Study of a class of compact complex manifolds

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Director: Dr. Marcel Nicolau Reig

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CERTIFICO que aquesta memòria ha estat realitzada per na Mònica Manjarín Arcas sota la direcció del Dr. Marcel Nicolau Reig. Bellaterra, maig de 2006.

Dr. Marcel Nicolau Reig.

Agraïments

Aquest treball no hagués estat possible sense la guia, la dedicació i el suport del meu director Marcel Nicolau. La seva contribució ha anat molt més enllà de suggerir un problema. Li agraeixo sincerament la seva disponibilitat per discutir matemàtiques, la seva paciència i l'esforç que ha fet per ensenyar-me no només l'ofici sinó també matemàtiques, estiguessin o no relacionades amb aquest treball.

La construcció del capítol 3 està inspirada en una idea de'n Jean Jacques Loeb i en Marcel Nicolau per a estructures complexes sobre grups de Lie compactes semisimples. El seu ajut per comprendre els punts delicats i les seves suggerències han estat molt útils. En Jean Jacques Loeb s'ha interessat pel meu treball i he pogut beneficiar-me dels seus comentaris en més d'una ocasió. Li vull agrair també haver acceptat estar al tribunal de la tesi i escriure'n l'informe preliminar.

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Table of Contents

Introduction

Most of the known examples of complex manifolds, in particular all projective manifolds, are of Kähler type. Since Erich Kähler's article of 1933 (see [Käh33]) the subject of Kählerian geometry has been transformed into a major area of mathematics and Kähler manifolds are pretty well understood. One can quote Hodge theory, which imposes strong topological conditions on compact Kähler manifolds in terms of the De Rham and Dolbeaut cohomology groups, or the Albanese torus, which provides a description of holomorphic vector fields that allows to distinguish those that admit zeros from the non-vanishing ones. Furthermore, Kodaira immersion theorem characterizes which compact Kähler manifolds are projective. Compact Riemann surfaces are always Kählerian. In dimension 2, it is known that a compact surface is Kählerian if and only if its first Betti number (that is, the dimension of the first real De Rham cohomology group) is even. In higher dimensions, deciding whether a given compact complex manifold is Kähler is far from being a solved question[1].

If only for the strong topological restrictions that must verify a compact complex manifold to be Kählerian one would expect that, besides dimension 2, Kähler manifolds are the exception rather than the rule. A corollary of a result by Taubes (see [Tau92]) implies that every finite presentation group is the fundamental group of a non-Kähler compact complex 3-manifold. Although it is not related to our main goal we will briefly outline the proof of this fact (see C. LeBrun's expository article, [LeB97], *Twistors for Tourists*) for it is a beautiful example of the interplay between complex and Riemannian geometry.

Given a Riemannian oriented 4-manifold (M, g) one can consider the S^2-bundle $\pi : \mathrm{Z} \to \mathrm{M}$ of almost complex structures on M compatible with g and the orientation. The total space Z carries naturally an almost-complex structure

[1] The most recent significant progress on that problem I am aware of is the work by M.Paun and J.P. Demailly on the Kähler cone of a compact complex manifold.

such that it acts by rotation by $-90°$ on the fibre and as the almost-complex structure corresponding to that point of Z on the horizontal subspace of $T\mathrm{Z}$ (with respect to the Levi-Civitta connection). When (M, g) is an anti-self-dual 4-manifold the previous almost-complex structure on Z, which depends only on the conformal class $[g]$ of g, is integrable and the complex 3-manifold Z is called the *twistor space* of M. Taubes proved that for any compact 4-manifold M there exists $n_0 \geq 0$ such that the connected sum of M and n copies of $\overline{\mathbb{P}^2}$ ($\mathbb{P}^2$ with the reversed orientation) admits an anti-self-dual metric for $n \geq n_0$. Moreover, a theorem by Hitchin states that up to a conformal isometry the only compact anti-self-dual 4-manifolds with Kählerian twistor spaces are S^4 and $\overline{\mathbb{P}^2}$. The fundamental group of the twistor space of $\mathrm{M}\#\overline{\mathbb{P}^2}$ being the same that the one of M, one concludes that any finite presentation group is the fundamental group of a twistor space. All this suggests that non-Kählerian geometry is an area of geometry worth exploring in which not only results but also easy-to-handle examples are lacking.

One can easily notice that some of the standard techniques to produce examples of compact smooth manifolds do not work in the complex case. For instance, the maximum principle implies that the only compact complex manifolds of $\mathbb{C}^n$ obtained as the zeros of a holomorphic function are points. If one is interested in non-Kählerian examples the situation becomes even more difficult. Compact complex manifolds obtained as zeros of holomorphic functions of $\mathbb{C}P^n$ turn out to be projective by Chow's theorem (and therefore Kählerian). Compact complex manifolds obtained as a quotient of a bounded domain of $\mathbb{C}P^n$ are also algebraic (see [Sun80] or [Wel80]). On the other hand the only compact complex Lie groups are complex torus.

Historically, the first examples of non-Kähler manifolds were constructed by H. Hopf as a quocient of $\mathbb{C}^n \backslash \{0\}$ for $n > 1$ by a contracting biholomorphism of $\mathbb{C}^n$ which fixes the origin. Later, E. Calabi and B. Eckmann described a class of non-Kähler complex structures on the product $S^{2n+1} \times S^{2m+1}$ for $n, m \geq 0$ such that the corresponding complex manifold is the total space of a holomorphic principal bundle over $\mathbb{P}^n \times \mathbb{P}^m$ with fiber an elliptic curve. In [LN96] J.J. Loeb and M. Nicolau generalized Calabi-Eckmann and Hopf structures by the construction of a class of complex structures on the product $S^{2n+1} \times S^{2m+1}$ that contains the precedents. Every complex manifold of this class is the subset of the orbit space of an holomorphic flow on $\mathbb{C}^{n+m}$ and admits a non-vanishing holomorphic vector field. A similar idea has been used by S. Lopez de Medrano and A. Verjovsky in [LdMV97] to construct another family of non-Kählerian compact manifolds and later generalized by L. Meersseman in [Mee00].

In this thesis we present a different approach, even though it is partially inspired by Loeb-Nicolau class of examples, to construct a new family of complex structures on some classes of compact manifolds. Namely, we depart from a class of odd-dimensional compact connected real manifolds equipped with a normal almost contact structure and by very elementary geometrical constructions (products and suitable S^1-principal bundles and suspensions) we produce a compact manifold together with a complex structure defined by means of the normal almost contact structure. The basic ingredient will be the class $\mathcal{T}$ of odd-dimensional real manifolds admitting a CR-structure of maximal dimension and a transverse CR-action, which is also known as a normal almost contact structure (see [Bla02] for a survey on these concepts or chapter 2). More precisely, we will consider three constructions: (A) products of odd-dimensional real manifolds in the class $\mathcal{T}$, (B) S^1-principal bundles over a manifold in the class $\mathcal{T}$ (with an extra restriction on the bundle) and (C) suspensions of a manifold in the class $\mathcal{T}$ by an automorphism preserving the normal almost contact structure. The common feature of all the manifolds so constructed is the existence of a holomorphic vector field without zeros. Conversely, we will prove that the complex structure of a compact Kähler manifold with a non-vanishing holomorphic vector field can be recovered by the construction of case (C).

The idea of trying to relate normal almost contact structures on odd-dimensional manifolds to complex structures and of exploiting the parallelisms between them is not new. In 1963 A. Morimoto showed how to define an integrable almost complex structure on a product of two normal almost contact manifolds $\mathrm{M}_1 \times \mathrm{M}_2$ (see [Mor63]). M. Capursi, in 1984, characterized when the product metric corresponding to some particular metrics adapted to the normal almost contact structure on M_1 and M_2 is Kähler (see [Cap84]). The complex structures that we describe on $\mathrm{M}_1 \times \mathrm{M}_2$, case (A), for the choice of the parameter $\tau = -\mathrm{i}$ are exactly those of Morimoto. On the contrary, the constructions of cases (B) and (C) had not been studied and we present a unified approach to the three cases, which is quite close to that of A. Haefliger and D. Sundararaman in [HS85] or M. Brunella in [Bru96], in which the goal is to complexify a transversely holomorphic flow.

The natural question at this point is whether it is possible to characterize when the previous complex structures admit a Kähler metric or, if this goal turns out to be beyond our reach, to find obstructions for the previous complex manifolds to admit a Kähler metric. Note that even for case (A) such a result is stronger than the one in [Cap84]. Indeed, we prove that there is an obstruction for the resulting manifolds to be Kähler which can be expressed in terms of a cohomological invariant of the departing normal almost contact structure (or structures): the Euler class of the flow $\mathcal{F}$ associated to the CR-action. When the flow $\mathcal{F}$ is isometric, the Euler class agrees with the classical one, as one would expect. We will see that no complex manifold obtained by the constructions of cases (A), (B) or (C) can possibly be Kähler unless the Euler classes of all the involved normal almost contact structures are zero. One of the principal interests of this result is that it relies very little on particular characteristics of the resulting complex manifold other than the existence of a holomorphic vector field without zeros. This allows us to deduce several consequences on elliptic principal bundles, $\mathbb{C}^*$-principal bundles and $(\mathbb{C}, +)$-principal bundles. Moreover, when the flows associated to the CR-action are isometric we give necessary and sufficient conditions for the complex manifold to be Kähler exploiting the fact that for isometric flows that are transversely Kähler it is possible to develop a transverse Hodge theory analogous to the one on compact Kähler manifolds (see [EKA90]). In cases (A) and (B) the characterization is complete for isometric flows and can be roughly stated by saying that the resulting complex manifold is Kähler if and only if the Euler classes are zero and the flows are transversely Kählerian. In case (C) a result in this spirit requires assuming that the automorphism corresponding to the suspension is an isometry. Since this hypothesis is too restrictive we will study the question under different assumptions.

This work has been done keeping the following general goals in mind as guidelines:

A. Construct explicit and easy-to-handle examples of non-Kähler compact complex manifolds.

B. Obtain new examples of manifolds in the class $\mathcal{T}$.

C. Study the interplay between complex and normal almost contact structures, with special emphasis on Kählerianity.

In practice the results obtained can be divided in three groups:

1. Description of a class of complex structures on certain compact manifolds obtained from manifolds in the class $\mathcal{T}$ (Chapter 4);

2. Geometrical description of left-invariant normal almost contact structures on compact connected semisimple Lie groups and a geometrical construction of new non-invariant normal almost contact structures (Chapter 3);

3. Obtention of necessary (and in some cases sufficient) conditions for the complex manifolds in **1.-** to admit a Kähler metric (Chapter 5).

The rest of the thesis is organized as follows. In chapter 1 we recall all preliminary concepts and results. In chapter 2 we define the class $\mathcal{T}$, we present an alternative characterization of $\mathcal{T}$ that will be more suitable for our purposes, we define the Euler class and we describe briefly several classical families of examples. Finally, in chapter 6, we classify compact 3-manifolds admitting a normal almost contact structure and study the compact complex surfaces that are produced by the constructions in chapter 3.

In the rest of the introduction we intend to outline the main results of the thesis. We begin by recalling some concepts and by fixing the notation.

One says that a topological space M is a complex manifold when it carries a complex structure, that is, an atlas $\{(U_i, \varphi_i)\}_{i\in I}$ where the U_i are open subsets homeomorphic to open subsets of $\mathbb{C}^n$ such that $\mathrm{M} = \bigcup_{i\in I} U_i$ and the functions $\varphi_j \circ \varphi_i^{-1}$ are holomorphic. Every complex manifold carries naturally an almost complex structure, that is, a tensor $J : T\mathrm{M} \to T\mathrm{M}$ of type (1,1) such that $J^2 = -\mathrm{Id}$ (the tensor J is simple the rotation by $90°$ from the real direction to the imaginary direction of a complex line). Since J can be extended linearly to $T^{\mathbb{C}}\mathrm{M}$ one can define the eigenspaces $T^{1,0}\mathrm{M}$ and $T^{0,1}\mathrm{M}$ of J of eigenvalues i and $-$i respectively. That allows us to speak of vector fields of type $(1,0)$ and $(0,1)$ and by duality of forms of type (p,q). Moreover, by a famous theorem by Newlander-Niremberg we know that the almost complex structures that arise from complex structures are exactly those satisfying $[T^{1,0}\mathrm{M}, T^{1,0}\mathrm{M}] \subset T^{1,0}\mathrm{M}$ (often called involutive or integrable). From now on, we will identify complex structures and involutive almost complex structures.

A Riemannian metric g on a complex manifold compatible with the complex structure in the sense that J is an isometry with respect to it (i.e. g is hermitian) and such that $\nabla_X J = 0$ for every vector field X on M (where ∇ denotes the Levi-Civitta connection) is called a Kähler metric. The last condition is equivalent to imposing that the 2-form Φ, defined by $\Phi(X,Y) := g(X,JY)$ for every pair of vectors X, Y on M, is closed. Essentially we will use two major results on compact Kähler manifolds: Hodge theorem and a well-known corollary of it, the $\partial\overline{\partial}$-lemma, and Carrell-Liebermann theorem (see chapter 1).

On a compact smooth manifold M of odd dimension $2n+1$ the closest structure to a complex one is a CR-structure (of maximal dimension), i.e. a complex bundle $\Phi^{1,0}$ of complex dimension n such that $\Phi^{1,0} \cap \overline{\Phi^{1,0}} = \{0\}$ and $[\Phi^{1,0}, \Phi^{1,0}] \subset \Phi^{1,0}$. The real subbundle $\mathcal{D}$ underlying to $\Phi^{1,0}$ defines a distribution of codimension 1 at every point of M (notice that $\mathcal{D}$ needs not be integrable in the real sense, that is, we cannot assume that $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$). If M^{2n+1} is a compact connected manifold the pair $(\Phi^{1,0}, T)$ of a CR-structure and a vector field T without zeros is called a normal almost contact structure on M if T is transverse to the distribution $\mathcal{D}$ and the action induced by T preserves the CR-structure $\Phi^{1,0}$. Under this hypothesis the flow $\mathcal{F}$ defined by T is transversely holomorphic. Moreover, the 1-form ω defined by $\omega(T) = 1$ and $\omega_{|\mathcal{D}} = 0$ verifies that $d\omega$ is a basic 1-form of type (1,1). Conversely, let $\mathcal{F}$ be a transversely holomorphic flow on a compact manifold M^{2n+1} generated by a real vector field T without zeros and a 1-form ω such that $\omega(T) = 1$. Set $\mathcal{D} = \ker \omega$ and let J be the almost complex structure on $\mathcal{D}$ induced by $\mathcal{F}$. Then $(\mathcal{D}, J)$ is a CR-structure on M of dimension n and T defines a transverse CR-action if and only if $i_T d\omega = 0$ and the basic form $d\omega$ is of type $(1,1)$ with respect to the complex structure transverse to $\mathcal{F}$.

We define the Euler class of a normal almost contact structure, which is a natural generalization of the Euler class of an isometric flow, as the basic cohomology class given by

$$e_{\mathcal{F}}(\mathrm{M}) = [d\omega] \in H^2(\mathrm{M}/\mathcal{F}).$$

Note that if $(\Phi^{1,0}, T)$ and $(\Psi^{1,0}, T)$ are two normal almost contact structures their Euler classes coincide. The vanishing of the Euler class is equivalent to the existence of an integrable distribution $\mathcal{D}$ transverse to the vector field T and invariant by the action of T. Moreover, as a consequence of Tischler's theorem (see theorem 2.1.7), if $e_{\mathcal{F}}(\mathrm{M}) = 0$ the compact manifold M is a fibre bundle over S^1 and in particular M cannot be simply connected. On the other side, when $\mathcal{D}$ is a contact distribution the Euler class is not zero.

We will denote by $\mathcal{T}$ the class of compact connected manifolds M of odd dimension which are endowed with a normal almost contact structure. In view of the previous discussion is not difficult to verify that circle principal bundles over a compact complex manifold B such that there exists a connection 1-form ω with associated curvature form $d\omega$ of type (1,1) with respect to the complex structure on B are examples of manifolds in the class $\mathcal{T}$. More generally, transversely holomorphic isometric flows on a compact manifold for which there exists a characteristic 1-form ω such that $d\omega$ is of type $(1,1)$ with respect to the transverse complex structure also belong to the class $\mathcal{T}$. This includes certain classes of Seifert fibrations over complex orbifolds, for instance Brieskorn manifolds (see section 2.3). The suspension of a compact complex manifold N by an automorphism $g \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$ carries a natural normal almost contact structure where $\Phi^{1,0} = T^{1,0}\mathrm{N}$. When the compact manifold M has real dimension 3 we have a complete classification of the manifolds in the class $\mathcal{T}$ (see section 2.2 and chapter 6) based on Brunella-Ghys classification of transversely holomorphic flows on compact connected 3-manifold. Let M be a compact connected manifold of dimension 3 in the class $\mathcal{T}$, then, up to diffeomorphism, the manifold M^3 and the vector field inducing the CR-action belong to the following list:

- **(i)** Seifert fibrations over a Riemann surface with a vector field tangent to the fibres such that the isometric flow of the action admits a characteristic 1-form ω such that $d\omega$ is of type $(1,1)$.
- **(ii)** Linear vector fields in $\mathbb{T}^3$.
- **(iii)** Foliations on S^3 induced by a singularity of a holomorphic vector field in $\mathbb{C}^2$ in the Poincaré domain and their finite quotients, i.e. foliations on the lens spaces $L_{p,q}$.
- **(iv)** Suspensions of a holomorphic automorphism of $\mathbb{P}^1$ with a vector field tangent to the flow associated to the suspension.

Moreover, all the previous manifolds admit a normal almost contact structure such that the CR-action is the one induced by the corresponding vector field.

In chapter 3 we consider normal almost contact structures on a particular family of manifolds: compact connected semisimple Lie groups of odd dimension. In the first part of the chapter we study left-invariant normal almost contact structures, that is, structures on a compact Lie group K such that the vector field T and the CR-structure $\Phi^{1,0}$ are invariant by the action on the left of elements of K. A classical theorem proved independently by Wang and Samelson states that every compact Lie group of even dimension admits a complex structure such that left translations are holomorphic maps. A quite recent result by Charbonnel-Khalgui states that every compact semisimple Lie group of odd dimension admits a left-invariant CR-structure of maximal dimension (see chapter 3 for more details and more antecedents). We prove that every compact connected Lie group of odd dimension admits a normal almost contact structure (as a matter of fact it is enough to prove the statement for compact connected semisimple Lie groups, one can afterwards remove the hypothesis using general properties of Lie groups and Samelson's result). The main interest of the previous result lies in the fact that we can describe all left-invariant normal almost contact structures on a compact connected semisimple Lie groups geometricaly. Before we give the precise statement one must note that a left invariant normal almost contact structure on a Lie group K with Lie algebra $\mathfrak{k}$ is defined by a pair of Lie complex subalgebras $\mathfrak{l} \subset \mathfrak{l}'$ of dimensions n and $n+1$ of $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ such that:

(a) $\mathfrak{l} \cap \mathfrak{k} = \{0\}$;

(b) $\dim_{\mathbb{R}} \mathfrak{l}' \cap \mathfrak{k} = 1$;

(c) $\mathfrak{l}$ is an ideal of $\mathfrak{l}'$, i.e. $[\mathfrak{l}, \mathfrak{l}'] \subset \mathfrak{l}$.

The CR-structure $\Phi^{1,0}$ is given by $\bar{\mathfrak{l}}$ (the conjugation is a convention that will simplify things later) and the vector field defining the CR-action by $\mathfrak{l}' \cap \mathfrak{k}$. Obviously the previous definition is valid only at the point $e \in \mathrm{K}$ but since we are considering a left-invariant structure the extension is unique.

