellipsoid method* due to Shor in 1977.

[3] Compositeness (and thus primality) are well-characterized properties: we can certify a composite number with two factors and we can also certify prime numbers! The structure of a certificate for prime numbers is quite more complex than that for composite numbers and it comes from a series of nice theorems in Number Theory.

Finding a polynomial-time algorithm for primality and compositeness had been an open problem for centuries in Number Theory. Recently, Agrawal *et al.* (2002) have provided a solution.

computationally solved in reasonable time. Moreover, NP-complete problems cannot be well-characterized (in fact, this is true if and only if $NP = co - NP$, which is implied by the unlikely fact that $P = NP$), although the proof of this fact is out of the scope of this introduction.

In practical terms, the natural way to show that a problem Π is NP-complete is to prove its membership in the class NP and then to choose some known NP-complete source problem that we must polynomially reduce to our problem Π. We only have to show a polynomial reduction from a single source NP-complete problem because, by the transitivity of $\propto$, all problems in NP will be consequently reducible to our problem. In this paper, the source problem will be the PARTITION problem in many cases (this problem will be explained in detail in the following sections). For simplicity, in order to prove the polynomial-time nature of our transformations, we will only claim that the resulting instance's length is polynomial in the original instance's length -this is a very usual practice in the literature concerning computational complexity- because in all cases, the operations involved in the transformation (copying players, copying coalitions, appending coalitions to a list, etc.) are easy from the computational perspective and hence, the polynomial-time nature of the transformation is implied by its polynomial-space nature.

2.2 Hedonic Games

A *hedonic game* is a particular type of cooperative game in which the preferences of each player can be completely summarized as preferences over the coalitions where she is included. Next, we show a formal description of a hedonic game. While in the previous section Π represented a computational problem, in the following, we will use the symbol Π to denote a partition of a set, as defined in the following paragraph.

Let $N = \{1, 2, ..., |N|\}$ be the set of players. Any non-empty subset $S \subseteq N$ is called a *coalition*. A *coalition partition*, *coalition structure*, *configuration* or *exact cover* is a collection $\Pi = \{S_k\}_{k=1}^{K}$ of subsets which partitions N, that is, $\cup_{k=1}^{K} S_k = N$ and $S_k \cap S_l = \emptyset$ for all $k \neq l$. Player i's *preferences* are represented by the rational preference relation $\succsim_i$ (a complete, reflexive and transitive binary relation) over the family $\{S \subseteq N : i \in S\}$. Denote the corresponding strict preference relation by $\succ_i$. And $\sim_i$ will be the notation used for the indifference relation. For a given coalition partition Π and a player i, we write $S_\Pi(i)$ to mean the unique set $S \in \Pi$ such that $i \in S$.

Definition 2.1 *A hedonic game is a tuple $\langle N, \{\succsim_i\}_{i \in N}\rangle$ (or simply $\langle N, \succsim\rangle$),* *where $N = \{1, 2, ..., |N|\}$ and every $\succsim_i$ is a rational preference relation over coalitions of N where i is included*[4].

Definition 2.2 *A coalition partition Π is individually rational in the hedonic game $\langle N, \succsim\rangle$ if $S_\Pi(i) \succsim_i \{i\}$ for all $i \in N$.*

Definition 2.3 *A coalition partition Π is core stable in the game $\langle N, \succsim\rangle$ if $\nexists T \subseteq N$ such that $T \succ_i S_\Pi(i)$ for all $i \in T$. The core is the set of all core stable coalition structures.*

We say that coalition T *blocks* partition Π if $T \succ_i S_\Pi(i)$ for all $i \in T$. That is, Π is in the core if and only if no coalition can block it.

Definition 2.4 *A coalition partition Π is Nash stable in the hedonic game $\langle N, \succsim\rangle$ if $\forall i \in N$ $S_\Pi(i) \succsim_i S_k \cup \{i\}$ for all $S_k \in \Pi \cup \{\emptyset\}$.*

That is, Π is Nash stable if no individual player would be strictly better off by moving into a different coalition inside the same structure (or by moving to her stand-alone coalition).

Definition 2.5 *A coalition partition Π is individually stable in the hedonic game $\langle N, \succsim\rangle$ if $\nexists i \in N, \nexists S_k \in \Pi \cup \{\emptyset\}$ such that $S_k \cup \{i\} \succ_i S_\Pi(i)$ and $S_k \cup \{i\} \succsim_j S_k$ for all $j \in S_k$.*

So individual stability also requires that the members of the new coalition do not accept the entering member because this would put some existing member into a worse situation.

Definition 2.6 *A coalition partition Π is contractually individually stable in the hedonic game $\langle N, \succsim\rangle$ if $\nexists i \in N, \nexists S_k \in \Pi \cup \{\emptyset\}$ such that $S_k \cup \{i\} \succ_i$*

[4] It is important to note that, in hedonic games, players only care about the coalitions where they are included.

$S_\Pi(i)$, $S_k \cup \{i\} \succsim_j S_k$ for all $j \in S_k$ and $S_\Pi(i)\backslash\{i\} \succsim_j S_\Pi(i)$ for all $j \in S_\Pi(i)\backslash\{i\}$.

In this context of contractual individual stability, the single player who wishes to change to another coalition in the structure may also ask for permission to the members of her old coalition in order to leave it, and these members should not permit this movement. It is not difficult to prove that every hedonic game has a contractually individually stable coalition structure[5]. Hence, we will not bother about the complexity of computationally deciding its existence in a given game.

We will loosely use these definitions not only for partitions, but also for individual players or for coalitions. For example, when we say that coalition S is individually rational for player i, we mean that $S \succsim_i \{i\}$. Or when we say that Π is not individually stable for $S \in \Pi$ we mean that at least one player in S wishes to move and is accepted in another coalition $T \in \Pi$.

It is not difficult to prove that Nash stability implies individual stability, which implies contractual individual stability. And it is also transparent that core stability, Nash stability or individual stability imply individual rationality.

An example with 3 players can be useful to view some of these stability definitions:

$$\begin{array}{c} \{1,3\} \succ_1 \{1,2,3\} \succ_1 \{1\} \succ_1 \{1,2\} \\ \{1,2\} \succ_2 \{1,2,3\} \succ_2 \{2\} \succ_2 \{2,3\} \\ \{1,3\} \succ_3 \{1,2,3\} \succ_3 \{3\} \succ_3 \{2,3\} \end{array} \tag{1}$$

In this game the preferences are *strict*: no player is indifferent for any two alternatives. Three partitions are of special interest in this game. First, there is a unique core stable partition, $\{\{1,3\},\{2\}\}$. It can be checked that no possible coalition can block it, so it is in the core. It is not Nash stable because player 2 wishes to join $\{1,3\}$. But it is individually stable because this move would make both 1 and 3 strictly worse off; and players 1 and 3 do not wish to move into another coalition because they are in their best respective positions. And it is contractu-

[5] Starting with any initial configuration, let us construct a path of configurations. Given an element of this path, find a coalition where some player would like to move and this movement is allowed by both the new and the old coalition. If this coalition cannot be found, the configuration is contractually individually stable. Otherwise, we get another configuration of the players and we follow the same procedure. It is easy to see that this process can have no cycles because of the transitivity of preferences and the fact that, in each movement, one player strictly improves and the rest are not harmed. The result follows because the set of all possible configurations for a game $\langle N, \succeq\rangle$ is finite and the path we have defined on this set is acyclic.

ally individually stable because of the implications explained before. Second, the structure $\{\{1,2,3\}\}$, where the *grand coalition* has formed, is Nash stable (and hence individually stable and contractually individually stable) because nobody is better off by being alone. But it is not in the core: players 1 and 3 can form a strictly better coalition for them, discarding player 2's preferences. Third, partition $\{\{1,2\},\{3\}\}$ is not core stable or Nash stable or individually stable because it is not individually rational for player 1. But it is contractually individually stable: player 2 will not ever let anyone in or out and she is currently in her best possible coalition. Finally, it can be checked that either $\{\{1\},\{2\},\{3\}\}$ or $\{\{1\},\{2,3\}\}$ do not satisfy any of the stability definitions explained above.

Before proving results for hedonic games, we describe one particular NP-complete combinatorial problem related to set partitioning, which will be used throughout this paper as the source of many of our reductions to decision problems in hedonic games.

Definition 2.7 *The PARTITION problem can be defined as follows:*

Given a finite set X and a collection C of subsets of X, is there a subcollection $\Pi \subseteq C$ such that Π is a partition of X?

The length function of this problem can be considered to be $Length_p = |C||X|$, where $|C|$ is the number of subsets belonging to C and $|X|$ is the size of the set X. An instance of this problem could be

$$\begin{aligned} X &= \{1,2,3,4,5\} \\ C &= \{\{4\},\{2,3\},\{1,3,4\},\{1,5,4\}\} \end{aligned}$$

for which the answer would be *yes*. In particular, C contains the partition $\Pi = \{\{1,5,4\},\{2,3\}\}$. The size assigned by $Length_p$ to this particular instance is $|C||X| = 4 \cdot 5 = 20$. On the other hand, if we had

$$\begin{aligned} X &= \{1,2,3,4,5,6,7\} \\ C &= \{\{2,3\},\{1,3,4\},\{5,6,7\}\} \end{aligned}$$

of size $3 \cdot 7 = 21$, then the solution of the decision problem for this instance would be *no*, since there is no partition of X contained in C.

Theorem 2.1 *PARTITION is NP-complete.*

Proof. Lovász and Gács (1999) offer a proof based on the k-PARTITION problem, which is a slight variation of this problem. An alternative approach to Lovász's proof is to see the PARTITION problem as a generalization of X3C (eXact Cover by 3-sets), which is a known NP-complete combinatorial problem (Karp (1972)). ■

2.3 NP-Completeness in Hedonic Games with Preferences over Coalitions

In the following sections, some of the proofs will have attached an example at the end, in order to provide some light on the application of the transformation rules.

Before defining the problem, we must specify the *representation*, that is, what type of *object* we are going to use to represent a particular instance. This representation, which we can call a *representation based on individually rational lists for coalitions (RIRLC)*, consists of N preference lists where agent i's list contains only her individually rational coalitions, that is, coalitions which are at least as good as being alone. The rest of coalitions are not included in her list. Thus, the representation used for the previous hedonic game 2.1 would be:

$$\begin{gathered}\{\mathbf{1},\mathbf{3}\} \succ_1 \{1,2,3\} \succ_1 \{1\} \\ \{1,2\} \succ_2 \{1,2,3\} \succ_2 \{2\} \\ \{1,3\} \succ_3 \{1,2,3\} \succ_3 \{3\}\end{gathered}$$

It is easy to see that this does not affect core stability, Nash stability or individual stability in any case because any configuration (coalition structure) based on these eliminated coalitions would be individually irrational for some player. On the one hand, it does not make the set of stable configurations smaller: a partition which is not individually rational for some player cannot be stable in any of these ways. On the other hand, it does not make the set of stable configurations bigger: removing a coalition which is not individually rational for some player could be problematic only for the hypothetical stability of configurations that are not individually rational for her and hence, for unstable configurations under the complete preferences.

We complement the representation of the problem with the *length function* of this RIRL for coalitions*:* $Length_c = K_c|N|$, where K_c is the number of coalitions appearing in the RIRLC's lists. For the previous instance, $N = 3$ and $K_c = 6$ because the coalitions that appear in the RIRLC are $\{1,3\}$, $\{1,2,3\}$,

$\{1\}$, $\{1,2\}$, $\{2\}$ and $\{3\}$. So, $Length_c = 18$.[6][7]

2.3.1 Arbitrary Preferences over Coalitions

In this section, we consider the simpler case of preferences where preferences among coalitions are arbitrary in the sense that *ties are allowed* in players' preferences.

Definition 2.8 *The HCORE problem with arbitrary preferences over coalitions: "Given a RIRLC representation of a hedonic game, does there exist (at least) one core stable partition in the game?"*

In the same fashion, we can define HNASH and HIS decision problems for the Nash stable set and the individually stable set, respectively.

Proposition 2.1 *HCORE with arbitrary preferences over coalitions is NP-complete.*

Proof. This problem is in NP. A natural certificate is a core stable partition. Given a core stable partition, Π, it is easy to check its core stability. First, check that it is in fact a partition from the RIRLC. Then, starting with player 1, check that every coalition S in her preference list that she strictly prefers to her coalition inside Π, $S_\Pi(i)$, is such that some other player $j \in S$ does not strictly improve in her preference list by being in S rather than in $S_\Pi(j)$. Then, follow the same procedure for the rest of the players. This operation has polynomial-time complexity: its cost is polynomial in $Length_c$.

Before proving completeness based on the PARTITION problem, it should be noted the following: HCORE with indifferences is a generalization of the *stable roommates problem*[8] with indifferences (and unacceptable partners), which

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[6] This length function is correct for our problems because it can be proven that it is *polynomially related* to the number of symbols written in order to represent a given RIRLC. For the same reason, the number of players, $|N|$, is *not* a reasonable length function for a RIRLC because the length of her list can be as big as an exponential amount in $|N|$. A good definition of polynomial relations and reasonability of length functions can be found in Garey and Johnson (1979).

[7] The length function $Length_c = K_c$ would also be valid because, given the RIRLC representation we have that $|N| \leq K_c$. Nevertheless, for reasons of compatibility with the length of the partition problem, $Length_p$, we use $K_c|N|$

[8] The *stable roommates problem* is just the same problem in the context of a hedonic game where the the preference profiles are made up of coalitions containing at most 2 players.

is NP-complete (Ronn (1986)). Hence, completeness follows immediately[9]. Nevertheless, we provide an alternative reduction which simplicity will help the reader understand the logic behind the rest of the proofs in this paper.

We make a polynomial transformation from PARTITION, that is, we will implicitly provide a mapping that gives us a RIRLC from any instance of PARTITION and which preserves the answer to both problems. Let C and X be, respectively, the subset collection and the set in the PARTITION instance. Initially, the set of players N of our RIRLC will be X (we will create new players later). The preferences of these initial players will be such that they must be indifferent among all coalitions (subsets) of C where they belong. But still, we may not have a proper representation for the game because, if some singleton subset $\{j\}$ is not in C, then the corresponding player j does not have the stand-alone coalition in her RIRLC list. Where should we put this coalition $\{j\}$ inside player j's preference list? For this matter, we propose the following additional modifications. For each j such that $\{j\} \notin C$, create players j_a and j_b with the following preference lists:

$$\{j, j_a\} \succ_{j_a} \{j_a, j_b\} \succ_{j_a} \{j_a\}$$
$$\{j_a, j_b\} \succ_{j_b} \{j, j_b\} \succ_{j_b} \{j_b\}$$

and append the following items at the end of player j's preference list: "$\succ_j \{j, j_b\} \succ_j \{j, j_a\} \succ_j \{j\}$"[10]. We say that players j, j_a and j_b belong to the same *cyclic relationship* where player j is the *parent,* and j_a and j_b are the *children*. So, we have that the final set of players is X plus the children from all cyclic relationships. The intuition about the use of this gadget is that we must avoid the existence of core stable partitions in the game whenever there is no underlying partition of X inside C, as we will explain in detail immediately. First, note that every player ranks all the coalitions in C above the rest of the coalitions, and that she is indifferent among all these coalitions of C.

If there is a partition in C then this partition together with the collection $\{\{j_a, j_b\} : \{j\} \notin C\}$ must be a core stable partition of the game: all players

[9] Unfortunately, there does not seem to be such a straightforward transformation from the stable roommates problem with indifferences into HIS or HNASH.

On the other hand, if we impose strict preferences, we have an added difficulty to find a straightforward proof: the stable roommates problem has polynomial time complexity when preferences are strict and a reduction of this kind would not prove NP-completeness.

[10] In the following, when we append a string to some player's initial preference string, we assume that this initial string is not empty. If this is not the case, then we should just add the same string but omitting the first relation token appearing in it, i.e., we would omit the first "$\succ_j$"in this case.

from X are at their best positions and children cannot improve without doing harm to someone.

If there is no partition in C, any partition of the whole set of players must consist of two disjoint families:

- A subpartition ”inside” C made up from some proper subset of X. This inner subpartition may be empty.
- A nonempty subpartition ”outside” C made up from some players in X and all the children from cyclic families.

But if this game partition were individually rational, the following properties should be satisfied:

- *The outer subpartition can only be formed from the set of players with no stand-alone coalitions inside C and their cyclic children*. Otherwise the players i with $\{i\} \in C$ would be strictly better off by leaving the current coalition and achieving their best situation $\{i\}$, which is inside C.
- *In the outer subpartition, there is at least one relationship $R = \{j, j_a, j_b\}$ (in fact, every relationship satisfies this property) such that none of its members is mixed with members from other families*. Suppose not. First, note that putting in the same coalition some child (j_a or j_b) with members of different cyclic families is not individually rational because this coalition does not even appear in the children's preference lists. So, suppose that every parent from a cyclic family is in coalition only with other parents in the outer subpartition. Since the game partition is individually rational, all these coalitions must belong to C, by construction of the game. So C contains a partition of X, leading to a contradiction.

Given this relationship R (whose members are in coalitions which are subsets of R) , it cannot induce a core stable partition in the game because preferences of every triplet j, j_a and j_b have been constructed to be cyclic. That is, j prefers to be with j_b than to be with j_a, but j_b prefers to be with j_a, who prefers to be with j. And it is not individually rational for any of them to form the coalition $\{j, j_a, j_b\}$.

We conclude that when there is no partition of X inside C, any individually rational partition of the expanded set of players, that is, any configuration extracted from the RIRLC, cannot be core stable.

Finally, it is easy to check that this transformation can be done in polynomial time in the size of the PARTITION instance. The set of players is $N = X \cup_{\{j\} \notin C} \{j_a, j_b\}$, and hence $|N| \leq 3|X|$. On the other hand, $K_c = |C| + \sum_{\{j\} \notin C} 6 \leq$

$|C| + 6|X|$. So $Length_c = K_c|N| \leq 3Length_p + 18|X|^2 \leq 3Length_p + 18Length_p^2$, which is polynomial in the size of PARTITION. ■

An example will help illustrate the essence of the construction. First, suppose we are given the following PARTITION instance:

$$\begin{aligned} X &= \{1, 2, 3\} \\ C &= \{\{\mathbf{1}\}, \{1, 3\}, \{\mathbf{2}, \mathbf{3}\}, \{3\}\} \end{aligned}$$

where there is a partition contained in C: $\Pi = \{\{1\}, \{2, 3\}\}$. In the resulting game, note that because $\{2\} \notin C$ we will need to create a cyclic relationship for player 2:

$$\begin{aligned} &\{1,3\} \sim_1 \{\mathbf{1}\} \\ &\{\mathbf{2},\mathbf{3}\} \succ_2 \{2,2_b\} \succ_2 \{2,2_a\} \succ_2 \{2\} \\ &\{2,2_a\} \succ_{2_a} \{\mathbf{2_a},\mathbf{2_b}\} \succ_{2_a} \{2_a\} \\ &\{\mathbf{2_a},\mathbf{2_b}\} \succ_{2_b} \{2,2_b\} \succ_{2_b} \{2_b\} \\ &\{1,3\} \sim_3 \{\mathbf{2},\mathbf{3}\} \sim_3 \{3\} \end{aligned}$$

where bold symbols indicate the corresponding core stable partition of the game: $\{\{1\}, \{2, 3\}, \{2_a, 2_b\}\}$. Now consider an opposite case, where the source PARTITION instance has no partition in C:

$$\begin{aligned} X &= \{1, 2, 3\} \\ C &= \{\{1\}, \{1, 3\}, \{3\}\} \end{aligned}$$

As expected, the corresponding game,

$$\begin{aligned} &\{1,3\} \sim_1 \{1\} \\ &\{2,2_b\} \succ_2 \{2,2_a\} \succ_2 \{2\} \\ &\{2,2_a\} \succ_{2_a} \{2_a,2_b\} \succ_{2_a} \{2_a\} \\ &\{2_a,2_b\} \succ_{2_b} \{2,2_b\} \succ_{2_b} \{2_b\} \\ &\{1,3\} \sim_3 \{3\} \end{aligned}$$

has no core stable partition. Note that the inclusion of cyclic relationships is crucial for our result. If we had omitted this fact from our transformation procedure we would not have obtained an answer preserving mapping. For example, discarding cycles in the last transformation would lead to the game:

$$\begin{array}{c}\{\mathbf{1},\mathbf{3}\} \sim_1 \{1\} \\ \{\mathbf{2}\} \\ \{\mathbf{1},\mathbf{3}\} \sim_3 \{3\}\end{array}$$

where $\{\{1,3\},\{2\}\}$ is stable.

Proposition 2.2 *HNASH with arbitrary preferences over coalitions is NP-complete*

Proof. It is intuitive that HNASH is in NP. Given a hedonic game and a Nash stable partition, Π, it takes polynomial time to check whether it is actually Nash stable or not: after checking individual rationality inside the lists, check that no player can individually improve in her list by moving into another coalition inside the same partition. This takes a polynomial number of steps in $|N|$, and consequently in the size of the game, $Length_c$.

The transformation again uses PARTITION with some minor changes. Begin constructing preferences where every player is indifferent among all coalitions of C where she belongs. For each player j such that $\{j\} \notin C$, add a single additional partner j^* (the *leader*) with the stand-alone coalition as her uniquely preferred coalition (player j^*'s preference profile is: "$\{j^*\}$"). And append this preference string to player j's preference profile: "$\succ_j \{j,j^*\} \succ_j \{j\}$" so that these two coalitions are ranked below every coalition in C that includes player j. If there is a partition in C then this partition in conjunction with every j^* alone will be Nash stable in the game. And if not, following the same argument as in HCORE, the only possibility is that any individually rational partition contains either $\{\{j\},\{j^*\}\}$ or $\{\{j,j^*\}\}$ for some j (no mixture of players with other relationships), and none of these collections can be part of a Nash stable partition: the first one induces player j to move with his leader, and the second one is not individually rational for the leader.

This shorter transformation must have polynomial time demand, as in HCORE.
■

Proposition 2.3 *HIS with arbitrary preferences over coalitions is NP-complete.*

Proof. This problem is in NP following a similar argument as in HNASH with the additional step of checking that the member willing to go is not accepted in the new coalition.

It can be easily proven that the same polynomial transformation used for HCORE is valid, because no cyclic relationship can induce an individually stable partition in the new game. ■

2.3.2 Strict Preferences over Coalitions

The fact that the previous proofs rely heavily on the assumption that indifferences are allowed in preferences, makes us question about the complexity of these problems when preferences are required to be strict. In particular, Ronn (1986) shows that the stable roommates problem is NP-complete when indifferences are allowed, whilst there are polynomial time algorithms for solving it when only strict references are allowed (Irving, 1985). This fact further motivates this restriction to strict preferences in hedonic games. We will show that requiring strict preferences does not make any of our problems easier.

First of all, it is obvious that all these problems remain in NP because the verification procedure for any certificate (a stable partition) is the same as in the case with arbitrary preferences.

Proposition 2.4 *HCORE with strict preferences over coalitions is NP-complete.*

Proof. Although a reduction from PARTITION is still possible, we will use the general setting of HCORE where indifferences are permitted, that is, we provide a polynomial-time computable mapping that will produce a strict preference profile, given any unrestricted preference profile, such that, one induces an empty core if and only if the other does.

Let $I_i(S)$, the *indifference family of S for player i*, be the set of coalitions for which player i is indifferent between any of them and S, that is, $I_i(S) = \{T : T \sim_i S\}$. It is obvious that $S \in I_i(S)$ and that if the game only allows for strict preferences then $I_i(S) = \{S\}$ for all S. Let $I_i^* = \{I_i(S) : |I_i(S)| \geq 2\}$, so that if the game only allows for strict preferences then I_i^* is empty for every player i. And let C_i be the largest collection inside I_i^*.

The set of players N will be constructed from scratch as follows. If a player i from the source game has no indifference families then it will be created in the new game with the label i_1 for the sake of the notation that will follow. If she does, instead of creating player i_1, we create $|C_i|$ players labeled $i_1, ..., i_{|C_i|}$. Let us define the following transformation f on the coalitions of the source game. Given a coalition S of the source RIRLC, $f(S)$ is the coalition of players in the target game that results from substituting each player i by her newly created players $i_1, i_2, ...i_{|C_i|}$.

Now, let us explore how to construct the preference lists of the new game. As required by the structure of a RIRLC, the stand-alone coalition of any player should be written at the end of the newly created RIRLC lists.

For players with no indifferences in the source RIRLC ($I_i^* = \emptyset$), their new lists are just translations (based on f) of their old ones, that is, each coalition in her list is substituted accordingly to f. For the rest of the players, in order to build the preferences of each i_k, we must translate (according to f) the RIRLC preferences of player i in the original HCORE problem with these rules in mind:

1. Keep the same ordering among indifference blocks as in the original game, that is, if $S \succ_i S'$ in the source game (so that S and S' belong to different indifference families of player i) then we must have that $f(S) \succ_{i_k} f(S')$ for all players i_k corresponding to i in the transformed game.
2. Whenever possible, player i_k ranks the f-image of the k-th coalition of any indifference family $I_i \in I_i^*$ before any other f-image from the same indifference family (the application of this rule may be impossible when a family I_i has less than k coalitions, in which case only the rest of the rules apply). These other f-images can be in any order without breaking any of the other rules.
3. When necessary, in order to complete the RIRLC, the stand-alone coalition will be appended at the end of the list of player i_k in the form "$\succ_{i_k} \{i_k\}$".

With this construction, it is not difficult to see that f is an into mapping from coalitions appearing in the source instance into coalitions of the new instance. It is into because given any coalition appearing in the new RIRLC lists, it can only be the image of at most one coalition of the source RIRLC. Moreover, if a coalition in the new lists is not in the image of f then it cannot induce a core stable configuration in the new game. The reason is that if a coalition S' that is chosen from the new lists is such that $S' \notin \mathrm{Im}(f)$, then it must be of the form $S' = \{i_k\}$ in which case the rest of the players created from i must remain alone to preserve their individual rationality (note that preferences have been constructed in a way that individual rationality implies that either all players created from i are all together -maybe with other players- or all alone, that is, they cannot be separately mixed with other players). But then $f(\{i\}) = \{i_1, ...i_{|C_i|}\}$ (the union of all players created from i) blocks S'.

Consequently, there exists another mapping g, $g(\Pi) = \{f(S) : S \in \Pi\}$, from partitions in the initial game's RIRLC to partitions in the new game's RIRLC such that it is into and if a partition is not in the image of g then it is not core stable in the new game. From these two properties of g, to prove that for any Π in the initial RIRLC, Π is core stable if and only if $g(\Pi)$ is core stable in the target game, is sufficient for our goal, which is to prove that the initial instance has a nonempty core if and only if the new instance does.

Suppose that a partition Π is not core stable in the original game. Then it must be blocked by some S that must be outside (on the left side of) the indifference blocks induced by Π for the players in S. Since the ordering among indifference blocks is preserved in the new game, the corresponding partition $g(\Pi)$ of the new game must be blocked by the counterpart of S, $f(S)$, in the new game.

If Π is core stable in the initial game, it is not blocked by any coalition outside the indifference blocks induced by Π. And neither is it in the transformed game, by the construction rules that preserve the ordering among indifference blocks. And it is not possible that the counterpart of Π, $g(\Pi)$, is blocked in the new game by some coalition S' for which some player i was indifferent in the original game ($f^{-1}(S') \sim_i S_\Pi(i)$), because, by construction, some player i_k is in her best position relative to the corresponding block of sets, and all players created out of i must be together in any core stable setting.

To conclude, this transformation requires polynomial time. In fact, the set of players of the new game, N', can be bounded by $|N| \max_{i \in N} |C_i|$. On the other hand, $K_c^{'} \leq K_c + |N'|$, implying that $Length_c^{'} = K_c^{'}|N'|$ is bounded above by a polynomial in $Length_c$. ■

For this case, let us explore an example that shows how this transformation works for an initial game. For the sake of space, we omit here the brackets and commas needed to write a coalition, as well as the relations' subindices (that is, we simply write $\succ$ instead of $\succ_1$) because the last element in the list already indicates whose list we are dealing with:

$$123 \sim \mathbf{12} \succ 13 \sim 1$$
$$123 \succ \mathbf{12} \sim 2$$
$$123 \succ 13 \succ \mathbf{3}$$

Player 1 has two indifference families and the largest one of them has two subsets, so we create players 1_1 and 1_2. The largest indifference family of player 2 contains two coalitions, so that 2_1 and 2_2 are created. As for player 3, only 3_1 is created since she has no indifferences. The set of players is $N' = \{1_1, 1_2, 2_1, 2_2, 3_1\}$ and the game is:

$$1_1 1_2 2_1 2_2 3_1 \succ \mathbf{1_1 1_2 2_1, 2_2} \succ 1_1 1_2 3_1 \succ 1_1 1_2 \succ 1_1$$
$$\mathbf{1_1 1_2 2_1, 2_2} \succ 1_1 1_2 2_1 2_2 3_1 \succ 1_1 1_2 \succ 1_1 1_2 3_1 \succ 1_2$$
$$1_1 1_2 2_1 2_2 3_1 \succ \mathbf{1_1 1_2 2_1 2_2} \succ 2_1 2_2 \succ 2_1$$
$$1_1 1_2 2_1 2_2 3_1 \succ 2_1 2_2 \succ \mathbf{1_1 1_2 2_1 2_2} \succ 2_2$$
$$1_1 1_2 2_1 2_2 3_1 \succ 1_1 1_2 3_1 \succ \mathbf{3_1}$$

Note that, by the second rule, for any coalition of the original game which was part of an indifference family, its image is preferred by some player to the rest of the images from the same indifference family. To see the correspondence between stable outcomes, $\Pi = \{\{1,3\},\{2\}\}$ is blocked by $S = \{1,2,3\}$ and $g(\Pi) = \{\{1_1, 1_2, 3_1\}, \{2_1, 2_2\}\}$ is blocked by $f(S) = \{1_1, 1_2, 2_1, 2_2, 3_1\}$. On the other hand, $\Pi = \{\{1,2\},\{3\}\}$ is core stable and $g(\Pi) = \{\{1_1, 1_2, 2_1, 2_2\}, \{3_1\}\}$ is stable in the new game, i.e, $\{1_1, 1_2, 2_1, 2_2, 3_1\}$ (which preimage is in the same indifference family as $\{1,2\}$ in player 1's preferences) cannot block it because $\{1_1, 1_2, 2_1, 2_2\}$ is preferred by player 1_2.

Proposition 2.5 *HNASH with strict preferences over coalitions is NP-complete.*

Proof. We provide a fast transformation from PARTITION.

The initial set of players will be X plus an additional *partner* i_1 for each member $i \in X$. All coalitions from C must be transformed to contain both their former players and the newly created partners for the members of the coalition: $f(S) = \cup_{i \in S}\{i, i_1\}$ (i.e., the subset $\{1,3\}$ becomes $\{1, 1_1, 3, 3_1\}$). For every player i, we start by creating an arbitrary strict preference list over transformed coalitions where she is included. Now we follow a similar procedure as in the general case of HNASH: for each pair of players (j, j_1) such that $\{j\} \notin C$, we create a pair of *leaders* (j^*, j_1^*) with the stand-alone coalition as their unique preferred coalition, and we append this string to the preferences of players j and j_1: "$\succ_j \{j, j^*\} \succ_j \{j, j_1\}$" (for player j) and "$\succ_{j_1} \{j_1, j_1^*\} \succ_{j_1} \{j, j_1\}$" (for player j_1). Finally, when necessary, we append the stand-alone coalition "$\succ_i \{i\}$" at the end of every player's preference list.

Hence, we have defined an into mapping f from subsets in X into individually rational coalitions of the game.Let the into mapping g be $g(\Pi) = \{f(S) : S \in \Pi\} \cup_{\{j\} \notin C} \{\{j^*\}, \{j_1^*\}\}$. As in the previous proof, if $\Pi' \notin \mathrm{Im}(g)$ then it cannot be Nash stable because the same problems with individual rationality arise. Similarly, Nash stability requires that players i and i_1 are together (maybe with more people) or both alone.

If $\Pi \subseteq C$ is a partition of X, then $g(\Pi)$ must be Nash stable. First, no individual player i or i_1 (players i and i_1 are together in $g(\Pi)$) in the new game has the possibility to improve by moving into another coalition because the preferences have been constructed so that her current partner appears in every strictly preferred coalition, and she is currently with her partner, so that these coalitions are not compatible with individual movements. She does not even want to move with her leader (because she would be strictly worse off) or with any other leader (because this is not in her individually rational list) or alone (this is always the worst individually rational alternative in the strict preferences case). Finally, no leader can improve by moving into any other coalition because she only wants to be alone.

On the other hand any Nash stable partition Π' of the game should satisfy these immediate properties:

- It must be individually rational, that is, it must be induced from the RIRLC of the new game.
- So, every leader must be alone.
- For $\{j\} \notin C$, the coalition $\{j, j_1\}$ cannot be in that partition because both j and j_1 would rather meet their leaders.
- Every pair of partners i and i_1 should be together in the same coalition.
- From all this, every coalition must be in the image of f.

Hence, Π' can be expressed as $\{S'_1, ...S'_2\} \cup_{\{j\} \notin C} \{\{j^*\}, \{j^*_1\}\}$ with the property that partners are together in coalitions. With this construction, we can conclude that the preimages of these coalitions S'_k must constitute a partition of X inside C.

Finally, it is clear that this transformation is polynomial: the number of players is bounded above by $4|X|$ and it is easy to check, as in the previous proofs, that the total number of coalitions appearing in the new game's RIRLC is of the order $O(|C|)$. ■

The following example may give some intuition of the proof. The PARTITION instance given by $X = \{1, 2, 3\}$ and $C = \{\{1, 2\}, \{2, 3\}, \{2\}\}$ contains no exact cover for X and it generates a hedonic game with player set $\{1, 1^*, 1_1, 1^*_1, 2, 2_1, 3, 3^*, 3_1, 3^*_1\}$:

$$
\begin{array}{l}
11_1 22_1 \succ 11^* \succ 11_1 \succ 1 \\
\quad 1^* \\
11_1 22_1 \succ 1_1 1_1^* \succ 11_1 \succ 1_1 \\
\quad 1_1^* \\
11_1 22_1 \succ 22_1 33_1 \succ 22_1 \succ 2 \\
11_1 22_1 \succ 22_1 33_1 \succ 22_1 \succ 2_1 \\
22_1 33_1 \succ 33^* \succ 33_1 \succ 3 \\
\quad 3^* \\
22_1 33_1 \succ 3_1 3_1^* \succ 33_1 \succ 3_1 \\
\quad 3_1^*
\end{array}
$$

which has no Nash stable partition.

Proposition 2.6 *HIS with strict preferences over coalitions is NP-complete.*

Proof. We do a similar reduction as in HNASH with strict preferences from PARTITION, that is, creating a partner i_1 for every i, and transforming subsets according to $f(S) = \cup_{i \in S}\{i, i_1\}$. This, as before, induces an incompatibility between sets, avoiding movement of players and guaranteeing stability when a partition is contained in C. But instead of adding leaders as in HNASH, create cyclic families $\{j, j_a, j_b\}$ and $\{j_1, j_{1a}, j_{1b}\}$ for each respective partner j and j_1 such that $\{j\} \notin C$. And we must append, at the end of this player j's list, the cyclic preferences over her children, whose preferences are cycled as in the general HCORE problem. Finally, when necessary, we append the stand-alone coalition at the end of the players' preference lists: "$\succ_i \{i\}$".

Now, g is defined as $g(\Pi) = \{f(S) : S \in \Pi\} \cup_{\{j\} \notin C} \{\{j_a, j_b\}, \{j_{1a}, j_{1b}\}\}$. As usual, if $\Pi' \notin \mathrm{Im}(g)$ then it cannot be individually stable: otherwise we must be inside a cycle (unstable).

If there is a partition in C then its g-image should be individually stable, since the coalitions inside the initial preferences of players make it incompatible to move into another coalition. In fact, individual stability is here implied by Nash stability. Hence, checking whether a movement hurts some of the current members of the new coalition is not necessary.

If there is no partition, the penalty imposed by these cycles will avoid undesirable individually stable partitions in the game.

Polynomial reducibility follows because these transformation has a similar complexity to the previous one. ■

2.4 NP-Completeness in Hedonic Games with Anonymous Preferences

Let us now consider a variation of hedonic games which complexity deserves our attention. A hedonic game with *anonymous* preferences is one in which every player is indifferent among coalitions of the same size. Hence, players' preferences can be written as preferences over sizes of coalitions. For example, in the following game $\{\{1,2\},\{3\}\}$ is a core stable partition:

$$
\begin{aligned}
&2 \succ_1 1 \succ_1 3 \\
&3 \succ_2 2 \sim_2 1 \\
&3 \sim_3 2 \succ_3 1
\end{aligned}
$$

We will show that if preferences are anonymous, then the core, the Nash stable set and the individually stable set have corresponding NP-complete decision problems, even when preferences over sizes are required to be strict.

In a similar way to the case of preferences over coalitions, the object that we will use is the *representation based on individually rational lists for sizes (RIRLS)*, with the property that no number in the representation should exceed the total number of players (it makes no sense to consider bigger sizes because no stable outcome can be obtained from them, since there would not be enough players). The RIRLS corresponding to the example is:

$$
\begin{aligned}
&2 \succ_1 1 \\
&3 \succ_2 2 \sim_2 1 \\
&3 \sim_3 2 \succ_3 1
\end{aligned}
$$

Finally, the length function will be simply $L_s = N$, that is, the number of players involved in the game[11]. Also, it must be pointed out that, given a partition of the players, identifying the corresponding numbers in the RIRLS

[11] As opposed to the case of RIRLCs, the number of elements (in this case, these elements are numbers) in the representation is bounded by N^2. Therefore, we can use N as the size of the problem because of the polynomial relation with N. In the case of RIRLCs, this was not a reasonable length function because the elements were coalitions, and the number of them could be exponential in N.

lists has low complexity. For example, in a game with 4 players, the partitions $\{\{1,2\},\{3,4\}\}$, $\{\{1,3\},\{2,4\}\}$ and $\{\{1,4\},\{2,3\}\}$ correspond to the tuple $(2,2,2,2)$, meaning that all players are in coalitions of size two, and finding this numbers in the preference profile is immediate.

Proposition 2.7 *HCORE with anonymous preferences is NP-complete, whether we allow for indifferences or not.*

Proof. To see that this problem is in NP, given a core stable partition we must only check that for every size k which is strictly preferred by some player in her list, there are in total less than k players who strictly prefer the size k to the size of their current coalition. This operation can be done in polynomial time in the size of the problem N.

We make a transformation from a hedonic game with preferences over coalitions. The set of players will be initially the same. Sort all distinct coalitions from the source game, in increasing order of their sizes (ties can be broken arbitrarily). The result is an ordered list $(S_1,...,S_L)$ of $L = K_c$ elements where the first N coalitions of this list must be the stand-alone coalitions of the source game's players. Assign to each coalition in the sorted list, S_l, the value $v(S_l)$, where:

$$\begin{array}{rcc} v(S_l) = 1 & \text{if} & |S_l| = 1 \\ v(S_l) = \max\{v(S_{l-1})+1, |S_l|\} & \text{otherwise} & \end{array}$$

so that there is a distinct value for each coalition. What we are doing is to assign a unique size to each possible coalition (of size greater than one) of the original game. Now, keeping the ordering of the source game's RIRLC preference lists, substitute every coalition S in the preference lists by the value $v(S)$. Additionally, we should create new players that "fill the space" that we created: for each S_l create $n_l = \max\{0, v(S_l) - |S_l|\}$ *redundant* players labeled $l_1,...,l_{n_l}$ with anonymous preferences represented by "$v(S_l) \succ_{l_i} 1$".

The value function v can be used to construct a function f between coalitions of both games:

$$\begin{array}{rcc} f(\{i\}) = \{i\} & \text{for all} & \{i\} \\ f(S_l) = S_l \cup \{l_1,...,l_{n_l}\} & \text{if} & |S_l| \geq 2 \end{array}$$

which is into. We can then construct the mapping g between partitions of both RIRLs, such that $g(\Pi) = \{f(S) : S \in \Pi\} \cup \{\{l_1\},...,\{l_{n_l}\} : S_l \notin \Pi\}$. Because g is one-to-one (every partition induced from the target RIRLS has a unique counterpart in the source RIRLC) it suffices to prove that Π is core stable if and only if $g(\Pi)$ is core stable.

If the individually rational partition Π is not core stable in the original game and it is blocked by S, then the partition $g(\Pi)$ must be blocked by $f(S)$.

If the individually rational partition Π' is not core stable in the new game and it is blocked by some individually rational coalition S of size v_0 (any coalition that blocks a RIRL partition must have size greater or equal than 2), then the preimage of S must block the preimage of Π'.

This transformation has polynomial time demand. In fact, the number of players in the new game is N' can be bounded polynomially because v can be: for all l, $v(S_l) \leq v(S_L) \leq |N| + K_c$ so that $N' \leq \sum_{l=1}^{K_c} v(S_l) \leq K_c(|N| + K_c) \leq 2Length_c^2$, polynomial in the size of the source problem, $Length_c$.

And because we have exactly used the same preference relations as in the original problem without adding any extra indifferences, the result remains valid even if preferences are required to be strict in these anonymous hedonic games. ■

The following source example with an empty core is quite illustrative for the proof.

$$
\begin{aligned}
12 \succ 13 \succ 123 \sim \mathbf{1} \\
123 \succ \mathbf{23} \succ 12 \succ 2 \\
13 \succ \mathbf{23} \sim 123 \succ 3
\end{aligned}
$$

where partition $\{\{1\}, \{2, 3\}\}$ is blocked by $\{1, 3\}$. From this, we can write the table

S	$v(S)$	$f(S)$
$S_a = \{1\}$	1	$\{1\}$
$S_b = \{2\}$	1	$\{2\}$
$S_c = \{3\}$	1	$\{3\}$
$S_d = \{1, 2\}$	2	$\{1, 2\}$
$S_e = \{1, 3\}$	3	$\{1, 3, e_1\}$
$S_f = \{2, 3\}$	4	$\{2, 3, f_1, f_2\}$
$S_g = \{1, 2, 3\}$	5	$\{1, 2, 3, g_1, g_2\}$

where $(S_a, S_b, ..., S_g)$ is the ordered list of coalitions. The resulting anonymous game with 8 players is:

$$2 \succ_1 3 \succ_1 5 \sim_1 \mathbf{1}$$
$$5 \succ_2 \mathbf{4} \succ_2 2 \succ_2 1$$
$$3 \succ_3 \mathbf{4} \sim_3 5 \succ_3 1$$
$$3 \succ_{e_1} \mathbf{1}$$
$$\mathbf{4} \succ_{f_1} 1$$
$$\mathbf{4} \succ_{f_2} 1$$
$$5 \succ_{g_1} \mathbf{1}$$
$$5 \succ_{g_2} \mathbf{1}$$

where the corresponding partition $\{\{1\}, \{2, 3, f_1, f_2\}, \{e_1\}, \{g_1\}, \{g_2\}\}$ is blocked by the unique coalition of size $v(\{1, 3\})$, that is, $\{1, 3, e_1\}$.

Consider the following hedonic game with anonymous preferences, where $m \geq 2$. Let us call it the Nash *penalizing component*:

$$m \succ_i m-1 \succ_i m-2 \succ_i ... \succ_i 2 \succ_i 1 \text{ for } i = 1, ..., m$$
$$2(m+1) - j \succ_j 1 \text{ for } j = m+1, ..., 2m-1$$

We have a total of $2m-1$ players divided in 2 *blocks* of m and $m-1$ players, respectively. It is clear that if we discard all players from the first block, every player of the second block in her stand-alone coalition is a Nash stable partition in the subgame induced by these players from the second block, because their preferred sizes are all different and greater than or equal to 3. Now, we show that this cannot be the case if we consider subgames which do not discard all players from the first block. Before that, let us show an example for $m = 3$:

$$3 \succ_1 2 \succ_1 1$$
$$3 \succ_2 2 \succ_2 1$$
$$3 \succ_3 2 \succ_3 1$$
$$4 \succ_4 1$$
$$3 \succ_5 1$$

Claim 2.1 *The Nash penalizing component ($m \geq 2$) has no Nash stable partition. Formally, given any subset S of the players that includes all players of the* *second block and at least one player from the first, there is no partition of S that is Nash stable in the subgame induced by S.*

Proof. Note the following properties for a Nash stable partition in the subgame induced by any S:

- Player $m+1$ must be alone because her preferred coalition of size $m+1$ is not individually rational for anyone else.
- Hence, no player in the first block can be alone because she would join player $m+1$. This means that all players of S that come from the first block must be in coalitions of size at least two (these coalitions may include or not some players from the second block).
- Let k be the maximum size of these coalitions of size at least two in a stable partition. It is obvious that k cannot be greater than or equal to $m+1$ because the maximum individually rational feasible size is m. Then, there is a player j in the second block such that her preferred size is $k+1$ and she is currently alone, willing to join this coalition of size k.

So, we have that some player in the second block will always be willing to join this biggest coalition, and this cannot be Nash stable. ■

The intuition about the use of this game is that by keeping the second block intact, there is a Nash stable partition if and only if we kick out from the game all the players from the first block. We will use this component for the following proof by creating a second block which penalizes undesirable stable situations in the case that some player lies inside the first block.

Proposition 2.8 *HNASH with anonymous preferences is NP-complete, whether we allow for indifferences or not.*

Proof. This decision problem is in NP. To check that Π is Nash stable, perform similar actions as in the general HNASH problem, but in the context of preference on sizes: Π is Nash stable if and only if no player i prefers a size k such that $k-1$ is the size of some other coalition of Π. This operations can be performed in polynomial time.

To prove completeness, we will make a reduction from PARTITION.

The initial set of players is X. Let m be the cardinality of the set $\{j \in X : \{j\} \notin C\}$. We will not consider the trivial case where $m \leq 1$.[12]

[12] For $m=0$, there is a partition in C where everyone is alone. For $m=1$, there is a partition in

Sorting the distinct subsets of C in increasing order of their sizes, we obtain the ordered list $(S_1, ...S_L)$ and we define the value v as:

$$v(S_l) = 1 \quad \text{if} \quad |S_l| = 1$$
$$v(S_l) = \max\{m+2, v(S_{l-1})+2, |S_l|\} \quad \text{otherwise}$$

For this initial set of players, player $i's$ preferences will be written as any arbitrary ordering (allowing indifferences or not: this is why this proof makes the result valid for both settings) over sizes $v(S_l)$ such that S_l contains player i (except for the case of the size 1, which must always appear as the tail of every RIRLS list). For every player j with $\{j\} \notin C$, her preferences will be appended "$\succ_j$" followed by a string like the preferences of players from the first block of the Nash penalizing component, according to m. After that, we will add to the game the second block of $m-1$ players $\{p_1, ..., p_{m-1}\}$, completing the penalizing component. We still need to fill some space with redundant players $l_1, ..., l_{n_l}$ for each coalition S_l, as we did in the case of the core. The preferences of each of these redundant players will be "$v(S_l) \succ_{l_i} 1$". Next, we prove that there is a partition in C if and only if there is a Nash stable partition in the game.

Suppose that $\Pi = (S_1, ...S_{|\Pi|})$ is partition inside C. Let Π' be the partition of the target game be such that,

- Each coalition $S_k \in \Pi$ has been augmented, when necessary, with its corresponding redundant players, and its size becomes $v(S_k)$.
- Players of the second block of the penalizing component and every redundant player l_i such that $S_l \notin \Pi$ are all alone.

Then Π' is Nash stable in the new game. The explanation of this comes from the fact that individual movements cannot improve welfare. We can see this from the following facts:

- *Players in coalitions of size not smaller than $m+2$ have no incentives to move*. Redundant players corresponding to these coalitions are always in their best position. For the rest of the players in these type of coalitions (that is, players from X), it is not possible to individually move to a coalition of size greater by one unit because it is not individually rational (all preferred coalitions have sizes with increments of at least two). Nor is it to move to coalitions of size less than $m+2$ because this is worse.
- *Player $i \in X$ such that $\{i\} \in \Pi$ (and $\{i\} \in \Pi'$) has no place to move*. Her

C if and only if the unique j with $\{j\} \notin C$ belongs to some subset of C, and this can be checked in linear time in the size of the PARTITION instance. This means that the NP-complete problem used for the reduction is the subproblem consisting of the restriction of PARTITION to the case $m \geq 2$.

preferred sizes are of size at least $m+2$, but the differences in size between all coalitions in this range makes the movement individually irrational for her.

- *Players of the second block or the remaining redundant players would not move, either*. Within this coalition structure, there cannot be any coalition of size $2, 3, ..., m$ or $m+1$: the fact that, by construction of Π', every player from X is at the left of the penalizing part of her preferences under Π', makes it impossible to have coalitions with these sizes. This means that players from the second block cannot improve because their preferred sizes range from 3 to $m+1$. On the other hand, those redundant players who are in their stand-alone coalitions cannot move to form their most preferred size, because their leaders (the set of players from X that they can only form coalitions with) are already spread among other coalitions that they cannot individually join (again, the difference between sizes is what makes this fact possible).

On the other hand, suppose that the game has a Nash stable partition $\Pi' = (S'_1, ...S'_{|\Pi'|})$. This partition must satisfy:

- Players from the first block ($\{j\} \notin C$) must be in coalitions of size not smaller than $m+2$ (so that, they are all "out" of the penalizing component). So, players of the second block must be all alone, otherwise we already proved that this cannot induce a Nash stable partition in the game.
- For every player with $\{i\} \in C$, she must be in a coalition of size in the image of v: these are the only sizes appearing in her list.
- For any redundant player created from $S \in C$, either she is alone or she is with all the members in S and all other redundant players created from S, because these are the only individually rational coalitions for her.