More generally a normal almost contact structure on a compact connected Lie group K of odd dimension $2n+1$ is determined by a complex subbundle V of $T^{\mathbb{C}}\mathrm{K}$ of rank n and a real vector field ξ on K such that $V' := V \oplus \langle \xi \rangle_{\mathbb{C}}$ is a complex subbundle of $T^{\mathbb{C}}\mathrm{K}$ of rank $n+1$ fulfilling:

(a) $V \cap T_p\mathrm{K} = \{e\}$;

(b) $[V, V] \subset V$;

(c) $[\xi, V] \subset V$;

for every $p \in \mathrm{K}$. By convention V corresponds to the distribution of vector fields of $T^{\mathbb{C}}_p\mathrm{K}$ of type $(0,1)$. The normal almost contact structure is then left-invariant if and only if the subbundle V and the vector field ξ are left-invariant by product of elements of K.

Theorem 1. *Let* K *be a semisimple compact connected Lie group of odd dimension* $2n+1$ *and rank* $2r+1$ *and let* G *be its universal complexification. Assume* *that* H *is a Cartan subgroup of* G *and* $\Lambda : \mathbb{C}^{r+1} \to$ H *a Lie group morphism verifying the transversality condition* (I). *If* B *is a Borel subgroup of* G *such that* H $\subset$ B *and* U *is its subgroup of unipotent elements then the Lie subalgebras* $\mathfrak{l}_\Lambda \subset \mathfrak{l}'_\Lambda$ *of* $\mathfrak{g}$ *associated to the complex Lie subgroups* $\mathrm{L'}_\Lambda = \Lambda(\mathbb{C}^{r+1}) \cdot \mathrm{U}$ *and* $\mathrm{L}_\Lambda = \Lambda(\{0\} \times \mathbb{C}^r) \cdot \mathrm{U}$ *of* G *define a left-invariant normal almost-contact structure* K_Λ *on* K. *Moreover, the Lie subgroup* L_Λ *is closed and the CR-structure on* K *determined by* L_Λ *agrees with the one induced by the embedding* $\mathrm{K} \hookrightarrow \mathrm{G}/\mathrm{L}_\Lambda$ *of* K *as a real hypersurface of the complex manifold* $\mathrm{G}/\mathrm{L}_\Lambda$. *Conversely, every left-invariant normal almost contact structure is induced by such a morphism* Λ *from* $\mathbb{C}^{r+1}$ *into a Cartan subgroup* H *of* G.

The transversality condition (I) (see lemma 3.1.20) is a condition on the ranks of some matrices associated to the Lie morphism Λ. To clarify the previous result it might be useful to recall that if $\rho : \mathrm{K} \to \mathrm{G}$ is the universal complexification of K, which is a non-compact complex algebraic group with Lie algebra $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \oplus i\mathfrak{k}$, then there is a decomposition of the Lie algebra $\mathfrak{g}$ which has as a consequence a decomposition in terms of Lie groups that we explote. Let us choose a maximal torus T of K with Lie algebra $\mathfrak{t}$, let H be the Cartan subgroup of G with Lie algebra $\mathfrak{r} = \mathfrak{t}^{\mathbb{C}}$ and a Borel subgroup B of G such that H $\subset$ B. We denote by U the subgroup of unipotent elements of B and by A the simply connected Lie subgroup of G with Lie algebra $i\mathfrak{t}$. Then the map

$$\Phi : (k, a, u) \mapsto k \cdot a \cdot u; \qquad k \in \mathrm{K},\ a \in \mathrm{A},\ u \in \mathrm{U}$$

is a diffeomorphism from the product manifold K $\cdot$ A $\cdot$ U into G (this is known as the Iwasawa decomposition). In particular $\mathrm{G}/\mathrm{U} \cong \mathrm{K} \cdot \mathrm{A}$ and note that $\mathrm{A} \cong \mathbb{R}^{\mathrm{rank}\,\mathrm{K}}$. In consequence, as the Lie group K is embedded in G/U and $\mathrm{U} \subset \mathrm{L}_\Lambda$, one can see the normal almost contact structure on K as induced by a locally free holomorphic $\mathbb{C}^{r+1}$-action $\varphi : \mathbb{C}^{r+1} \times \mathrm{G}/\mathrm{U} \to \mathrm{G}/\mathrm{U}$. Indeed, if we define

$$F_x := \varphi(\{0\} \times \mathbb{C}^r, [x]); \qquad F'_x := \varphi(\mathbb{C}^{r+1}, [x]),$$

that is, the leaves through $[x] \in \mathrm{G}/\mathrm{U}$ of the foliations defined by the actions of $\mathbb{C}^r \cong \{0\} \times \mathbb{C}^r$ and $\mathbb{C}^{r+1}$ respectively, then one has:

Lemma 2. *A locally free holomorphic* $\mathbb{C}^{r+1}$*-action* $\varphi : \mathbb{C}^{r+1} \times \mathrm{G}/\mathrm{U} \to \mathrm{G}/\mathrm{U}$ *fulfilling:*

- **(i)** $\dim_{\mathbb{R}}(F_p \cap \mathrm{K} \cdot a) = 0$*, for* $a = e$ *and each* $p \in \mathrm{K} \cdot e = \mathrm{K}$*,*

(ii) $\dim_{\mathbb{R}}(F'_p \cap \mathrm{K} \cdot a) = 1$, *for* $a = e$ *and each* $p \in \mathrm{K} \cdot e = \mathrm{K}$,

(iii) *there exists* $\lambda \in \mathbb{C}$ *such that* $\xi = d\varphi\big(\mathrm{Re}\,\big(\lambda\frac{\partial}{\partial z_0}\big)\big)$ *is tangent to* $\mathrm{K} \cdot a$, *for* $a = e$ *and each* $p \in \mathrm{K} \cdot e = \mathrm{K}$,

induces a normal almost contact structure on K. *We refer to the previous conditions as the transversality hypothesis* (II).

In our case the $\mathbb{C}^{r+1}$-action is induced by the Lie group morphism $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H}$ composed with the action of H by product on the right (that is well defined because $N(\mathrm{U}) = \mathrm{B}$).

The previous discussion allows us, in the second part of the chapter 3, to construct non-invariant normal almost contact structures on a compact connected semisimple Lie group K. The first remark is that the action of $(\mathbb{C}^*)^{4r+2} \cong \mathrm{H} \times \mathrm{H} = \mathrm{B}/\mathrm{U} \times \mathrm{B}/\mathrm{U}$ on the homogeneous space G/U given by

$$\mathrm{H} \times \mathrm{H} \times \mathrm{G}/\mathrm{U} \to \mathrm{G}/\mathrm{U}, \quad (h_1, h_2, [g]) \mapsto [h_1 \cdot g \cdot h_2]$$

is well defined. Using the same idea as in the left invariant case, one can construct normal almost contact structures on K exploiting this action of H×H on G/U. As before, it implies the existence of non-invariant normal almost contact structures on every compact connected Lie group of odd dimension.

Theorem 3. *Let* K *be a semisimple compact connected Lie group of odd dimension* $2n + 1$ *and rank* $2r + 1$ *and let* G *be its universal complexification. Assume that* $\mathrm{H} \subset \mathrm{B}$ *are a Cartan subgroup and a Borel subgroup of* G *respectively. Then every morphism of Lie groups* $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H} \times \mathrm{H}$ *inducing a locally free holomorphic action* $\varphi_\Lambda : \mathbb{C}^{r+1} \times \mathrm{G}/\mathrm{U} \to \mathrm{G}/\mathrm{U}$ *verifying* (II) *determines a normal almost contact structure in a natural way by lemma 3.2.4. Moreover, such a normal almost contact structure is left-invariant if and only if* $\Lambda = (e, \Lambda_2)$ *where* $\Lambda_2 : \mathbb{C}^{r+1} \to \mathrm{H}$ *is a morphism verifying the transversality hypothesis* (I). *In particular, there exist small deformations of the previous ones obtained by deforming* Λ *which induce suitable* $\mathbb{C}^{r+1}$*-actions defining normal almost contact structures on* K *generically non-invariant.*

The previous result implies that every compact connected Lie group of odd dimension is in the class $\mathcal{T}$, not only equipped with left-invariant normal almost contact structures but also provided with non-invariant (and in particular new) normal almost contact structures.

After discussing normal almost contact structures and classical and new examples of manifolds in the class $\mathcal{T}$ we intend to discuss how to obtain complex structures on the three classes of compact connected manifolds (A), (B) and (C) described above. Namely, we prove the following results:

Proposition 4 (Case (A)). *Let* M_1 *and* M_2 *be two manifolds in the class* $\mathcal{T}$*. There exists a 1-parametric family of integrable almost complex structures* K_τ *on the product* $\mathrm{M}_1 \times \mathrm{M}_2$*, for* $\tau \in \mathbb{C}\backslash\mathbb{R}$*, so that the complex manifold* $\mathrm{M}_1 \times \mathrm{M}_2$ *admits a non-vanishing holomorphic vector field* v*.*

Morimoto complex structures mentioned before are particular cases of this construction for $\tau = -i$. Calabi-Eckmann complex structures on the products of spheres of odd-dimension $S^{2n+1} \times S^{2m+1}$ are also examples.

Proposition 5 (Case (B)). *Let* M *be a manifold in the class* $\mathcal{T}$*. Denote by* T *the vector field inducing the CR-action and by* $\mathcal{F}_T$ *the transversely holomorphic flow induced by* T*. Let* $\pi : \mathrm{X} \to \mathrm{M}$ *be a* S^1*-principal bundle over* M *with Chern class* $[d\beta]$*, where* β *is a 1-form on* X *such that* $d\beta \in \pi^*\Omega^{1,1}(M/\mathcal{F}_T)$*. Then there exists a 1-parametric family of integrable almost complex structures* K_τ *on* X *for* $\tau \in \mathbb{C}\backslash\mathbb{R}$ *so that the complex manifold* X *admits a non-vanishing holomorphic vector field* v*.*

The Iwasawa manifold or, more generally, elliptic principal bundles over a compact complex manifold are examples of this situation.

Proposition 6 (Case (C)). *Let* M^{2n+1} *be a manifold in the class* $\mathcal{T}$ *with a CR-structure* $\Phi^{1,0}$ *and a vector field* T *inducing a transverse CR-action. Given* $f \in \mathrm{Aut}_{\mathrm{CR}}(\mathrm{M})$ *such that* $f_*T = T$ *the suspension* X *of* M *by* f *admits a 1-parametric family of integrable almost complex structures* K_τ *for* $\tau \in \mathbb{C}\backslash\mathbb{R}$ *so that the complex manifold* X *admits a non-vanishing holomorphic vector field* v *induced by* $T - \tau\frac{\partial}{\partial s}$*.*

An example of this situation is what we will refer to as a *double suspension* of a compact complex manifold. Namely, let N be a compact complex manifold and $f, g \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$ so that $f \circ g = g \circ f$, the quotient X of $\mathrm{N} \times \mathbb{C}$ by $F(x, z) = (f(x), z+1)$ and $G(x, z) = (g(x), z+\tau)$ for $\tau \in \mathbb{C}\backslash\mathbb{R}$ is a complex manifold with a non-vanishing vector field induced by $\frac{\partial}{\partial z}$.

Even though the last construction might seem rather particular it is not in view of the following theorem that we prove:

Theorem 7. *Every compact Kähler manifold* X *admitting a non-vanishing holomorphic vector field v can be obtained by the construction of proposition 6.*

Indeed, the example of the double suspension is also quite remarkable:

Theorem 8. *Every compact Kähler manifold* X *admitting a holomorphic vector field v without zeros admits a complex structure on the underlying smooth manifold* X *arbitrarily close to the original one that can be obtained by a double suspension.*

The previous results constitute the core of the chapter 4. To avoid presenting a different proof of the integrability of the almost complex structures defined in cases (A), (B) and (C) the first section of the chapter tries to describe an unified approach. In the three constructions the initial data allows to construct a smooth compact manifold X together with a locally free action of $\mathbb{R}^2$ (given by two linearly independent vector fields T_1 and T_2) inducing a transversely holomorphic foliation $\mathcal{F}$. Moreover, the real distribution of the normal almost contact structures gives rise to a distribution $\mathcal{D}$ transverse to $\mathcal{F}$. The integrability condition of the almost complex structure on X defined by imposing that it is compatible with the transverse holomorphic structure of $\mathcal{F}$ and that $v = T_1 - \tau T_2$ is a vector field of type (1,0) for any $\tau \in \mathbb{C}\backslash\mathbb{R}$ can be explicitly written and we will make use of this in the proof of the propositions. Moreover, when the distribution $\mathcal{D}$ is invariant by the vector fields T_1 and T_2, which is always the case for our constructions, v is a holomorphic non-vanishing vector field. Loeb-Nicolau complex structures on the product of odd-dimensional spheres (see [LN96]) can be produced by this construction for an invariant distribution $\mathcal{D}$. Indeed, the complex structure of every compact complex manifold with a holomorphic vector field without zeros can be recovered in the previous way, maybe for a non-invariant distribution $\mathcal{D}$.

We will next state some of the main results regarding the question of whether the complex manifolds obtained by the constructions of cases (A), (B) and (C) admit a Kähler metric (see chapter 5 for the complete discussion). We will begin by exhibiting obstructions in terms of the Euler class for the different situations. To simplify the exposition we introduce the notion of complexification of a pair (M, T), where M is a manifold in the class $\mathcal{T}$ and T the vector field defining the CR-action, which includes cases (A) and (C). We denote by $\mathcal{F}$ the transversely holomorphic flow induced by T. We say that a compact complex manifold X endowed with a non-singular holomorphic vector field v is a *complexification* of the pair (M, T) if:

(i) M is a real submanifold of X.

(ii) The transversely holomorphic structure on $\mathcal{F}$ is induced by the complex structure on X.

(iii) There exists $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda v) = T$.

Theorem 9. *Let* M *be a manifold in the class* $\mathcal{T}$ *and* T *the vector field inducing the CR-action. If* $e_{\mathcal{F}}(\mathrm{M}) \neq 0$ *then* (M, T) *admits no Kähler complexification.*

Corollary 10. *Let* M *be a manifold in the class* $\mathcal{T}$ *and* T *the vector field inducing the CR-action. If* M *admits a normal contact structure compatible with the CR-action induced by* T *then* (M, T) *admits no Kähler complexification.*

Corollary 11. *Let* M *be a manifold in the class* $\mathcal{T}$ *and* T *the vector field inducing the CR-action. If* $b_1(\mathrm{M}) = 0$*, in particular if* M *is simply connected, then* (M, T) *admits no Kähler complexification.*

A remarkable particular case of this situation is when M is a compact connected semisimple real Lie group of odd dimension, for its first Betti number is zero.

The previous results apply to the constructions of the cases (A) and (C). For the construction of the case (B) we prove the following:

Theorem 12. *Assume that* X *is a compact complex manifold constructed as in proposition* 5 *(case B) from a manifold* M *in the class* $\mathcal{T}$*. If* X *is Kählerian then* $e_{\mathcal{F}}(\mathrm{M}) = 0$ *and the* S^1*-principal bundle* $\pi : \mathrm{X} \to \mathrm{M}$ *is flat. In particular, if* X *is Kähler and* $H^2(\mathrm{M}, \mathbb{Z})$ *has no torsion then the* S^1*-principal bundle is topologically trivial.*

Theorem 13. *Assume that* X *is a compact complex manifold constructed as in proposition* 6 *(case C) from a manifold* M *in the class* $\mathcal{T}$*. If* X *is Kählerian then* $e_{\mathcal{F}}(\mathrm{M}) = 0$.

The main tool used in the proof of the previous results is a theorem by Carrell-Liebermann (see [CL73]) that states that a holomorphic vector field v over a compact Kähler manifold X has zeros if and only if for every holomorphic 1-form α on X we have $\alpha(v) = 0$. As all the complex structures that we are considering admit a non-vanishing vector field v if the compact complex manifold X is Kähler we can assume that there exists a closed holomorphic 1-form α such that $\alpha(v) = 1$. This implies the existence of a Levi-flat complex distribution on X invariant by the vector field v, which can be translated into a condition on the departing manifold M and on its normal almost contact structure.

Under more restrictive hypothesis we can obtain necessary and sufficient conditions for the resulting complex manifold to be Kählerian:

Theorem 14. *Let* $\mathrm{X} = \mathrm{M}_1 \times \mathrm{M}_2$ *be a complex manifold obtained by proposition 4 (case A) from two manifolds* M_1 *and* M_2 *in the class* $\mathcal{T}$ *such that the flows* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *in* M_1 *and* M_2*, respectively, induced by the vector fields of the normal almost contact structures are Riemannian. Then* X *is Kählerian if and only if* $e_{\mathcal{F}_1}(\mathrm{M}_1) = e_{\mathcal{F}_2}(\mathrm{M}_2) = 0$ *and the flows* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are isometric and transversely Kählerian (on* M_1 *and* M_2*, respectively).*

Theorem 15. *Let* X *be a complex manifold obtained by proposition 5 (case B) from a manifold* M *in the class* $\mathcal{T}$ *such that the flow* $\mathcal{F}_T$ *on* M *is Riemannian. Then* X *is Kähler if and only if the* S^1*-principal bundle* $\pi : \mathrm{X} \to \mathrm{M}$ *is flat,* $e_{\mathcal{F}_T}(\mathrm{M}) = 0$ *and the flow* $\mathcal{F}_T$ *is isometric and transversely Kählerian.*

The two last theorems are in fact two particular cases of a more general statement and its proof explotes strongly the fact that given a normal almost contact structure such that the flow $\mathcal{F}$ induced by the vector field of the CR-action is isometric and transversely Kählerian one can apply Hodge theory on the transverse part of the flow $\mathcal{F}$. The control on the part tangent to $\mathcal{F}$ is achieved thanks to the hypothesis on the Euler class. The difficulty in obtaining a complete characterization of whether the complex structures admit Kähler metrics or not when the flows are not isometric lies in the lack of control on the transverse part of the flow $\mathcal{F}$.

A different approach, based on a result of A.Blanchard (see [Bla56]) and a theorem of D.Liebermann (see [Lie78]), allows us to prove the following necessary and sufficient conditions for some of the constructions of case (C). The first theorem deals with double suspensions and characterize when the complex structure is Kähler (we also give a characterization of when a double suspension is a projective manifold, see chapter 5) and the second is a generalization of the first that is proved by the same arguments with the only difference that transverse Hodge theory plays the role of Hodge theory on a compact Kähler manifold.

Theorem 16. *Let* N *be a compact complex manifold and* $f, g \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$ *such that* $f \circ g = g \circ f$ *and let* X *be the suspension* $\mathrm{N} \times \mathbb{C}/\langle F, G\rangle$ *where* $F(x, z) = (f(x), z{+}1)$, $G(x, z) = (g(x), z{+}\tau)$ *and* $\mathrm{Im}(\tau) \neq 0$. *Then the following conditions are equivalent:*

- **(i)** X *is Kähler.*
- **(ii)** *There is a Kähler form* ω *on* N *such that* $[f^*\omega] = [g^*\omega] = [\omega]$.
- **(iii)** N *is Kähler and there are integers* $n, m > 0$ *such that* $f^n, g^m \in \mathrm{Aut}_0(\mathrm{N})$.