Hence, let us remove all redundant players (if any) from every S'_k in Π'. Because Π' is a partition of the game, the non-empty subsets obtained by these removals, without the stand-alone coalitions of players from the second block, must constitute a partition of X inside C.

Once more, the complexity of this transformation turns to be polynomial. The size of the target game, N, is polynomial in the size of the source instance, $|C||X|$ because v can be bounded polynomially in a similar way to the previous proof. ■

Next we show how this transformation works for a specific yes-instance of PARTITION: $X = \{1, 2, 3\}$, $C = \{\{\mathbf{1}, \mathbf{2}\}, \{1, 3\}, \{\mathbf{3}\}\}$. This results in the following data, where $m = 2$:

S	$v(S)$	Players
$S_a = \{3\}$	1	$\{3\}$
$S_b = \{1, 2\}$	4	$\{1, 2, b_1, b_2\}$
$S_c = \{1, 3\}$	6	$\{1, 3, c_1, c_2, c_3, c_4\}$
		penalizing player p_1

The anonymous preferences of players can be written taking into account the part of the preferences which are in the penalizing component (pc.). Note that we are free to use some indifferences outside the penalizing area for players which are in X (i.e, player 3):

Player	Belongs to pc	Preferences of each player	
		Outside pc.	Inside pc.
1	Yes	$\mathbf{4} \succ_1 6$	$\succ_1 2 \succ_1 1$
2	Yes	$\mathbf{4}$	$\succ_2 2 \succ_2 1$
3	No	$6 \sim_3 \mathbf{1}$	
b_1	No	$\mathbf{4} \succ_{b_1} 1$	
b_2	No	$\mathbf{4} \succ_{b_2} 1$	
c_1	No	$6 \succ_{c_1} \mathbf{1}$	
c_2	No	$6 \succ_{c_2} \mathbf{1}$	
c_3	No	$6 \succ_{c_3} \mathbf{1}$	
c_4	No	$6 \succ_{c_4} \mathbf{1}$	
p_1	Yes		$3 \succ_{p_1} \mathbf{1}$

where the Nash stable configuration can be uniquely identified by looking at the bold numbers.

Before proceeding to prove that HIS is NP-complete for anonymous preferences, we provide a different version of a problem from Bogomolnaia and Jackson (2002)[13]:

[13] In their work, they provide the smallest game they could find where no individually stable partition existed:

$$57 \succ_1 2 \succ_1 7 \succ_1 6 \succ_1 1$$
$$7 \succ_2 2 \succ_2 57 \succ_2 56 \succ_2 1$$
$$2 \succ_3 7 \succ_3 6 \succ_3 1$$
$$7 \succ_i 6 \succ_i 5 \succ_i 4 \succ_i 3 \succ_i 2 \succ_i 1 \text{ for } i = 4, ..., 8$$
$$63 \succ_i 62 \succ_i 61 \succ_i 60 \succ_i ... \succ_i 2 \succ_i 1 \text{ for } i = 9, ..., 63$$

$$
\begin{aligned}
56+m \succ_1 2 \succ_1 7 \succ_1 6 \succ_1 1 \\
7 \succ_2 2 \succ_2 56+m \succ_2 55+m \succ_2 1 \\
2 \succ_3 7 \succ_3 6 \succ_3 1 \\
7 \succ_i 6 \succ_i 5 \succ_i 4 \succ_i 3 \succ_i 2 \succ_i 1 \text{ for } i=4,...,8 \\
62+m \succ_i 61+m \succ_i ... \succ_i 2 \succ_i 1 \text{ for } i=9,...,62+m
\end{aligned}
$$

In this game, we have a total of $62 + m$ players and no individually stable partition exists for $m \geq 1$ as we prove in the following claims, using similar arguments to those of the original paper. In this way, we will use this game as a penalizing component for the next proposition. For $m = 0$, it can be easily verified that the partition

$$
\begin{aligned}
&\{ \\
&\{1,9,...,14\},\{2,15,...,20\},\{3,21,...,26\}, \\
&\{4,27,..,32\},\{5,33,...,38\},\{6,39,...,44\}, \\
&\{7,45,...,50\},\{8,51,...56\},\{57,...62\} \\
&\}
\end{aligned}
$$

is individually stable. Now, we show a series of claims that should be satisfied by any individually stable partition $\Pi = \{S_1, ..., S_{|\Pi|}\}$ for $m \geq 1$, reaching a contradiction.

Claim 2.2 *There are no two S_k and S_l in Π such that $S_k, S_l \subset \{9, ..., 62+m\}$.*

Proof. Otherwise, any member of the smallest coalition between S_k and S_l would be accepted in the other coalition. ■

Claim 2.3 *Suppose that $S_1 = \{9, ..., 62+m\}$. Then, no configuration of players 1 to 8 can be part of an individually stable partition of the game.*

Proof. First, a coalition of size 8 is not individually rational for players 1 to 8. Similarly, no coalition of size 7 can be stable because players 1 and 3 prefer size 2 to size 7 and one of them would join the player who is alone. For size 6, player 2 would always like to join this coalition and would be accepted ($7 \succ 6$ for all players). A size of 5 would either make 1 and 3 willing to join it, or player 2 willing to join coalition.$\{9, ..., 62 + m\}$. Sizes 4 and 3, which only correspond to players from $\{4, .., 8\}$, are not possible because the rest of the players from $\{4, ..., 8\}$ would be admitted in the coalition. So, only sizes 1 or 2 are reasonable. But then two of the players from $\{4, ..., 8\}$ must be together in a coalition of size 2, because by being alone they could join. From this, the rest of the players from $\{4, ..., 8\}$ would be admitted in this coalition. ■

Claim 2.4 *The coalition $S_1 = \{9, ..., 62 + m\}$ is in the individually stable partition* Π.

Proof. By claim 2, $|\Pi| \leq 9$. Letting S^* be the biggest coalition in Π, then $|S^*| \geq \left\lceil \frac{62+m}{|\Pi|} \right\rceil \geq 7$ because $m \geq 1$. But if $|\Pi| = 9$ and $m = 1$, we have 63 players divided in 9 coalitions and players 1 to 8 are separated in eight different coalitions by claim 2. In this case, if $|S^*| = 7$ then $|S_k| = 7$ for all S_k in Π and this means that there is a coalition with exactly 7 players from the last block, reaching a contradiction with individual stability because new members are accepted. And if $|S^*| > 7$, since players 1 or 2 cannot be in it (we must have 9 coalitions), it must be the case that $S^* \subseteq \{9, ..., 62 + m\}$. But if $S^* \subset \{9, ..., 62 + m\}$, we reach the same contradiction with individual stability. We are left with $S^* = \{9, ..., 62 + m\}$, the desired result.

Now, consider the case where $|\Pi| \leq 9$ and $m \geq 1$, but both do not hold with equality simultaneously, which implies that $|S^*| > 7$. If $|S^*| < 54 + m$, then it must be the case that $S^* \subset \{9, ..., 62 + m\}$ and the remaining players from $\{9, ..., 62 + m\}$ are admitted in the biggest coalition S^*. So, consider the cases where $|S^*| \geq 54 + m$. If $|S^*| = 55 + m$, that is, $S^* = \{2, 9, ..., 62 + m\}$, player 1 can join S^*. If $|S^*| = 56 + m$ ($S^* = \{1, 2, 9, ..., 62 + m\}$), it is easy to see that players 3 to 8 cannot be arranged in order to induce an individually stable partition, because they must stay together in a coalition of size 6 and thus player 2 would be admitted. It is clear that sizes strictly greater than $56 + m$ are not possible, because there are not enough players for which these sizes would be individually rational. Therefore, $|S^*| = 54 + m$, and the only individually rational possibility is that $S^* = \{9, ..., 62 + m\}$. So, $S_1 = S^*$. ■

Claim 2.5 *The game has no individually stable partition for $m \geq 1$.*

Proof. Because $S_1 = \{9, ..., 62 + m\}$, we obtain the impossibility result from claim 3. ■

Proposition 2.9 *HIS with anonymous preferences is NP-complete, whether we allow for indifferences or not.*

Proof. This is an NP problem by following the same checking procedure as in the anonymous HNASH problem, and checking the status of the new coalition members.

Let X and C be the instance of PARTITION. We make a similar (although slower) transformation than with anonymous HNASH.

First, we must count the elements j with $\{j\} \notin C$. Let this number be m (we discard the trivial case where $m = 0$). Sorting the distinct subsets of C in increasing order of their sizes, we obtain the ordered list $(S_1, ...S_L)$ and we define v as:

$$\begin{array}{rcc} v(S_l) = 1 & \text{if} & |S_l| = 1 \\ v(S_l) = \max\{64 + m, v(S_{l-1}) + 2, |S_l|\} & \text{otherwise} & \end{array}$$

As we see, v is similar to the case of HNASH, but the relevant sizes start from at least $64 + m$ (the number of extra players that we will need to create will be very large). The initial set of players will be X, and player $i's$ preferences will be written as an arbitrary ordering (using indifferences or not: this is why this proof makes the result valid for both settings) over sizes $v(S_l)$ for all S_l containing player i. For the non-trivial case that $m \geq 1$, we create 62 players $\{p_1, ..., p_{62}\}$ with preferences identical (according to m) to players $\{1, 2, ..., 62\}$ of the penalizing component. For the m players j with no stand-alone coalition in C, we add them to this penalizing component, by adding this string to the right-most region of their preferences: "$\succ_j 62 + m \succ_j 61 + m \succ_j ... \succ_j 1$". That is, the tails of their preferences are like the preferences of the players in the last block of the penalizing component described before. The preferences of the necessary redundant players will be "$v(S_l) \succ_{l_i} 1$".

It can be checked, using the usual approach, that the penalizing component avoids undesired individually stable partitions when there is no partition in C. And a partition in C induces an individually stable partition in the new game where the initial 62 players in the penalizing block $\{p_1, ..., p_{62}\}$ are grouped as in the individually stable partition of 62 players shown for $m = 0$.

This transformation requires a lot of players, but still it is polynomial in the size of the PARTITION problem because v can be polynomially bounded. ■

There does not seem to be a much simpler transformation than the one obove described. Its considerable length is imposed by the fact that non-trivial games exist only for a great amount of players.

2.5 Interpretation of the Results and Conclusions

The main conclusion of this work sums up to saying that the general decision problems inherent to some of the stability definitions in hedonic games are hard from a computational perspective. In particular, they are shown to be NP-complete. In a natural way, this implies that finding a stable configuration is difficult, in general.

Not only this is bad news from the computational point of view, but proofs provide us with partial ideas on why these problems are difficult. For instance, a quick inspection of the transformation rule given in Proposition 2.1 enables us to say that finding a stable outcome remains difficult even when there are lots of indifferences at the top of all rankings. This, in principle, may lead us to the wrong conclusion that a stable outcome would be easy to find inside this part of the preferences. More intricate implications can be obtained from the structure of the proofs in the anonymous case: from Proposition 2.4, the HCORE problem with preferences over coalitions (proven to be NP-complete in Propositions 2.1 and 2.4) remains difficult even if each player can only submit preferences over coalitions of different sizes. In the same fashion, we could further analyze the complexity hidden inside all these problems with the help of the transformations provided in this paper. This is why proving an NP-complete result in different ways may be attractive in order to give more arguments for the origin of the problem's complexity. For instance, based on an alternative proof which we omit for the sake of space limitations, we have proven that it is still difficult to find a core stable partition when agents are distributed in a rectangular lattice and they are restricted to submit preferences over their immediate neighbors[14].

From another perspective, the relevance of this work arises because we have been able to classify all these problems into a class of equivalent problems which are of special interest in discrete mathematics, operations research and computer science. The accepted conjecture that $P \neq NP$ is the reason why all NP-complete problems are considered intractable, that is, the time required by an algorithm solving any of them can become unreasonable, even for examples of moderate size. Moreover, finding a fast algorithm for a single NP-complete problem would imply a fast solution to all the others (so that $P = NP$!), and viceversa, proving that such fast algorithm is unfeasible would prove that all NP-complete problems are intractable ($P \neq NP$). Efforts are being put by researchers to demonstrate the accepted conjecture $P \neq NP$, since no one has ever found a fast algorithm for any NP-complete problem, and the implications

[14] The alternative polynomial reduction for proposition 1 has been made from the *Minesweeper consistency* problem Kaye (2000). The complexity of HCORE arises only if several players have preferences over the same neighbors, that is, when their preferences "overlap".

of this would be somewhat stunning.

Economic implications are of special interest in contexts of bounded rationality, as it happens in Gilboa (1988) and Ben-Porath (1990). For instance, for a planner with bounded computational resources, her role in trying to implement a stable configuration can become unfruitful, given the complexity of these problems. In another way, take into consideration the question of how coalitions form in large societies and groups: then, the results of this paper might affect the strength of some coalition formation models. For instance, if a model handles coalition formation as a process that can be simulated by a computer, then it can be believed that stable configurations, if any, will be reached eventually, but this process may take too long.

But these results do not constitute a total critique to hedonic games or to the stable solutions studied. In fact, these solutions are still important. Even if they can be difficult to find or to reach, configurations which are not stable can be easily identified, i.e., given a configuration, any individual is able to identify the coalition she would like to move to, so that individual stability and Nash stability can be sensible solutions in this context. In the case of the core, although stable outcomes can also be verified in polynomial time as core stable, it seems to be a bit more difficult because agents have to inspect more deeply their preferences in order to find a better coalition to propose which also benefits the rest of its members. In any event, this subtle difference in the complexity of checking a given configuration could give rise to further investigation on how boundedly rational agents organize themselves in a model of coalition formation.

Some other conclusions can be obtained regarding on how ”good” theorems that one constructs for hedonic games can be. In fact, we can say that none of the stability concepts studied in this paper can be *well-characterized*. A decision problem is well-characterized if both it and its complementary problem (the problem which asks exactly the opposite question) are NP problems. The difficulty with NP-complete problems is that they can have good characterizations if and only if $NP = co - NP$ and this is very unlikely to happen. In fact, consider the case of a hedonic game with an empty core. Is there an efficient way to certify that it is empty? We could span the whole space of possible partitions and check that none of them is stable (using polynomial time for each of these partitions). But this sounds quite inefficient (slow) because the total number of partitions is exponential in the size of the underlying set N. What the theory of NP-completeness is telling us is that there cannot be any nice certificate for checking the emptiness of the core in hedonic games. Hence, the strategy of trying to find good general theorems for any of these stability concepts might not be a promising task.

Further research on this field includes the study of subclasses of hedonic games whose decision problems involve polynomial time, that is, subproblems which are easy. Additionally, other related problems like the search problem (find a stable partition), the enumeration problem (find the cardinality of the stable set), the problem of representing a whole stable set, or some optimization problems attached to these sets, may deserve some attention in the future. Nonetheless, the simplest problems (decision problems) are shown to be difficult, and this was the main purpose of this work.

Chapter 3
Who is Who in Networks. Wanted: The Key Player

with
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and
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3.1 Introduction

The dependence of individual outcomes on group behavior is often referred to as *peer effects* in the literature.[15] In standard peer effects models, this dependence is homogeneous across members, and corresponds to an *average* group influence. Technically, the marginal utility to one person of undertaking an action is a function of the average amount of the action taken by her *peers*. Generative models of peer effects, though, suggest that this intragroup externality is, in fact, heterogeneous across group members,[16] and varies across individuals with their level of group exposure.

In this paper, we allow for a general pattern of bilateral influences, and analyze the resulting dependence of individual outcome on group behavior.

More precisely, consider a finite population of players with linear-quadratic interdependent utility functions. Take the matrix of cross derivatives in these players' utilities. Our first task is to decompose additively this matrix of cross effects into an idiosyncratic component, a global interaction component, and a local interaction network. The idiosyncratic effect reflects (part of) the concavity of the payoff function in own efforts. The global interaction effect is uniform across all players, and reflects a strategic substitutability in efforts across all pairs of players. Finally, the local interaction component reflects a (relative) strategic complementarity in efforts that varies across pairs of players. The population wide pattern of these local complementarities is well-captured by a network. This

[15] Durlauf (2004) offers an exhaustive survey of the theoretical and empirical literature on peer effects.

[16] For instance, when job information flows through friendship links, employment outcomes vary across otherwise identical agents with their location in the network of such links (Calvó-Armengol and Jackson, 2004).

description allows for a clear view of global and local externalities and their sign for a given general pattern of interdependencies.

Based on this reformulation, the paper provides three main results. First, we relate individual equilibrium outcomes to the players' positions in the network of local interactions. Second, we show that the aggregate equilibrium outcome increases with the density and size of the local interaction network. Finally, we characterize an optimal network disruption policy that exploits the geometric intricacies of this network structure.

In network games, the payoff interdependence is, at least in part, rooted in the network structure of players' links.[17] In these games, equilibrium strategies, that subsume the payoff interdependence in a consistent manner, should naturally reflect the players' network embeddedness. When the relative magnitude of global and local externalities for our decomposition of cross effects scale adequately, our network game has a unique and interior Nash equilibrium, proportional to the Bonacich network centrality. This measure has been proposed for nearly two decades in sociology by Bonacich (1987), and counts the number of all paths[18] emanating from a given node, weighted by a decay factor that decreases with the length of these paths.[19] This is intuitively related to the equilibrium behavior, as the paths capture all possible feedbacks. In our case, the decay factor depends on how others' actions enter into own action's payoff.

The sociology literature on social networks is well-established and extremely active (see, in particular, Wasserman and Faust, 1994). One of the focus of this literature is, precisely, to propose different measures of network centralities and to assert the descriptive and/or prescriptive suitability of each of these measures to different situations.[20] This paper provides a behavioral foundation to the Bonacich's index, thus singling it out from the vast catalogue of network measures.

The relationship between equilibrium strategic behavior and network topology given by the Bonacich measure allows for a general comparative statics exercise. We show that a denser and larger network of local interactions increases the aggregate equilibrium outcome. This is simply because the aggregate number of network paths increase with the number of available connections.

When the Nash-Bonacich linkage holds, the variance of equilibrium actions reflects the variance of network centralities. In this case, a planner may want to

[17] See, in particular, the recent literature survey by Jackson (2004).
[18] Not just shortest paths.
[19] It was originally interpreted as an index of influence or power of the actors of a social network. Katz (1953) is a seminal reference.
[20] See Borgatti (2003) for a discussion on the lack of a systematic criterium to pick up the "right" network centrality measure for each particular situation.

remove a few suitably selected targets from the local interaction network, so as to alter the whole distribution of outcomes. To characterize the network optimal targets, we propose a new measure of network centrality, the *inter-centrality measure*, that does not exist in the social network literature. Players with the highest inter-centrality are the *key players* whose removal results in the maximal decrease in overall activity.

Contrary to the Bonacich centrality measure, this new centrality measure does not derive from strategic (individual) considerations, but from the planner's optimality (collective) concerns. Bonacich centrality fails to internalize all the network payoff externalities agents exert on each other, while the inter-centrality measure internalizes them all. Indeed, removing a player from a network has two effects. First, less players contribute to the aggregate activity level (direct effect), and second, the network topology is modified, the remaining players thus adopting different actions (indirect effect). As such, the inter-centrality measure accounts not only for individual Bonacich centralities but also for cross-contributions across them. In particular, the key player is not necessarily the player with the highest equilibrium outcome.

Section 3.2 introduces notation. Section 3.3 presents the model. Sections 3.4 contains the equilibrium analysis, and Section 3.5 the network-based policy. Section 3.6 contains a number of applications, including crime networks, R&D collaboration links in oligopoly markets, and conformist behavior. Section 3.7 discusses a number of extensions.

3.2 Definitions and Notation.

The primitive set of our model is a set of agents $N = \{1, 2, ..., n\}$.

3.2.1 Matrix and Vector Notation

Matrices and vectors will be denoted in bold letters, like $\mathbf{A}$ and $\mathbf{x}$, respectively. When necessary, we use $\mathbf{A}_{l\times m}$ to specify that $\mathbf{A}$ has l rows and m columns. Generally, $l = m = n$. The *entries* of $\mathbf{A}$ and $\mathbf{x}$ are written like a_{ij} and x_i.

The *transpose* of $\mathbf{A}$ and $\mathbf{x}$ are $\mathbf{A}^T$ and $\mathbf{x}^T$. Matrix *multiplication* is written like $\mathbf{A}\cdot\mathbf{x}$, $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{x}^T\cdot\mathbf{y}$. The matrix $\mathbf{A}^k$ is the k-th *power* of $\mathbf{A}$, and its (i, j)-entry is written $a_{ij}^{[k]}$.

Given a set $S \subseteq N$, $\mathbf{A}_S$ and $\mathbf{x}_S$ are the *restrictions* of $\mathbf{A}$ and $\mathbf{x}$ to the set S. Also, $\mathbf{A}_{-S} \equiv \mathbf{A}_{N\setminus S}$ and $\mathbf{x}_{-S} \equiv \mathbf{x}_{N\setminus S}$.

The identity matrix is $\mathbf{I}$. The symbols $\mathbf{O}$ and $\mathbf{0}$ will be used for the *zero* matrix and vector. The symbols $\mathbf{U}$ and $\mathbf{1}$ will be used for the *one* matrix and vector, where every entry is 1. Given a vector $\mathbf{x}$, the scalar $x \equiv \mathbf{1}^T\cdot\mathbf{x}$ is the sum of all its entries, and $x_S \equiv \mathbf{1}^T\cdot\mathbf{x}_S$.

We write $\mathbf{A} \leq \mathbf{B}$, whenever $a_{ij} \leq b_{ij}$ for all i and j in N. Also, $\mathbf{A} \lneq \mathbf{B}$ if $\mathbf{A} \leq \mathbf{B}$ and there exists $a_{ij} < b_{ij}$. Finally, $\mathbf{A} < \mathbf{B}$ if $a_{ij} < b_{ij}$ for all i and j. The symbols $\geq$, $\gneq$ and $>$ are defined accordingly.

An *eigenvalue* of a matrix $\mathbf{A}$ is a complex number μ satisfying $\mathbf{A} \cdot \mathbf{v} = \mu \mathbf{v}$ for some vector $\mathbf{v}$. Let $\mathcal{S}(\mathbf{A})$ (called the *spectrum* of the matrix $\mathbf{A}$) be the set of eigenvalues of $\mathbf{A}$.

3.2.2 Networks

A *network* (*graph*) g consists of a set of *agents* (*vertices* or *nodes*) $V(g)$ and a set of weighted *links* (*edges*) among them, where $g_{ij} \geq 0$ is the weight assigned to the link (i, j). The link (i, j) is also written ij, for short. For this reason, we can represent a network by means of a nonnegative square *adjacency matrix* $\mathbf{G} = (g_{ij})_{i,j \in V(g)}$. Without loss of generality, we will consider networks where $g_{ij} \in [0, 1]$. Generally, the set of agents of the network g, $V(g)$, will be N. In some cases, it may be different, but this will be easily inferred from the notation that we introduce later. A network g_{ij} is *un-weighted* when $g_{ij} \in \{0, 1\}$, for all $i, j \in V(g)$.

The network g is *undirected* when its adjacency matrix $\mathbf{G}$ is *symmetric*, that is, $g_{ij} = g_{ji}$ for all $i, j \in N$. We say that a network *has no self-loops* whenever $g_{ii} = 0$ for all $i \in V(g)$. We will assume that networks have no self-loops, so that $\mathbf{G}$ is a zero-diagonal matrix.

We refer to the agents i and j as being *directly linked* in the network g, whenever $g_{ij} > 0$. We also denote this by $ij \in g$. The *number of links* of g is $|g| \equiv |ij \in g|$.

A link $ij \in g$ is *incident* with the vertex $v \in V(g)$ whenever $i = v$ or $j = v$.

A *path* in g of length k from i to j is a sequence $p = \langle i_0, i_1, ..., i_k \rangle$ of agents such that $i_0 = i$, $i_k = j$, $i_p \neq i_{p+1}$, and i_p and i_{p+1} are directly linked, for all $0 \leq p \leq k - 1$. Agents i and j are said to be *indirectly linked* in g if there exists a path from i to j in g. An agent $i \in V(g)$ is *isolated* in g if $g_{ij} = 0$ for all j. The network g is said to be *empty* when all its agents are isolated, that is, $\mathbf{G} = \mathbf{O}$.

We say that a path p *traverses* or *hits* agent i if i is in the sequence defined by the path. The path p *covers* the set $S \subseteq V(g)$ if p traverses every agent $i \in S$.

We say that network g' *is a (proper) subnetwork of* g, written $g' \subseteq g$ ($g' \subset g$), whenever $V(g') \subseteq V(g)$ and $\mathbf{G}' \leq \mathbf{G}_{V(g')}$ ($\mathbf{G}' \lneq \mathbf{G}_{V(g')}$). We also say that g' is (*strictly*) *contained* in g. Given two networks g and g', their *union* is written $h = g \cup g'$, where $V(h) = V(g) \cup V(g)$ and $h_{ij} = g_{ij} + g'_{ij}$.