Theorem 17. *Let* M^{2n+1} *be a manifold in the class* $\mathcal{T}$ *with CR-structure* $\Phi^{1,0}$ *and vector field* T. *Assume that the flow* $\mathcal{F}$ *induced by the vector field* T *on* M *is isometric. Assume that* X *is a compact complex manifold constructed as in proposition* 6 *(case C) from* M. *If* X *is Kähler then the following conditions hold:*

- **(i)** *The Euler class* $e_{\mathcal{F}}(\mathrm{M})$ *is zero.*
- **(ii)** *The flow* $\mathcal{F}$ *is transverselly Kähler and there exists a Kähler transversal form* Φ *such that* $[f^*\Phi] = [\Phi] \in H^{1,1}(\mathrm{M}/\mathcal{F})$.

Moreover if the CR-structure is Levi-flat (so in particular the Euler class $e_{\mathcal{F}}(\mathrm{M})$ *is zero) and* $f^* = \mathrm{id}$ *acting on* $H^1(\mathrm{M}, \mathbb{C})$ *then* X *is Kähler if and only* ***(ii)*** *holds.*

Finally in chapter 6 we discuss examples of compact complex surfaces S that can be obtained by the previous discussions. Using that S admits a holomorphic vector field without zeros a well-known classification implies that such a surface must belong to the following list:

- **(I)** Complex tori.
- **(II)** Principal Seifert fibre bundles over a Riemann surface of genus $g \geq 1$ with fiber an elliptic curve.
- **(III)** Ruled surfaces over an elliptic curve.
- **(IV)** Almost-homogeneous Hopf-surfaces.

On the other hand the classification of transversely holomorphic flows on a compact connected 3-manifold by M.Brunella and E.Ghys plus a small discussion gives the list of possibilities for compact connected 3-manifolds in the class $\mathcal{T}$ stated above. In most cases we can determine precisely which compact complex surface is obtained by each construction.

Chapter 1

Preliminaries

In this chapter we briefly recall all the classical notions and results that we use hereinafter. While the first six sections deal with concepts that appear throughout all the thesis the last section is only used in chapter 3.

1.1 CR-structures and complex manifolds

Let M be a smooth manifold and suppose that $\Phi^{1,0}$ is a complex subbundle of dimension m of the complexified tangent bundle $T^{\mathbb{C}}\mathrm{M} = T\mathrm{M} \otimes \mathbb{C}$. We recall that the pair $(\mathrm{M}, \Phi^{1,0})$ is called a CR-*manifold* or that the bundle $\Phi^{1,0}$ is called a CR-*structure* on M of dimension m (cf. [KN69] or [Bog91]) if:

- **(i)** $\Phi^{1,0} \cap \overline{\Phi^{1,0}} = \{0\}$;
- **(ii)** $\Phi^{1,0}$ is involutive, i.e. $[\Phi^{1,0}, \Phi^{1,0}] \subset \Phi^{1,0}$.

The complex bundle $\Phi^{1,0}$ induces a real subbundle $\mathcal{D}$ of $T\mathrm{M}$ defined as $\mathcal{D} = T\mathrm{M} \cap (\Phi^{1,0} \oplus \overline{\Phi^{1,0}})$. We define an endomorphism $J : \mathcal{D} \to \mathcal{D}$ imposing that $v - \mathrm{i}Jv \in \Phi^{1,0}$ for every $v \in \mathcal{D}$. Note that we can determine the CR-structure by giving $(\mathrm{M}, \mathcal{D}, J)$. Setting $\Phi^{0,1} = \overline{\Phi^{1,0}}$ we have a decomposition $\mathcal{D}\otimes\mathbb{C} = \Phi^{1,0}\oplus\Phi^{0,1}$ where $\Phi^{1,0}$ and $\Phi^{0,1}$ are the eigenspaces of J (extended by complex linearity to $\mathcal{D} \otimes \mathbb{C}$) of eigenvalue i and $-$i respectively. We denote by $\mathrm{Aut}_{\mathrm{CR}}(\mathrm{M})$ the subset of Diff(M) of maps f such that df preserves $\mathcal{D}$ and commutes with J.

Now let $\{\varphi_t : t \in \mathbb{R}\}$ be the flow induced by a smooth $\mathbb{R}$-action on M. We say that $\{\varphi_t\}$ defines a CR-*action* if $\varphi_t \in \mathrm{Aut}_{\mathrm{CR}}(\mathrm{M})$ for each t. When $\dim_{\mathbb{R}} \mathrm{M} = 2m+1$ we call the action *transverse* to the CR-structure if the smooth vector field $T = (d\varphi_t/dt)_{t=0}$ is everywhere transverse to $\mathcal{D}$, i.e. $\langle T\rangle \oplus \mathcal{D}$ has real dimension $2m+1$ at every point.

An *almost complex structure* on a manifold M^{2n} is a complex subbundle $T^{1,0}\mathrm{M}$ of $T^{\mathbb{C}}\mathrm{M}$ of dimension n such that $T^{1,0}\mathrm{M} \cap \overline{T^{1,0}\mathrm{M}} = \{0\}$. We say that M is an *almost complex manifold*. Equivalently an almost complex structure on M is given by a tensor $J : T\mathrm{M} \to T\mathrm{M}$ of type $(1,1)$ such that $J^2 = -\mathrm{Id}$. Notice that a complex manifold M carries a canonical almost complex structure which can be easily defined, if $z_j = x_j + \mathrm{i}y_j$ are holomorphic coordinates, by $J\big(\frac{\partial}{\partial x_j}\big) = \frac{\partial}{\partial y_j}$ and $J\big(\frac{\partial}{\partial y_j}\big) = -\frac{\partial}{\partial x_j}$. If M is an almost complex manifold we can decompose the complexified tangent space $T^{\mathbb{C}}\mathrm{M}$ into $T^{1,0}\mathrm{M}\oplus T^{0,1}\mathrm{M}$ where $T^{1,0}\mathrm{M}$ and $T^{0,1}\mathrm{M}$ are the eigenspaces of J (extended by complex linearity to $T^{\mathbb{C}}\mathrm{M}$) of eigenvalue i and $-$i respectively. In particular $T^{1,0}\mathrm{M}\cap T^{0,1}\mathrm{M} = \{0\}$ and $T^{1,0}\mathrm{M}\oplus T^{0,1}\mathrm{M} = T^{\mathbb{C}}\mathrm{M}$. We also obtain a decomposition of the complexified exterior algebra: we denote by $\Omega^{p,q}(\mathrm{M})$ the space of sections of the bundle $\Lambda^p(T^{1,0}\mathrm{M})^* \otimes \Lambda^q(T^{0,1}\mathrm{M})^*$ so that $\Omega^r(\mathrm{M}) \otimes \mathbb{C} = \sum_{p+q=r} \Omega^{p,q}(\mathrm{M})$. We say that an r-form ω is of *type* (p,q) if $\omega \in \Omega^{p,q}(\mathrm{M})$. An almost complex structure induced by a complex structure is *involutive* (also called *integrable*), that is $[T^{1,0}\mathrm{M}, T^{1,0}\mathrm{M}] \subset T^{1,0}\mathrm{M}$. The reciprocal, that is, an integrable almost complex structure is the canonical almost complex structure associated to some complex structure, is true by a deep theorem by Newlander-Nirenberg (see [NN57]):

Theorem 1.1.1. *With the above notation the following conditions are equivalent:*

- **(i)** J *is induced by a complex structure on* M.
- **(ii)** $T^{1,0}\mathrm{M}$ *is involutive, i.e.,* $[T^{1,0}\mathrm{M}, T^{1,0}\mathrm{M}] \subset T^{1,0}\mathrm{M}$.
- **(iii)** $d\Omega^{p,q}(\mathrm{M}) \subset \Omega^{p+1,q}(\mathrm{M}) \oplus \Omega^{p,q+1}(\mathrm{M})$.
- **(iv)** *The Nijenhuis tensor* $N(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY]$ *vanishes identically.*

Consequently, a possible way to endow a real manifold M of dimension $2n$ with a complex structure is to define an almost complex structure on M and to check the involutivity of the distribution $T^{1,0}\mathrm{M}$ or any other equivalent condition.

1.2 Kähler manifolds

We say that a Riemannian metric g on an almost complex manifold (M, J) is *hermitian* if $g(X,Y) = g(JX,JY)$ for all pairs X,Y of vector fields on M. If a complex manifold M admits a hermitian metric g such that its Levi-Civitta connection ∇ verifies $\nabla_X J = 0$ for every vector field X on M we say that M is a *Kähler manifold* and that g is a *Kähler metric*. A classical example of Kähler manifold is $\mathbb{P}^n$ with the Fubini-Study metric and consequently all projective manifolds are also Kählerian (restricting the Fubini-Study metric).

Given an hermitian metric g on a complex manifold (M, J) we define the fundamental 2-form Φ associated to g as $\Phi(X,Y) = g(X,JY)$ for every $X,Y \in \mathfrak{X}(\mathrm{M})$. The real form Φ is of type $(1,1)$. Kähler metrics are characterized in terms of Φ as follows:

Proposition 1.2.1. *Let g be a hermitian metric on a complex manifold* M*. Then g is Kählerian if and only if $d\Phi = 0$.*

Therefore Φ represents a cohomology class in $H^2(\mathrm{M}, \mathbb{R})$ which is not zero if the manifold M is compact. Indeed, if M^n is a compact Kähler manifold then even Betti numbers $b_{2k} = \dim_{\mathbb{R}} H^{2k}(\mathrm{M}, \mathbb{R})$ are positive for $0 \leq k \leq n$. Clearly every 1-dimensional complex manifold is Kählerian because as $d\Phi$ is a 3-form it must vanish. A compact complex surface S is Kählerian if and only if $b_1(\mathrm{S}) = \dim_{\mathbb{R}} H^1(\mathrm{S}, \mathbb{R})$ is even (see [Lam99] or [Buc99]). For higher dimensions we are far from having such a simple characterization, however compact Kähler manifolds verify strong topological conditions (see [ABC$^+$96] for restrictions on the fundamental group and a survey of related topological questions).

If M is a complex manifold we denote by $\Omega^r(\mathrm{M})$ the sheaf of germs of holomorphic r-forms on M. If M is compact and Kählerian then the holomorphic q-forms $H^0(\mathrm{M}, \Omega^q)$ inject into $H^q(\mathrm{M}, \mathbb{C})$, that is every holomorphic form $\nu \neq 0$ is closed and non-exact. Let us denote by $H^{p,q}(\mathrm{M})$ *Dolbeaut's cohomology groups.* Recall that $H^{p,q}(\mathrm{M}) \cong H^q(\mathrm{M}, \Omega^p(\mathrm{M}))$ and set $h^{p,q} = \dim_{\mathbb{R}} H^{p,q}(\mathrm{M})$. One of the fundamental results of the theory of compact Kähler manifolds is the Hodge decomposition theorem (see [GH78] for details):

Theorem 1.2.2. *Let* M^n *be a compact Kähler manifold. Then*

$$H^r(\mathrm{M}, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(\mathrm{M}), \qquad for\ 0 \leq r \leq n$$

and $H^{p,q}(\mathrm{M}) \cong \overline{H^{q,p}(\mathrm{M})}$. *In particular* $b_k = \sum_{p+q=k} h^{p,q}$ *and* $h^{q,p} = h^{p,q}$.

It follows that if M^n is a compact Kähler manifold odd Betti numbers $b_{2k+1} = \dim_{\mathbb{R}} H^{2k+1}(\mathrm{M}, \mathbb{R})$ are even for $0 \leq k \leq n$. Note also that the class of the 2-form Φ belongs to $H^{1,1}(\mathrm{M}, \mathbb{C}) \cap H^2(\mathrm{M}, \mathbb{R})$.

In [HL83] Harvey-Lawson give a characterization in terms of currents of those compact complex manifolds which admit Kähler metrics, namely they prove that a compact complex manifold admit a Kähler metric if and only if it does not carry any non-trivial positive current which is a $(1,1)$-component of a boundary. More restrictions for a compact complex manifold M to be Kählerian expressed in terms of currents can be found in the recent work of Demailly-Paun (c.f. [DP04]) on the Kähler cone of a compact manifold (the set of $H^{1,1}(\mathrm{M}, \mathbb{C}) \cap H^2(\mathrm{M}, \mathbb{R})$ of all class of $(1,1)$-forms associated with Kähler metrics on M)).

An important consequence of the Hogde theorem that will be used later on is the so-called $\partial\overline{\partial}$-lemma:

Proposition 1.2.3. *Let* η *be a* d*-closed form of type* (p,q) *on a compact Kähler manifold* M *which is* d, ∂ *or* $\overline{\partial}$*-exact. Then there exists a form* γ *of type* $(p-1, q-1)$ *such that* $\eta = \partial\overline{\partial}\gamma$. *Furthermore, if* $p = q$ *and* η *is real we can choose* $i\gamma$ *to be real.*

1.3 The Albanese torus

Let M be a compact Kähler manifold and keep fixed the notation of the preceding section. The *Albanese torus* Alb(M) of M is defined as $H^0(\mathrm{M}, \Omega^1)^*/H_1(\mathrm{M}, \mathbb{Z})$. Assume $h^{1,0} = k$ and fix a basis $\omega_1, ..., \omega_k$ of $H^0(\mathrm{M}, \Omega^1)$. Then

$$\Delta = \left\{ \left(\int_\gamma \omega_1, ..., \int_\gamma \omega_k \right) \in \mathbb{C}^k \,|\gamma \in H_1(\mathrm{M}, \mathbb{Z}) \right\}$$

is a lattice in $\mathbb{C}^k$ and $\mathrm{Alb(M)} \cong \mathbb{C}^k/\Delta$, which is a complex torus. In particular if M is a complex torus then Alb(M) $\cong$ M. The *Jacobi map* J is the holomorphic map from M to Alb(M) defined by

$$p \mapsto \left(\alpha \mapsto \left(\int_{p_0}^p \alpha \bmod \Delta \right) \right)$$

where $\alpha \in H^0(\mathrm{M}, \Omega^1)$ and p_0 is an arbitrary base point of M. One of the most remarkable properties of the Jacobi map is its behaviour with respect to holomorphic vector fields on M. Indeed every holomorphic vector field v on M admits a projection to Alb(M) which vanishes identically if and only if v has zeros.

Let $\mathrm{Aut}_0(\mathrm{M})$ be the identity component of the group Aut(M) of holomorphic automorphisms of M and choose $f \in \mathrm{Aut}_0(\mathrm{M})$. There exists $f! \in \mathrm{Aut}(\mathrm{Alb}(\mathrm{M}))$ such that $J \circ f = f! \circ J$. Note that for every holomorphic 1-form α on M we have $f^*\alpha = \alpha$, then:

$$\int_{p_0}^{f(p)} \alpha = \int_{p_0}^{f(p_0)} \alpha + \int_{f(p_0)}^{f(p)} \alpha = \int_{p_0}^{f(p_0)} \alpha + \int_{p_0}^{p} f^*\alpha = \int_{p_0}^{f(p_0)} \alpha + \int_{p_0}^{p} \alpha$$

Therefore we can define $f!(y) = y + J(f(p_0))$. Let us denote by $\mathfrak{h}$ the Lie algebra of holomorphic vector fields over M and by $\mathfrak{g}$ the Lie algebra of holomorphic vector fields over Alb(M). The map $f \mapsto f!$ induces a homomorphism of Lie algebras between $\mathfrak{h}$ and $\mathfrak{g}$.

Theorem 1.3.1. *Let* M *be a compact Kähler manifold and* v *an holomorphic vector field on* M. *Then* v *has zeros if and only if* v *is tangent to the fibres of the Jacobi map.*

We refer the reader to [Mat71] for the proof. Equivalently, a vector field v has zeros if and only if it belongs to the kernel of the previous homomorphism $\mathfrak{h} \to \mathfrak{g}$. We denote this ideal by $\mathfrak{h}_0$. The above theorem is a reformulation of the Carrell-Liebermann theorem, that we will use later on:

Theorem 1.3.2 ([CL73])**.** *A holomorphic vector field* v *over a compact Kähler manifold* M *has zeros if and only if for every holomorphic 1-form* α *on* M *we have* $\alpha(v) = 0$.

1.4 Flows with transverse structures

Recall that a *foliation* $\mathcal{F}$ on a manifold M is given by an atlas $\{U_i, f_i, \gamma_{ij}\}_{i \in I}$ where:

(a) $\{U_i\}_{i \in I}$ is an open covering of M,

(b) f_i is a submersion from U_i onto a manifold V called the *transverse manifold*,

(c) γ_{ij} is a local diffeomorphism of V such that $f_i(x) = (\gamma_{ij} \circ f_j)(x)$ for every $x \in U_i \cap U_j$,

(d) the diffeomorphisms γ_{ij} verify the cocycle condition $\gamma_{ik}(x) = (\gamma_{ij} \circ \gamma_{jk})(x)$ for all $x \in f_k(U_i \cap U_j \cap U_k)$.

The *leaves* of $\mathcal{F}$ are defined on each open set U_i as the fibers of the submersion f_i.

If M and V are complex manifolds and f_i and γ_{ij} are holomorphic maps we say that $\mathcal{F}$ is a *holomorphic foliation*. A foliation $\mathcal{F}$ is said to be *orientable* if the plane field tangent to $\mathcal{F}$ is orientable and *transversely orientable* if there is a field complementary to the tangent field to $\mathcal{F}$ continuous and orientable. An orientable nonsingular foliation $\mathcal{F}$ of dimension 1 is always defined by a nonsingular vector field T, we call such a foliation together with the vector field a *flow* and we write $\mathcal{F}$ when the vector field is implicitly understood.

Defining a *transverse structure* for the foliation $\mathcal{F}$ is equivalent to imposing conditions on the pseudo-group Γ generated by $\{\gamma_{ij}\}$. For instance, a foliation $\mathcal{F}$ is *transversely holomorphic* if V has a complex structure invariant by Γ.

We denote by $T\mathcal{F}$ the tangent bundle of $\mathcal{F}$. A form α on M is called *basic* with respect to a foliation $\mathcal{F}$ if $i_S\alpha = i_S d\alpha = 0$ for every vector field S tangent to the leaves of $\mathcal{F}$. Thus we can consider the *basic de Rham complex* $\Omega^r(\mathrm{M}/\mathcal{F}, \mathbb{R})$ and the *basic cohomology* $H^*(\mathrm{M}/\mathcal{F}, \mathbb{R})$ (in an analogous way we define $\Omega^*(\mathrm{M}/\mathcal{F}, \mathbb{C})$ and $H^*(\mathrm{M}/\mathcal{F}, \mathbb{C})$).

1.5 Riemannian and isometric flows

A foliation $\mathcal{F}$ is *Riemannian* if there exists a Riemannian metric h on V invariant by Γ. This is equivalent to the existence of a Riemannian metric g on M whose transverse part is invariant along the leaves of $\mathcal{F}$. Such a Riemannian metric on M is called *bundle-like*.

Let (M, g) be a Riemannian manifold. A vector field T is said to be *Killing* if the associated 1-parameter group φ_t is an isometry for every t, equivalently if $L_T g \equiv 0$. We say that a one-dimensional orientable foliation $\mathcal{F}$ on a compact manifold M is *isometric* if there exist a Riemannian metric g on M and a non-vanishing Killing vector field T such that the integral curves of T are the leaves of $\mathcal{F}$. Rescaling the metric we can always assume that T is of constant length one. We will say that $\mathcal{F}$ together with a Killing vector field of length one is an *isometric flow*. An alternative (and classical) characterization of isometric flows is the following one:

Proposition 1.5.1. *Let $\mathcal{F}$ a flow on a compact smooth manifold* M *generated by a non-vanishing vector field T. Then T is a Killing vector field with respect to some Riemannian metric g on* M *if and only if $H = \overline{\{\varphi_t\}}_{t\in\mathbb{R}}$ is a compact subgroup of* Diff(M)*, where φ_t is the 1-parameter group associated to T.*

We say that a 1-form χ is a *characteristic form* for $\mathcal{F}$ and T if $\chi(T) = 1$ and $i_T d\chi = 0$. In particular $d\chi$ is basic and $L_T\chi = 0$, so χ is invariant by T. Actually, for an isometric flow there always exists at least one characteristic form that can be defined by imposing $\chi(T) = 1$ and $\chi_{|T^\perp} = 0$. Conversely, if $\mathcal{F}$ is an Riemannian flow induced by a vector field T and there exists a 1-form χ such that $\chi(T) = 1$ and $i_T d\chi = 0$ then the flow $\mathcal{F}$ is isometric. The basic cohomology class $e_g(\mathcal{F}) = [d\chi] \in H^2(\mathrm{M}/\mathcal{F})$ does not depend on the characteristic 1-form chosen, for if χ_1 and χ_2 are characteristic 1-forms then $\chi_1 - \chi_2 \in \Omega^1(\mathrm{M}/\mathcal{F})$. It is called the *Euler class* of $\mathcal{F}$ with respect to g and it does not depend on the metric up to a non-zero factor (see [Sar85]).

Example 1.5.2. An example of isometric flow on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ with the standard metric is given by the Killing vector field $T = \mathrm{Re}(\mathrm{i}R)$ for $R = z_1\frac{\partial}{\partial z_1} + ... + z_n\frac{\partial}{\partial z_n}$. A characteristic form is given by

$$\omega = \big(\mathrm{i}\sum_{j=1}^{n}(z_j d\bar{z}_j - \bar{z}_j dz_j)\big)_{|S^{2n-1}} = \big(2\sum_{j=1}^{n}(x_j dy_j - y_j dx_j)\big)_{|S^{2n-1}}.$$

It defines a contact structure on S^{2n-1}, i.e. $\omega \wedge d\omega^n \neq 0$. The Euler class is the basic cohomology class of $d\omega = \big(4\sum_{j=1}^{n} dx_j \wedge dy_j)\big)_{|S^{2n-1}}$, which is the class of the curvature form of the Hopf S^1-principal bundle $\pi : S^{2n+1} \to \mathbb{P}^n$.

A foliation $\mathcal{F}$ of codimension n on a compact manifold M is called *homologically orientable* if $H^n(\mathrm{M}/\mathcal{F}, \mathbb{C}) \neq 0$.

Theorem 1.5.3 ([MS85])**.** *An orientable Riemannian foliation $\mathcal{F}$ of dimension 1 on a compact manifold M^{n+1} is a flow of isometries if and only if $H^n(\mathrm{M}/\mathcal{F}, \mathbb{C}) \neq 0$.*

For the sake of clarity we prove the direct implication, that is, that every isometric flow is homologically orientable. Let T be a Killing vector field for a Riemannian metric g on a compact manifold M^{n+1}, let χ be a characteristic 1-form and η the volume form. Since $\mathcal{F}$ is isometric $L_T\eta = 0$ so $i_T\eta$ is closed and basic. Therefore we can write $\eta = \frac{1}{n+1} i_T\eta \wedge \chi$. If $i_T\eta$ were exact, i.e. $i_T\eta = d\alpha$, then $\eta = d\big(\frac{1}{n+1}\alpha \wedge \chi\big)$, which is a contradiction. Thus $i_T\eta$ defines a non-zero class of $H^n(\mathrm{M}/\mathcal{F}, \mathbb{C})$.

1.6 Transversely Kählerian flows

Let $\mathcal{F}$ be a transversely holomorphic foliation of complex codimension n. We will assume that $\mathcal{F}$ is homologically orientable, that is, $H^n(\mathrm{M}/\mathcal{F}, \mathbb{C}) \neq 0$. We consider the complex $\Omega^{p,q}(\mathrm{M}/\mathcal{F})$ of a smooth *basic forms of type* (p, q) and the operator

$$\overline{\partial} : \Omega^{p,q}(\mathrm{M}/\mathcal{F}) \to \Omega^{p,q+1}(\mathrm{M}/\mathcal{F})$$

inducing the *basic Dolbeaut cohomology* of $\mathcal{F}$, which we denote by $H^{p,q}(\mathrm{M}/\mathcal{F})$.

We say that $\mathcal{F}$ is *transversely hermitian* if there exists a hermitian metric h on the transverse manifold V invariant by Γ. In particular $\mathcal{F}$ is Riemannian. If there exists a *closed* real form Φ on M whose transverse part corresponds to a transversely hermitian metric h (analogously as in the Kähler case) we say that $\mathcal{F}$ is *transversely Kählerian*. We call such a form Φ a *transverse Kähler form* and note that the transverse part of Φ is of type $(1, 1)$. The analogous to Hodge decomposition theorem in this context is the following:

Theorem 1.6.1 ([EKA90])**.** *Let $\mathcal{F}$ be a homologically orientable and transversely Kählerian foliation on a compact manifold* M *of complex codimension n. Then:*

$$\begin{cases} H^r(\mathrm{M}/\mathcal{F}, \mathbb{C}) = \oplus_{p+q=r} H^{p,q}(\mathrm{M}/\mathcal{F}) & \quad 0 \leq r \leq n \\ H^{p,q}(\mathrm{M}/\mathcal{F}) = \overline{H^{q,p}(\mathrm{M}/\mathcal{F})}. \end{cases}$$

As a corollary one obtains the so-called $\partial\overline{\partial}$-lemma :

Lemma 1.6.2. *Let η be a d-closed basic form of type (p, q) on a compact manifold* M *with a homologically orientable and transversely Kählerian foliation such that η is d, ∂ or $\overline{\partial}$-exact as a basic form. Then there exists a basic form γ of type $(p - 1, q - 1)$ such that $\eta = \partial\overline{\partial}\gamma$. Furthermore, if $p = q$ and η is real we can choose* iγ *to be real.*

1.7 Lie groups

Unless otherwise specified K will denote a Lie group which can be real or complex, in general though we will reserve the notation K for real Lie groups and G for complex Lie groups. We refer the reader to [Hel78], [Che68], [Die75] and [OV94] for most proofs.

1.7.1 Basic concepts

A Lie group K is called *semisimple* if its Lie algebra $\mathfrak{k}$ is *semisimple*, i.e. if $[\mathfrak{k},\mathfrak{k}] = \mathfrak{k}$. A Lie group K is called *quasisimple* if its Lie algebra $\mathfrak{k}$ is *simple*, i.e. if it is not abelian and it does not contain other ideals besides $\{0\}$ and $\mathfrak{k}$. Every semi-simple Lie algebra can be represented in an unique way as finite direct sum of simple ideals $\mathfrak{k} = \sum_{j=1}^{s} \mathfrak{i}_j$ and every ideal of $\mathfrak{k}$ if a finite sum of ideals $\mathfrak{i}_j$. A semisimple Lie algebra has *center* $C(\mathfrak{g}) = \{X \in \mathfrak{g} : [X,Y] = 0, \ \forall Y \in \mathfrak{g}\}$ equal to $\{0\}$, therefore the *center* $Z(\mathrm{K})$ of a semisimple Lie group is discrete. Every semisimple Lie group K is, up to a covering, a product $\mathrm{K}_1 \times ... \times \mathrm{K}_r$ of quasisimple Lie groups. Besides, the only connected normal subgroups of a simply connected semisimple Lie group K are products of a certain number of K_j, therefore they are closed in K (cf. [Die75], p.39). It follows that every normal connected subgroup H of a semisimple compact Lie group K is closed because the universal covering $\widetilde{\mathrm{K}}$ of K is a finite covering (see section 1.7.5).

If $X \in \mathfrak{k}$ the adjoint of X is the endomorphism of K defined by $ad_X : Y \mapsto [X,Y]$. The map $X \mapsto ad_X$ is called *adjoint representation* and it sends every $X \in \mathfrak{k}$ to a derivation of $\mathfrak{k}$. Let $\mathfrak{k}$ be a Lie algebra over a field of characteristic zero. The *Killing form* κ of $\mathfrak{k}$ is defined as the bilinear form $\kappa(X,Y) = Tr(ad_X ad_Y)$ over $\mathfrak{k} \times \mathfrak{k}$ where Tr denotes the trace of an endomorphism of vector spaces.

Theorem 1.7.1. *A Lie algebra $\mathfrak{k}$ over a field of characteristic zero is semisimple if and only if the Killing form κ of $\mathfrak{k}$ is non-degenerate.*

The *n-th derived group* of K (for n a positive integer), which we will denote by $\mathcal{D}^n\mathrm{K}$, is the subgroup of K defined inductively in the following way: $\mathcal{D}^0\mathrm{K} = \mathrm{K}$ and $\mathcal{D}^{n+1}\mathrm{K} = [\mathcal{D}^n\mathrm{K}, \mathcal{D}^n\mathrm{K}]$ for all $n > 0$ is the commutator subgroup of $\mathcal{D}^n\mathrm{K}$. The group K is called *solvable* if there exists $n \geq 0$ such that $\mathcal{D}^n\mathrm{K} = \{e\}$. Equivalently a Lie group is solvable if its Lie algebra is *solvable*, i.e. if there exists a sequence

$$0 = \mathfrak{k}_l \subset ... \subset \mathfrak{k}_0 = \mathfrak{k} \qquad (*)$$

of subalgebras such that every quotient $\mathfrak{k}_i/\mathfrak{k}_{i+1}$ is abelian. If we set $\mathfrak{k}_0 := \mathfrak{k}$ and $\mathfrak{k}_{n+1} := [\mathfrak{k}_n, \mathfrak{k}_n]$ for every positive integer n the algebra $\mathfrak{k}$ is solvable if there exists l such that $\mathfrak{k}_l = 0$, in this case these $\mathfrak{k}_i$ provide a sequence verifying $(*)$.

Recall that every connected compact abelian Lie group is isomorphic to a torus. In particular, given a real compact Lie group K, every Lie subgroup T which is closed, connected and abelian is a torus. We say that T is a *maximal* *torus* of K if there does not exist any torus in K different from T and containing it. A necessary and sufficient condition for a connected Lie subgroup H to be a maximal torus is that its Lie algebra $\mathfrak{h}$ is an abelian maximal subalgebra of $\mathfrak{k}$. Every abelian connected Lie subgroup H of a compact connected Lie group K is contained in a maximal torus of K and every compact connected Lie group K is the reunion of its maximal torus (as a consequence of the exhaustivity of the exponential map on a compact connected Lie group).

Theorem 1.7.2. *Let* K *be a compact connected real Lie group,* T *a maximal torus of* K *and* A *a torus in* K*. There exists* $s \in \mathrm{K}$ *so that* $s\mathrm{A}s^{-1} \subset \mathrm{T}$ *(therefore* $s\mathrm{A}s^{-1} = \mathrm{T}$ *if* A *is a maximal torus).*

In particular all maximal torus are conjugate and we can define *rank*(K) as the dimension of a maximal torus.

Let G be a complex Lie group, we say that G is a *complex algebraic group* if it admits a structure of complex affine algebraic variety such that the map $\mu : \mathrm{G} \times \mathrm{G} \to \mathrm{G}$ defined by $\mu(x, y) = x \cdot y^{-1}$ is a morphism of algebraic varieties. Examples of complex algebraic groups are $\mathbb{C}^l$, $(\mathbb{C}^*)^l$ and $\mathrm{GL}(l, \mathbb{C})$ for $l \geq 1$. On $\mathbb{C}^n$ we define *Zariski's topology* by imposing that affine algebraic varieties are closed sets. An *algebraic subgroup* of G is a Lie subgroup of G closed with respect to the Zariski topology.

Theorem 1.7.3. *Every connected complex semisimple Lie group* G *admits a unique structure of complex algebraic group.*

1.7.2 The universal complexification of a real Lie group

The *universal complexification* of a compact real Lie group K is a couple (G, γ) where G is a complex Lie group and $\gamma : \mathrm{K} \to \mathrm{G}$ is a Lie group morphism such that for every complex Lie group $\widetilde{\mathrm{G}}$ and every morphism $u : \mathrm{K} \to \widetilde{\mathrm{G}}$ of real Lie groups there exists a unique complex analytic morphism $u^+ : \mathrm{G} \to \widetilde{\mathrm{G}}$ such that $u = u^+ \circ \gamma$. The couple (G, γ) is uniquely determined by K up to isomorphism. The universal complexification $\mathrm{G} = \mathrm{K}^{\mathbb{C}}$ of a compact Lie group K can be constructed in the following way. Since every compact Lie group admits a faithful complex linear representation

$$\rho : \mathrm{K} \hookrightarrow \mathrm{GL}(n, \mathbb{C})$$

(cf. [Die75], p.90) we can consider the Zariski closure G of $\rho(\mathrm{K})$ in $\mathrm{GL}(n, \mathbb{C})$. Then G is a complex linear algebraic Lie group (i.e. a subgroup of $\mathrm{GL}(n, \mathbb{C})$ defined by polynomial equations) with Lie algebra $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}} := \mathfrak{k} \otimes \mathbb{C}$. It contains K as completely real submanifold and verifies $Z(\mathrm{K}) = \mathrm{K} \cap Z(\mathrm{G})$. Moreover G is reductive, i.e. it has a finite number of connected components and the connected component G^0 of $\{e\}$ has a real compact form. Recall that a closed subgroup K with Lie algebra $\mathfrak{k}$ of a complex connected Lie group G with Lie algebra $\mathfrak{g}$ is called a *real form* of G (or of $\mathfrak{g}$) if $\mathfrak{k} \to \mathfrak{g}$ induces a Lie algebras isomorphism $\mathfrak{k}^{\mathbb{C}} \cong \mathfrak{g}$. If K is compact we then say that K is a *real compact form* of G. Every semisimple complex Lie algebra $\mathfrak{g}$ admits a real compact form (cf. [Hel78]). If G is a complex semisimple Lie group and K is a real compact form of G, the group K is semisimple, it is a maximal compact subgroup of G and we have Z(G)⊂K, thus Z(G)=Z(K) (cf. [Lee02]).

The following result, due to Cartan, Malcev and Iwasawa (cf. [Iwa49]), states that every Lie group admits a Lie subgroup which is a compact deformation retract.

Theorem 1.7.4. *Let* K *be a real connected Lie group, then* K *has a maximal compact subgroup* T *unique up to conjugacy and* K *is homeomorphic to the product* $\mathrm{T} \times \mathbb{R}^m$. *In particular* K *and* T *have the same homotopy groups.*

Let $\mathrm{G} = \mathrm{K}^{\mathbb{C}}$ be the universal complexification of a compact connected real Lie group K. Since K is a maximal compact subgroup of G we conclude from the previous theorem that G is homeomorphic to $\mathrm{K} \times \mathbb{R}^m$.

1.7.3 Cartan, Iwasawa and Levi decompositions

Let $\mathrm{G} = \mathrm{K}^{\mathbb{C}}$ be the universal complexification of a compact connected real Lie group K. A *Borel subgroup* B of G is a maximal solvable irreducible algebraic subgroup. All Borel subgroups of a complex algebraic group G are closed and conjugate to each other (cf. [Hum75]). A *Borel subalgebra* $\mathfrak{b}$ of $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ is a maximal solvable subalgebra. A *Cartan subgroup* H of G is the centralizer of a maximal torus. All Cartan subgroups are abelian, connected and conjugated to each other. The subalgebra of a Cartan subgroup is called a *Cartan subalgebra*. Given a Cartan subalgebra $\mathfrak{r}$ there exists a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ such that $\mathfrak{r} = \mathfrak{t}^{\mathbb{C}}$ (see [Bor91]). Fix a Borel subalgebra $\mathfrak{b}$ such that $\mathfrak{r} \subset \mathfrak{b}$ and set $\mathfrak{u} := [\mathfrak{b}, \mathfrak{b}]$.

The endomorphisms of $\mathfrak{g}$ defined by $\mathrm{ad}_{\mathfrak{g}}\mathfrak{r}$, i.e. $\phi_R(X) = [R, X]$ where $R \in \mathfrak{r}$ and $X \in \mathfrak{g}$, are diagonalizable due to the fact that $\mathfrak{g}$ is semisimple (c.f. [Hum78] and [Hal03]). Therefore, by a standard result in linear algebra they diagonalize simultaneously. We have thus a decomposition in proper spaces $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{r}^*} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [R, X] = \alpha(R) \cdot X, \ \forall R \in \mathfrak{r}\}$. Moreover $\mathfrak{g}_0 = \mathrm{C}(\mathfrak{r}) = \mathfrak{r}$. We denote by Φ the finite subset of $\alpha \in \mathfrak{r}^*$, $\alpha \neq 0$, such that $\mathfrak{g}_\alpha \neq 0$ and its elements will be called *roots* of $\mathfrak{g}$ relative to $\mathfrak{r}$. With this notation there is a decomposition in root spaces or *Cartan decomposition*:

$$\mathfrak{g} = \mathfrak{r} \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Proposition 1.7.5. *With the above notation:*

(a) *Φ spans $\mathfrak{r}^*$ and if $\alpha \in \Phi$ then $-\alpha \in \Phi$.*

(b) *If $\alpha, \beta \in \Phi$ then $\alpha + \beta \in \Phi$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.*

(c) *If $\alpha \in \Phi$, $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$ then $[X, Y] = \kappa(X, Y) \cdot T_\alpha$ where κ is the Killing form on $\mathfrak{g} \times \mathfrak{g}$ and T_α is defined imposing $\kappa(T_\alpha, R) = \alpha(R)$ for all $R \in \mathfrak{r}$. In particular $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ has dimension 1.*

(d) *For every $\alpha \in \Phi$ and $X_\alpha \in \mathfrak{g}_\alpha$, $X_\alpha \neq 0$, there exists $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $X_\alpha, Y_\alpha, H_\alpha = [X_\alpha, Y_\alpha] \in \mathfrak{r}$ span a simple subalgebra $\mathfrak{s}_\alpha$ of dimension 3 isomorphic to $sl(2, \mathbb{C})$ via*

$$X_\alpha \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad Y_\alpha \mapsto \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \quad H_\alpha \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$

(e) *If $\alpha \in \Phi$ then $\dim \mathfrak{g}_\alpha = 1$. In particular if $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ then $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{h}_\alpha$.*

We can choose a subset $\widetilde{\Phi}$ of Φ maximal with respect to the subsets Ψ such that $\alpha \in \Psi$ if and only if $-\alpha \notin \Psi$ and if $\alpha, \beta \in \Psi$ then $\alpha + \beta \in \Psi$. In that case $\mathfrak{r} = \oplus_{\alpha \in \widetilde{\Phi}} \mathfrak{h}_\alpha$ and $\mathfrak{b} = \mathfrak{r} \oplus_{\alpha \in \widetilde{\Phi}} \mathfrak{g}_\alpha$ is a Borel subalgebra (and every Borel subalgebra containing $\mathfrak{r}$ is of this type for a proper choice of the subset $\widetilde{\Phi}$). Moreover $\mathfrak{u} := [\mathfrak{b}, \mathfrak{b}] = \oplus_{\alpha \in \widetilde{\Phi}} \mathfrak{g}_\alpha$ and therefore $\mathfrak{b} = \mathfrak{r} \oplus \mathfrak{u}$.

Let H be the Cartan subgroup of G associated to the subalgebra $\mathfrak{r}$. It is the universal complexification of $(S^1)^{\mathrm{rank}\,\mathrm{K}}$ and therefore $\mathrm{H} \cong (\mathbb{C}^*)^{\mathrm{rank}\,\mathrm{K}}$. If U and B are the Lie subgroups of G associated to $\mathfrak{u}$ and $\mathfrak{b}$ respectively then, as a consequence of the Cartan decomposition, $\mathrm{B} \cong \mathrm{H} \cdot \mathrm{U}$.