Given a network g and a set $S \subseteq V(g)$, we say that g_S is the *subnetwork of* g *induced by* S whenever the adjacency matrix of g_S is $\mathbf{G}_S$. Note that $V(g_S) = S$. We write g_{-S} to denote the network $g_{N \setminus S}$, that is g_{-S} is the network that results after eliminating all the agents in S.

A *cut* in a network g is a partition of the vertices into two disjoint subsets S and $V(g)\backslash S$. Every cut in a network *divides* it into two different networks g_S and g_{-S}. A network is *bipartite* if it has a cut dividing it into two empty networks.

The *spectral radius* of a network g is defined as:

$$\mu_1(g) = \max_{\mu \in \mathcal{S}(\mathbf{G})} |\mu|$$

where $|\mu|$ is the modulus of the (complex) eigenvalue μ of the matrix $\mathbf{G}$. When g is undirected, all the eigenvalues of $\mathbf{G}$ are real and $\mu_1(g)$ is called the *index* of the network g.

3.3 The Model

3.3.1 The Game

Each player $i = 1, ..., n$ selects an effort $x_i \geq 0$, and gets a payoff $u_i(x_1, ..., x_n)$. We focus on bilinear payoff functions of the form:

$$u_i(x_1, ..., x_n) = \alpha_i x_i + \frac{1}{2}\sigma_{ii} x_i^2 + \sum_{j \neq i} \sigma_{ij} x_i x_j, \tag{3.1}$$

strictly concave in one's effort, that is, $\partial^2 u_i / \partial x_i^2 = \sigma_{ii} < 0$. We set $\alpha_i = \alpha > 0$ and $\sigma_{ii} = \sigma$, identical for all players. Net of bilateral influences, players have thus the same payoffs.

Bilateral influences are captured by the cross derivatives $\partial^2 u_i / \partial x_i \partial x_j = \sigma_{ij}$, $i \neq j$. They depend on the pair of players considered, and can be of either sign. When $\sigma_{ij} > 0$, an increase in effort from j triggers a downwards shift in i's response. We say that efforts are strategic complements from i's perspective within the pair (i, j). Reciprocally, when $\sigma_{ij} < 0$, efforts are strategic substitutes from i's perspective within the pair (i, j).

Let $\underline{\sigma} = \min\{\sigma_{ij} \mid i \neq j\}$ and $\overline{\sigma} = \max\{\sigma_{ij} \mid i \neq j\}$.

We assume that $\sigma < \min\{\underline{\sigma}, 0\}$. When $\underline{\sigma} \geq 0$, this is simply the concavity of payoffs in own efforts. When $\underline{\sigma} < 0$, this requires that own marginal returns decrease with the level of x_i at least as much as cross marginal returns do.

Let $\mathbf{\Sigma} = [\sigma_{ij}]$ be the square matrix of cross effects:

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}.$$

We use $\mathbf{\Sigma}$ as a short-hand for the simultaneous move $n-$player game with payoffs (3.1) and strategy spaces $\mathbb{R}_+$.

3.3.2 The Decomposition of Cross Effects

We decompose the matrix $\mathbf{\Sigma}$ additively into an idiosyncratic concavity component, a global (uniform) substitutability component, and a local complementarity component, in the following way.

Let $\gamma = -\min\{\underline{\sigma}, 0\} \geq 0$. If efforts are strategic substitutes for some pair of players, then $\underline{\sigma} < 0$ and $\gamma > 0$. Otherwise, $\underline{\sigma} \geq 0$ and $\gamma = 0$. As we shall see below, the parameter γ accounts for the global substitutability of efforts across all pairs of players.

Let $\lambda = \overline{\sigma} + \gamma \geq 0$. We assume that $\lambda > 0$. This is a generic property, as $\lambda = 0$ if and only if $\underline{\sigma} = \overline{\sigma}$, and this equality is not robust to small perturbations of the coefficients $\underline{\sigma}$ and $\overline{\sigma}$.[21] Let also $g_{ij} = (\sigma_{ij} + \gamma)/\lambda$, for $i \neq j$, and set $g_{ii} = 0$. By construction, $0 \leq g_{ij} \leq 1$. The parameter g_{ij} measures the relative complementarity in efforts from i's perspective within the pair (i, j) with respect to the benchmark value $-\gamma \leq 0$. This measure is expressed as a fraction of λ, that corresponds to the highest possible relative complementarity for all pairs.

The decomposition is depicted in Figure 3.1. This is just a centralization (β and λ are defined with respect to γ) followed by a normalization (the g_{ij}s are in $[0, 1]$) of the cross effects. The figure in the upper panel corresponds to σ_{ij} of either sign (the case $\sigma_{ij} \leq 0$, for all $i \neq j$ is similar) while the figure in the lower panel corresponds to $\sigma_{ij} \geq 0$, for all $i \neq j$ (recall that we assume $\sigma < 0$).

Consider the matrix $\mathbf{G} = [g_{ij}]$. This is a zero diagonal and non-negative square matrix. We interpret it as the adjacency matrix of a network g that reflects the pattern of existing payoff (relative) complementarities across all pairs of players. When $\sigma_{ij} = \sigma_{ji}$, for all i, j, the matrix $\mathbf{G}$ is symmetric and the network g is un-directed. When, moreover, cross effects only take two values, that is, $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$, for all $i \neq j$ with $\underline{\sigma} \leq 0$, then $\mathbf{G}$ is a symmetric $(0, 1)-$matrix and the network g is un-directed and un-weighted. In this case, g can be represented by a graph without loops nor multiple links,[22] where nodes stand for players and two nodes i and j are directly linked if and only if efforts are relative strategic complements across these two players, that is, $\sigma_{ij} = \sigma_{ji} = \overline{\sigma}$.

Finally, we write the (common) second order derivative in own efforts as

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[21] The set of parameters σ_{ij}s for which $\underline{\sigma} = \overline{\sigma}$ has Lebesgue measure zero in $\mathbb{R}^{n(n-1)}$.

[22] A loop is a single direct link starting at i and ending at i, that is, $g_{ii} = 1$. A direct link between i and j is multiple when $g_{ij} \in \{2, 3, ...\}$. The matrix $\mathbf{G}$ is zero diagonal, thus ruling out loops. Besides, $0 \leq g_{ij} \leq 1$ by construction, thus ruling out multiple links. It is important to note, though, that our results also hold for networks with loops and/or multiple links, but the economic intuitions are less appealing in this case.

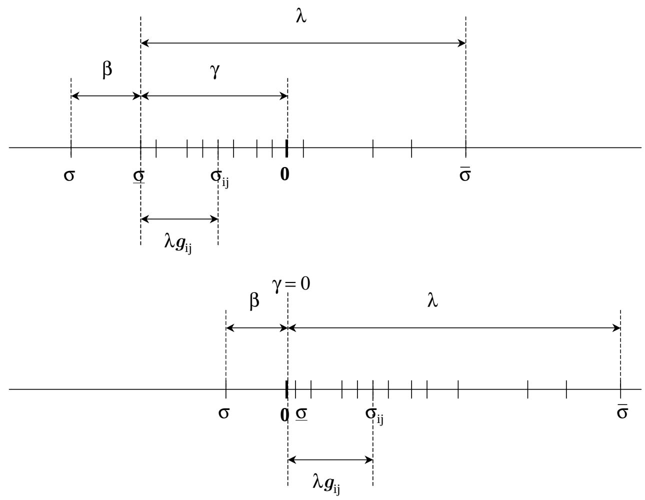

Figure 3.1 The decomposition of cross effects

$\partial^2 u_i / \partial x_i^2 = \sigma = -\beta - \gamma$, where $\beta > 0$. Given our assumption that $\sigma < \min\{\underline{\sigma}, 0\}$, this is without loss of generality.

Proposition 3.1 *Let* $\mathbf{I}$ *be the* $n-$*square identity matrix, and* $\mathbf{U}$ *the* $n-$*square matrix of ones. Then:*

$$\mathbf{\Sigma} = -\beta\mathbf{I} - \gamma\mathbf{U} + \lambda\mathbf{G}, \tag{3.2}$$

with $\beta > 0$*,* $\gamma \geq 0$ *and* $\lambda > 0$*.*

Proof. From the definition of β, γ, λ and $\mathbf{G}$. ■

The pattern of bilateral influences results from the combination of an idiosyncratic effect, a global interaction effect, and a local interdependence component.

The idiosyncratic effect reflects (part of) the concavity of the payoff function in own efforts. The global interaction effect is uniform across all players (matrix $\mathbf{U}$) and corresponds to a substitutability effect across all pairs of players with value $-\gamma \leq 0$. The local interaction component captures the (relative) strategic complementarity in efforts that varies across pairs of players, with maximal strength λ and population pattern reflected by $\mathbf{G}$. The whole heterogeneity in $\mathbf{\Sigma}$ is thus captured by $\mathbf{G}$.

Let $\mathbf{\Sigma}$ be a matrix of cross effects for some bilinear payoff functions (3.1). Following the decomposition of $\mathbf{\Sigma}$ in (3.2), we rewrite these payoffs as:

$$u_i(x_1, ..., x_n) = \alpha x_i - \frac{1}{2}(\beta - \gamma) x_i^2 - \gamma \sum_{j=1}^{n} x_i x_j + \lambda \sum_{j=1}^{n} g_{ij} x_i x_j, \ i = 1, \ldots, n. \tag{3.3}$$

Let $\lambda^* = \lambda/\beta$ denote the strength of local interactions relative to self-concavity.

3.3.3 The Bonacich Network Centrality Measure

Before turning to the equilibrium analysis, we define a network centrality measure due to Bonacich (1987) that proves useful for our analysis.

The $n-$square adjacency matrix $\mathbf{G}$ of a network g keeps track of the direct connections in this network. Indeed, two players i and j are directly connected in g if and only if $g_{ij} > 0$, in which case $0 \leq g_{ij} \leq 1$ measures the weight associated to this direct connection.

Let $\mathbf{G}^k$ be the kth power of this adjacency matrix with coefficients $g_{ij}^{[k]}$, where k is some non-zero integer. This matrix keeps track of the indirect connections in the network. Indeed, $g_{ij}^{[k]} \geq 0$ measures the number of paths of length $k \geq 1$ in g between i and j. In particular, $\mathbf{G}^0 = \mathbf{I}$, that is, $g_{ii}^{[0]} = 1$ and $g_{ij}^{[0]} = 0$ for all $i \neq j$.

Given a scalar $a \geq 0$ and a network g, we define the following matrix:

$$\mathbf{M}(g, a) = [\mathbf{I} - a\mathbf{G}]^{-1} = \sum_{k=0}^{+\infty} a^k \mathbf{G}^k.$$

Note that these expressions are all well-defined for low enough values of a.[23] The parameter a is a decay factor that scales down the relative weight of longer paths.

Provided the matrix $\mathbf{M}(g, a)$ is non-negative, its coefficients $m_{ij}(g, a) = \sum_{k=0}^{+\infty} a^k g_{ij}^{[k]}$ count the number of paths in g starting at i and ending at j, where paths of length k are weighted by a^k.

Let $\mathbf{1}$ be the $n-$dimensional vector of ones.

Definition 3.1 *Consider a network g with adjacency $n-$square matrix $\mathbf{G}$ and a scalar a such that $\mathbf{M}(g, a) = [\mathbf{I} - a\mathbf{G}]^{-1}$ is well-defined and non-negative. The*

[23] Take a smaller than the norm of the inverse of the largest eigenvalue of $\mathbf{G}$.

vector of Bonacich centralities of parameter a in g is:

$$\mathbf{b}(g,a) = [\mathbf{I}{-}a\mathbf{G}]^{-1} \cdot \mathbf{1}.$$

The Bonacich centrality of node i is $b_i(g,a) = \sum_{j=1}^{n} m_{ij}(g,a)$, and counts the $total$ number of paths in g starting from i.[24] It is the sum of all loops $m_{ii}(g,a)$ starting from i and ending at i, and all outer paths $\sum_{j\neq i} m_{ij}(g,a)$ that connect i to every other player $j \neq i$, that is:

$$b_i(g,a) = m_{ii}(g,a) + \sum_{j\neq i} m_{ij}(g,a).$$

By definition, $m_{ii}(g,a) \geq 1$, and thus $b_i(g,a) \geq 1$, with equality when $a = 0$.

3.4 Nash Equilibrium and Bonacich Centrality

3.4.1 Characterization and Uniqueness

We now characterize a Nash equilibrium of the game $\mathbf{\Sigma}$, where equilibrium strategies are proportional to Bonacich centralities in the network of local complementarities g derived from $\mathbf{\Sigma}$. We provide conditions such that this equilibrium is unique and interior.

Consider a matrix of cross effects $\mathbf{\Sigma}$ decomposed as in Proposition 3.1. From now on, we focus on symmetric matrices such that $\sigma_{ij} = \sigma_{ji}$, for all $j \neq i$. Then, the largest eigenvalue $\mu_1(g)$ of $\mathbf{G}$, sometimes referred to as the index of the network g, is well-defined. Also, $\mu_1(g) > 0$ as long as $\sigma_{ij} \neq 0$, for some $j \neq i$.[25]

For all vector $\mathbf{y} \in \mathbb{R}^n$, denote by $y = y_1 + ... + y_n$ the sum of its coordinates.

Theorem 3.1 *The matrix $[\beta \mathbf{I} - \lambda \mathbf{G}]^{-1}$ is well-defined and non-negative if and only if $\beta > \lambda \mu_1(g)$. Then, the game $\mathbf{\Sigma}$ has a unique Nash equilibrium $\mathbf{x}^*(\mathbf{\Sigma})$,*

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[24] In fact, $\mathbf{b}(g,a)$ is obtained from Bonacich (1987)'s measure by an affine transformation. Bonacich defines the following network centrality measure:

$$\mathbf{h}(g,a,b) = b\,[\mathbf{I} - a\mathbf{G}]^{-1}\,\mathbf{G} \cdot \mathbf{1}.$$

Therefore, $\mathbf{b}(g,a) = \mathbf{1}{+}a\mathbf{h}(g,a,1) = \mathbf{1} + \mathbf{k}(g,a)$, where $\mathbf{k}(g,a)$ is an early measure of network status introduced by Katz (1953). See also Guimerà $et\ al.$ (2001) for a related network centrality measures.

[25] Note that $\mathbf{G}$ is symmetric from the symmetry of $\mathbf{\Sigma}$. By the Perron-Frobenius theorem, the eigenvalues of a symmetric matrix $\mathbf{G}$ are all real numbers. Also, the matrix $\mathbf{G}$ with all zeros in the diagonal has a trace equal to zero. Therefore, $\mu_1(g) > 0$ whenever $\mathbf{G} \neq \mathbf{0}$.

which is interior and given by:

$$\mathbf{x}^*(\mathbf{\Sigma}) = \frac{\alpha}{\beta + \gamma b(g, \lambda^*)} \mathbf{b}(g, \lambda^*). \tag{3.4}$$

Proof. The necessary and sufficient condition for $[\beta \mathbf{I} - \lambda \mathbf{G}]^{-1}$ to be well-defined and non-negative derives from Theorem III*, page 601 in Debreu and Herstein (1953).

An interior Nash equilibrium in pure strategies $\mathbf{x}^* \in \mathbb{R}^n_+$ is such that $\partial u_i / \partial x_i(\mathbf{x}^*) = 0$ and $x_i^* > 0$, for all $i = 1, ..., n$. If such an equilibrium exists it then solves:

$$-\mathbf{\Sigma} \cdot \mathbf{x} = [\beta \mathbf{I} + \gamma \mathbf{U} - \lambda \mathbf{G}] \cdot \mathbf{x} = \alpha \mathbf{1}. \tag{3.5}$$

The matrix $\beta \mathbf{I} + \gamma \mathbf{U} - \lambda \mathbf{G}$ is generically non-singular, and (3.5) has a unique generic solution in $\mathbb{R}^n$, denoted by $\mathbf{x}^*$.[26] Using the fact that $\mathbf{U} \cdot \mathbf{x}^* = x^* \mathbf{1}$, (3.5) is equivalent to:

$$[\mathbf{I} - \lambda^* \mathbf{G}] \cdot \mathbf{x}^* = \frac{\alpha - \gamma x^*}{\beta} \mathbf{1} \Leftrightarrow \mathbf{x}^* = \frac{\alpha - \gamma x^*}{\beta} \mathbf{b}(g, \lambda^*),$$

and (3.4) follows by simple algebra. Given that $\alpha > 0$ and $b_i(g, \lambda^*) \geq 1$, for all i, it follows that $\mathbf{x}^*$ is interior.

We now establish uniqueness. First, the previous argument shows that $\mathbf{x}^*$ is a unique interior equilibrium. We deal with corner solutions.

Denote by $\beta(\mathbf{\Sigma}), \gamma(\mathbf{\Sigma}), \lambda(\mathbf{\Sigma})$ and $\mathbf{G}(\mathbf{\Sigma})$ the elements from the decomposition of $\mathbf{\Sigma}$ in Proposition 3.1. We omit the dependence in $\mathbf{\Sigma}$ when there is no confusion. For all matrix $\mathbf{Y}$, vector $\mathbf{y}$ and set $S \subset \{1, ..., n\}$, $\mathbf{Y}_S$ is the sub-matrix of $\mathbf{Y}$ with rows and columns in S, and $\mathbf{y}_S$ is the sub-vector of $\mathbf{y}$ with rows in S.

Let $S \subset \{1, ..., n\}$. Then $\gamma(\mathbf{\Sigma}_S) \leq \gamma(\mathbf{\Sigma}), \beta(\mathbf{\Sigma}_S) \geq \beta(\mathbf{\Sigma})$ and $\lambda(\mathbf{\Sigma}_S) \leq \lambda(\mathbf{\Sigma})$. Also, $\lambda \mathbf{G} = \mathbf{\Sigma} + \gamma(\mathbf{U} - \mathbf{I}) - \sigma \mathbf{I}$, and the coefficients in the S rows and columns of $\lambda \mathbf{G}$ are at least as high as the corresponding coefficients in $\lambda(\mathbf{\Sigma}_S)\mathbf{G}_S$. Theorem I*, page 600 in Debreu and Herstein (1953) then implies that $\mu_1(\lambda(\mathbf{\Sigma}_S)\mathbf{G}_S) \leq \mu_1(\lambda(\mathbf{\Sigma})\mathbf{G})$. Therefore, $\beta(\mathbf{\Sigma}) > \lambda(\mathbf{\Sigma})\mu_1(\mathbf{G})$ implies $\beta(\mathbf{\Sigma}_S) > \lambda(\mathbf{\Sigma}_S)\mu_1(\mathbf{G}_S)$.

Let $\mathbf{y}^*$ be a non-interior Nash equilibrium of the game $\mathbf{\Sigma}$. Let $S \subset \{1, ..., n\}$ such that $y_i^* = 0$ if and only if $i \in N \backslash S$. Therefore, $y_i^* > 0$, for all $i \in S$. Note

[26] The set of parameters β, γ, λ for which $\det(\beta \mathbf{I} + \gamma \mathbf{U} - \lambda \mathbf{G}) = 0$ has Lebesgue measure zero in $\mathbb{R}^3$.

that $S \neq \emptyset$, as $\partial u_i/\partial x_i(\mathbf{0}) = \alpha > 0$, and $\mathbf{0}$ cannot be a Nash equilibrium. Then,

$$-\mathbf{\Sigma}_S \cdot \mathbf{y}_S^* = [\beta \mathbf{I}_S + \gamma \mathbf{U}_S - \lambda \mathbf{G}_S] \cdot \mathbf{y}_S^* = \alpha \mathbf{1}_S.$$

Given that $\beta(\mathbf{\Sigma}_S) > \lambda(\mathbf{\Sigma}_S)\mu_1(\mathbf{G}_S)$, we have:

$$\mathbf{y}_S^* = \frac{\alpha - \gamma y_S^*}{\beta} \mathbf{b}(g_S, \lambda^*). \tag{3.6}$$

Now, every player $i \in N \backslash S$ is best-responding with $y_i^* = 0$, so that:

$$\frac{\partial u_i}{\partial x_i}(\mathbf{y}^*) = \alpha + \sum_{j \in S} \sigma_{ij} y_j^* = \alpha - \gamma y_S^* + \lambda \sum_{j \in S} g_{ij} y_j^* \leq 0, \text{ for all } i \in N \backslash S,$$

where the last equality uses the decomposition of $\mathbf{\Sigma}$. Using (3.6), we rewrite this inequality as:

$$(\alpha - \gamma y_S^*)(1 + \lambda^* \sum_{j \in S} g_{ij} b_j(g_S, \lambda^*)) \leq 0, \text{ equivalent to } \alpha - \gamma y_S^* \leq 0.$$

Using (3.6), we conclude that $y_i^* \leq 0$, for all $i \in S$, which is a contradiction. ■

When the matrix of cross effects $\mathbf{\Sigma}$ reduces to $\lambda \mathbf{G}$ (that is, $\beta = \gamma = 0$), the game $\mathbf{\Sigma}$ is supermodular and we have a multiplicity of Nash equilibria. If, instead, this matrix reduces to $-\beta \mathbf{I} - \gamma \mathbf{U}$ (that is, $\lambda = 0$), the equilibrium is generically unique. The condition in Theorem 3.1 requires that the parameter for own-concavity β to be high enough to counter the payoff complementarity captured by $\lambda \mu_1(g)$. Here, λ has to do with the level and $\mu_1(g)$ with the population-wide pattern of positive cross effects. Note that this condition does not impose any bound on the absolute values for these cross effects, but only on their relative magnitude.

The Bonacich-equilibrium expression (3.4) also implies that:

$$x_i^*(\mathbf{\Sigma}) = \frac{b_i(g, \lambda^*)}{b(g, \lambda^*)} x^*(\mathbf{\Sigma}).$$

Each player contributes to the aggregate equilibrium outcome in proportion to her network centrality. The dependence of individual outcomes on group behavior is referred to as *peer effects*. Here, this intragroup externality is not an average influence. It is *heterogeneous* across members, with a variance related to the Bonacich network centrality.

Remark 3.1 *Consider the general game* $(\mathbf{\Sigma}, \boldsymbol{\alpha})$*, where* $\alpha_i > 0$ *differs across* *players. Then, (3.4) still holds where* $\alpha\mathbf{b}(g,\lambda^*)$ *is replaced by the* weighted *Bonacich centrality measure* $\mathbf{b}_{\boldsymbol{\alpha}}(g,\lambda^*) = [\mathbf{I}-\lambda^*\mathbf{G}]^{-1}\cdot\boldsymbol{\alpha}$.

Remark 3.2 *In the proof of Theorem 3.1, the symmetry of* $\boldsymbol{\Sigma}$ *does not play any explicit role but for the existence and positivity of the largest eigenvalue of* $\mathbf{G}$. *Therefore, the Bonacich-Nash linkage holds for any asymmetric matrix* $\boldsymbol{\Sigma}$ *of cross effects with a well-defined positive spectral radius.*[27]

When g is an un-directed and un-weighted network, the condition in Theorem 3.1 can be expressed directly in terms of the number of nodes and links in g, thus dispensing with computing the network index.

Let $g = \sum_{i,j} g_{ij}$ be the sum of all direct links in g. When $\mathbf{G}$ is a $(0,1)-$matrix, this is twice the number of direct links in the un-directed and un-weighted network g.

Corollary 3.1 *Suppose that* $\sigma_{ij} \in \{\underline{\sigma},\overline{\sigma}\}$, *for all* $i \neq j$ *with* $\underline{\sigma} \leq 0$, *and that the network* g *induced by the decomposition of* $\boldsymbol{\Sigma}$ *in (3.2) is connected. If* $\beta > \lambda\sqrt{g+n-1}$, *the only Nash equilibrium of the game* $\boldsymbol{\Sigma}$ *is given by (3.4).*

Proof. From the upper bound on the index of a connected graph in Theorem 1.5, page 5 in Cvetković and Rowlinson (1990). ■

3.4.2 Comparative Statics

The previous results relate individual equilibrium outcomes to the Bonacich centrality in the network g of local complementarities. The next result establishes a positive relationship between the aggregate equilibrium outcome and the pattern of local complementarities.

For any two matrices $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}'$, we write $\boldsymbol{\Sigma}' \gneq \boldsymbol{\Sigma}$ if $\sigma'_{ij} \geq \sigma_{ij}$, for all i, j, with at least one strict inequality.

Theorem 3.2 *Let* $\boldsymbol{\Sigma}$ *and* $\boldsymbol{\Sigma}'$ *symmetric such that* $\boldsymbol{\Sigma}' \gneq \boldsymbol{\Sigma}$. *If* $\beta > \lambda\mu_1(g)$ *and* $\beta' > \lambda'\mu_1(g')$ *for the decompositions (3.2) of* $\boldsymbol{\Sigma}$ *and* $\boldsymbol{\Sigma}'$, *then* $x^*(\boldsymbol{\Sigma}') > x^*(\boldsymbol{\Sigma})$.

[27] Debreu and Hernstein (1953) provide some general conditions for a matrix to have a well-defined and positive spectral radius.