From the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{r}$ one also obtains the following decomposition of $\mathfrak{g}$ as a real Lie algebra:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathrm{i}\mathfrak{t} \oplus \mathfrak{u},$$

which is known as *Iwasawa decomposition*. Note that in particular $\mathfrak{b} \cap \mathfrak{k} = \mathfrak{t}$. Moreover it follows that

$$\dim_{\mathbb{C}} \mathfrak{u} = (\dim_{\mathbb{R}} \mathrm{K} - \operatorname{rank} \mathrm{K})/2; \quad \dim_{\mathbb{C}} \mathfrak{b} = (\dim_{\mathbb{R}} \mathrm{K} + \operatorname{rank} \mathrm{K})/2.$$

We also conclude that $\dim_{\mathbb{R}} \mathrm{K}$ and $\operatorname{rank} \mathrm{K}$ have the same parity.

Theorem 1.7.6. *Let* G *be the universal complexification of a semisimple compact connected Lie group* K *with Lie algebra* $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$*. Set* $\mathfrak{r} = \mathfrak{t}^{\mathbb{C}}$ *where* $\mathfrak{t}$ *is an abelian maximal subalgebra of* $\mathfrak{k}$*,* $\mathfrak{b}$ *a Borel subalgebra such that* $\mathfrak{r} \subset \mathfrak{b}$ *and* $\mathfrak{u} = [\mathfrak{b}, \mathfrak{b}]$*. Let* A *and* U *be the subgroups of* G *with Lie algebras* $\mathrm{i}\mathfrak{t}$ *and* $\mathfrak{u}$ *respectively. Then the map*

$$\Phi : (k, a, u) \mapsto k \cdot a \cdot u; \qquad k \in \mathrm{K},\ a \in \mathrm{A},\ u \in \mathrm{U}$$

is a diffeomorphism from the product manifold $\mathrm{K} \cdot \mathrm{A} \cdot \mathrm{U}$ *into* G*. Moreover the groups* A *and* U *are simply connected.*

With the notation of the above theorem let T be the connected Lie subgroup of K corresponding to $\mathfrak{t}$ and B the Borel subgroup of G corresponding to $\mathfrak{b}$. Then $\mathrm{K}/\mathrm{T} \cong \mathrm{G}/\mathrm{B}$. In particular K/T admits a left invariant complex structure.

We finally recall Levi-Malcev's theorem on the existence of Levi decompositions. A representation of a Lie algebra $\mathfrak{g}$ as a sum $\mathfrak{r} + \mathfrak{s}$ of a solvable ideal $\mathfrak{r}$ and a semisimple subalgebra $\mathfrak{s}$ is called a *Levi decomposition* of the Lie algebra $\mathfrak{g}$.

Theorem 1.7.7 (Levi-Malcev)**.** *Let* $\mathfrak{g}$ *be a Lie algebra over a field of characteristic 0, then* $\mathfrak{g}$ *admits a Levi decomposition.*

1.7.4 Classification of abelian complex Lie groups

Every compact complex Lie group is abelian, in particular it is a complex torus. On the other hand there is a classification of abelian connected complex Lie groups due to A.Morimoto.

We say that a complex Lie group is a *Stein group* if G is Stein as a complex manifold. It is known that every abelian connected Stein group is isomorphic to

$\mathbb{C}^m \times (\mathbb{C}^*)^n$ for $m, n \geq 0$ (cf. [MM96]). A complex Lie group is called a *(HC)-group* if holomorphic functions over G are constant. It is known that every (HC)-group is abelian and that given a connected complex Lie group G there exists a unique normal closed connected complex subgroup G^0 such that G/G^0 is a Stein group and G^0 is a (HC)-group. A.Morimoto then proves the following:

Theorem 1.7.8 ([Mor66]). *Let* G *be an abelian connected complex Lie group. Then* G *is isomorphic to the product* $\mathrm{G}^0 \times \mathbb{C}^m \times (\mathbb{C}^*)^n$ *where* $m, n \geq 0$ *and* G^0 *is a (HC)-group. Moreover if* G_1 *and* G_2 *are abelian connected complex Lie groups and* $\mathrm{G}_i = \mathrm{G}_i^0 \times \mathbb{C}^{m_i} \times (\mathbb{C}^*)^{n_i}$ *are the previous decompositions then* $\mathrm{G}_1 \cong \mathrm{G}_2$ *(isomorphism of complex Lie groups) if and only if* $\mathrm{G}_1^0 \cong \mathrm{G}_2^0$*,* $m_1 = m_2$ *and* $n_1 = n_2$*.*

1.7.5 The topology of compact Lie groups

Theorem 1.7.9. (H. Weyl) *Let* K *be a semisimple compact connected Lie group. The universal covering* $\widetilde{\mathrm{K}}$ *of* K *is compact.*

Let K be a compact connected real Lie group and $\mathfrak{k}$ its Lie algebra. Then $\mathfrak{k} = [\mathfrak{k}, \mathfrak{k}] \oplus C(\mathfrak{k})$ and $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$ is a semisimple subalgebra of $\mathfrak{k}$. Recall that $C(\mathfrak{k})$ denotes the center of $\mathfrak{k}$ and that it is an abelian subalgebra of $\mathfrak{k}$. The universal covering $\widetilde{\mathrm{K}}$ of K is isomorphic to a product $\mathbb{R}^n \times \mathrm{K}'$ where K' is a semisimple simply connected compact Lie group. Moreover the center $Z(\mathrm{K}')$ of K' is finite and K is isomorphic to $(\mathbb{R}^n \times \mathrm{K}')/\mathrm{D}$ where D is a discrete subgroup of $Z(\mathbb{R}^n \times \mathrm{K}') = \mathbb{R}^n \times Z(\mathrm{K}')$. Consider the subgroup $\mathrm{D}' = \mathrm{D} \cap (\mathbb{R}^n \times \{e\})$, the compact Lie group $(\mathbb{R}^n \times \mathrm{K}')/\mathrm{D}'$ is a finite covering of K isomorphic to $(S^1)^n \times \mathrm{K}'$. Thus we have obtained:

Theorem 1.7.10 ([Die75]). *Let* K *be a compact connected real Lie group. Then* K *admits a finite covering of the form* $(S^1)^n \times \mathrm{K}'$ *where* K' *is a simply connected semisimple compact Lie group. Moreover* $\mathrm{K} \cong ((S^1)^n \times \mathrm{K}')/\Gamma$ *where* Γ *is a discrete subgroup of* $\{e\} \times Z(\mathrm{K}')$*.*

Corollary 1.7.11. *If* K *is a semisimple compact connected real Lie group then* $\pi_1(\mathrm{K})$ *is finite and* $b_1(\mathrm{K}) = 0$*.*

We will now see that the computation of the De Rham cohomology over a compact connected Lie group can be reduced to the computation of the cohomology of its Lie algebra. Let K be a compact connected real Lie group and $\mathfrak{k}$

its Lie algebra. We consider the complex of differentiable forms $\Omega^*(\mathrm{K})^{\mathrm{K}}$ over K invariants by the action of K on the left. Notice that $\Omega^*(\mathrm{K})^{\mathrm{K}} \cong \mathfrak{k}^*$. A theorem by Cartan (cf. [Hae85]) states that $H^*_{DR}(\mathrm{K},\mathbb{R})$ is isomorphic to the cohomology of the complex of left invariant forms, that we will denote by $H^*(\mathfrak{k}^*,\mathbb{R})$. The natural inclusion $\Omega^*(\mathrm{K})^{\mathrm{K}} \hookrightarrow \Omega^*(\mathrm{K})$ induces an isomorphism between $H^*(\mathfrak{k}^*,\mathbb{R})$ and $H^*(\mathrm{K},\mathbb{R})^{\mathrm{K}}$ (De Rham cohomology classes on K invariant by the action of K on the left). Since K is connected the action of K over $H^*(\mathrm{K},\mathbb{R})$ is trivial, therefore

$$H^*(\mathrm{K},\mathbb{R}) \cong H^*(\mathfrak{k}^*,\mathbb{R}).$$

On the other hand we can define the cohomology $H^*(\mathfrak{k},\mathbb{R})$ of a Lie algebra $\mathfrak{k}$ (cf. [CE48], [Kos50]), [Jac62]) using the complex of cochains $\mathcal{C}^*(\mathfrak{k})$ where a p-cochain c is a alternated linear p-form on $\mathfrak{k}$ with values on $\mathbb{R}$ and the differential d is defined by

$$\begin{aligned}(\mathrm{d}c)(X_0,...,X_p) &= \sum(-1)^i X_i c(X_0,...,\widehat{X}_i,...,X_p)\\ &\quad + \sum_{r<s}(-1)^{r+s}c([X_r,X_s],X_0,...,\widehat{X}_r,...,\widehat{X}_s,...,X_p).\end{aligned}$$

The restriction to left-invariant vector fields induces an isomorphism between $\Omega(\mathrm{K})^{\mathrm{K}}$ and $\mathcal{C}(\mathfrak{k})$ which yields

$$H^*(\mathfrak{k},\mathbb{R}) \cong H^*(\mathfrak{k}^*,\mathbb{R}).$$

Theorem 1.7.12. *If $\mathfrak{k}$ is a semisimple real Lie algebra of finite dimension then $H^1(\mathfrak{k},\mathbb{R}) = H^2(\mathfrak{k},\mathbb{R}) = 0$.*

Corollary 1.7.13. *If* K *is a compact connected semisimple Lie group then $b_1(\mathrm{K}) = b_2(\mathrm{K}) = 0$.*

Chapter 2

Normal almost contact structures. The class $\mathcal{T}$.

This chapter is devoted to the class of normal almost contact manifolds, which we will denote by $\mathcal{T}$. We introduce normal almost contact structures and describe several classical families of manifolds in the class $\mathcal{T}$. All manifolds are supposed to be smooth and connected and all differentiable objects (differentiable structures, tensors,...) to be of class $\mathcal{C}^\infty$, unless it is otherwise specified.

2.1 Generalities

In this section we introduce the class $\mathcal{T}$ of normal almost contact manifolds (see [Bla02]) and we give some equivalent definitions which are better adapted to our purposes.

Definition 2.1.1. We denote by $\mathcal{T}$ the class of compact connected manifolds M of odd dimension which are endowed with a CR-structure $\Phi^{1,0}$ of maximal dimension (i.e. $\dim_{\mathbb{C}} \Phi^{1,0} = n$ if $\dim_{\mathbb{R}} \mathrm{M} = 2n+1$) and a transverse CR-action induced by a flow $\{\varphi_t\}$.

Let M^{2n+1} be a smooth manifold and suppose that there are given an endomorphism φ on the tangent space, a vector field T and a 1-form ω. We say that (φ, T, ω) is an *almost contact structure* on M if: **(1)** $\omega(T) = 1$, **(2)** rank $\varphi = 2n$, **(3)** $\varphi(T) = 0$, **(4)** $\omega(\varphi(X)) = 0$ and $\varphi^2(X) = -X + \omega(X)T$ for every tangent vector field X on M. There is an almost contact structure on $\mathbb{R}$ given by $(0, \frac{\partial}{\partial t}, dt)$. If M_1 has an almost contact structure $(\varphi_1, T_1, \omega_1)$ then there is an almost complex structure K on $\mathrm{M}_1 \times \mathbb{R}$ defined by $K(X_1, a\frac{\partial}{\partial t}) = (\varphi_1(X_1) - aT_1, \omega_1(X_1)\frac{\partial}{\partial t})$. We say that an almost contact structure on M_1 is *normal* if K is integrable (cf. [Bla02]).

Proposition 2.1.2. *The class $\mathcal{T}$ is the class of normal almost contact manifolds.*

Proof. Let M^{2n+1} be a smooth manifold with a CR-structure $(\mathcal{D}, J)$ of dimension n and T a transverse vector field inducing a CR-action, let ω be the 1-form defined by $\ker\omega = \mathcal{D}$ and $\omega(T) = 1$ and using $T\mathrm{M} \cong \mathcal{D} \oplus \langle T\rangle$ set $\varphi = (J, 0)$. Then (φ, T, ω) defines a normal almost contact structure on M. The proof of the converse is analogous. □

We discuss now another equivalent way to determine a CR-structure of maximal dimension and a transverse CR-action on a manifold M^{2n+1}. Notice first that if M^{2n+1} is a manifold in the class $\mathcal{T}$ the flow $\mathcal{F}$ defined by the CR-action is transversely holomorphic.

Let $\mathcal{F}$ be a transversely holomorphic flow on a compact manifold M^{2n+1} generated by a real vector field T without zeros and let $\mathcal{D}$ be a distribution such that $T\mathrm{M} = \mathcal{D} \oplus T\mathcal{F}$. We will denote by $\Phi^{1,0}$ the vectors in $\mathcal{D}^{\mathbb{C}}$ of type $(1,0)$ with respect to the transverse complex structure. We assume that:

- **(i)** The vector field T preserves $\mathcal{D}$, i.e. $[T, \mathcal{D}] \subset \mathcal{D}$.
- **(ii)** $\Phi^{1,0}$ defines a CR-structure, i.e. $[\Phi^{1,0}, \Phi^{1,0}] \subset \Phi^{1,0}$.

Then T defines a transverse CR-action.

Note that in the above situation if ω is the 1-form on M defined by $\omega(T) = 1$ and $\mathcal{D} = \ker\omega$ then **(i)** holds if and only if $i_T d\omega = 0$. In that case $d\omega$ is basic and then **(ii)** holds if and only if $d\omega$ is of type $(1,1)$ with respect to the complex structure transverse to $\mathcal{F}$. Therefore we obtain the following characterization of normal almost contact structures:

Proposition 2.1.3. *Let $\mathcal{F}$ be a transversely holomorphic flow on a compact manifold M^{2n+1} generated by a real vector field T without zeros and a 1-form ω such that $\omega(T) = 1$. Set $\mathcal{D} = \ker\omega$ and J the almost-complex structure on $\mathcal{D}$ induced by $\mathcal{F}$. Then $(\mathcal{D}, J)$ is a CR-structure on M of dimension n and T defines a transverse CR-action if and only if $i_T d\omega = 0$ and the basic form $d\omega$ is of type $(1,1)$ with respect to the complex structure transverse to $\mathcal{F}$.*

Let M be a compact manifold in the class $\mathcal{T}$ and let $\mathcal{F}$ be the transversely holomorphic flow induced by the vector field T. Let ω be the 1-form associated to the normal almost contact structure.

Definition 2.1.4. With the above notation, we define the *Euler class* of the pair $(\mathrm{M}, \mathcal{F})$ as the basic cohomology class given by

$$e_{\mathcal{F}}(\mathrm{M}) = [d\omega] \in H^2(\mathrm{M}/\mathcal{F}).$$

This definition generalizes the classical notion of Euler class of an isometric flow. Note that the class $e_{\mathcal{F}}(\mathrm{M})$ only depends on the flow $\mathcal{F}$ (in particular, on the vector field T inducing the CR-action). Clearly the Euler class does not depend on the CR-structure. Let ω' be another 1-form inducing a normal almost contact structure on M, i.e. $\omega'(T) = 1$, $i_T d\omega' = 0$ and $d\omega'$ is of type $(1,1)$ with respect to the complex structure transverse to $\mathcal{F}$, then $\omega - \omega' \in \Omega^1(M/\mathcal{F})$. Therefore $[d\omega] = [d\omega']$. Nevertheless, to define the Euler class we have used the fact that there exists a distribution of maximal dimension transverse to T and invariant by the flow, which is equivalent to state that there exists a 1-form χ such that $\chi(T) = 1$ and $i_T d\chi = 0$. Note that $e_{\mathcal{F}}(\mathrm{M}) \in H^{1,1}(\mathrm{M}/\mathcal{F})$ but the class in $H^{1,1}(\mathrm{M}/\mathcal{F})$ might depend on the CR-structure.

Proposition 2.1.5. *With the above notation the following conditions are equivalent:*

(a) $e_{\mathcal{F}}(\mathrm{M}) = 0$.

(b) *There exists a* closed *1-form* χ *on* M *such that* $\chi(T) = 1$.

(c) *There exists a distribution transverse to* T *of maximal dimension and invariant by the flow which is* integrable.

The proof is straightforward. The distribution is given by $\ker \chi$ and since $L_T\chi = 0$ the form χ is invariant by the flow. Conversely one defines the 1-form χ imposing that it vanishes on the distribution and $\chi(T) = 1$.

Corollary 2.1.6. *With the above notation if* $e_{\mathcal{F}}(\mathrm{M}) = 0$ *then* M *is a fiber bundle over* S^1. *In particular* $b_1(\mathrm{M}) \neq 0$ *and* M *is not simply connected.*

The first statement of the previous corollary is a consequence of the following theorem by D. Tischler :

Theorem 2.1.7 ([Tis70]). *Let* M *be a compact manifold. If* M *admits a non-vanishing closed 1-form then* M *is a fibre bundle over* S^1.

The second statement of the corollary is an immediate consequence of the homotopy exact sequence associated to a fibration.

Proposition 2.1.8. *With the above notation, if there exists a contact form* χ *on* M *such that* $\chi(T) = 1$ *and* $i_T d\chi = 0$ *then* $e_{\mathcal{F}}(\mathrm{M}) \neq 0$.

We include the proof for the sake of clarity, however it must be noted that follows from an argument of Saralegui for isometric flows (cf. [Sar85]).

Proof. Suppose that $\dim \mathrm{M} = 2n+1$ and $e_{\mathcal{F}}(\mathrm{M}) = 0$. Then there exists a 1-form β basic with respect to $\mathcal{F}$ such that $d\beta = d\chi$. Therefore

$$d(\beta \wedge \chi \wedge d\chi^{n-1}) = \chi \wedge d\chi^n - \beta \wedge d\chi^n = \chi \wedge d\chi^n$$

since $\beta \wedge d\chi^n$ is a basic $(2n+1)$-form. It follows, by Stoke's theorem, that $\int_{\mathrm{M}} \chi \wedge d\chi^n = 0$, which contradicts the hypothesis $\chi \wedge d\chi^n \neq 0$. □

The rest of this chapter and the following one will be devoted to discuss some families of manifolds in the class $\mathcal{T}$. Essentially we will consider manifolds in the class $\mathcal{T}$ of the following types:

- *Isometric flows*, which include circle principal bundles over complex manifolds and Seifert fibratrions over complex orbifolds (in particular real hypersurfaces in complex manifolds);
- *Suspensions of complex manifolds*;
- *Lie groups* (next chapter).

2.2 Isometric flows.

The next result follows immediately from the previous discussion:

Corollary 2.2.1. *Let* M *be a compact manifold and* $\mathcal{F}$ *a transversely holomorphic isometric flow on* M. *Let* T *be a Killing vector field,* ω *a characteristic 1-form and* J *the induced almost-complex structure on* $\mathcal{D} = \ker \omega$. *Then* $(\mathcal{D}, J)$ *is a* CR*-structure on* M *if and only if* $d\omega$ *is of type* $(1,1)$. *In that case* T *defines a transverse* CR*-action.*

Remark 2.2.2. Recall that a Riemannian real flow $\mathcal{F}$ induced by a vector field T without zeros is isometric if and only if there exits an invariant transverse distribution, equivalently if and only if there exists a basic form η such that $\eta(T) = 1$. Therefore if M is a compact manifold endowed with a normal almost contact structure such that the flow $\mathcal{F}$ induced by the CR-action is Riemannian then $\mathcal{F}$ is an isometric flow.