Proof. When $\mathbf{\Sigma}' > \mathbf{\Sigma}$, we write $\mathbf{\Sigma}' = \mathbf{\Sigma} + \lambda \mathbf{D}$, with $d_{ij} \geq 0$ with at least one strict inequality, and λ given by the decomposition (3.2) of $\mathbf{\Sigma}$. If $\beta > \lambda \mu_1(\mathbf{G})$ and $\beta' > \lambda' \mu_1(\mathbf{G}')$, Theorem 3.1 holds, so that $-\mathbf{\Sigma} \cdot \mathbf{x}^*(\mathbf{\Sigma}) = -\mathbf{\Sigma}' \cdot \mathbf{x}^*(\mathbf{\Sigma}') = \alpha \mathbf{1}$, and $\mathbf{x}^*(\mathbf{\Sigma}), \mathbf{x}^*(\mathbf{\Sigma}') > \mathbf{0}$. We compute $-\mathbf{x}^{*t}(\mathbf{\Sigma}') \cdot \mathbf{\Sigma} \cdot \mathbf{x}^*(\mathbf{\Sigma})$ in two different ways. First, $-\mathbf{x}^{*t}(\mathbf{\Sigma}') \cdot \mathbf{\Sigma} \cdot \mathbf{x}^*(\mathbf{\Sigma}) = \alpha \mathbf{x}^{*t}(\mathbf{\Sigma}') \cdot \mathbf{1} = \alpha x^*(\mathbf{\Sigma}')$. Second, using the symmetry of $\mathbf{\Sigma}'$, we have:

$$\begin{aligned} -\mathbf{x}^{*t}(\mathbf{\Sigma}') \cdot \mathbf{\Sigma} \cdot \mathbf{x}^*(\mathbf{\Sigma}) &= -\mathbf{x}^{*t}(\mathbf{\Sigma}') \cdot \mathbf{\Sigma}' \cdot \mathbf{x}^*(\mathbf{\Sigma}) + \lambda \mathbf{x}^{*t}(\mathbf{\Sigma}') \cdot \mathbf{D} \cdot \mathbf{x}^*(\mathbf{\Sigma}) \\ &= \alpha x^*(\mathbf{\Sigma}) + \lambda \mathbf{x}^{*t}(\mathbf{\Sigma}') \cdot \mathbf{D} \cdot \mathbf{x}^*(\mathbf{\Sigma}). \end{aligned}$$

Using the fact that $\alpha > 0$, we conclude that $x^*(\mathbf{\Sigma}') > x^*(\mathbf{\Sigma})$. ■

In words, the denser the pattern of local complementarities, the higher the aggregate outcome, as players can rip more complementarities in g' than in g. The geometric intuition for this result is clear. Recall that $b(g, \lambda^*)$ counts the total number of weighted paths in g. This is obviously an increasing function in g (for the inclusion ordering), as more links imply more such paths.

Remark 3.3 *When the decompositions (3.2) of* $\mathbf{\Sigma}$ *and* $\mathbf{\Sigma}'$ *are such that* $(\alpha, \beta, \gamma, \lambda) = (\alpha', \beta', \gamma', \lambda')$ *and* $\mathbf{G}' \gneq \mathbf{G}$*, then* $\beta' > \lambda' \mu_1(g')$ *implies that* $\beta > \lambda \mu_1(g)$.[28]

3.5 A Network-Based Policy

3.5.1 Finding the Key Player

In our model, individual equilibrium behavior is tightly rooted in the network structure through (3.4). The removal of a player from the population, holding the pattern of social interactions among the other players fixed, alters the whole distribution of outcomes.

We provide a simple geometric criterion to identify the optimal target in the population when the planner wishes to reduce (or to increase) optimally the aggregate group outcome.[29]

In what follows, we suppose that $\mathbf{\Sigma}$ is symmetric with $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$ for all $i \neq j$, and $\underline{\sigma} \leq 0$. In this case, the decomposition of $\mathbf{\Sigma}$ in (3.2) yields a

[28] From the monotonicity of the largest eigenvalue with the coefficients of the matrix in Theorem I*, page 600 in Debreu and Hernstein (1953).

[29] Bollobás and Riordan (2003) contains a mathematical analysis of the relative network disruption effects of a topological attack versus random failures in large networks. See also Albert $et\ al.$ (2000) for a numerical analysis for the case of the World Wide Web.

$(0,1)-$adjacency matrix $\mathbf{G}$ and an un-weighted and un-directed network g, with its corresponding graph representation.

Let's eliminate some player i from the current population. Suppose that for each possible value $v \in \{\underline{\sigma}, \overline{\sigma}\}$ for the cross effects, there exists at least two different pairs of players (i,j) and (i',j'), differing two-by-two, such that $\sigma_{ij} = \sigma_{i'j'} = v$. This is a mild requirement that guarantees that the values of β, γ and λ in the decomposition (3.2) of $\mathbf{\Sigma}$ do not change for any such single player removal. The adjacency matrix becomes $\mathbf{G}_{-i}$, obtained from $\mathbf{G}$ by setting to zero all of its ith row and column coefficients. The resulting network is g_{-i}.[30]

Similarly, denote by $\mathbf{\Sigma}_{-i}$ the matrix that results from removing the ith row and column from $\mathbf{\Sigma}$.

The planner's problem is to reduce $x^*(\mathbf{\Sigma})$ optimally by picking the adequate player from the population.[31] Formally, she solves $\max\{x^*(\mathbf{\Sigma}) - x^*(\mathbf{\Sigma}_{-i}) \mid i = 1, ..., n\}$, equivalent to:

$$\min\{x^*(\mathbf{\Sigma}_{-i}) \mid i = 1, ..., n\} \tag{3.7}$$

This is a finite optimization problem, that admits at least one solution.

Let i^* be a solution to (3.7). We call player i^* the *key player*. Removing i^* from the initial network g has the highest overall impact on the aggregate equilibrium level. We provide a simple and direct geometric characterization of the key player.

Definition 3.2 *Consider a network g with adjacency $n-$square matrix $\mathbf{G}$ and a scalar a such that $\mathbf{M}(g,a) = [\mathbf{I} - a\mathbf{G}]^{-1}$ is well-defined and non-negative. The inter-centrality of player i of parameter a in g is:*

$$c_i(g,a) = \frac{b_i^2(g,a)}{m_{ii}(g,a)}.$$

The Bonacich centrality of player i counts the number of paths in g stemming from i, while the inter-centrality computes the total number of such paths that hit i at some time. It is the sum of i's Bonacich centrality and i's contribution to

[30] If the primitive of our model is the bilinear expression for the payoffs in (3.1), the key player analysis applies to matrix of cross effects $\mathbf{\Sigma}$ symmetric and for which the σ_{ij}s only take two possible values, that is, $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$ with $\underline{\sigma} \leq 0$. In this case, $\mathbf{G}$ is a $(0,1)-$matrix. If, instead, the primitive of our model is the expression for the payoffs in (3.3), the key player analysis carries over to any symmetric adjacency matrix $\mathbf{G}$ with $0 \leq g_{ij} \leq 1$.

[31] Corollary 3.1 below considers the symmetric case where the planner wishes to increase $x^*(\mathbf{\Sigma})$ optimally.

every other player Bonacich centrality. Holding $b_i(g,\lambda^*)$ fixed, $c_i(g,\lambda^*)$ decreases with the proportion m_{ii}/b_i of i's Bonacich centrality due to self-loops.

Theorem 3.3 *If $\beta > \lambda\mu_1(g)$, the key player i^* that solves $\min\{x^*(\mathbf{\Sigma}_{-i}) \mid i = 1,...,n\}$ is the one with the highest inter-centrality measure of parameter λ^* in g, that is, $c_{i^*}(g,\lambda^*) \geq c_i(g,\lambda^*)$, for all $i = 1,...,n$.*[32]

Proof. We first prove a useful result. ■

Lemma 3.1 *Let $a > 0$ such that $\mathbf{M}(g,a) = [\mathbf{I}-a\mathbf{G}]^{-1}$ is well-defined and non-negative. Then, $m_{ij}(g,a)m_{ik}(g,a) = m_{ii}(g,a)\left[m_{jk}(g,a) - m_{jk}(g_{-i},a)\right]$.*

Proof. First note that the symmetry of $\mathbf{\Sigma}$ implies that $m_{jk}(g,a) = m_{kj}(g,a)$, for all j,k and g. We have:

$$\begin{aligned}
m_{ii}(g,a)\left[m_{jk}(g,a) - m_{jk}(g_{-i},a)\right] &= \sum_{p=1}^{+\infty} a^p \sum_{\substack{r+s=p \\ r\geq 0, s\geq 1}} g_{ii}^{[r]}\left(g_{jk}^{[s]} - g_{j(i^0)k}^{[s]}\right) \\
&= \sum_{p=1}^{+\infty} a^p \sum_{\substack{r+s=p \\ r\geq 0, s\geq 2}} g_{ii}^{[r]} g_{j(i)k}^{[s]} \\
&= \sum_{p=1}^{+\infty} a^p \sum_{\substack{r'+s'=p \\ r'\geq 1, s'\geq 1}} g_{ji}^{[r']} g_{ik}^{[s']} \\
&= m_{ji}(g,a)m_{ik}(g,a),
\end{aligned}$$

where $g_{j(i^0)k}^{[s]}$ (resp. $g_{j(i)k}^{[s]}$) is the compound weight of length$-s$ paths from j to k not containing i (resp. containing i), and $g_{ii}^{[0]} = 1$. ■

First, note that $\mu_1(g) \geq \mu_1(g_{-i})$. Therefore, if $\mathbf{M}(g,\lambda^*)$ is well-defined and non-negative (as implied by the condition in Theorem 3.1), so is $\mathbf{M}(g_{-i},\lambda^*)$, for all $i = 1,...,n$.

With $\alpha > 0$, $x^*(\mathbf{\Sigma}_{-i})$ increases in $b(g_{-i},\lambda^*)$, and (3.7) is equivalent to $\min\{b(g_{-i},\lambda^*) \mid i = 1,...,n\}$. Define:

$$b_{ji}(g,\lambda^*) = b_j(g,\lambda^*) - b_j(g_{-i},\lambda^*), \text{ for all } j \neq i.$$

[32] Note that there may be more than just one key player.

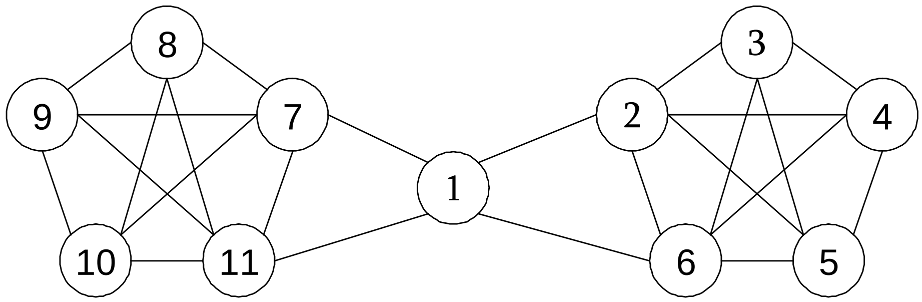


Figure 3.2 A network with eleven players, player 1 being a bridge between two cliques of size 5.

This is the contribution of i to j's Bonacich centrality in g. Summing over all $j \neq i$, and adding $b_i(g, \lambda^*)$ to both sides gives:

$$b(g, \lambda^*) - b(g_{-i}, \lambda^*) = b_i(g, \lambda^*) + \sum_{j \neq i} b_{ji}(g, \lambda^*) \equiv d_i(g, \lambda^*).$$

The solution of (3.7) is i^* such that $d_{i^*}(g, \lambda^*) \geq d_i(g, \lambda^*)$, for all $i = 1, ..., n$. We have:

$$\begin{aligned} d_i(g, \lambda^*) &= b_i(g, \lambda^*) + \sum_{j \neq i} \left[b_j(g, \lambda^*) - b_j(g_{-i}, \lambda^*) \right] \\ &= b_i(g, \lambda^*) + \sum_{j \neq i} \sum_{k=1}^{n} \left[m_{jk}(g, \lambda^*) - m_{jk}(g_{-i}, \lambda^*) \right]. \end{aligned}$$

Using Lemma 3.1, this becomes:

$$\begin{aligned} d_i(g, \lambda^*) &= b_i(g, \lambda^*) + \sum_{j \neq i} \sum_{k=1}^{n} \frac{m_{ij}(g, a) m_{ik}(g, a)}{m_{ii}(g, a)} \\ &= b_i(g, \lambda^*) \left[1 + \sum_{j \neq i} \frac{m_{ij}(g, a)}{m_{ii}(g, a)} \right] \\ &= \frac{b_i^2(g, \lambda^*)}{m_{ii}(g, \lambda^*)}. \end{aligned}$$

∎

Example 3.1 *Consider the network g in Figure 3.2 with eleven players.*

There are three different locations in this network: player 1, players 2, 6, 7 and 11, and players 3, 4, 5, 8, 9, and 10. Type-1 and type-3 players have four direct links, while type -2 players have five. Player 1 bridges together two fully intra-connected communities of five players each. By removing player 1, the network is maximally disrupted. By removing a type-2 player, we get a network with the lowest total number of links.

Table 1 computes the Bonacich and inter-centrality measures for different values of the decay factor a. A superscript star identifies the highest column value.[33]

a	0.1		0.2	
Player Type	b_i	c_i	b_i	c_i
1	1.75	2.92	8.33	41.67*
2	1.88*	3.28*	9.17*	40.33
3	1.72	2.79	7.78	32.67

Table 1

Type-2 players have the highest Bonacich centrality. They have the highest number of direct connections and are directly connected to the bridge player 1, who gives them access to a wide span of indirect connections. When a is low, they are also the key players. When a is high, though, the most active players are not the key players anymore. Now, indirect effects matter, and eliminating player 1 has the highest joint direct and indirect effect on aggregate outcome.

Corollary 3.2 *If $\beta > \lambda\mu_1(\mathbf{G})$, the key player i^* that solves $\max\{x^*(\mathbf{\Sigma}_{-i}) \mid i = 1, ..., n\}$ is the one with the lowest inter-centrality measure of parameter λ^* in g, that is, $c_{i^*}(g, \lambda^*) \leq c_i(g, \lambda^*)$, for all $i = 1, ..., n$.*

Remark 3.4 *When $\mathbf{\Sigma}$ is not symmetric, Theorem 3.3 and Corollary 3.1 still hold where the intercentrality measure is now given by:*

$$\widetilde{c}_i(g, a) = b_i(g, a)(\sum_{j=1}^{n} m_{ji}(g, a))/m_{ii}(g, a).$$

33 Here, the highest value for a compatible with our definition of centrality measures is $\frac{2}{3+\sqrt{41}} \simeq 0.213$.

3.6 Applications

In this section, we propose three different applications of the previous results

3.6.1 Crime Networks

There are n criminals, each exerting a level of crime x_i that results from a trade off between the costs and benefits of criminal activities. The expected utility of criminal i is:

$$u_i(\mathbf{x},r) = y_i(\mathbf{x}) - p_i(\mathbf{x}, r)f, \tag{3.8}$$

where $y_i(\mathbf{x})$ are the proceeds, $p_i(\mathbf{x}, r)$ the apprehension probability, and f the corresponding fine. Following Calvó-Armengol and Zenou (2004), the cost of committing crime $p_i(\mathbf{x}, r)f$ increases with x_i, as the apprehension probability increases with one's involvement in crime, hitherto, with one's exposure to deterrence.

Also, and consistent with standard criminology theories, criminals improve illegal practice through interactions with their direct criminal mates.[34] Formally, criminals are connected through a friendship network r, where $r_{ij} = 1$ when i and j are directly related to each other. For instance, let:

$$\begin{cases} y_i\left(\mathbf{x}\right) = x_i\left[1 - \eta \sum_{j=1}^n x_j\right] \\ p_i(\mathbf{x},r) = p_0 x_i \left[1 - \nu \sum_{j=1}^n r_{ij} x_j\right] \end{cases}.$$

The expected utility then becomes:

$$u_i(\mathbf{x},r) = (1-\pi)x_i - \eta \sum_{j=1}^n x_i x_j + \pi\nu \sum_{j=1}^n r_{ij} x_i x_j, \tag{3.9}$$

where $\pi = p_0 f$ is the marginal expected punishment cost for an isolated criminal, and $-\eta < 0$ captures a congestion in the crime market. The utility function (3.9) coincides with the expression in (3.3) with $\alpha = 1 - \pi$, $\beta = \gamma = \eta$, $\lambda = \pi\nu$ and $g = r$. When $\pi\nu\mu_1(r) < \eta$, the unique Nash equilibrium of the crime game with payoffs (3.9) is:

$$\mathbf{x}^* = \frac{1-\pi}{\eta} \frac{1}{1 + b(r, \pi\frac{\nu}{\eta})} \mathbf{b}(r, \pi\frac{\nu}{\eta}).$$

Here, the key player policy in Theorem 3.3 has both a direct and an indirect effect on crime reduction. On the contrary, a standard deterrence policy (an increase in π) has a positive direct impact on crime reduction, but a negative indirect effect,

34 See, e.g., Sutherland (1947).

as criminals now counter the extra deterrence they face by strengthening their network interactions.

3.6.2 R&D Collaboration Networks

Consider a standard Cournot game with n (ex ante) identical firms, each of them choosing the quantity q_i. As in Goyal and Moraga-González (2001) and Goyal and Joshi (2003), firms can form bilateral agreements to jointly invest in cost-reducing R&D activities. We set $c_{ij} = 1$ when firms i and j set up a collaboration link. Firm i's marginal cost is $\lambda_0 - \lambda \sum_{j \neq i} r_{ij} q_j$. Here, $\lambda_0 > 0$, represents the marginal cost of an isolated firm, while $\lambda > 0$ is the cost reduction induced by each link it forms. With a linear inverse demand, the profit function of firm i is:

$$\begin{aligned} u_i(\mathbf{x}, r) &= \left[\phi - \sum_{j=1}^{n} q_j\right] q_i - \left[\lambda_0 - \lambda \sum_{j \neq i} r_{ij} q_j\right] q_i \\ &= (\phi - \lambda_0)\, q_i - \sum_{j=1}^{n} q_i q_j + \lambda \sum_{j \neq i} r_{ij} q_i q_j. \end{aligned} \tag{3.10}$$

Again, this objective function is a particular case of (3.3), where $\alpha = \phi - \lambda_0 > 0$, $\beta = \gamma = 1$ and $g = r$. Using Corollary 3.1, we conclude that the Cournot game with payoffs (3.10) has a unique Nash equilibrium in pure strategies:

$$\mathbf{q}^* = \frac{\phi - \lambda_0}{1 + b(r, \lambda)} \mathbf{b}(r, \lambda),$$

when $1 > \lambda \sqrt{g + n - 1}$. In particular, Theorem 3.2 implies that the overall industry output increases when the network of collaboration links expands, irrespective of this network geometry and the number of additional links. For the case of a linear inverse demand curve, this generalizes the findings in Goyal and Moraga-González (2001) and Goyal and Joshi (2003), where monotonicity of industry output is established for the case of regular collaboration networks, where each firm forms the same number of bilateral agreements. For such regular networks, links are added as multiples of n, as all firms' connections are increased simultaneously.

3.6.3 Conformism and Social Norms

There are n players whose well-being depends on their behavior compared to that of their reference group. More precisely, each player chooses an action $x_i \geq 0$ and loses utility when failing to conform to the social norm of her reference group, equal to the average action of its members. This framework encompasses a variety of issues where conformism is the driving force for individual behavior.[35] Here, contrarily to previous models, we allow for the reference group, and its induced social norm, to vary with the friendship and community ties of each player.

Formally, when i and j are friends we set $f_{ij} = 1$. Let also $f_{ii} = 0$ for all i. This collection of links constitutes a network f. Player i has $f_i = \sum_{j=1}^{n} f_{ij}$ direct links in f, whose average action is $\overline{x}_i = \sum_{j=1}^{n} f_{ij} x_j / f_i$. This is the social norm of player i. We assume that $f_i > 0$, for all i.

Consider the following utility function, with $\xi, \alpha, \theta, d > 0$:

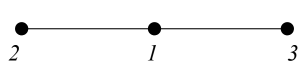
$$u_i(\mathbf{x}, f) = \xi + \alpha x_i - \theta x_i^2 - d(x_i - \overline{x}_i)^2. \tag{3.11}$$

In words, non-conformist behavior entails a quadratic utility loss. When f is the complete network with self-loops, this is equation (5) in Akerlof (1997), page 1009. We have:

$$\frac{\partial^2 u_i(\mathbf{x}, f)}{\partial x_i \partial x_j} = \begin{cases} -2(\theta + d), \text{ when } i = j \\ 0, \text{ when } i \neq j \text{ and } f_{ij} = 0 \\ 2d/f_i > 0, \text{ when } i \neq j \text{ and } f_{ij} = 1 \end{cases} .$$

This utility function (3.11) thus coincides with (3.3) with $\beta = 2(\theta + d)$, $\gamma = 0$, $\lambda = 2d$ and $g_{ij} = f_{ij}/f_i$. Note that g is a row-normalization of the initial friendship network f, as illustrated in the following example, where $\mathbf{F}$ and $\mathbf{G}$ are the adjacency matrices of, respectively, f and g.

Example 3.2 *Consider the following friendship network f:*



Then,

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad \mathbf{G} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

[35] Different issues have been explored in the literature. For example, and to name a few, (i) peer pressures and partnerships, when individuals are penalized for working less than the group norm (Kandel and Lazear 1992), (ii) religion, when the benefits of praying increase with the number of participants (Iannaccone 1992, Berman 2000), (iii) social status and social distance, when deviations from the social norm imply a loss of reputation and status (Akerlof 1980 and 1997 and Bernheim 1994, among others).

Observe that $\mathbf{G}$ is a stochastic matrix, that is $g_{ij} \geq 0$ and $\sum_j g_{ij} = 1$. This implies that $\mu_1(\mathbf{G}) = 1$ and $\mathbf{G}^k$ is also a stochastic matrix, that is $g_{ij}^{[k]} \geq 0$ and $\sum_j g_{ij}^{[k]} = 1$, $\forall k$. Applying Theorem 3.1, it is easy to see that this conformity game with payoffs (3.11) has a unique Nash equilibrium in pure strategies and, whatever the structure of the network, this equilibrium is always symmetric, that is $x^* = x_1^* = ... = x_n^*$ and $\overline{x}^* = \overline{x}_1^* = ... = \overline{x}_n^*$, and is given by:[36]

$$x^* = \overline{x}^* = \frac{\alpha}{2\theta} \tag{3.12}$$

In particular, the equilibrium Bonacich-centrality measure is the same for all individuals and is equal to:

$$b_1(g, \frac{d}{\theta + d}) = ... = b_n(g, \frac{d}{\theta + d}) = \frac{\theta + d}{\theta}$$

The equilibrium value (3.12) is exactly the value found by Akerlof (1997), page 1010. So, even if individuals are ex ante heterogeneous because of their location in the network, in a conformist equilibrium where each individual would like to conform as much as possible to the norm of her reference group, all individuals will exert the same effort level. In other words, the distribution of population does not matter in equilibrium even if it matters ex ante. The only relevant statistics is the average.

3.7 Discussion and Extensions

We discuss a number of possible extensions of this work.

First, our analysis is restricted to linear-quadratic utility functions that capture linear externalities in players' actions. First order conditions for interior equilibria then produce a system of linear equations that leads to the Bonacich-Nash linkage. Suppose, instead, that externalities are non-linear, and that utility

[36] To prove this result, one has to calculate the Bonacich vector since it is the only source of heterogeneity between players. In a conformist game, we have:

$$\begin{aligned} b_i(g,a) &= m_{ii}(g,a) + \sum_{j \neq i} m_{ij}(g,a) \\ &= a\sum_{j=1}^{n} g_{ij} + ... + a^k \sum_{j=1}^{n} g_{ij}^{[k]} + ... \\ &= \sum_{j=1}^{+\infty} a^k = \frac{1}{1-a} \end{aligned}$$

functions $\mathbf{u}$ are C^2. Let $\mathbf{\Sigma}(\mathbf{x}^*)$ be the (symmetric) Jacobian of $\nabla\mathbf{u}$ evaluated at an interior Nash equilibrium $\mathbf{x}^* > \mathbf{0}$. Decompose $\mathbf{\Sigma}(\mathbf{x}^*)$ as in (3.2). Then, by a simple continuity argument, the first order approximation of $\mathbf{x}^*$ corresponds to the Bonacich centrality vector for this decomposition.

Second, Theorem 3.3 characterizes the key player when the planner's objective function is the aggregate group outcome $x^*(\mathbf{\Sigma})$. Suppose, instead, that the planner's objective is to maximize the welfare function $W^*(\mathbf{\Sigma}) = \sum_{i=1}^n u_i(\mathbf{x}^*(\mathbf{\Sigma}))$. Simple algebra gives $2W^*(\mathbf{\Sigma}) = (\beta + \gamma)\sum_{i=1}^n x_i^*(\mathbf{\Sigma})^2$ and, when $\gamma = 0$, this becomes $2\beta W^*(\mathbf{\Sigma}) = \alpha^2 \sum_{i=1}^n b_i(g, \lambda^*)^2$. A geometric characterization of the key player is also possible in this case. The building block is provided by Lemma 3.1, which characterizes all the path changes in a network when a node is removed. The network index obtained in this case is less intuitive than the inter-centrality measure, as it now accounts for individual direct contributions to the aggregate outcome, indirect contributions, and the variance of the latter.

Third, Theorem 3.3 characterizes geometrically single player targets, but the inter-centrality measure can be generalized to a group index.[37] Note that the group target selection problem is not amenable to a sequential key player problem. For instance, the key group of size 2 in Example 1 when $a = 0.2$ is $\{2, 7\}$, rather than the sequential optimal pair $\{1, 2\}$.

Fourth, beyond the optimal player removal problem, the network policy analysis can also accommodate more general optimal targeted tax or subsidy problems. Consider a population of $n + 1$ agents $i = 0, 1, ..., n$ and a matrix of cross effects $\mathbf{\Sigma}$ with associated network g in (3.2). Suppose that the planner holds the outcome of player $i = 0$ to some fixed exogenous value $s \in \mathbb{R}$. The case $s > 0$ (resp. $s < 0$) is a subsidy (resp. tax), while $s = 0$ corresponds to the key player problem solved above. Players $i = 1, ..., n$ then play an $n-$player game with interior Bonacich-Nash equilibrium $\mathbf{x}^*_{-0}(\mathbf{\Sigma}_{-0}, s)$. Denote by g_0 the $n-$dimensional column vector with coordinates $g_{01}, ..., g_{0n}$ that keeps track of player 0's direct contacts in g, let $\alpha^* = \alpha/\beta$ and $\gamma^* = \gamma/\beta$. Then, the total equilibrium population outcome is $s + x^*_{-0}(\mathbf{\Sigma}_{-0}, s)$, where:

$$x^*_{-0}(\mathbf{\Sigma}_{-0}, s) = \frac{1}{1 + \gamma^* b(g_{-0}, \lambda^*)}[(\alpha^* - s)b(g_{-0}, \lambda^*) + \lambda^* s b_{g_0}(g_{-0}, \lambda^*)].$$

Here, $b_{g_0}(g_{-0}, \lambda^*)$ is the aggregate weighted Bonacich centrality defined in Remark 1. Given an objective function related to the total population output $s + x^*_{-0}(\mathbf{\Sigma}_{-0}, s)$, and a set of constraints, the planner's problem is to fix optimally the value of s and the target identity i. Holding s constant, the choice of the optimal target is a simple finite optimization problem. In particular, when $s = 0$

[37] See Ballester *et al.* (2005b).

and the planner wants to minimize the overall output, the solution to this problem is $i^* \in \arg\max\{c_i(g, \lambda^*) \mid i = 1, ..., n\}$.

Finally, the analysis so far deals with a fixed network. When $\mathbf{G}$ is a $(0, 1)-$matrix, we can easily endogenize the network with a two-stage game the following way. In the first stage, players decide simultaneously to stay in the network or to drop out of it (and get their outside option). This is modelled as a simple binary decision. In the second stage, the players that stay play the network game on the resulting network. Uniqueness of the second-stage Nash equilibrium and its closed-form expression crucially simplify the analysis of this two-stage game. See, e.g, Ballester *et al.* (2005b) and Calvó-Armengol and Jackson (2004) for analysis along this vein.

Chapter 4
Optimal Targets in Peer Networks

with
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4.1 Introduction

The study of *peer effects*, or the dependence of individual outcomes on group behavior, has been a subject of research and debate for decades. This field is not only important from the point of view of explaining the behavior of the economic actors in a society, but also to design effective policies in order to change the outcome of such interactions.

Although the traditional approach relies on a homogeneous dependence on group behavior, some authors have also made efforts to go beyond this assumption. Ballester *et al.* (2005a) propose a model of social interactions where this dependence is not homogeneous across individuals. Calvó-Armengol and Jackson (2004) explain how employment outcomes vary across otherwise identical agents with their informational location in the social setting.

The manipulation of the framework of interactions among individuals may be justified in some specific situations that would permit a planner to increase the welfare of the society. For instance, when agents are delinquents that acquire information through a network[38] of relationships, it is important to identify criminals that should be neutralized in order to minimize the harm done by crime in a particular town. A specific model of delinquent behavior where this type of policy emerges is Calvó-Armengol and Zenou (2004).

There are two basic ways of designing a policy in a peer network. On the one hand, a policy can be scattered, and all agents are affected with equal footing. On the other hand, a policy might discriminate across network types and implement a differentiated policy for each type. This is precisely, the kind of policy we are considering in this paper: one which targets specific agents in the social setting depending on their position in the network. Some other works have also considered the network architecture in order to assess the effects of network design, like Albert *et al.* (2000) and Bollobás and Riordan (2003).

[38] Jackson (2005) offers a complete survey of network formation in economic environments.

Particularly, given a game with interactions of any kind between pairs of agents, we study, among other related problems, the *key group problem*[39]. This problem consists of choosing optimally the group whose removal from the game disrupts optimally the aggregate activity. If it turns out that the pattern of interactions is homogeneous across agents, the optimal policy can be naturally summarized in terms of the number of agents that should be targeted (eliminated). On the contrary, if the architecture of the social network is not symmetric, the policy must be described for each of the agents in the optimal target set.

Any policy tool should always be evaluated in terms of its applicability to real-life situations. For this reason, we first provide an analysis of the complexity of the key group problem. Complexity analysis is becoming a useful tool in modeling bounded rationality, in studying the applicability of theoretical solutions, and, generally, in describing the potential difficulties that one may encounter in the implementation of specific policies or mechanisms. Some examples that explain the relevant role of computational complexity in games are Ballester (2004) and Ben-Porath (1990).

We review notation in section 4.2. In section 4.3, we describe the game and we summarize the characterization of the equilibrium, from Ballester *et al.* (2005a). This characterization relates the outcomes of the game with a network centrality measure proposed by Bonacich (1987). The re-interpretation of the game in terms of this graph-theoretical, path-based measure, allows us to provide useful analytical results throughout the paper. In section 4.4, we propose the key group policy. In section 4.5, we provide useful tools for solving the key group problem. Particularly, we make use of the existing tools for maximizing submodular functions. In section 4.6, we focus on an extended game with voluntary participation and we describe some of the properties of the key group policy, which turns out to be different on this framework. In section 4.7, we describe some related problems and show that they can also be dealt with by the use of approximation heuristics. We conclude and leave some open questions in Section 4.8.

4.2 Definitions, Notation and Preliminary Results

The primitive set of our model is a set of agents $N = \{1, 2, ..., n\}$.

4.2.1 Matrix and Vector Notation

Matrices and vectors will be denoted in bold letters, like $\mathbf{A}$ and $\mathbf{x}$, respectively. When necessary, we use $\mathbf{A}_{l \times m}$ to specify that $\mathbf{A}$ has l rows and m columns. Generally, $l = m = n$. The *entries* of $\mathbf{A}$ and $\mathbf{x}$ are written like a_{ij} and x_i.

[39] Borgatti (2003) proposes the problem of the key node, where a node (or a set of nodes) is removed from a graph in order to optimize a particular combinatorial function.

The *transpose* of $\mathbf{A}$ and $\mathbf{x}$ are $\mathbf{A}^T$ and $\mathbf{x}^T$. Matrix *multiplication* is written like $\mathbf{A} \cdot \mathbf{x}$, $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{x}^T \cdot \mathbf{y}$. The matrix $\mathbf{A}^k$ is the k-th *power* of $\mathbf{A}$, and its (i, j)-entry is written $a_{ij}^{[k]}$.

Given a set $S \subseteq N$, $\mathbf{A}_S$ and $\mathbf{x}_S$ are the *restrictions* of $\mathbf{A}$ and $\mathbf{x}$ to the set S. Also, $\mathbf{A}_{-S} \equiv \mathbf{A}_{N \setminus S}$ and $\mathbf{x}_{-S} \equiv \mathbf{x}_{N \setminus S}$.

The identity matrix is $\mathbf{I}$. The symbols $\mathbf{O}$ and $\mathbf{0}$ will be used for the *zero* matrix and vector. The symbols $\mathbf{U}$ and $\mathbf{1}$ will be used for the *one* matrix and vector, where every entry is 1. Given a vector $\mathbf{x}$, the scalar $x \equiv \mathbf{1}^T \cdot \mathbf{x}$ is the sum of all its entries, and $x_S \equiv \mathbf{1}^T \cdot \mathbf{x}_S$.

We write $\mathbf{A} \leq \mathbf{B}$, whenever $a_{ij} \leq b_{ij}$ for all i and j in N. Also, $\mathbf{A} \lneq \mathbf{B}$ if $\mathbf{A} \leq \mathbf{B}$ and there exists $a_{ij} < b_{ij}$. Finally, $\mathbf{A} < \mathbf{B}$ if $a_{ij} < b_{ij}$ for all i and j. The symbols $\geq$, $\gneq$ and $>$ are defined accordingly.

An *eigenvalue* of a matrix $\mathbf{A}$ is a complex number μ satisfying $\mathbf{A} \cdot \mathbf{v} = \mu \mathbf{v}$ for some vector $\mathbf{v}$. Let $\mathcal{S}(\mathbf{A})$ (called the *spectrum* of the matrix $\mathbf{A}$) be the set of eigenvalues of $\mathbf{A}$.

4.2.2 Networks

A *network* (*graph*) g consists of a set of *agents* (*vertices* or *nodes*) $V(g)$ and a set of weighted *links* (*edges*) among them, where $g_{ij} \geq 0$ is the weight assigned to the link (i, j). The link (i, j) is also written ij, for short. For this reason, we can represent a network by means of a nonnegative square *adjacency matrix* $\mathbf{G} = (g_{ij})_{i,j \in V(g)}$. Without loss of generality, we will consider networks where $g_{ij} \in [0, 1]$. Generally, the set of agents of the network g, $V(g)$, will be N. In some cases, it may be different, but this will be easily inferred from the notation that we introduce later. A network g_{ij} is *un-weighted* when $g_{ij} \in \{0, 1\}$, for all $i, j \in V(g)$.

The network g is *undirected* when its adjacency matrix $\mathbf{G}$ is *symmetric*, that is, $g_{ij} = g_{ji}$ for all $i, j \in N$. We say that a network *has no self-loops* whenever $g_{ii} = 0$ for all $i \in V(g)$. We will assume that networks have no self-loops, so that $\mathbf{G}$ is a zero-diagonal matrix.

We refer to the agents i and j as being *directly linked* in the network g, whenever $g_{ij} > 0$. We also denote this by $ij \in g$. The *number of links* of g is $|g| \equiv |ij \in g|$.

A link $ij \in g$ is *incident* with the vertex $v \in V(g)$ whenever $i = v$ or $j = v$.

A *path* in g of length k from i to j is a sequence $p = \langle i_0, i_1, ..., i_k \rangle$ of agents such that $i_0 = i$, $i_k = j$, $i_p \neq i_{p+1}$, and i_p and i_{p+1} are directly linked, for all $0 \leq p \leq k - 1$. Agents i and j are said to be *indirectly linked* in g if there exists a path from i to j in g. An agent $i \in V(g)$ is *isolated* in g if $g_{ij} = 0$ for all j. The network g is said to be *empty* when all its agents are isolated, that is, $\mathbf{G} = \mathbf{O}$.

We say that a path p *traverses* or *hits* agent i if i is in the sequence defined by the path. The path p *covers* the set $S \subseteq V(g)$ if p traverses every agent $i \in S$.

We say that network g' *is a (proper) subnetwork of* g, written $g' \subseteq g$ ($g' \subset g$), whenever $V(g') \subseteq V(g)$ and $\mathbf{G}' \leq \mathbf{G}_{V(g')}$ ($\mathbf{G}' \lneq \mathbf{G}_{V(g')}$). We also say that g' is (*strictly*) *contained* in g. Given two networks g and g', their *union* is written $h = g \cup g'$, where $V(h) = V(g) \cup V(g)$ and $h_{ij} = g_{ij} + g'_{ij}$.

Given a network g and a set $S \subseteq V(g)$, we say that g_S is the *subnetwork of* g *induced by* S whenever the adjacency matrix of g_S is $\mathbf{G}_S$. Note that $V(g_S) = S$. We write g_{-S} to denote the network $g_{N \backslash S}$, that is g_{-S} is the network that results after eliminating all the agents in S.

A *cut* in a network g is a partition of the vertices into two disjoint subsets S and $V(g) \backslash S$. Every cut in a network *divides* it into two different networks g_S and g_{-S}. A network is *bipartite* if it has a cut dividing it into two empty networks.

The *spectral radius* of a network g is defined as:

$$\mu_1(g) = \max_{\mu \in \mathcal{S}(\mathbf{G})} |\mu|$$

where $|\mu|$ is the modulus of the (complex) eigenvalue μ of the matrix $\mathbf{G}$. When g is undirected, all the eigenvalues of $\mathbf{G}$ are real and $\mu_1(g)$ is called the *index* of the network g.

We adapt some results from spectral graph theory[40] and algebra into our framework.

Lemma 4.1 *The following properties hold for any network g:*

1. $\mu_1(g)$ *is a real nonnegative number.*
2. $\mu_1(g) > 0$ *if and only if g is not the empty network.*
3. *If $g' \subseteq g$, then $\mu_1(g') \leq \mu_1(g)$.*
4. $\mu_1(g_S) \leq \mu_1(g)$ *for all $S \subseteq V(g)$.*

Proof. 1, 2 and 3 are trivial consequences of Theorem I* in Debreu and Herstein (1953), given that g has an associated nonnegative square adjacency matrix $\mathbf{G}$. The case of 4 is similar: Let us define the partitioned n-square matrix

$$\mathbf{H} = \begin{pmatrix} \mathbf{G}_S & \mathbf{O}_{s \times (n-s)} \\ \mathbf{O}_{(n-s) \times s} & \mathbf{O}_{(n-s) \times (n-s)} \end{pmatrix},$$

40 Cvetković et al. (1997) is the main reference for spectral graph theory.

where $s = |S|$. Let h be its associated network, where all agents in $N \backslash S$ are isolated. By 3, $\mu_1(h) \leq \mu_1(g)$. Given that $\mathcal{E}(\mathbf{G}_S) = \mathcal{E}(\mathbf{H})$, we have that $\mu_1(g_S) = \mu_1(h)$ and the result follows. ■

4.3 The Game $\mathbf{\Sigma}$ and its Equilibrium

Each player $i \in N$ selects an effort $x_i \geq 0$, and gets a payoff $u_i(x_1, ..., x_n)$. We focus on linear-quadratic payoff functions of the form:

$$u_i(\mathbf{x}) = \alpha_i x_i + \frac{1}{2}\sigma_{ii} x_i^2 + \sum_{j \neq i} \sigma_{ij} x_i x_j, \tag{4.1}$$

strictly concave in one's effort, that is, $\partial^2 u_i / \partial x_i^2 = \sigma_{ii} < 0$. We set $\alpha_i = \alpha > 0$ and $\sigma_{ii} = \sigma$, identical for all players. Net of bilateral influences, players have thus the same payoffs.

Bilateral influences are captured by the cross derivatives $\partial^2 u_i / \partial x_i \partial x_j = \sigma_{ij}$, $i \neq j$. They depend on the pair of players considered, and can be of either sign. When $\sigma_{ij} > 0$, an increase in effort from j triggers a downwards shift in i's response. We say that efforts are strategic complements from i's perspective within the pair (i, j). Reciprocally, when $\sigma_{ij} < 0$, efforts are strategic substitutes from i's perspective within the pair (i, j).

Let $\underline{\sigma} = \min\{\sigma_{ij} \mid i \neq j\}$ and $\overline{\sigma} = \max\{\sigma_{ij} \mid i \neq j\}$. In the following, we will assume that $\underline{\sigma} < \overline{\sigma}$. When $\underline{\sigma} = \overline{\sigma}$, we are left with a trivial game where all players are identical.

We assume that $\sigma < \min\{\underline{\sigma}, 0\}$. When $\underline{\sigma} \geq 0$, this is simply the concavity of payoffs in own efforts. When $\underline{\sigma} < 0$, this requires that own marginal returns decrease with the level of x_i at least as much as cross marginal returns do.

Let $\mathbf{\Sigma} = (\sigma_{ij})_{ij \in N}$ be the square matrix of cross effects among players. We also use $\mathbf{\Sigma}$ as a short-hand for the simultaneous move $n-$player game with payoffs (4.1) and strategy spaces $\mathbb{R}_+$. Sometimes, it will be useful to denote the utility function in more detail as $u_i(\mathbf{\Sigma}, \mathbf{x})$.

In order to refer to the game *induced* by a subset of players $S \subseteq N$, we use $\mathbf{\Sigma}_S$. In this *induced game*, the set of players is S with generic strategy profile $\mathbf{x} \in \mathbb{R}_+^{|S|}$, and the utilities are $u_i(\mathbf{\Sigma}_S, \mathbf{x})$ for all $i \in S$. In words, in the induced game $\mathbf{\Sigma}_S$, the players in $N \backslash S$ are no longer in the game, and thus, interactions with any of them have vanished.

4.3.1 The Decomposition of Cross Effects

Following, Ballester *et al.* (2005a), we decompose the matrix $\mathbf{\Sigma}$ additively into an idiosyncratic concavity component, a global (uniform) substitutability component, and a local complementarity component, in the following way:

$$\mathbf{\Sigma} = -\beta\mathbf{I} - \gamma\mathbf{U} + \lambda\mathbf{G}, \tag{4.2}$$

where

$$\begin{aligned} \gamma &= -\min\{\underline{\sigma}, 0\} \geq 0 \\ \beta &= -\gamma - \sigma > 0 \\ \lambda &= \overline{\sigma} + \gamma > 0 \\ g_{ij} &= \frac{\sigma_{ij} + \gamma}{\lambda} \in [0, 1] \end{aligned}$$

By construction, the zero-diagonal matrix $\mathbf{G}$ is the adjacency matrix of a network g that reflects the pattern of existing payoff (relative) complementarities across all pairs of players. Note that g is undirected if and only if $\mathbf{\Sigma}$ is symmetric ($\sigma_{ij} = \sigma_{ji}$, for all i, j). When, moreover, cross effects only take two values, that is, $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$, for all $i \neq j$ with $\underline{\sigma} \leq 0$, then $\mathbf{G}$ is a symmetric $(0, 1)$-matrix and the network g is undirected and un-weighted.

This transformation is just a centralization (β and λ are defined with respect to γ) followed by a normalization (the g_{ij}'s are in $[0, 1]$) of the cross effects.

Following the decomposition of $\mathbf{\Sigma}$ in (4.2), we may rewrite these payoffs as:

$$u_i(x_1, ..., x_n) = \left(\alpha - \gamma \sum_{j=1}^{n} x_j + \lambda \sum_{j=1}^{n} g_{ij} x_j\right) x_i - \frac{1}{2}(\beta - \gamma)\, x_i^2, \text{ for all } i = 1, ..., n. \tag{4.3}$$

Let $\lambda^* = \lambda/\beta$ denote the strength of local interactions relative to self-concavity.

4.3.2 The Bonacich Network Centrality Measure

Before turning into the equilibrium analysis, we define a network centrality measure due to Bonacich (1987) that proves useful for our analysis.

The $n-$square adjacency matrix $\mathbf{G}$ of a network g keeps track of the direct connections in this network. For $k \geq 1$, the matrix $\mathbf{G}^k$ keeps track of the indirect connections in the network. Indeed, $g_{ij}^{[k]} \geq 0$ measures the number of paths of length $k \geq 1$ in g between i and j.[41]

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[41] In fact, $g_{ij}^{[k]}$ accounts for the total weight of all paths of length k, from i to j. When the

network is unweighted, that is, $\mathbf{G}$ is a $(0, 1)-$matrix, $g_{ij}^{[k]}$ is simply the number of paths of length k from i to j.

In particular, $\mathbf{G}^0 = \mathbf{I}$, that is, $g_{ii}^{[0]} = 1$ and $g_{ij}^{[0]} = 0$ for all $i \neq j$.

Given a scalar $a \geq 0$ and a network g, we define the following matrix:

$$\mathbf{M}(g,a) = [\mathbf{I}-a\mathbf{G}]^{-1} = \sum_{k=0}^{+\infty} a^k \mathbf{G}^k.$$

Note that these expressions are all well-defined for low enough values of a: In particular, take a smaller than the spectral radius of the network g. The parameter a is a decay factor that scales down the relative weight of longer paths.

Provided the matrix $\mathbf{M}(g,a)$ is non-negative, its coefficients $m_{ij}(g,a) = \sum_{k=0}^{+\infty} a^k g_{ij}^{[k]}$ count the number of paths in g starting at i and ending at j, where paths of length k are weighted by a^k.

Let $\mathbf{1}$ be the $n-$dimensional vector of ones.

Definition 4.1 *Consider a network g with adjacency $n-$square matrix $\mathbf{G}$ and a scalar a such that $\mathbf{M}(g,a) = [\mathbf{I}-a\mathbf{G}]^{-1}$ is well-defined and non-negative. The vector of Bonacich centralities of parameter a in g is:*

$$\mathbf{b}(g,a) = [\mathbf{I}-a\mathbf{G}]^{-1} \cdot \mathbf{1}.$$

The Bonacich centrality of node i is $b_i(g,a) = \sum_{j=1}^{n} m_{ij}(g,a)$, and counts the *total* number of paths in g starting from i.[42] It is the sum of all loops $m_{ii}(g,a)$ starting from i and ending at i, and all outer paths $\sum_{j \neq i} m_{ij}(g,a)$ that connect i to every other player $j \neq i$, that is:

$$b_i(g,a) = m_{ii}(g,a) + \sum_{j \neq i} m_{ij}(g,a).$$

By definition, $m_{ii}(g,a) \geq 1$, and thus $b_i(g,a) \geq 1$, with equality when $a = 0$. The parameter a is the *radius of influence* and controls how much "distant" agents are taken into account when measuring the centrality of agent i. The lower a, the less relative weight is put on long paths.

[42] In fact, $\mathbf{b}(g,a)$ is obtained from Bonacich (1987)'s measure by an affine transformation. Bonacich defines the following network centrality measure:

$$\mathbf{h}(g,a,b) = b\,[\mathbf{I}-a\mathbf{G}]^{-1}\,\mathbf{G} \cdot \mathbf{1}.$$

Therefore, $\mathbf{b}(g,a) = \mathbf{1} + a\mathbf{h}(g,a,1)$.

For the case of directed networks (asymmetric $\mathbf{\Sigma}$), let us define the measure

$$\widetilde{b}_i(g,a) = m_{ii}(g,a) + \sum_{j \neq i} m_{ji}(g,a),$$

that counts the number of paths in g *ending* at i.

4.3.3 The Nash-Bonacich Linkage: Characterization and Uniqueness

The following result is due to Ballester et al. (2005a) and it relates the unique outcome of the game $\mathbf{\Sigma}$ as a function of the centrality scores in the network of complementarities. Recall that $\mu_1(g)$ is the spectral radius of the network g.

Theorem 4.1 *(Ballester, Calvó-Armengol and Zenou, 2005) The matrix* $[\beta\mathbf{I} - \lambda\mathbf{G}]^{-1}$ *is well-defined and non-negative if and only if* $\beta > \lambda\mu_1(g)$*. Then, the game* $\mathbf{\Sigma}$ *has a unique Nash equilibrium* $\mathbf{x}^*(\mathbf{\Sigma})$*, which is interior and given by:*

$$\mathbf{x}^*(\mathbf{\Sigma}) = \frac{\alpha}{\beta + \gamma b(g,\lambda^*)} \mathbf{b}(g,\lambda^*). \tag{4.4}$$

The condition in Theorem 4.1 requires that the parameter for own-concavity β to be high enough to counter the payoff complementarity captured by $\lambda\mu_1(g)$. Here, λ has to do with the level and $\mu_1(g)$ with the population-wide pattern of positive cross effects. Note that this condition does not impose any bound on the absolute values for these cross effects, but only on their relative magnitude.

4.4 A Network-Based Policy

4.4.1 Finding the Key Group of Players

In our model, individual equilibrium behavior is tightly rooted in the network structure through (4.4). The removal of a set of players from the population, holding the pattern of social interactions among the other players fixed, alters the whole distribution of outcomes.

We will devote this section to identifying the optimal target set in the population when the planner wishes to reduce (or to increase) optimally some function $F(\mathbf{\Sigma})$.[43]

We wish to eliminate a group of s players from the current population. If we remove a set S of players, the interaction matrix becomes $\mathbf{\Sigma}_{-S}$.

[43] Bollobás and Riordan (2003) contains a mathematical analysis of the relative network disruption effects of a topological attack versus random failures in large networks. See also Albert *et al.* (2000) for a numerical analysis for the case of the World Wide Web.