Circle principal bundles are a particular case of isometric flows. Let B be a compact complex manifold, $\pi : \mathrm{M} \to \mathrm{B}$ a S^1-principal bundle, T the fundamental vector field of the action and ω a connection form. We can endow M with a Riemannian metric so that the flow $\mathcal{F}$ generated by T is isometric. Then T is a Killing vector field and ω is a characteristic form. Furthermore the flow $\mathcal{F}$ is transversely holomorphic with respect to the complex structure induced by B. Then $(\mathcal{D} = \ker \omega, J)$ is a CR-structure on M if and only if the curvature form $d\omega$ is of type $(1,1)$ on B. It is known that a S^1-principal bundle admits such a connection form if and only if it is the unit bundle associated to a hermitian metric on a holomorphic line bundle. When this hypothesis is verified the vector field T induces a transverse CR-action. Notice that if $\dim_{\mathbb{R}} \mathrm{M} = 3$ this condition is always fulfilled since every 2-form on a compact Riemann surface is of type $(1,1)$.

More generally, compact Seifert fibrations over a complex orbifold also provide examples of transversely holomorphic isometric flows. Let Γ be a pseudogroup of complex automorphisms of $\mathbb{C}^n$. One defines a *complex orbifold* of dimension n to be a Hausdorff paracompact space which is locally homeomorphic to the quotient space of $\mathbb{C}^n$ by a finite group of automorphisms belonging to Γ (see [Sat56] for a precise definition). Assume now that M is a smooth manifold such that S^1 acts freely on M on the right, the quotient space of M by the action of S^1 is a complex orbifold B and the canonical projection $\pi : \mathrm{M} \to \mathrm{B}$ is differentiable. We say that M is a *Seifert fibration* over B if M is locally homeomorphic to the quotient space of $\mathbb{C}^n \times S^1$ by a finite subgroup of automorphisms of $\mathbb{C}^n \times S^1$ belonging to $\Gamma \times S^1$ and the projection p is the one induced by $\pi_1 : \mathbb{C}^n \times S^1 \to \mathbb{C}^n$ where $\pi_1(z,t) = z$ (again see [Sat56] for the precise definition). The S^1-action on M is given by a global vector field without zeros and since all the orbits are closed if we consider the 1-parametric flow φ_t corresponding to T it generates a compact subgroup of Diff(M). Therefore if M is compact the flow $\mathcal{F}$ induced by T is isometric and it is clearly transversely holomorphic. We are thus under the hypothesis of the corollary 2.2.1 and we will obtain a CR-structure on M provided that there exists a characteristic 1-form ω such that $d\omega$ is of type $(1,1)$.

Let now $(\mathcal{D}, J)$ be a CR-structure on M^{2n+1} and suppose that the distribution $\mathcal{D}$ is a contact structure, i.e., if ω is a 1-form such that $\ker\omega = \mathcal{D}$ then $\omega\wedge(d\omega)^n \neq 0$. Then $(\mathcal{D}, J)$ is called a strictly pseudo-convex CR-structure on M. The couple of a strictly pseudo-convex CR-structure of maximal dimension and a transverse CR-action on an odd-dimensional manifold is also known as a *normal contact structure*. The following result is well-known, we include the proof for the sake of clarity.

Proposition 2.2.3. *Let* M *be a compact connected 3-manifold. If* $(\mathcal{D}, J)$ *is a strictly pseudo-convex* CR*-structure and* T *a vector field inducing a transverse* CR*-action then* T *is Killing for a Riemannian metric* g*.*

Proof. The hypothesis $\omega \wedge d\omega \neq 0$, $i_T d\omega = 0$ and $\omega(T) = 1$ imply that $d\omega$ is a non-degenerate form on $\mathcal{D}$, thus we can assume that $d\omega(X, JX) > 0$ for every $X \in \mathcal{D}$ such that $X \neq 0$ (note that $\omega \wedge d\omega(T, X, JX) = 2d\omega(X, JX) \neq 0$ for every $X \in \mathcal{D}$ such that $X \neq 0$). Moreover $d\omega$ is a closed real-valued 2-form of type $(1,1)$ invariant by the action of the vector field T. Therefore $d\omega$ defines a hermitian metric on $\mathcal{D}^{\mathbb{C}}$ invariant by the action of T. Since $d\omega$ is real-valued $g = \omega \otimes \omega + \tilde{g}$, where $\tilde{g}(X, Y) := d\omega(X, JY)$, is a Riemannian metric on M such that T is a Killing vector field with respect to g. □

Corollary 2.2.4. *If* $(\mathrm{M}, \mathcal{D}, J)$ *is a strictly pseudo-convex CR-structure,* T *a vector field on* M *inducing a transverse CR-action and* $\mathcal{F}$ *the flow induced by* T*, then* $e_{\mathcal{F}}(\mathrm{M}) \neq 0$*.*

The corollary is a consequence of proposition 2.1.8.

When M is a compact manifold of dimension 3 there is a classification due to H.Geiges of the manifolds admitting a normal almost contact structure based on the classification of compact complex surfaces:

Theorem 2.2.5 ([Gei97])**.** *A compact 3-manifold admits a normal almost contact structure if and only if it is diffeomorphic to one of the following manifolds:*

(a) $\Gamma\backslash S^3$ *with* $\Gamma \subset \mathrm{SO}(4) \cong \mathrm{Isom}_0(S^3)$*;*

(b) $\Gamma\backslash\widetilde{SL}_2$ *with* $\Gamma \subset \mathrm{Isom}_0(\widetilde{SL}_2)$*;*

(c) $\Gamma\backslash\mathrm{Nil}^3$ *with* $\Gamma \subset \mathrm{Isom}_0(\mathrm{Nil}^3)$*;*

(d) $\Gamma\backslash\mathbb{H}^2 \times \mathbb{R}$ *with* $\Gamma \subset \mathrm{Isom}_0(\mathbb{H}^2 \times \mathbb{R})$*;*

(e) $\mathbb{T}^2$*-bundles over* S^1 *with periodic monodromy;*

(f) $S^2 \times S^1$*;*

where $\widetilde{SL}_2$ *denotes the universal covering of* $PSL_2(\mathbb{R})$*,* Nil^3 *the Heisenberg group of upper triangular* (3×3)*-matrices and* $\mathbb{H}^2$ *the hyperbolic plane.* $\mathrm{Isom}_0(\mathrm{X})$ *stands for the identity component of the isometry group a Riemannian manifold* X *and* Γ *denotes a discrete subgroup of* $\mathrm{Isom}_0(\mathrm{X})$ *acting freely on* X*.*

For normal almost contact structures on a compact connected 3-manifold M such that the flow induced by the CR-action is isometric we can give a more explicit classification. Note that as a consequence of proposition 2.2.3 this case includes all normal contact structures. From the classification of isometric flows on compact 3-manifolds (see [Car84]) we conclude that if a compact connected 3-manifold admits a normal almost contact structure such that the flow induced by the CR-action is isometric then it is diffeomorphic to one of the following manifolds:

(i) Seifert fibrations.

(ii) Linear foliations of $\mathbf{T}^3$.

(iii) Lens spaces $L_{p,q} = S^3/\langle\gamma_{p,q}\rangle$ for $p, q \in \mathbb{Z}$ with action $\gamma_{p,q}(z_1, z_2) = (e^{2\pi i/p}z_1, e^{2\pi i/q}z_2)$ and flow given by $\varphi_t[z_1, z_2] = [e^{i\mu_1 t}z_1, e^{i\mu_2 t}z_2]$ where $\mu_1, \mu_2 \in \mathbb{R}$. Notice that $L_{1,1} = S^3$.

(iv) $S^2 \times S^1$ and the flow given by the suspension of an irrational rotation of S^2, i.e. if we identify S^2 with $\mathbb{C}P^1$ we consider the suspension of $f(z) = e^{i\alpha}z$ where $\alpha \notin \mathbb{Q}$.

Moreover the flow associated to the CR-action is the one specified in each case and by proposition 2.1.8 we conclude that we can obtain a normal contact structure only in cases **(iii)** and **(i)**. Note that when $\dim_{\mathbb{R}} \mathrm{M} = 3$ the base B of a Seifert fibration $\pi : \mathrm{M} \to \mathrm{B}$ is a Riemann surface and the Euler class of the isometric flow induced by a Seifert fibration is zero if and only if the fibration is flat. On the other hand every isometric flow on a 3-manifold is transversely holomorphic, since we can define a transverse operator J on the distribution $\mathcal{D}$

orthogonal to the leaves and the pair $(\mathcal{D}, J)$ defines a CR-structure on M such that the vector field T induces a transverse CR-action with respect to it. Then from the corollary 2.2.1 it is clear that all the previous isometric flows admit a normal almost contact structure.

In the last chapter we will prove that actually the only compact connected 3-manifolds which are in the class $\mathcal{T}$ are the ones in the previous list plus foliations on S^3 induced by a singularity of a holomorphic vector field in $\mathbb{C}^2$ in the Poincaré domain and their finite quotients, i.e. foliations on the lens spaces $L_{p,q}$ (note that the flows in **(iii)** are a particular case of these ones) and suspensions of a holomorphic holomorphism of $\mathbb{P}^1$ (i.e. case **(iv)** with $f \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^1)$ instead of an irrational rotation), regardless of whether the vector fields induce an isometric flow or not.

2.3 Real hypersurfaces in complex manifolds.

Let Ω be a real hypersurface of a compact complex manifold V with a transverse holomorphic vector field S such that $T = \mathrm{Re}(S)$ is tangent to Ω. Set $\mathcal{D} = T\Omega \cap JT\Omega$ where J denotes the almost complex structure on Ω. Then $(\mathcal{D}, J)$ is a CR-structure on Ω. Moreover if the vector field T preserves $\mathcal{D}$ then T defines a transverse CR-action. This is the case for instance for $S^{2n-1} \subset \mathbb{C}^n$ and $S = \mathrm{i}a_1 z_1 \frac{\partial}{\partial z_1} + ... + \mathrm{i}a_n z_n \frac{\partial}{\partial z_n}$ with $a_j \in \mathbb{R}^+$ for $j = 1, ..., n$. Set now $(q_1, ..., q_n) \in (\mathbb{N}\backslash\{0\})^n$. We say that $p(z_1, ..., z_n)$ is a weighted homogeneous polynomial of type $(q_1, ..., q_n)$ if $p(t^{q_1} z_1, ..., t^{q_n} z_n) = t^d p(z_1, ..., z_n)$ for some $d \in \mathbb{N}$. Let us assume that $\mathrm{V} = \{p = 0\} \subset \mathbb{C}^n$ is a smooth manifold or that it has an isolated singularity at the origin. Set $\widehat{R} = q_1 z_1 \frac{\partial}{\partial z_1} + ... + q_n z_n \frac{\partial}{\partial z_n}$ and let J be the almost complex structure of $\mathbb{C}^n$. Then $\widehat{R}$ is transverse to the unit sphere S^{2n-1} and we can define a CR-structure of maximal dimension on $\mathrm{M}(p) = \mathrm{V} \cap S^{2n-1}$ setting $(\mathcal{D} = T\mathrm{M}(p) \cap J(T\mathrm{M}(p)), J_{|\mathcal{D}})$. Moreover the vector field $T = Re(\mathrm{i}\widehat{R})$ induces a CR-action on $\mathrm{M}(p)$. A particular case of this situation are Brieskorn manifolds, which are given by $p(z_1, ..., z_n) = (z_1)^{a_1} + ... + (z_n)^{a_n}$ where $a_j \in \mathbb{Z}$ and $a_j \geq 2$. All these examples are S^1-Seifert fibre bundles over a complex orbifold for which the S^1-action is given by the real vector field T induces a projection $\pi : \mathrm{M}(p) \to \mathrm{B}$ over a complex orbifold B.

2.4 Suspensions of complex manifolds.

The opposite situation to a strictly pseudo-convex CR-structure from the point of view of the real integrability of the distribution $\mathcal{D}$ is Levi-flatness, that is, the condition $\omega \wedge d\omega \equiv 0$. In this case we can easily construct examples of CR-manifolds with a vector field T inducing a CR-action and such that the flow generated by T is not isometric.

Definition 2.4.1. The *suspension* of a compact manifold N by $g \in \mathrm{Diff}(\mathrm{N})$ is the compact manifold M given by $\mathrm{M} = \mathrm{N} \times \mathbb{R}/\sim$ where $(z, s) \sim (g(z), s + 1)$. We denote $\mathrm{M} = \mathrm{N} \times_g \mathbb{R}$.

Proposition 2.4.2. *Let* N *be a compact complex manifold and* $g \in Aut_{\mathbb{C}}(\mathrm{N})$*. The suspension* M *of* N *by* g *carries a natural Levi-flat* CR*-structure defined by* $T\mathrm{N}$ *and a transverse* CR*-action induced by* $\frac{\partial}{\partial s}$*. In particular if* $\mathcal{F}$ *denotes the flow induced by the CR-action then* $e_{\mathcal{F}}(\mathrm{M}) = 0$*.*

Remark 2.4.3. If we choose g such that it is not an isometry for any metric on N, for instance $\mathrm{N} = \mathbb{C}P^1$ and $g(z) = \lambda z$ with $\lambda \in \mathbb{C}$ such that $|\lambda| \neq 1$, the flow $\mathcal{F}$ generated by $\frac{\partial}{\partial s}$ is clearly not isometric for any Riemannian metric on M.

Chapter 3

Normal almost contact structures on Lie groups

By a classical result of Samelson it is known that every compact Lie group of even dimension admits a complex structure such that left translations are holomorphic maps (c.f. [Sam53]). Independently Wang proved that quotient spaces of even dimension K/P, where K is a compact semisimple Lie group and P a parabolic Lie subgroup, admit a left-invariant complex structure and that every homogenous complex compact manifold is of this type (c.f. [Wan54]). Wang's theorem includes Samelson's result as a particular case. Charbonnel and Khalgui studied in [CK04] left-invariant complex and CR structures of maximal dimension on a compact Lie group by means of the Lie subalgebras associated to them (for CR-structures see also [GT92]). Finally CR-structures on homogeneous manifolds have been studied in [AHR85] and [AS03].

We start this chapter by giving a new geometrical construction proving the existence of *left-invariant* CR-structures of maximal dimension on a compact semisimple Lie group of odd dimension, with and without a transverse CR-action. We then prove that every left-invariant normal almost contact structure on a compact semisimple Lie group can be recovered by this construction. More precisely:

Theorem 3.0.4. *Let* K *be a semisimple compact connected Lie group of odd dimension* $2n+1$ *and rank* $2r+1$ *and let* G *be its universal complexification. Assume that* H *is a Cartan subgroup of* G *and* $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H}$ *a Lie group morphism verifying the transversality condition* (I)*. If* B *is a Borel subgroup of* G *such that* $\mathrm{H} \subset \mathrm{B}$ *and* U *is its subgroup of unipotent elements then the Lie subalgebras* $\mathfrak{l}_\Lambda \subset \mathfrak{l}'_\Lambda$ *of* $\mathfrak{g}$ *associated to the complex Lie subgroups* $\mathrm{L}'_\Lambda = \Lambda(\mathbb{C}^{r+1})\cdot\mathrm{U}$ *and* $\mathrm{L}_\Lambda = \Lambda(\{0\}\times\mathbb{C}^r)\cdot\mathrm{U}$ *of* G *define a left-invariant normal almost-contact structure* K_Λ *on* K*. Moreover, the Lie subgroup* L_Λ *is closed and the CR-structure on* K *determined by* L_Λ *agrees with the one induced by the embedding* $\mathrm{K} \hookrightarrow \mathrm{G}/\mathrm{L}_\Lambda$ *of* K *as a real hypersurface of the complex manifold* $\mathrm{G}/\mathrm{L}_\Lambda$*. Conversely, every left-invariant normal almost contact structure is induced by such a morphism* Λ *from* $\mathbb{C}^{r+1}$ *into a Cartan subgroup* H *of* G*.*

Condition (I) is stated precisely in lemma 3.1.20. Using Samelson-Wang result we can conclude the following:

Corollary 3.0.5. *Let* K *be a compact connected real Lie group of odd dimension, then* K *admits a left invariant normal almost contact structure (and in particular a left-invariant CR-structure of maximal dimension).*

We next show how we can use the previous geometrical construction to obtain by deformation normal almost contact structures on compact semisimple Lie groups of odd dimension which are not left invariant (we will call them *non-invariant*). More precisely we prove:

Theorem 3.0.6. *Let* K *be a semisimple compact connected Lie group of odd dimension* $2n+1$ *and rank* $2r+1$ *and let* G *be its universal complexification. Assume that* $\mathrm{H} \subset \mathrm{B}$ *are a Cartan subgroup and a Borel subgroup of* G *respectively. Then every morphism of Lie groups* $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H}\times\mathrm{H}$ *inducing a locally free holomorphic action* $\varphi_\Lambda : \mathbb{C}^{r+1}\times \mathrm{G}/\mathrm{U} \to \mathrm{G}/\mathrm{U}$ *verifying* (II) *determines a normal almost contact structure in a natural way by lemma 3.2.4. Moreover, such a normal almost contact structure is left-invariant if and only if* $\Lambda = (e, \Lambda_2)$ *where* $\Lambda_2 : \mathbb{C}^{r+1} \to \mathrm{H}$ *is a morphism verifying the transversality hypothesis* (I)*. In particular there exist small deformations of the previous ones obtained deforming* Λ *which induce suitable* $\mathbb{C}^{r+1}$*-actions defining normal almost contact structures on* K *generically non-invariant.*

Condition (II) is stated in lemma 3.2.4. From this result we deduce:

Corollary 3.0.7. *Let* K *be a compact connected real Lie group of odd dimension, then* K *admits a non-invariant normal almost contact structure (and in particular a non-invariant CR-structure of maximal dimension).*

The construction of non-invariant normal almost contact structures is based on an analogous of the following lemma for normal almost contact structures, which we will discuss in section §3.2:

Lemma 3.0.8. *Let $\mathcal{F}$ be a holomorphic foliation on a complex manifold X. A real submanifold $M^{cod\mathcal{F}}$ of X transverse to $\mathcal{F}$ inherits a natural complex structure.*

This view-point has been adopted by Loeb and Nicolau (c.f. [LN96]), López de Medrano and Verjovsky (c.f. [LdMV97]) and Meerseeman (c.f. [Mee00]) to construct families of non-kählerian compact complex manifolds which include Hopf and Calabi-Eckmann manifolds as very particular cases.

3.1 Left-invariant normal almost contact structures

Let K be a compact connected real Lie group such that $\dim_{\mathbb{R}}\mathrm{K} = 2n+1$ and $\operatorname{rank}\mathrm{K} = 2r+1$ and let $\mathfrak{k}$ be its Lie algebra.

Proposition 3.1.1. *A left-invariant CR-structure of maximal dimension over K is defined by a complex subalgebra $\mathfrak{l}$ of $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ such that $\dim_{\mathbb{C}}\mathfrak{l} = n$ and $\mathfrak{l} \cap \mathfrak{k} = \{0\}$.*

Proof. Notice that a left-invariant CR-structure on K is determined by a complex subspace $\mathfrak{l}$ of $\mathfrak{g}$ defining the vectors of $T^{\mathbb{C}}\mathrm{K}$ of type $(0,1)$. The hypothesis $\mathfrak{l} \cap \mathfrak{k} = \{0\}$ is equivalent to $\mathfrak{l} \cap \bar{\mathfrak{l}} = \{0\}$ and $\mathfrak{l}$ is involutive, i.e. $[\mathfrak{l},\mathfrak{l}] \subset \mathfrak{l}$, if and only if $\mathfrak{l}$ is a subalgebra of $\mathfrak{g}$. □

Remark 3.1.2. By convention the subalgebra $\mathfrak{l}$ will always correspond for us to the distribution of vector fields of type $(0,1)$ of the CR-structure.

From a result by Charbonnel-Khalgui if follows that every complex subalgebra $\mathfrak{l}$ of $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ defining a CR-structure is solvable:

Theorem 3.1.3 ([CK04])**.** *Let* K *be a compact connected real Lie group and $\mathfrak{l}$ be a complex subalgebra of $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ of maximal dimension such that $\mathfrak{l} \cap \mathfrak{k} = \{0\}$. Then the subalgebra $\mathfrak{l}$ is solvable.*

As a consequence we obtain:

Corollary 3.1.4. *Let* K^{2n+1} *be a compact connected real Lie group and let* $\mathfrak{l}$ *define a left-invariant CR-structure of maximal dimension. Then the subalgebra* $\mathfrak{l}$ *is solvable.*

This last result plays a crucial role in the proof of the main theorem, for it will allow us to conclude that all normal almost contact structures on a semisimple compact Lie group can be obtained by our construction. We will give an alternative and independent proof of this result. Nevertheless, we include here an sketch of the proof of the corollary following the ideas of Charbonnel-Khalgui under more restrictive hypothesis for the sake of clarity. We assume that K is a compact connected semisimple Lie group, which will be our situation. We denote by G the universal complexification of K and by L the Lie subgroup of G with Lie algebra $\mathfrak{l}$. Let $\overline{\mathrm{L}}$ be the clausure of L in G. One of the main elements of the proof is the following result (c.f. [Hel78], Ch. I, Thm. 13.5):

Theorem 3.1.5 (Cartan)**.** *Let* M *be a simply connected complete Riemann manifold with negative curvature. Let* K *be a compact Lie group of transformations of* M *such that its elements are isometries on* M*. Then the elements of* K *have a common fixed point.*

Proof. (Corollary 3.1.4). Let $\tilde{\mathfrak{l}}$ be the Lie algebra of $\overline{\mathrm{L}}$. Then one can prove that the subalgebra $\mathfrak{l}$ is an ideal of $\tilde{\mathfrak{l}}$. Let X be the left coset manifold G/K. We denote by θ the canonical map from G to X and by $\mathfrak{g}_{\mathbb{R}}$ the real Lie algebra underlying $\mathfrak{g}$. As K is semisimple the Killing form κ on $\mathfrak{g}_{\mathbb{R}} \times \mathfrak{g}_{\mathbb{R}}$ is non degenerate and defines on X a G-invariant Riemann structure of negative curvature (cf. [Hel78], Ch.V, Thm. 3.1). Let Y be the image of $\overline{\mathrm{L}}$ by θ. Then Y is a closed connected submanifold of X. Moreover the restriction to Y of the Riemannian structure over X is a Riemannian structure of negative curvature. By Levi-Malcev theorem we can decompose $\mathfrak{l}$ as the sum of a solvable ideal $\mathfrak{r}$ and a semisimple Lie algebra $\mathfrak{s}$. If $\mathfrak{l}$ is not solvable then $\mathfrak{s} \neq 0$ and it admits a real compact form T, i.e. there exists a real compact Lie subgroup T of L with Lie algebra $\mathfrak{t}$ such that $\mathfrak{s} = \mathfrak{t}^{\mathbb{C}}$. In particular T is a compact connected semisimple subgroup of $\overline{\mathrm{L}}$. Let us see that if we find an element $l \in \overline{\mathrm{L}}$ such that $l^{-1}\mathrm{T}l \subset \mathrm{K}$, or equivalently $l \in \overline{\mathrm{L}}$ such that the point $\theta(l)$ of the submanifold Y of X is fixed by the action of T on X, then we have a contradiction. Indeed, since $\mathfrak{l}$ is an ideal of $\tilde{\mathfrak{l}}$ the subgroup L is normal in $\overline{\mathrm{L}}$, thus $l^{-1}\mathrm{L}l = \mathrm{L}$ and $l^{-1}\mathrm{T}l \subset \mathrm{L} \cap \mathrm{K}$, but as $\mathfrak{l} \cap \mathfrak{k} = \{0\}$ this is not possible. Since T is a subgroup of $\overline{\mathrm{L}}$, Y is invariant by the action of T. We denote by $\widetilde{\mathrm{Y}}$ the universal covering of Y and by $\widetilde{\mathrm{T}}$ the universal covering of T. There exists a unique action of $\widetilde{\mathrm{T}}$ over $\widetilde{\mathrm{Y}}$ which defines on the quotient the action of T over Y. There exists a unique Riemannian structure on $\widetilde{\mathrm{Y}}$ which defines on the quotient the Riemannian structure on Y. By the previous proposition we know that the Riemannian structure on $\widetilde{\mathrm{Y}}$ is invariant by $\widetilde{\mathrm{T}}$, complete and of negative curvature. As T is connected, compact and semisimple, $\widetilde{\mathrm{T}}$ is compact. Therefore the action of $\widetilde{\mathrm{T}}$ on $\widetilde{\mathrm{Y}}$ has a fixed point, by Cartan's theorem, and it follows that the action of T over Y has a fixed point. □

Proposition 3.1.6. *A left-invariant normal almost contact structure over* K *is determined by a pair of complex subalgebras* $\mathfrak{l} \subset \mathfrak{l}'$ *of* $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ *of complex dimension* n *and* $n+1$ *respectively such that:*

(a) $\mathfrak{l} \cap \mathfrak{k} = \{0\}$*;*

(b) $\dim_{\mathbb{R}} \mathfrak{l}' \cap \mathfrak{k} = 1$;

(c) $\mathfrak{l}$ *is an ideal of* $\mathfrak{l}'$*, i.e.* $[\mathfrak{l}, \mathfrak{l}'] \subset \mathfrak{l}$.

Proof. Because of the proposition 3.1.1 the subalgebra $\mathfrak{l}$ defines a left-invariant CR-structure on K. Note that $\mathfrak{l}' \cap \mathfrak{k} = \langle \xi \rangle_{\mathbb{R}}$ corresponds to the left-invariant vector field defining the CR-action. Clearly the vector field ξ is transverse to the CR-structure determined by $\mathfrak{l}$ and **(c)** implies that it induces a CR-action. Conversely, given a normal almost contact structure, if $\mathfrak{l}$ defines the CR-structure and ξ the CR-action it is enough to set $\mathfrak{l}' = \mathfrak{l} \oplus \langle \xi \rangle_{\mathbb{C}}$. Notice that $\mathfrak{g} = \mathfrak{l} \oplus \bar{\mathfrak{l}} \oplus \langle \xi \rangle_{\mathbb{C}}$, therefore $\dim_{\mathbb{R}} \mathfrak{l}' \cap \mathfrak{k} = 1$. □

Remark 3.1.7. If the complex subalgebras $\mathfrak{l} \subset \mathfrak{l}' = \mathfrak{l} \oplus \langle \xi \rangle_{\mathbb{C}}$ (with the same notation as in the above proof) of $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ determine a left-invariant normal almost contact structure over K then we can define a natural left-invariant complex structure on $\mathrm{K} \times S^1$ by imposing that the subalgebra $\mathfrak{l} \oplus \langle \xi + i\frac{\partial}{\partial t} \rangle$ is the distribution of vector fields of type $(0,1)$ where $\frac{\partial}{\partial t}$ is a tangent vector field on S^1 corresponding to the S^1-action (note that the integrability of the complex structure follows from the fact that the almost contact structure is normal, see definition 2.1.1).

Definition 3.1.8. Let $\mathfrak{l} \subset \mathfrak{l}'$ and $\mathfrak{m} \subset \mathfrak{m}'$ be left-invariant normal almost contact structures over the compact Lie groups K and M respectively. We say that $f : \mathrm{K} \to \mathrm{M}$ is an *isomorphism of left-invariant normal almost contact structures* if f is a Lie group isomorphism between K and M and the complexified linear map $f_* : \mathfrak{k}^{\mathbb{C}} \to \mathfrak{m}^{\mathbb{C}}$ verifies $f_*\mathfrak{l} = \mathfrak{m}$ and $f_*\mathfrak{l}' = \mathfrak{m}'$.

Theorem 3.1.9. *Let* K *be a compact connected real semisimple Lie group with Lie algebra* $\mathfrak{k}$ *endowed with a left-invariant normal almost contact structure defined by a pair of subalgebras* $\mathfrak{l} \subset \mathfrak{l}'$ *of* $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$*. Let* G *be the universal complexification of* K *and* L *the connected complex Lie subgroup of* G *associated to* $\mathfrak{l}$*. Then* L *is closed and the map* $\mathrm{K} \hookrightarrow \mathrm{G}/\mathrm{L}$ *induced by the inclusion* $\mathrm{K} \subset \mathrm{G}$ *is a totally real embedding such that the left-invariant CR-structure on* K *defined by* $\mathfrak{l}$ *agrees with the induced by the embedding and the natural complex structure on* G/L*. Moreover the subalgebras* $\mathfrak{l}$ *and* $\mathfrak{l}'$ *are solvable.*

Since $\mathfrak{l}' = \mathfrak{l} \oplus \langle \xi \rangle_{\mathbb{C}}$ and $\mathfrak{l}$ is an ideal of $\mathfrak{l}'$ we have $[\mathfrak{l}', \mathfrak{l}'] \subset \mathfrak{l}$, thus $\mathfrak{l}'$ is solvable if and only if $\mathfrak{l}$ is solvable. By the previous theorem by Charbonnel and Khalgui we know that the subalgebra $\mathfrak{l}$ is solvable. Nevertheless we present here an alternative proof since its approach will allow us (see the next section) to construct non-invariant normal almost contact structures. Note also that we prove that the connected Lie subgroup L of G corresponding to the subalgebra $\mathfrak{l}$ is *closed*, which could not be concluded from Charbonnel-Khalgui proof. In the following argument we use strongly the existence of a normal almost contact structure, not only of a CR-structure.

Proof. We can define on $\mathrm{K} \times S^1$ a left-invariant complex structure such that if $\tilde{\mathfrak{l}}$ is the distribution of vector fields of type $(0, 1)$ of $\mathrm{K} \times S^1$ then $\tilde{\mathfrak{l}} \cap \mathrm{T}^{\mathbb{C}}\mathrm{K} = \mathfrak{l}$ (see remark 3.1.7). First of all we will prove that there is a closed subgroup L of the universal complexification G of K and a totally real embedding $\mathrm{K} \hookrightarrow \mathrm{G}/\mathrm{L}$ such that the left-invariant CR-structure on K defined by $\mathfrak{l}$ agrees with the induced by the embedding and the natural complex structure on G/L. Next we will see that L is the connected complex Lie subgroup of G associated to the Lie subalgebra $\mathfrak{l}$. We define $\widehat{\mathrm{G}} = \mathrm{Aut}_{\mathbb{C}}(\mathrm{K} \times S^1)$ and $\widehat{\mathrm{L}} = \mathrm{I}_e$, the isotropy group of e, i.e. $\mathrm{I}_e = \{f \in \widehat{\mathrm{G}} : f(e) = e\}$. It is a well-known fact that $\widehat{\mathrm{G}}$ is a complex Lie group (cf. [Kob72], p. 77). Note that $\widehat{\mathrm{L}}$ is closed and we have an embedding $\mathrm{K} \hookrightarrow \widehat{\mathrm{G}}/\widehat{\mathrm{L}}$ where the elements of K act as left-translations on K and fix S^1. Note that as $\mathrm{K} \times S^1$ acts transitively by left-translations on $\mathrm{K} \times S^1$ the complex manifold $\mathrm{K} \times S^1$ is naturally identified to $\widehat{\mathrm{G}}/\widehat{\mathrm{L}}$. Therefore the CR-structure on K agrees with the one induced by the embedding $\mathrm{K} \hookrightarrow \widehat{\mathrm{G}}/\widehat{\mathrm{L}}$ and the complex structure on $\widehat{\mathrm{G}}/\widehat{\mathrm{L}}$. Let $\widehat{\mathfrak{g}}$ be the Lie algebra of $\widehat{\mathrm{G}}$, next we will see that $\mathfrak{k}$ is completely real in $\widehat{\mathfrak{g}}$. Consider the ideal of $\mathfrak{k}$ defined by $\mathfrak{m} = \mathfrak{k} \cap \mathrm{i}\mathfrak{k}$, where i denotes the complex product by $\sqrt{-1}$ corresponding to the complex structure on $\widehat{\mathfrak{g}}$. Since K is semisimple we have $\mathfrak{k} \cong \mathfrak{i}_1 \oplus ... \oplus \mathfrak{i}_s$ where $\mathfrak{i}_1, ..., \mathfrak{i}_s$ are simple ideals of $\mathfrak{k}$ and we can write $\mathfrak{m}$ as a sum of some of these ideals. In particular $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}$. On the other hand $\mathfrak{m}$ is a complex subalgebra and there exists a Lie subgroup M of K with a complex Lie group structure such that its Lie algebra is $\mathfrak{m}$. Since $\mathfrak{m}$ is an ideal of a semisimple compact Lie group K the subgroup M is normal and therefore closed. It follows that M is compact (for K is compact) and consequently abelian, which contradicts $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}$. We have thus proved that $\mathfrak{m} = 0$ o equivalently that $\mathfrak{k}$ is totally real in $\widehat{\mathfrak{g}}$. Let now G be the connected complex subgroup of $\widehat{\mathrm{G}}$ associated to the complex Lie algebra $\mathfrak{g} := \mathfrak{k} \oplus \mathrm{i}\,\mathfrak{k}$, we will next show that G is the universal complexification of K. Recall that K is totally real in G and that the Lie algebra of G is a complexification of the Lie algebra of K, therefore K is a real compact form of the complex semisimple Lie group G. We denote by $\rho : \mathrm{K} \hookrightarrow \mathrm{K}^{\mathbb{C}}$ the universal complexification of K. The semisimple complex Lie group $\mathrm{K}^{\mathbb{C}}$ also admits K as a real compact form. In particular K is maximal compact subgroup both of $\mathrm{K}^{\mathbb{C}}$ and of G, therefore K is a deformation retract of both complex Lie groups. On the other hand G and $\mathrm{K}^{\mathbb{C}}$ differ at most in a finite quotient but, as they have the same deformation retract, one has $\mathrm{G} \cong \mathrm{K}^{\mathbb{C}}$. We define now the closed subgroup $\mathrm{L} := \widehat{\mathrm{L}} \cap \mathrm{G}$ of G and check that there is an embedding $\mathrm{K} \hookrightarrow \mathrm{G}/\mathrm{L}$ and that $\dim_{\mathbb{C}} \mathrm{G}/\mathrm{L} = n+1$. Notice that there is a totally real embedding $\mathrm{K} \hookrightarrow \widehat{\mathrm{G}}/\widehat{\mathrm{L}}$ and that by construction $\mathrm{K} \subset \mathrm{G}$, therefore $\mathrm{K} \hookrightarrow \mathrm{G}/\mathrm{L}$ and $\dim_{\mathbb{C}} \mathrm{G}/\mathrm{L} \geq n+1$. Moreover $\dim_{\mathbb{C}} \mathrm{G}/\mathrm{L} \leq \dim_{\mathbb{C}} \widehat{\mathrm{G}}/\widehat{\mathrm{L}} = n+1$, so the equality follows. Therefore $\mathrm{G}/\mathrm{L} \cong \widehat{\mathrm{G}}/\widehat{\mathrm{L}}$. In addition the CR-structure on K corresponding to $\mathfrak{l}$ agrees with the one induced by the embedding of K as a real hypersurface of G/L (because then the CR-structure was compatible by construction with the embedding $\mathrm{K} \hookrightarrow \widehat{\mathrm{G}}/\widehat{\mathrm{L}}$ and the complex structure of $\widehat{\mathrm{G}}/\widehat{\mathrm{L}}$), i.e. the Lie algebra of L coincides with $\mathfrak{l}$. Now Levi-Malcev theorem states that $\mathfrak{l}$ can be decomposed as sum of a solvable ideal $\mathfrak{r}$ and a semisimple subalgebra $\mathfrak{s}$. If $\mathfrak{l}$ is not solvable then the subalgebra $\mathfrak{s} \neq 0$ admits a real compact form T, i.e. there exists a compact real Lie subgroup T of L with Lie algebra $\mathfrak{t}$ such that $\mathfrak{s} = \mathfrak{t}^{\mathbb{C}}$. As K is a maximal compact subgroup of G there exists an element $g \in \mathrm{G}$ such that $\mathrm{T} \subset g\mathrm{K}g^{-1} = \mathrm{K}'$, in particular, $\mathrm{K}' \cap \mathrm{L} \supseteq \mathrm{T} \neq \{e\}$. Recall that $\mathrm{K} \times S^1$ is naturally identified to $\widehat{\mathrm{G}}/\widehat{\mathrm{L}}$. It follows that the action of K over $\widehat{\mathrm{G}}/\widehat{\mathrm{L}} \cong \mathrm{G}/\mathrm{L}$ by left-translations is free and therefore the orbits have real dimension $2n+1$.

We conclude that the action of K' by left-translations over G/L is free and the orbits of this action have real dimension $2n+1$, since if $[x \cdot \mathrm{L}] \in \mathrm{G/L}$ and $k \in \mathrm{K}$ then $[g^{-1} \cdot k \cdot g \cdot x \cdot \mathrm{L}] = [x \cdot \mathrm{L}]$ if and only if $[k \cdot (g \cdot x) \cdot \mathrm{L}] = [(g \cdot x) \cdot \mathrm{L}]$. Setting $x = e$ it follows that $\mathrm{K}' \cap \mathrm{L} = \{e\}$. Thus we have reached a contradiction so $\mathfrak{l}$ must be solvable. □

Remark 3.1.10. Given a complex subalgebra $\mathfrak{l}$ with $\dim_{\mathbb{C}} \mathfrak{l} = n$ and $\mathfrak{l} \cap \mathfrak{k} = \{0\}$ it does not always exist a subalgebra $\mathfrak{l}' \supset \mathfrak{l}$ defining a normal almost contact structure. The Lie groups SO(3) and SU(2) provide examples of this situation, as we will see in the last section.

Proposition 3.1.11. *Let K^{2n+1} be a compact connected semisimple Lie group with Lie algebra $\mathfrak{k}$ and assume that the pair $\mathfrak{l} \subset \mathfrak{l}'$ defines a left-invariant normal almost contact structure on K^{2n+1}. Then there exists a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{r} = \mathfrak{t}^{\mathbb{C}} \subset \mathfrak{b}$ such that $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{k}$ and $\mathfrak{l}' \subset \mathfrak{b}$. Moreover, if $\mathfrak{u} = [\mathfrak{b}, \mathfrak{b}]$ we have $\mathfrak{l}' = (\mathfrak{l}' \cap \mathfrak{r}) \oplus \mathfrak{u}$ and $\mathfrak{l} = (\mathfrak{l} \cap \mathfrak{r}) \oplus \mathfrak{u}$. In particular $\mathfrak{u} \subset \mathfrak{l} \subset \mathfrak{l}'$.*

Before we start the proof we establish some results that we will need.

Lemma 3.1.12. *There is a unique maximal torus in $(\mathbb{C}^*)^k$ given by $(S^1)^k$.*

Note that since $(\mathbb{C}^*)^k$ is abelian the conjugation plays no role.