The problem is to minimize $F(\mathbf{\Sigma}_{-S})$, by picking the adequate set S from the population. The case in which the planner maximizes $F(\mathbf{\Sigma}_{-S})$ is analogous. Formally, she solves maximizes the total change in F

$$\max_{|S|\leq s} F(\mathbf{\Sigma}) - F(\mathbf{\Sigma}_{-S}),$$

equivalent to:

$$\min_{|S|\leq s} F(\mathbf{\Sigma}_{-S}) \tag{4.5}$$

This is a finite optimization problem, that admits at least one solution. Let S^* be a solution to (4.5). We call the set S^* a *key group* of the game $\mathbf{\Sigma}$. Removing S^* from the game has the highest overall impact on the value of F.

In the following, we assume that the condition of Theorem 4.1 holds in the game, guaranteeing the uniqueness of Bonacich solutions in any game induced by a subset of players. The reason is that, by Lemma 4.1, $\mu_1(g) \geq \mu_1(g_{-S})$. As a consequence, if $\mathbf{b}(g, \lambda^*)$ is well-defined and non-negative (as implied by the condition in Theorem 4.1), so is $\mathbf{b}(g_{-S}, \lambda^*)$.

Let us consider the problem of optimizing the aggregate activity at equilibrium[44], that is: $F(\mathbf{\Sigma}) = x^*(\mathbf{\Sigma})$, where

$$x^*(\mathbf{\Sigma}) = \frac{\alpha}{\beta + \gamma b(g, \lambda^*)} b(g, \lambda^*).$$

Given the form of $x^*(\mathbf{\Sigma})$ and $\alpha > 0$, it is clear that $x^*(\mathbf{\Sigma})$ increases in aggregate centrality $b(g, \lambda^*)$, and the problem is:

$$\max_{|S|=s} \left\{ b(g, \lambda^*) - b(g_{-S}, \lambda^*) \right\}, \tag{4.6}$$

where $|S| = s$ because $b(g_{-S}, \lambda^*)$ is clearly decreasing in S.

Given the Bonacich centralities $\mathbf{b}(g, a)$, define

$$b_{jS}(g, a) = b_j(g, a) - b_j(g_{-S}, a), \text{ for all } j \in N.$$

This is the contribution of S to j's Bonacich centrality in the network g. When $j \in S$, this quantity is just the centrality $b_j(g, a)$ because,after S's removal, $b_j(g_{-S}, a) = 0$ for all $j \in S$. Summing over all $j \in N$, we obtain the contribution of S to the total centrality in g:

44 This problem may arise in decisions related to crime reduction, where the objective is to choose the set of criminals whose removal decreases crime activity in society. Calvó-Armengol and Zenou (2004) provide an economic model of crime decisions where this type of optimal choices could be applied.

Definition 4.2 *The group inter-centrality of S in the network g is:*

$$\begin{aligned} d_S(g,a) &\equiv \sum_{j \in N} b_{jS}(g,a) \\ &= b(g,a) - b(g_{-S},a) \end{aligned}$$

In fact, $d_S(g,a)$ is the weighted number of paths in g traversing some agent in S. Then, our problem reduces to choose the set with highest group inter-centrality:

$$\max_{|S|=s} d_S(g,\lambda^*), \tag{4.7}$$

that is, the solution of (4.6) is $S^* \subseteq N$ such that $d_{S^*}(g,\lambda^*) \geq d_S(g,\lambda^*)$, for all $S \subseteq N$ with $|S| = s$.

The version of the problem with $s = 1$ (the *key player* problem) is analyzed in Ballester *et al.* (2005a). They prove that individual contributions (*individual inter-centralities* of players) can be directly computed as:

$$d_i(g,a) = \frac{b_i(g,a)\widetilde{b}_i(g,a)}{m_{ii}(g,a)} \tag{4.8}$$

and the choice of the key player corresponds to that with highest inter-cetrality measure, which need not coincide with the most central agent in the network.

Remark 4.1 *An equivalent formulation of the key group problem (4.7) is:*

$$\max_{\{i_1,\dots,i_s\} \subseteq N} d_{i_1}(g,a) + d_{i_2}(g_{-\{i_1\}},a) + \dots + d_{i_s}(g_{-\{i_1,\dots,i_{s-1}\}},a), \tag{4.9}$$

In words, the key group maximizes the sum of the individual inter-centrality measures of its members across the networks obtained through sequential removal of these members.[45] The idea behind this expression is the following. We must eliminate a set of players $S = \{i_1, \dots, i_s\}$ in order to minimize the total number of weighted walks in the network, $b(g_{-S},a)$. After deleting player i_1, the resulting number of paths is $b(g,a) - d_{i_1}(g,a)$. Now, the expression $d_{i_2}(g_{-i_1},a)$ counts the number of walks that hit agent i_2 *once agent* i_1 *has been eliminated*, so that we are not counting the same path twice. Thus, $b(g,a) -$

[45] Note that this sum is independent of the order in which nodes are removed.

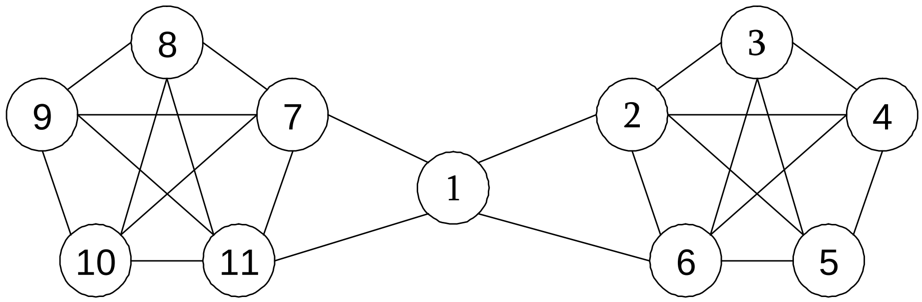

Figure 4.1 A network with eleven players, player 1 being a bridge between two cliques of size 5.

$d_{i_1}(g, a) - d_{i_2}(g_{-i_1}, a)$ is the remaining set of walks after eliminating players i_1 and i_2, keeping in mind that we only want to count each walk once. By the previous argument, also note that the remaining set of weighted paths is the same if we change the order of deletion of these two players, that is:

$$b(g, a) - d_{i_1}(g, a) - d_{i_2}(g_{-i_1}, a) = b(g, a) - d_{i_2}(g, a) - d_{i_1}(g_{-i_2}, a)$$

Extending this argument to the rest of the players in S, we obtain an expression (4.9).

4.4.2 Example

Consider the network g in Figure 4.1 with eleven players and a radius of influence $a = 0.2$.

There are three different locations in this network: player 1, players 2, 6, 7 and 11, and players 3, 4, 5, 8, 9, and 10. Type-1 and type-3 players have four direct links, while type -2 players have five. Player 1 bridges together two fully interconnected communities of five players each. By removing player 1, the network is maximally disrupted. By removing a type-2 player, we get a network with the lowest total number of links.

When $s = 1$, note that Bonacich *centrality* and our *individual inter-centrality* measure are different concepts. The first accounts for the influence of one agent from his position, in terms of the number of agents that he can reach. The second adds the contribution of this agent to the Bonacich centrality of the others. Hence, individual inter-centrality captures the role of each agent as a broker in the interactions among the others. For instance, it is easy to check that the key player is 1 because he has the highest individual inter-cetrality $d_1(g, a) = 41.67$. But the player with highest contribution need not be the one with highest Bonacich centrality. In particular, vertex 2 is more (Bonacich) central than vertex

1: $b_2(g,a) = 9.17 > 8.33 = b_1(g,a)$.

Consider the case where the required group size is $s = 2$. The next table shows the values of *group inter-centrality* $d_S(g,a)$ for each possible subset S of size two when $a = 0.2$. For the sake of simplicity, subsets that yield the same network architecture when they are removed are considered as equivalent:

Removed Group S	$d_S(g,a)$
$\{2,7\}^*$	67.22
$\{2,8\}$	64.01
$\{3,8\}$	59.39
$\{1,2\}$	56.66
$\{2,6\}$	50.41
$\{2,3\}$	46.96
$\{3,4\}$	42.15

The key group is $\{2,7\}$, that is, a set of two maximally connected nodes (with five direct contacts each), both connected to the inter-central player 1, and each at a different side of this player. This subset solves the following optimization problem:

$$\max_{i,j} d_{\{i,j\}}(g,a) = \max_{i,j} \left(d_i(g,a) + d_j(g_{-i},a)\right)$$

Suppose that we were to approximate the solution to this optimization problem with a greedy heuristic procedure that *sequentially* picks up the player that maximizes the individual inter-centrality at each step. Formally, let

$$i_1^* = \arg\max_{i\in N} d_i(g,a)$$

and then, at each step $2 \le t \le s$, choose the player i_t^* with maximum inter-centrality in the network where the previous players have been deleted, that is,

$$i_t^* \in \arg\max \left\{ d_i(g_{-\{i_1^*,\ldots,i_{t-1}^*\}}), a : i \in N \setminus \left\{i_1^*, ..., i_{t-1}^*\right\} \right\}$$

breaking possible ties arbitrarily. This greedy algorithm first eliminates player 1, and then any other remaining player (after player 1 has been removed, all the other players have identical positions in the network). Thus, the algorithm returns a group which is not optimal: there are other groups that are better candidates than $\{1,2\}$. Indeed, in this example, player 1 is not only very inter-central, but also its inter-centrality is very much correlated with the inter-centrality of others. Hence, being greedy and eliminating it at the first stage reduces the chance of finding highly central players at further stages. And, in fact, player 1 is not part of the key group!

Nevertheless, we have obtained a relatively accurate approximation for the result by a simple greedy algorithm, instead of choosing among all possible pairs of agents. Note that the error of this approximation is:

$$\frac{d_{\{2,7\}}(g,a) - d_{\{1,2\}}(g,a)}{d_{\{2,7\}}(g,a)} \approx 16\%$$

In fact, when $s = 2$, this error can be at most 25%. In the section devoted to algorithmic considerations, we discuss this issue more generally.

4.4.3 Group Inter-centrality as the Cardinality of a Set of Paths

Since $d_S(g,a)$ is the weighted number of paths in g traversing some agent in S, let $D_S^{[k]}(g)$ be the set of paths of length k traversing some element of S in the network g. Note that the contribution of S can be written as:

$$d_S(g,a) = \sum_{k=0}^{+\infty} a^k \left| D_S^{[k]}(g) \right|$$

Also, define $E_S^{[k]}(g)$ as the set of paths of length k that traverse all the elements of S in g. And let $e_S(g,a)$ be:

$$e_S(g,a) = \sum_{k=0}^{+\infty} a^k \left| E_S^{[k]}(g) \right|$$

The quantity $e_S(g,a)$ measures the total number of paths covering S in the network g.

Now, we characterize the key group in terms of the function e.

Proposition 4.1 *The impact on the total Bonacich centrality in g by the removal of the set of vertices S is given by:*

$$d_S(g,a) = \sum_{T \subseteq S} (-1)^{|T|+1} e_T(g,a)$$

Proof. Given that we are dealing with the cardinality measure of sets,

$$\begin{aligned} d_S(g,a) &= \sum_{k=0}^{+\infty} a^k \left| D_S^{[k]}(g) \right| \\ &= \sum_{k=0}^{+\infty} a^k \left| \cup_{i\in S} D_i^{[k]}(g) \right| \\ &= \sum_{k=0}^{+\infty} a^k \sum_{T\subseteq S} (-1)^{|T|+1} \left| \cap_{j\in T} D_j^{[k]}(g) \right| \\ &= \sum_{k=0}^{+\infty} a^k \sum_{T\subseteq S} (-1)^{|T|+1} \left| E_T^{[k]}(g) \right| \\ &= \sum_{T\subseteq S} (-1)^{|T|+1} e_T(g,a) \end{aligned}$$

■

4.5 Algorithmic Considerations

The optimal choice of the group of players requires, at least potentially, the study of all possible combinations of subsets of N. Thus, a computational approach is required.

4.5.1 Submodular Functions and The Greedy Algorithm

Let us define the set function $z(S) = F(\boldsymbol{\Sigma}) - F(\boldsymbol{\Sigma}_{-S})$ for all $S \subseteq N$. Consider the problem of solving (4.5), that is,

$$\max_{S\subseteq N}\{z(S) : |S| \leq s\}. \tag{4.10}$$

Before providing our results, we need some preliminary definitions regarding a set function $z : 2^N \to \mathbb{R}$.

Definition 4.3 *The set function $z : 2^N \to \mathbb{R}$ is submodular (supermodular) if for all $S, T \subseteq N$,*

$$z(S) + z(T) \underset{(\leq)}{\geq} z(S \cup T) + z(S \cap T)$$

Without loss of generality we can normalize z such that $z(\varnothing) = 0$. We only consider nondecreasing functions:

$$z(S) \le z(T) \text{ for all } S \subseteq T \subseteq N$$

although the following results can be adapted to non-monotonic functions. Let us denote individual contributions by:

$$\rho_i(S) = z(S \cup \{i\}) - z(S)$$

In fact, the set function z is submodular if individual contributions are increasing with respect to set containment.

Remark 4.2 *The set function $z : 2^N \to \mathbb{R}$ is submodular if and only if for all $S \subseteq T \subseteq N$ and $i \in N \backslash T$:*

$$\rho_i(S) \underset{(\le)}{\ge} \rho_i(T)$$

The problem of maximizing a submodular function, or equivalently, minimizing a supermodular function, is NP-hard, in general. Nemhauser *et al.* (1978) propose a polynomial-time greedy heuristic for approximating this kind of problems. At each step, the algorithm augments the solution set with the agent with highest contribution.

Algorithm 4.1 *Let $S_0 = \varnothing$. At step t set $S_t = S_{t-1} \cup i_t$, where:*

$$i_t \in \arg \max_{i \in N \backslash S_{t-1}} \rho_i(S_{t-1}).$$

Stop whenever $\rho_{i_t}(S_{t-1}) \le 0$ or $|S_t| = s$.

We summarize part of their results in the following proposition. Let Z be the optimal value of (4.10) and Z^G be the value obtained by applying the greedy algorithm.

Proposition 4.2 *If the greedy heuristic is applied to the problem (4.10), where z is submodular, then the approximation error is bounded like:*

$$\varepsilon \equiv \frac{Z - Z^G}{Z} \le \left(\frac{s - s}{s} \right)^s < \frac{1}{e} \approx 36.79\% \tag{4.11}$$

4.5.2 An Approximation for the Key Group

We prove that the key group problem has an inherent complexity that suggests the use of approximation algorithms. In particular, we will study the performance of a greedy procedure, where the optimal group is constructed by iteratively choosing an optimal vertex from the network. For a description of the problem NP problems and properties, see Ballester (2004) and Garey and Johnson (1978).

Now, we show that the key group problem is NP-hard, even when we want to completely disrupt the game.

A *vertex cover* of a network g is a subset of vertices $S \subseteq V(g)$ such that every link $ij \in g$ is incident with some vertex in S. A maximum vertex cover is a vertex cover of maximum size. The problem of finding a maximum vertex cover in a network is known to be NP-hard (Karp, 1972).

Proposition 4.3 *The problem of finding a key group in a network g is NP-hard.*

Proof. Given the maximum vertex cover problem, we can solve it by means of a polynomial-time parsimonious reduction to our key group problem. Given a maximum vertex cover S^* in g of size s^*, it is obvious that S^* is a key group of size s^*, because the removal of players in S^* results in a network of completely isolated vertices (empty network). Starting with $s = 1$ we can solve the key group problem, by increasing s iteratively. When the removal of the key group S results in an empty network, we can stop iterating and conclude that S is a vertex cover of g. ■

Since the computational complexity inherent to the key group selection is high, it is suitable to use algorithmic approximations in order to solve real-life problems with large networks. The following result shows that the use of the greedy procedure can be guaranteed to provide fairly good approximations for the problem.

Lemma 4.2 *The function $d_S(g, a)$ is submodular in S.*

Proof. Take $S \subseteq T \subseteq N$. Let $b_{ji}^{[k]}(g)$ denote the number of k-paths starting at

j and hitting i in the network g. Then, for all $i \in N \backslash T$:

$$\begin{aligned}
d_{S\cup\{i\}}(g,a) - d_S(g,a) &= \big(b(g,a) - b(g_{-(S\cup\{i\})},a)\big) - (b(g,a) - b(g_{-S},a)) \\
&= b(g_{-S},a) - b(g_{-(S\cup\{i\})},a) \\
&= d_i(g_{-S},a) \\
&= \sum_{k=0}^{\infty} a^k \sum_{j\in N\backslash S} b_{ji}^{[k]}(g_{-S}) \\
&\geq \sum_{k=0}^{\infty} a^k \sum_{j\in N\backslash T} b_{ji}^{[k]}(g_{-T}) \\
&= d_i(g_{-T},a) \\
&= d_{T\cup\{i\}}(g,a) - d_T(g,a)
\end{aligned}$$

■

Proposition 4.4 *The key group problem can be approximated in polynomial-time by the use of a greedy algorithm, where, at each step, expression (4.8) is used to find the agent i (with highest $d_i(g,a)$) who will become a member of the approximated key group. The error of the approximation can be bounded as in (4.11).*

Remark 4.3 *The use of the greedy algorithm is suitable also for other functions. For instance, suppose that we are to minimize the total welfare $\sum_{i\in S} u_i(\mathbf{\Sigma}_{-S})$, instead of the total activity $x(\mathbf{\Sigma}_{-S})$. Then, a greedy procedure would consist of a sequential elimination of the players with higher impact on aggregate welfare. The drawback of this approach is two-fold. On the one hand, we do not have an explicit expression like (4.8) in order to rapidly identify the player with higher impact on welfare at each step. Second, the change in welfare need not be submodular in S and , hence, the bound in (4.11) is no longer valid to guarantee a minimum error of approximation.*

4.6 Voluntary Participation

4.6.1 The Game

In this section, we extend the game $\mathbf{\Sigma}$ in order to allow the players to choose whether they want to participate in it in a first stage, as in Calvó-Armengol and Zenou (2004). Let ω be a nonnegative scalar.

Definition 4.4 *The extended game $(\omega, \mathbf{\Sigma})$ is a two stage game where:*

- *At stage 1, each player $i \in N$ decides whether to participate ($c_i = 1$) or not ($c_i = 0$).*
- *At stage 2, let S be the set of players who decided to participate. Then, these players play the game $\mathbf{\Sigma}_S$.*
- *The final utilities are:*

$$U_i(S, \mathbf{x}_S) = \begin{cases} u_i(\mathbf{\Sigma}_S, \mathbf{x}_S) & \text{if} \quad i \in S \\ \omega & \text{otherwise} \end{cases}$$

We study the *subgame perfect equilibrium in pure strategies* of the extended game $(\omega, \mathbf{\Sigma})$.

Definition 4.5 *The set S is supported at equilibrium if there exists ω and a subgame perfect equilibrium where the players who decide to participate is precisely S, given the outside option ω. S is also called an (equilibrium) participation pool of the game at the wage level ω.*

Let $\mathcal{E}(\mathbf{\Sigma}, \omega)$ the family of sets supported by ω at equilibrium in the game $(\mathbf{\Sigma}, \omega)$.

The following result characterizes the class of sets that can be supported by some ω.

Proposition 4.5 *Let $S \subseteq N$ and $\beta > \lambda\mu_1(g_{S\cup\{j\}})$ for all $j \in N\backslash S$. Then, the set S is supported at equilibrium by the outside option ω if and only if:*

$$\max_{j\in N\backslash S} \frac{b_j(g_{S\cup\{j\}}, \lambda^*)}{\beta + \gamma b(g_{S\cup\{j\}}, \lambda^*)} \leq \frac{1}{\alpha}\sqrt{\frac{2\omega}{\beta+\gamma}} \leq \min_{i\in S} \frac{b_i(g_S, \lambda^*)}{\beta + \gamma b(g_S, \lambda^*)}$$

Proof. The conditions $\beta > \lambda\mu_1(g_{S\cup\{j\}})$ for all $j \in N\backslash S$ imply that $b_j(g_{S\cup\{j\}}, a)$ is well-defined for all $j \in N\backslash S$. Given that $\mu_1(g_{S\cup\{j\}}) \geq \mu_1(g_S)$, $b_i(g_S, \lambda^*)$

is all also well-defined for all $i \in S$. On the other hand, by Theorem 4.1, this also implies uniqueness of Nash equilibrium in the second stage games $\mathbf{\Sigma}_S$ and $\mathbf{\Sigma}_{S\cup\{j\}}$, for all $j \in N\backslash S$:

$$x_i^*(\mathbf{\Sigma}_S) = \frac{\alpha b_i(g_S, \lambda^*)}{\beta + \gamma b(g_S, \lambda^*)} \text{ for all } i \in S \tag{4.12}$$

$$x_j^*(\mathbf{\Sigma}_{S\cup\{j\}}) = \frac{\alpha b_j(g_{S\cup\{j\}}, \lambda^*)}{\beta + \gamma b(g_{S\cup\{j\}}, \lambda^*)} \text{ for all } j \in N\backslash S. \tag{4.13}$$

First, we prove that $u_i(\mathbf{\Sigma}_S, \mathbf{x}^*(\mathbf{\Sigma}_S)) = [x_i^*(\mathbf{\Sigma}_S)]^2 (\beta+\gamma)/2$. First-order conditions for (4.1) yield

$$\alpha + \sigma x_i + \sum_{j\neq i} \sigma_{ij} x_j = 0$$

Then, (4.1) can be rewritten as:

$$\begin{aligned} u_i(\mathbf{\Sigma}, \mathbf{x}) &= \left(\alpha + \frac{1}{2}\sigma x_i + \sum_{j\neq i} \sigma_{ij} x_j\right) x_i \\ &= \left(\alpha + \sigma x_i + \sum_{j\neq i} \sigma_{ij} x_j - \frac{1}{2}\sigma x_i\right) x_i, \end{aligned}$$

and the result follows because of the first order condition and $-\sigma = \beta + \gamma$.

Now, uniqueness in the second-stage, allows us to concentrate on the pure strategy Nash equilibria of the whole game, where no agent j outside a sustainable S would be willing to enter the game $\mathbf{\Sigma}_S$ to obtain $u_j\left(\mathbf{\Sigma}_{S\cup\{j\}}, \mathbf{x}^*(\mathbf{\Sigma}_{S\cup\{j\}})\right)$; and no agent $i \in S$ would be better off by obtaining ω, rather that $u_i\left(\mathbf{\Sigma}_S, \mathbf{x}^*(\mathbf{\Sigma}_S)\right)$. Formally, a set S is supported by ω at equilibrium if and only if:

$$\max_{j\in N\backslash S} u_j\left(\mathbf{\Sigma}_{S\cup\{j\}}, \mathbf{x}^*(\mathbf{\Sigma}_{S\cup\{j\}})\right) \leq \omega \leq \min_{i\in S} u_i\left(\mathbf{\Sigma}_S, \mathbf{x}^*(\mathbf{\Sigma}_S)\right)$$

The result follows by using (4.12) and (4.13) and applying simple algebra. ■

Remark 4.4 *Whenever*

$$\omega > \frac{\alpha^2}{2(\beta+\gamma)},$$

all agents outside the game is an equilibrium, that is, $\varnothing$ is supported as an equilibrium by ω.

Proof. The result follows by picking a wage higher than the utility of an isolated agent, which is easily calculated given that she has centrality 1. ■

Remark 4.5 *It is an open problem whether any $\omega \in [0, \alpha^2/(2(\beta+\gamma))]$ can support some S at equilibrium.*

Whenever an equilibrium exists, multiplicity of equilibria is a natural outcome of the extensive form game. This multiplicity can arise, for instance, from the symmetric role of some agents in a network[46].

4.6.2 Entry Game Without Global Substitutability

Suppose that $\mathbf{\Sigma} \geq \mathbf{O}$ (so that $\gamma = 0$), that is, we have that the second-stage game $\mathbf{\Sigma}$ has strategic complementarities. Let S be a participation pool (not necessarily an equilibrium pool) at some wage ω. In this case, the payoff that an agent $i \in N \backslash S$ gets by joining S is precisely

$$x_i^*(\mathbf{\Sigma}_{S\cup\{i\}}) = \frac{\alpha}{\beta} b_i(g_{S\cup\{i\}}, \lambda^*),$$

that is, its centrality (times α/β) in the network $g_{S\cup\{i\}}$.

Given that the outside option is fixed ω, it is clear that the two-stage game is *supermodular*[47], in the sense that the payoffs of player i are increasing with respect to participation decisions of other agents. Formally, for all $S \subseteq T \subseteq N$ and $i \in N \backslash T$, it is clear that:

$$b_i(g_{S\cup\{i\}}, \lambda^*) \leq b_i(g_{T\cup\{i\}}, \lambda^*)$$

because the right-hand side measures a higher number of paths.

This property ensures existence of equilibrium for any wage ω, as summarized by the following proposition.

---

46 Two agents i and j are *symmetric* in a network whenever the network remains the same by exchanging their labels. In this case, if S is supported at equilibrium, $i \in S$ and $j \in N \backslash S$, so is S' where i has been interchanged with j.

47 Informally, a game is submodular if, for every player

- her action set is compact and,
- her utility has increasing differences in own actions and in each rival's actions.

For a broad description of supermodular games and their applications, see Amir (2003), Topkis (1998) and Vives (1990).

Proposition 4.6 *When* $\mathbf{\Sigma} \geq \mathbf{O}$*, the game* $(\mathbf{\Sigma}, \omega)$ *has at least one equilibrium participation pool.*

Proof. We provide an instance participation pool, by construction. Starting with an empty pool $S_0 = \varnothing$, at each step t, set $S_t = S_{t-1} \cup i_t$, where i_t is any player such that:

$$\sqrt{\frac{2\omega}{\beta}} \leq \frac{\alpha}{\beta} b_{i_t}(g_{S_t}, \lambda^*)$$

(This means that, at each step, players that want to enter the pool do so). Stop whenever there is no such agent i_t. It is clear that S_t (probably empty) is an equilibrium pool. The main implication of supermodularity is that this sequential decisions cannot be rolled-back: if an agent decided to enter the pool, then it must be profitable for her to stay after more agents decide participate. ■

One may be interested in providing all the possible equilibria of the game when supermodularity holds. Echenique (2005) provides a useful tool to list all the equilibria of a game with complementarities.

Remark 4.6 *Whenever the competition parameter* γ *is low enough, supermodularity still survives in the game, as well as the result in the proposition.*

Thus, the results for supermodularity that hold for $\gamma = 0$ are robust to small perturbations of γ. The intuitive idea is that substitutability is low enough to allow for increasing differences in utility of agents in their decisions to enter the participation pool.

4.6.3 The Key Group Problem with Voluntary Participation

In the simple case without an outside option, the choice of the key group was based on the contribution of that group to the connectivity (total Bonacich centrality) of the complementarity network. In the context of games with voluntary participation, an additional criterion should be taken into account: the fact that a removal of some players may induce further voluntary moves of other players in the network. Thus, the choice of the optimal target van change accordingly, and differ from the usual key group prescription when all players participate. We analyze the interplay between the optimal target and an outside option that acts as a participation threshold in the new game.

The issue of existence will be relaxed in this section by assuming that we are dealing with a wage ω such that, for any subgame $\mathbf{\Sigma}_T$, with $T \subseteq N$, there exists an equilibrium participation pool S supported by ω. On the other hand, multiplicity of equilibria makes it difficult to adopt a particular approach in order to assess the efficacy of a particular policy. In this paper, we focus on a extreme approach where the removal of a set of players from the network is evaluated by comparing the maximum equilibrium of the original game and the resulting game, that is, outcomes with maximum total activity x^*.

Definition 4.6 *Given a game $(\mathbf{\Sigma}, \omega)$ and $T \subseteq N$ define the remaining family after eliminating S as*

$$P(\mathbf{\Sigma}, \omega, S) = \left\{T \subseteq N \backslash S, T \in \mathcal{E}(\mathbf{\Sigma}_{N\backslash S}, \omega)\right\}$$

In words, a set $T \subseteq N \backslash S$ is in the remaining family after eliminating S whenever T is a participating pool in the restricted game $(\mathbf{\Sigma}_{-S}, \omega)$. This definition is just capturing the fact that of the posterior behavior of players after S's removal.

Let g be the graph associated with the decomposition (4.2) of the game $\mathbf{\Sigma}$. For a candidate set S to be eliminated, let

$$P_m(\mathbf{\Sigma}, \omega, S) \in \arg \max_{T \in P(\mathbf{\Sigma}, \omega, S)} \{b(g_T, a)\}$$

be a *maximum equilibrium* participation pool when the set S is eliminated. It is a pool where the maximum activity is achieved. Then, the choice of a key group S^* of size s is precisely:

$$S^* \in \arg \min_{|S|=s} b(g_{P_m(\mathbf{\Sigma}, \omega, S)}, \lambda^*)$$

4.6.4 Example

Consider a game where $\alpha = \beta = \gamma = 1$, $\lambda = 0.2$ and g is the network in Figure 4.1. Let us study the problem of eliminating one player ($s = 1$). The following table summarizes centralities and inter-centralities of different types of players:

Player Type	$b_i(g, \lambda^*)$	$d_{\{i\}}(g, \lambda^*)$
1	8.33	41.67^*
2	9.17^*	40.33
3	7.78	32.67

First, note that player 2 is more central than 1, but it is less inter-central. Second, the optimal choice without a participation stage (or, equivalently, $\omega = 0$), is player 2, who has the highest contribution to the connectivity of the network.

The situation is different when we allow for voluntary participation. The next table summarizes the sets S that are sustainable at equilibrium for the games $(\mathbf{\Sigma}_{-1}, \omega)$ and $(\mathbf{\Sigma}_{-2}, \omega)$, specifying the range $[w_L, w_H]$ of wages that support each S an equilibrium participating pool. We just specify distinct (up to isomorphism) equilibrium pools:

	$(\mathbf{\Sigma}_{-1}, \omega)$			$(\mathbf{\Sigma}_{-2}, \omega)$		
Pool S	w_L	w_H	$b(g_S, a)$	$w_L(S)$	w_L	$b(g_S, a)$
$\{2,3,4,5,6,7,8,9,10,11\}$	0.00	12.50	50.00	–	–	–
$\{2,3,4,5,6\}$	0.50	12.50	25.00	–	–	–
$\{1,6,7,8,9,10,11\}$	–	–	–	1.02	1.70	39.47
$\{1,3,4,5,6,7,8,9,10,11\}$	–	–	–	0.00	4.17	51.33
$\{1,7,8,9,10,11\}$	–	–	–	1.70	7.03	36.25
$\{7,8,9,10,11\}$	–	–	–	7.03	12.50	25.00
$\{1,3,4,5,6\}$	–	–	–	0.95	1.25	12.37
$\{3,4,5,6\}$	–	–	–	1.25	3.12	10.00

When $\omega = 5$, the highest equilibrium pool after removing player 1 is $N \backslash \{1\}$. This is the same choice as the simple selection based on group inter-centrality. However, when removing player 2, the maximum equilibrium pool becomes $\{1, 7, 8, 9, 10, 11\}$: the players on the right have lost their incentives to remain in the game. Hence, the key player is player 2, because its deletion causes a higher impact on the incentives of other players to leave for the outside option ω.

4.7 Related Problems

We discuss some problems that are closely related to the optimal choice of a group of players. In some of them, the nature of the problem will also allow us to approximate the solution with a greedy procedure. First, we list these problems and offer a brief description:

1. **Separation of Agents**: The problem is to partition the players in order to maximize the global outcome of the different games.
2. **The Key Interaction Set**: Find a set of links that reduces maximally the activity in the network.
3. **Optimal Attachment**: Where should new links be placed in order to maximize the overall activity?

4.7.1 Separation of Agents

Suppose that we partition the player set N as $\mathcal{P} = \{S_k\}_{k=1}^{K}$, in a way such that any player $i \in S_k$ only interacts with players in S_k. The new game is denoted by

$\mathbf{\Sigma}^{\mathcal{P}}$, where

$$\sigma_{ij}^{\mathcal{P}} = \begin{cases} \sigma_{ij} & \text{if} \quad i, j \in S_k \text{ for some } k = 1, ..., K \\ 0 & \text{otherwise.} \end{cases}$$

One could argue that, since all these independent games $\mathbf{\Sigma}_{S_k}$ may have less competition than the original game $\mathbf{\Sigma}$, then it would be good to find a balanced partition maximizing the total activity of all agents. In general, it will be the case that partitioning the game will increase total activity when the number of agents is large.

As usual, let us assume that $\beta > \lambda\mu_1(g)$ in order to be able to compare unique well-defined outcomes. Note that

$$x^*(\mathbf{\Sigma}^{\mathcal{P}}) = \sum_{k=1}^{K} x^*(\mathbf{\Sigma}_{S_k})$$

We say that a partition $\mathcal{P}$ *dominates* $\mathcal{P}'$ if $x^*(\mathbf{\Sigma}^{\mathcal{P}}) \geq x^*(\mathbf{\Sigma}^{\mathcal{P}'})$. A partition $\mathcal{P}$ is *efficient* if it dominates every other partition.

Proposition 4.7 *Consider a game $\mathbf{\Sigma}$ with $\beta > \lambda\mu_1(g)$ and total activity given by $x^*(\mathbf{\Sigma})$.*

1. *If $\mathbf{\Sigma} \geq \mathbf{O}$ (no competition) then $\{N\}$ is the unique maximal partition.*
2. *Otherwise, suppose that there is $s > 1$ such that $(\gamma/\beta)\sqrt{s-1} > 1$. Then, all blocks of an efficient partition must have strictly less than s elements. In particular:*
   (a) *If $(\gamma/\beta)\sqrt{n-1} > 1$ the partition $\{N\}$ is dominated.*
   (b) *if $\gamma > \beta$ (high competition) the unique efficient partition is the trivial stand-alone partition $\{\{i\}\}_{i=1}^{n}$.*

Proof. When $\mathbf{\Sigma} \geq \mathbf{O}$, $\gamma = 0$ and $x^*(\mathbf{\Sigma}_S) = (\alpha/\beta)b(g_S, a)$ which is clearly superadditive in S, given the interpretation of $b(g_S, a)$ as the total number of paths in g_S.

Consider $\mathbf{\Sigma} \ngeq \mathbf{O}$. Let us consider the case of a split of any non-singleton set $S \neq \varnothing$ into $\mathcal{P} = \{S_1, S_2\}$. In this case, the total activity $x^*(\mathbf{\Sigma}_S^{\mathcal{P}})$ is greater than $x^*(\mathbf{\Sigma}_S)$ if and only if:

$$\frac{b(g_S, \lambda^*)}{\beta + \gamma b(g_S, \lambda^*)} \leq \frac{b(g_{S_1}, \lambda^*)}{\beta + \gamma b(g_{S_1}, \lambda^*)} + \frac{b(g_{S_2}, \lambda^*)}{\beta + \gamma b(g_{S_2}, \lambda^*)},$$

equivalent to:

$$\begin{aligned} b(g_S, \lambda^*) &\leq b(g_{S_1}, \lambda^*) + b(g_{S_2}, \lambda^*) + \frac{2\gamma}{\beta} b(g_{S_1}, \lambda^*) b(g_{S_2}, \lambda^*) \\ &+ \frac{\gamma^2}{\beta^2} b(g_{S_1}, \lambda^*) b(g_{S_2}, \lambda^*) b(g_S, \lambda^*) \end{aligned}$$

A sufficient condition for this to hold is $(\gamma^2/\beta^2) b(g_{S_1}, \lambda^*) b(g_{S_2}, \lambda^*) \geq 1$. Given that

$$\begin{aligned} b(g_{S_1}, \lambda^*) b(g_{S_2}, \lambda^*) &\geq s_1 s_2 \\ &= s_1(s - s_1) \\ &\geq s - 1, \end{aligned}$$

a sufficient condition for inefficiency of S is $(\gamma/\beta)\sqrt{s-1} \geq 1$ and the result follows. ■

4.7.2 The Key Interaction Set

In some situations, the limitation of resources or the nature of the problem requires to choose optimally among the set of dependences among players. For instance, a social planner would like to optimally reduce the (communication) externalities among delinquents, subject to a restriction in the r number of bilateral influences that can be targeted. This situation is interpreted as a problem of optimally removing a set of links from the network.

More formally, for $g' \subseteq g$, let $l_{g'}(g, a)$ be the number of paths in g (weighted with a) that use some edge in g'. That is, this is the contribution of g' to the total connectivity of g.

Suppose that we need to maximize the change in the network after removing r links. Our best choice will consist of r links from the set of all possible links not present in g. Formally, we need to solve:

$$\max_{g' \subseteq g} \{ l_{g'}(g, a) : |g'| \leq r \}$$

We deal with the case of *directed* networks first, since it provides an easier proof of our result. In this case, the planner has more degrees of freedom: because he can target specific directed links. Let $h \equiv g \backslash \{ij\}$ be the network g where g_{ij} is set to zero. The following relation holds in this class of networks, for all pair of agents $k, l \in V(g)$:

$$m_{kl}(g, a) - m_{kl}(h, a) = a m_{ki}(h, a) m_{jl}(g, a) \tag{4.14}$$

That is, paths that cross kl arrive at k for a first time in the network h, cross the link kl and then continue from l in g. Let $l_{ij}(g,a)$ be the total contribution of link ij to the centrality of g:

$$l_{ij}(g,a) \equiv \sum_{k,l \in V(g)} (m_{kl}(g,a) - m_{kl}(h,a))$$

Lemma 4.3 *The contribution of a single directed link $ij \in g$ to the total Bonacich centrality of the network g is given by:*

$$l_{ij}(g,a) = a\frac{\widetilde{b}_i(g,a)b_j(g,a)}{1 + am_{ji}(g,a)} \tag{4.15}$$

Proof. We can specialize (4.14) to compute $m_{ki}(h,a)$ as:

$$\begin{aligned} m_{ki}(g,a) - m_{ki}(h,a) &= am_{ki}(h,a)m_{ji}(g,a) \\ m_{ki}(h,a) &= \frac{m_{ki}(g,a)}{1 + am_{ji}(g,a)} \end{aligned}$$

and, substituting it back in (4.14),

$$m_{kl}(g,a) - m_{kl}(h,a) = a\frac{m_{ki}(g,a)m_{jl}(g,a)}{1 + am_{ji}(g,a)} \text{ for all } k,l \in V(g).$$

Summing over all k and l, the result follows. ■

When networks are *undirected*, the following expression allows us to compute the contribution of a single link $ij \in g$ to the total Bonacich centrality of the network g. The proof is omitted, being similar to the case of directed links.

Lemma 4.4 *The contribution of a single undirected link $ij \in g$ to the total Bonacich centrality of the network g is given by:*

$$l_{ij}(g,a) = a\left(b_i(h,a)b_j(g,a) + b_i(g,a)b_j(h,a)\right) \tag{4.16}$$

where the centralities of the critical vertices i and j is computed as follows:

$$\begin{aligned} b_i(h,a) &= \frac{(1 + am_{ij}(g,a))b_i(g,a) - bm_{ii}(g,a)b_j(g,a)}{(1 + am_{ij}(g,a))^2 - a^2 m_{ii}(g,a)m_{jj}(g,a)} \\ b_j(h,a) &= \frac{(1 + am_{ij}(g,a))b_j(g,a) - bm_{jj}(g,a)b_i(g,a)}{(1 + am_{ij}(g,a))^2 - a^2 m_{ii}(g,a)m_{jj}(g,a)} \end{aligned}$$

Expressions (4.15) and (4.16) have obvious advantages, as (4.8) does in the case of the key player. We can compute the contribution of any link ij from the current data $\mathbf{M}(g,a)$ without having to recompute an inverse $\mathbf{M}(g\backslash\{ij\},a)$ for each $ij \in g$. These operations are clearly cheaper than the computation of an inverse. This fact becomes critical if we are to approximate the optimal interaction set. The reason is that the function $l_{g'}(g,a)$ is submodular in g', so that we can iteratively find the maximum of $l_{ij}(g,a)$ using (4.15) or (4.16) to get very fast to a good approximation of the problem:

Proposition 4.8 *The key interaction set problem can be approximated in polynomial-time by the use of a greedy algorithm, where, at each step, expression (4.15) or (4.16) is used to find the link ij (with highest $l_{ij}(g,a)$) that will become a member of the approximated key interaction set. The error of the approximation can be bounded as in (4.11).*

4.7.3 Optimal Attachments

Consider a planner that, in order to increase the quantity produced in an oligopolistic industry, may wish to optimally allocate an additional firm with collaboration links with others.

Given two disjoint[48] networks h' and h'' with $V(h') = S$ and $V(h'') = T = N\backslash S$ and a set of r links, let us construct a new network f by joining h' and h'' with r edges, in order to optimize a certain function. Formally, we wish to construct a bipartite network h between the vertices of the networks h' and h'', so as to *maximize* the overall centrality of $f = h' \cup h'' \cup h$. The adjacency matrix $\mathbf{H}$ is:

$$\mathbf{H} = \begin{pmatrix} \mathbf{O}_{s\times s} & \mathbf{V}_{s\times(n-s)} \\ \mathbf{W}_{(n-s)\times s} & \mathbf{O}_{(n-s)\times(n-s)} \end{pmatrix}$$

The problem is to find $\mathbf{V}$ and $\mathbf{W}$ in order to maximize $b(f,a)$ with the restriction $\mathbf{1}^T\cdot\mathbf{V}\cdot\mathbf{1}+\mathbf{1}^T\cdot\mathbf{W}\cdot\mathbf{1} = r$. Note that the matrix $\mathbf{M}(f,a) \equiv [\mathbf{I} - a\mathbf{F}]^{-1}$ can be written as

$$\mathbf{M}(f,a) = \begin{pmatrix} \mathbf{S}_{s\times s} & \mathbf{A}_{s\times(n-s)} \\ \mathbf{B}_{(n-s)\times n} & \mathbf{T}_{(n-s)\times(n-s)} \end{pmatrix}$$

48 The results can be extended in a straightforward way to the case where there are existing links among the agents.

Remark 4.7 *The following relation holds in any network f partitioned with the set of agents S.*

$$\begin{aligned}
\mathbf{S} &= \mathbf{M}(h',a) + a\mathbf{M}(h',a)\cdot\mathbf{V}\cdot\mathbf{B}\\
\mathbf{T} &= \mathbf{M}(h'',a) + a\mathbf{M}(h'',a)\cdot\mathbf{W}\cdot\mathbf{A}\\
\mathbf{A} &= a\mathbf{M}(h',a)\cdot\mathbf{V}\cdot\mathbf{T}\\
\mathbf{B} &= a\mathbf{M}(h'',a)\cdot\mathbf{W}\cdot\mathbf{S}
\end{aligned}$$

For instance, the matrix $\mathbf{S}$ summarizes the way in which one can travel in the network f by departing and ending at agents in S. This can be done as follows:

1. On the one hand, one can freely walk within h' and finish. This is the term $\mathbf{M}(h',a)$.
2. On the other hand, one can freely walk within h', cross the cut for a first time, and walk freely through the *whole* network f to end up in S. This is $a\mathbf{M}(h',a)\cdot\mathbf{V}\cdot\mathbf{B}$.

We can solve for $\mathbf{S}$ and $\mathbf{T}$:

$$\begin{aligned}
\mathbf{S} &= \left[\mathbf{I} - a^2\mathbf{M}(h',a)\cdot\mathbf{V}\cdot\mathbf{M}(h'',a)\cdot\mathbf{W}\right]^{-1}\cdot\mathbf{M}(h',a)\\
\mathbf{T} &= \left[\mathbf{I} - a^2\mathbf{M}(h'',a)\cdot\mathbf{W}\cdot\mathbf{M}(h',a)\cdot\mathbf{V}\right]^{-1}\cdot\mathbf{M}(h'',a)
\end{aligned}$$

This expression provides us with some of the structure of paths in the solution network f, given the previous information from h' and h'', and a particular attachment policy defined by $\mathbf{V}$ and $\mathbf{W}$. In order to get an intuition of the optimal choice, let us discard $a^2\mathbf{M}(h',a)\cdot\mathbf{V}\cdot\mathbf{M}(h'',a)\cdot\mathbf{W}$ and $a^2\mathbf{M}(h'',a)\cdot\mathbf{W}\cdot\mathbf{M}(h',a)\cdot\mathbf{V}$ inside the inverse terms. This eliminates from our analysis those paths that cross the cut at least twice. For simplicity, also consider that we deal with undirected networks so that $\mathbf{V} = \mathbf{W}^T$. We thus obtain an approximation for the result:

$$\begin{aligned}
\mathbf{S} &\approx \mathbf{M}(h',a)\\
\mathbf{T} &\approx \mathbf{M}(h'',a)\\
\mathbf{A} &\approx a\mathbf{M}(h',a)\cdot\mathbf{V}\cdot\mathbf{M}(h'',a)\\
\mathbf{B} &\approx a\mathbf{M}(h'',a)\cdot\mathbf{W}\cdot\mathbf{M}(h',a)
\end{aligned}$$

and noting that $b(f,a)$ sums all the entries in $\mathbf{S}$, $\mathbf{T}$, $\mathbf{A}$ and $\mathbf{B}$,

$$b(f,a) = b(h'a) + b(h'',a) + 2a\mathbf{b}(h'a)^T\cdot\mathbf{V}\cdot\mathbf{b}(h'',a) + o(a).$$

Our objective is to maximize the post-attachment centrality $b(f, a)$. Given the previous expression, we get our approximation by maximizing

$$\mathbf{b}(h'a)^T \cdot \mathbf{V} \cdot \mathbf{b}(h'', a) = \sum_{i,j} h_{ij} b_i(h', a) b_j(h'', a),$$

that is, the sum of products of the centralities from both sides of the bipartite network h. An iterative procedure clearly obtains this approximation for the problem, as we summarize in the next remark.

Remark 4.8 *Up to counting paths that cross the optimal connection network h at least twice, the following algorithm provides the optimal solution: At each step, among the unmatched agents in h, match those players $i \in V(h')$ and $j \in V(h'')$ with maximum b_i and b_j.*

Although we have not provided specific bounds for the approximation error, one can expect that when a is small enough, the number of such paths that cross h twice should not be high, compared to all the other paths in the final network f.

Nevertheless, we can still apply our result regarding submodular functions to obtain guarantees of approximation. The following result is similar to that for the key interaction set and the proof is omitted.

Given a network g and a disjoint g' ($g' \cap g = \varnothing$, meaning that new edges from g' are being incorporated to g), let $r_{g'}(g, a)$ be the number of new paths in $g \cup g'$ (weighted with a) that use some edge in g'. That is, this is the contribution of the edge set g' to the total connectivity of $g \cup g'$. Then, $r_{ij}(g, a)$ denotes the increase in total centrality by joining i and j in the network g:

$$r_{ij}(g, a) \equiv \sum_{k,l \in V(g)} (m_{kl}(g \cup \{ij\}, a) - m_{kl}(g, a))$$

Lemma 4.5 *The contribution of the new link $ij \notin g$ to the increase in the centrality of g is:*

1. *If ij is directed:*

$$r_{ij}(g, a) = a \frac{\widetilde{b}_i(g, a) b_j(g, a)}{1 - a m_{ji}(g, a)} \tag{4.17}$$

2. *If ij is undirected:*

$$r_{ij}(g,a) = a\left(b_i(g,a)b_j(g\cup\{ij\},a) + b_i(g\cup\{ij\},a)b_j(g,a)\right), \qquad (4.18)$$

where the post-linked centralities can be computed from g as

$$\begin{aligned} b_i(g\cup\{ij\},a) &= \frac{(1-am_{ij}(g,a))b_i(g,a) + bm_{ii}(g,a)b_j(g,a)}{(1-am_{ij}(g,a))^2 - a^2 m_{ii}(g,a)m_{jj}(g,a)} \\ b_j(g\cup\{ij\},a) &= \frac{(1-am_{ij}(g,a))b_j(g,a) + bm_{jj}(g,a)b_i(g,a)}{(1-am_{ij}(g,a))^2 - a^2 m_{ii}(g,a)m_{jj}(g,a)} \end{aligned}$$

In fact, it can be proven that $r_{g'}(g,a)$ is submodular in g', which allows us to get guarantees of approximation by applying a greedy algorithm.

Proposition 4.9 *The optimal attachment problem can be approximated in polynomial-time by the use of a greedy algorithm, where, at each step, expression (4.17) or (4.18) is used to find the link ij (with highest $r_{ij}(g,a)$) that should be added to the network. The error of the approximation can be bounded as in (4.11).*

4.8 Discussion and Open Problems

We have described the problem of optimally changing the outcome of a game with bilateral influences, by removing/adding agents/links. The fact that a network of complementarities emerges from our particular payoff structure, allows us to derive interesting properties of the optimal choice, based on graph theoretical concepts.

In order to deal with the applicability of our problem, we also show that, under some circumstances, a solution can be approximated within reasonable error bounds, given the inherent hardness attached to the computation of the key group. In particular, when the optimal policy can be written as a maximization program of a submodular set function, a simple greedy procedure that picks/removes the best element at each stage guarantees a fair approximation to the optimal policy.

We also model the optimal choice of the key group of players when the participation in the game becomes voluntary at a first stage. This completely changes the prescription of the key group, because an optimal choice has to take into account the fact that the elimination of a player may induce further voluntary exit from the game.

Most of our results also hold when we want to minimize submodular functions. But, in this case, the greedy algorithm is *not* adequate for our situation because guarantees of approximation are only possible for submodular functions. The problem of minimizing submodular functions *with* size constraints on the sets is NP-hard, in general. Nonetheless, we cannot provide a proof of hardness for our particular problems. On the other hand, we do not know whether the problem of choosing the network $g' \subseteq g$ that maximizes $l_{g'}(g, a)$ is hard or not. A reduction from the *edge cover* problem is straightforward, but this problem is polynomial-time solvable. The same situation arises when dealing with the complexity of maximizing the function $r_{g'}(g, a)$.

As we mentioned in the case of the two-stage game, it is not clear whether every ω admits a stable separation of the players. On top of that, we could not find an iterative fast procedure for finding maximum equilibrium in this game.

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