Lemma 3.1.13. *Every connected complex subgroup* M *of $(\mathbb{C}^*)^k$ is isomorphic to $\mathbb{C}^l \times (\mathbb{C}^*)^s$ for $l, s \geq 0$ and $l + s \leq k$. Moreover:*

(a) *If $\dim_{\mathbb{R}}(\mathrm{M} \cap (S^1)^k) = 0$ then* M *is isomorphic to $\mathbb{C}^l$ and $0 \leq l \leq k/2$.*

(b) *If $\dim_{\mathbb{R}}(\mathrm{M} \cap (S^1)^k) = 1$ then* M *is isomorphic to $\mathbb{C}^l \times \mathbb{C}^*$ or $\mathbb{C}^{l+1}$ and $0 \leq l \leq (k-1)/2$.*

Proof. By a theorem by Morimoto (see theorem 1.7.8) a Lie subgroup M of $(\mathbb{C}^*)^k$ is isomorphic to $\mathrm{M}^0 \times \mathbb{C}^l \times (\mathbb{C}^*)^s$ where M^0 is a (HC)-group. As $(\mathbb{C}^*)^k$ is Stein all its subgroups admit non-constant holomorphic functions (since for every $x, y \in \mathrm{M}$ there exists a holomorphic function f on $(\mathbb{C}^*)^k$ such that $f(x) \neq f(y)$), therefore $\mathrm{M}^0 = \{e\}$. Moreover since $(\mathbb{C}^*)^k$ and M are abelian their maximal tori are unique and it follows that the maximal torus $(S^1)^s$ of M is included in the maximal torus $(S^1)^k$ of $(\mathbb{C}^*)^k$. Therefore if $\dim_{\mathbb{R}}(\mathrm{M} \cap (S^1)^k) = 0$ then M is isomorphic to $\mathbb{C}^l$. Finally note that if an injective map $\Omega : \mathbb{C}^l \to (\mathbb{C}^*)^k$ verifies $\dim_{\mathbb{R}}(\mathrm{Im}(\Omega) \cap (S^1)^k) = 0$ then $l \leq k/2$. The argument of **(b)** is analogous to the previous one. □

Lemma 3.1.14. *If* M *is a complex subgroup of* $(\mathbb{C}^*)^{2k+1}$ *isomorphic to* $\mathbb{C}^{k+1}$ *and such that* $\dim_{\mathbb{R}}(\mathrm{M} \cap (S^1)^{2k+1}) = 1$ *then* $\overline{\mathrm{M}}^{Zar} = (\mathbb{C}^*)^{2k+1}$.

Proof. Every irreducible (or connected) algebraic subgroup of $(\mathbb{C}^*)^n$ is of the form $(\mathbb{C}^*)^m$ for $m \leq n$ where the $\mathbb{C}^*$ are some of the factors of $(\mathbb{C}^*)^n$ (c.f. [Oni90]). Since $\overline{\mathrm{M}}^{Zar}$ is an algebraic subgroup of $(\mathbb{C}^*)^{2k+1}$ it is isomorphic to $(\mathbb{C}^*)^l$ with $l \leq 2k+1$. It is enough to notice that $\mathrm{M} \cong \mathbb{C}^{k+1}$ can not be immersed into $(\mathbb{C}^*)^l$ for $l < 2k+1$ since it would contradict the inequality

$$2k+2+l = \dim_{\mathbb{R}} \mathbb{C}^{k+1} + \dim_{\mathbb{R}}(S^1)^l \leq \dim_{\mathbb{R}}(\mathbb{C}^*)^l + \dim_{\mathbb{R}}(\mathbb{C}^{k+1} \cap (S^1)^l) = 2l+1.$$

□

Lemma 3.1.15. *If* M *is a complex subgroup of* $(\mathbb{C}^*)^{2k}$ *isomorphic to* $\mathbb{C}^k$ *and such that* $\dim_{\mathbb{R}} \mathrm{M} \cap (S^1)^{2k} = 0$ *then* $\overline{\mathrm{M}}^{Zar} = (\mathbb{C}^*)^{2k}$.

The proof is analogous to the previous one.

Proposition 3.1.16. *Let* K^{2n+1} *be a compact connected semisimple Lie group with Lie algebra* $\mathfrak{k}$ *and assume that the pair* $\mathfrak{l} \subset \mathfrak{l}'$ *defines a left-invariant normal almost contact structure on* K^{2n+1}. *Let* $\mathfrak{b}$ *be a Borel subalgebra of* $\mathfrak{g}$ *such that* $\mathfrak{l}' \subset \mathfrak{b}$. *Then* $\mathfrak{t} := \mathfrak{b} \cap \mathfrak{k}$ *is a maximal abelian subalgebra of* k *and for the Cartan subalgebra* $\mathfrak{r} = \mathfrak{t}^{\mathbb{C}} \subset \mathfrak{b}$ *we have* $\mathfrak{l}' + \mathfrak{r} = \mathfrak{b}$ *and* $\dim_{\mathbb{R}}(\mathfrak{l}' \cap \mathfrak{r}) = 2r + 2$.

Proof. Let us begin by proving that $\mathfrak{t} := \mathfrak{b} \cap \mathfrak{k}$ is a maximal abelian subalgebra of $\mathfrak{k}$. As the identity component of $\mathrm{B} \cap \mathrm{K}$ is a closed connected Lie subgroup of a compact Lie group K, it is compact and it follows that $\mathfrak{b} \cap \mathfrak{k} = [\mathfrak{b} \cap \mathfrak{k}, \mathfrak{b} \cap \mathfrak{k}] \oplus C(\mathfrak{b} \cap \mathfrak{k})$, where $[\mathfrak{b} \cap \mathfrak{k}, \mathfrak{b} \cap \mathfrak{k}]$ is semisimple and $C(\mathfrak{b} \cap \mathfrak{k})$ is abelian (see section 1.7.5). On the other hand, $\mathfrak{b} \cap \mathfrak{k}$ is solvable, therefore it cannot admit a semisimple subalgebra and we conclude that the subalgebra $\mathfrak{b} \cap \mathfrak{k}$ is abelian. Recall now that $\dim_{\mathbb{R}} \mathfrak{k} = 2n+1$ and $\dim_{\mathbb{R}} \mathfrak{b} = 2n+2r+2$ (the last equality is a consequence of Cartan decomposition, see section 1.7.3). Thus,

$$4n+2r+3 = \dim_{\mathbb{R}} \mathfrak{b} + \dim_{\mathbb{R}} \mathfrak{k} = \dim_{\mathbb{R}}(\mathfrak{b} + \mathfrak{k}) + \dim_{\mathbb{R}}(\mathfrak{b} \cap \mathfrak{k}) \leq 4n+2+\dim_{\mathbb{R}}(\mathfrak{b} \cap \mathfrak{k}).$$

Therefore $\dim_{\mathbb{R}}(\mathfrak{b} \cap \mathfrak{k}) \geq 2r+1$ and $\mathfrak{b} \cap \mathfrak{k}$ must be a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. Then $\mathfrak{r} = \mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$ and $\dim_{\mathbb{R}} \mathfrak{r} = 4r+2$. Note that $\langle \xi \rangle_{\mathbb{R}} = \mathfrak{l}' \cap \mathfrak{k} \subset \mathfrak{b} \cap \mathfrak{k} = \mathfrak{t}$, therefore $\mathfrak{l}' \cap \mathfrak{t} = \langle \xi \rangle_{\mathbb{R}}$. Under the above hypothesis $\mathfrak{l}' + \mathfrak{r} \subset \mathfrak{b}$. Recall that

$$\dim_{\mathbb{R}} \mathfrak{r} = 4r+2; \quad \dim_{\mathbb{R}} \mathfrak{l}' = 2n+2; \quad \dim_{\mathbb{R}} \mathfrak{b} = 2n+2r+2.$$

Therefore

$$2n+2r+2=\dim_{\mathbb{R}}\mathfrak{b}\geq\dim_{\mathbb{R}}(\mathfrak{l}'+\mathfrak{r})=2n+4+4r-\dim_{\mathbb{R}}(\mathfrak{l}'\cap\mathfrak{r})$$

so $2r+2\leq\dim_{\mathbb{R}}(\mathfrak{l}'\cap\mathfrak{r})$. On the other hand $\mathfrak{l}'\cap\mathfrak{r}$ is a real subspace of dimension at least $2r+2$ of $\mathfrak{t}\oplus i\mathfrak{t}$ intersecting $\mathfrak{t}$ with dimension 1 (for $\mathfrak{l}'\cap\mathfrak{t}=\langle\xi\rangle_{\mathbb{R}}$), by Grassman formula one concludes then that $\dim_{\mathbb{R}}(\mathfrak{l}'\cap\mathfrak{r})=2r+2$. Finally from the previous inequality we see that $\dim_{\mathbb{R}}(\mathfrak{l}'+\mathfrak{r})=2n+2r+2$. Consequently $\mathfrak{l}'+\mathfrak{r}=\mathfrak{b}$. □

Before we begin the proof of the proposition 3.1.11 we recall a couple of results of linear algebra that we use throughout it.

Lemma 3.1.17. *Let f, g be diagonalizable endomorphisms of a finite dimensional vector space E such that $f\circ g=g\circ f$. There exists a basis in which f and g diagonalize simultaneously.*

Lemma 3.1.18. *Let f be a diagonalizable endomorphism on a finite dimensionl vector space E and F a f-invariant subspace of E, i.e. $f(F)\subset F$. Then $f_{|F}$ is diagonalizable and if $E=E_{\lambda_1}\oplus...\oplus E_{\lambda_s}$ is a decomposition of E in eigenspaces of eigenvalue λ_i with $\lambda_i\neq\lambda_j$ for $i\neq j$ then $F=(E_{\lambda_1}\cap F)\oplus...\oplus(E_{\lambda_s}\cap F)$ is a decomposition of F in eigenspaces.*

Proof. It is enough to prove that $F\subset(E_{\lambda_1}\cap F)\oplus...\oplus(E_{\lambda_s}\cap F)$. Any $v\in F$ admits a decomposition $v=v_1+...+v_s$ where $v_i\in E_i$ for every $i=1,...,s$. Consider now

$$\begin{array}{lcl} f(v) & = & \lambda_1 v_1+...+\lambda_s v_s\in F \\ f^2(v) & = & \lambda_1^2 v_1+...+\lambda_s^2 v_s\in F \\ ... & & \\ f^{s-1}(v) & = & \lambda_1^{s-1}v_1+...+\lambda_s^{s-1}v_s\in F \end{array}$$

It is enough to notice that

$$\left|\begin{array}{ccc} 1 & ... & 1 \\ \lambda_1 & ... & \lambda_s \\ & ... & \\ \lambda_1^{s-1} & ... & \lambda_s^{s-1} \end{array}\right| = \prod_{i>j}(\lambda_i-\lambda_j)\neq 0$$

to conclude that $v_i\in F$ for every i, so $v_i\in E_i\cap F$ for every i. □

Proof. (Proposition 3.1.11) Since $\mathfrak{l}'$ is solvable there exists a Borel subalgebra $\mathfrak{b}$ such that $\mathfrak{l}' \subset \mathfrak{b}$. In proposition 3.1.16 we have seen that $\mathfrak{t} = \mathfrak{b} \cap \mathfrak{k}$ is a maximal abelian subalgebra of $\mathfrak{k}$. Then $\mathfrak{r} = \mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra such that $\mathfrak{r} \subset \mathfrak{b}$. We must then show that $\mathfrak{l}' = (\mathfrak{l}' \cap \mathfrak{r}) \oplus \mathfrak{u}$ and $\mathfrak{l} = (\mathfrak{l} \cap \mathfrak{r}) \oplus \mathfrak{u}$. Recall that using the root decomposition of $\mathfrak{g}$ by respect to $\mathfrak{r}$ (see section 1.7.3) we obtain

$$\mathfrak{g} = \mathfrak{r} \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \quad \mathfrak{b} = \mathfrak{r} \oplus_{\alpha \in \widetilde{\Phi}} \mathfrak{g}_\alpha, \quad \mathfrak{u} = \oplus_{\alpha \in \widetilde{\Phi}} \mathfrak{g}_\alpha,$$

where Φ is the set of roots of $\mathfrak{g}$ relative to $\mathfrak{r}$, $\widetilde{\Phi}$ is a subset of Φ and $\mathfrak{g}_\alpha$ are proper spaces. We consider now the action on $\mathfrak{l}'$ of its abelian subalgebra $\mathfrak{l}' \cap \mathfrak{r}$. As a consequence of lemma 3.1.18 the endomorphisms of $\mathfrak{l}'$ defined by the elements of $\mathfrak{l}' \cap \mathfrak{r}$ diagonalize. Then lemma 3.1.17 assures us that they diagonalize simultaneously. Thus we obtain a decomposition of $\mathfrak{l}'$ as a direct sum of eigenspaces:

$$\mathfrak{l}' = \mathfrak{l}'_0 \oplus_{\alpha \in (\mathfrak{l}')^*} \mathfrak{l}'_\alpha,$$

where $\mathfrak{l}'_\alpha = \{X \in \mathfrak{l}' : [R, X] = \alpha(R) \cdot X, \quad \forall R \in \mathfrak{l}' \cap \mathfrak{r}\}$. Note that $\mathfrak{l}'_\alpha \subset \mathfrak{u}$ for $\alpha \neq 0$. Indeed if there exists $R \in \mathfrak{r} \cap \mathfrak{l}'$ such that $\alpha(R) \neq 0$ for $X \in \mathfrak{l}'_\alpha$ we have

$$X = \frac{1}{\alpha(R)}[R, X] \in [\mathfrak{l}', \mathfrak{l}'] \subset [\mathfrak{b}, \mathfrak{b}] = \mathfrak{u}.$$

Moreover it is clear that $\mathfrak{l}' \cap \mathfrak{r} \subset \mathfrak{l}'_0$. Now we want to prove that $\mathfrak{l}' = (\mathfrak{l}' \cap \mathfrak{r}) \oplus \mathfrak{u}$ with $\mathfrak{l}'_0 = \mathfrak{l}' \cap \mathfrak{r}$ and $\oplus_{\alpha \in (\mathfrak{l}')^*} \mathfrak{l}'_\alpha = \mathfrak{u}$. This will end the proof because then $\mathfrak{l} = (\mathfrak{l} \cap \mathfrak{r}) \oplus \mathfrak{u}$. Indeed since $[\mathfrak{l}', \mathfrak{l}'] \subset \mathfrak{l}$ we conclude that $\mathfrak{l}'_\alpha \subset \mathfrak{l}$ for $\alpha \neq 0$, therefore $\mathfrak{u} \subset \mathfrak{l}$, in particular $\mathfrak{l} = (\mathfrak{l} \cap \mathfrak{r}) \oplus \mathfrak{u}$.

We will start by seeing that $\mathfrak{l}'_0 \subset \mathfrak{r}$, which yields $\mathfrak{l}' \cap \mathfrak{r} = \mathfrak{l}'_0$ and we will then conclude by an argument of dimensions. We must check that given $X \in \mathfrak{l}'$ such that $[R, X] = 0$ for every $R \in \mathfrak{r} \cap \mathfrak{l}'$ then $[R, X] = 0$ for every $R \in \mathfrak{r}$. Let L' and H be the connected Lie subgroups of G corresponding to the Lie subalgebras $\mathfrak{l}'$ and $\mathfrak{r}$ respectively. Recall that $\mathrm{H} \cong (\mathbb{C}^*)^{2r+1}$. Define S' as the connected component of the id of $\mathrm{L}' \cap \mathrm{H}$. Applying lemma 3.1.13 we conclude that there are two possibilities. In the first case $\mathrm{S}' \cong \mathbb{C}^r \times \mathbb{C}^*$ and if we denote by M the subgroup of S' isomorphic to $\mathbb{C}^r$ then $\dim_{\mathbb{R}} \mathrm{M} \cap (S^1)^{2r+1} = 0$. Then by lemma 3.1.15 we obtain $\overline{\mathrm{S}'}^{Zar} = \mathrm{H}$. In the second one, $S' \cong \mathbb{C}^{r+1}$ and $\dim_{\mathbb{R}}(\mathbb{C}^{r+1} \cap (S^1)^{2r+1}) = 1$. Then by lemma 3.1.14 we also obtain $\overline{\mathrm{S}'}^{Zar} = \mathrm{H}$. By hypothesis $\mathrm{ad}_{\mathrm{R}} \cdot X = 0$ for every $R \in \mathrm{L}' \cap \mathrm{H}$ and $X \in \mathrm{L}'$ and we want to verify that $\mathrm{ad}_{\mathrm{R}} X = 0$ for every $R \in \mathrm{H}$ and $X \in \mathrm{L}'$. Since the hypothesis $\mathrm{ad}_{\mathrm{R}} X = 0$ is algebraic and $\overline{\mathrm{S}'}^{Zar} = \mathrm{H}$

it follows that $\mathfrak{l}'_0 \subset \mathfrak{r}$. We conclude that $\mathfrak{l}' = (\mathfrak{l}' \cap \mathfrak{r}) \oplus_{\alpha \in \mathfrak{l}'^*} \mathfrak{l}'_\alpha$ and $\mathfrak{l}'_\alpha \subset \mathfrak{u}$. Set $\mathfrak{u}' = \oplus_{\alpha \in (\mathfrak{l}')^*} \mathfrak{l}'_\alpha \subset \mathfrak{u}$ and notice that by proposition 3.1.16

$$\dim_{\mathbb{C}} \mathfrak{u}' = n + 1 - r - 1 = n - r = \dim_{\mathbb{C}} \mathfrak{u}.$$

□

The above result is not true when one considers a complex subalgebra $\mathfrak{l}$ defining a left-invariant CR-structure of maximal dimension. The Lie groups SO(2) and SU(3) provide a counterexample. Indeed, as we will see in the last section, they admit CR-structures that can not be completed to a normal almost contact structure, then the statement is a consequence of the following proposition:

Proposition 3.1.19. *With the above notation, if $\mathfrak{l}$ is a left-invariant CR-structure of maximal dimension on a semisimple compact connected Lie group K^{2n+1} then there exists a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ such that $\mathfrak{u} = [\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{l} \subset \mathfrak{b}$ if and only if there exists a subalgebra $\mathfrak{l}'$ of $\mathfrak{g}$ such that the pair $\mathfrak{l} \subset \mathfrak{l}'$ defines a left-invariant normal almost contact structure on K^{2n+1}. Moreover, in this case, we can choose $\mathfrak{l}' \subset \mathfrak{b}$.*

Proof. By the previous theorem it is enough to prove the direct implication. Recall that $\dim_{\mathbb{C}} \mathfrak{r} = 2r+1$, $\dim_{\mathbb{C}} \mathfrak{u} = n - r$, $\dim_{\mathbb{C}} \mathfrak{b} = n + r + 1$, $\dim_{\mathbb{C}} \mathfrak{l} = n$ and $\dim_{\mathbb{C}} \mathfrak{l}' = n + 1$. Moreover $\mathfrak{b} = \mathfrak{r} \oplus \mathfrak{u}$ and $\mathfrak{l} = \mathfrak{l} \cap \mathfrak{r} \oplus \mathfrak{u}$. Therefore $\dim_{\mathbb{C}}(\mathfrak{l} \cap \mathfrak{r}) < \dim_{\mathbb{C}} \mathfrak{r}$. Therefore it exists at least a vector $\xi \in \mathfrak{r} \backslash (\mathfrak{l} \cap \mathfrak{r})$ and if we define $\mathfrak{l}' = \mathfrak{l} \oplus \langle \xi \rangle$ then $\dim_{\mathbb{R}}(\mathfrak{l}' \cap \mathfrak{k}) = 1$. We claim that the pair $\mathfrak{l} \subset \mathfrak{l}'$ defines a normal almost contact structure. Clearly $\mathfrak{l}$ is an ideal of $\mathfrak{l}'$ since for every $X \in \mathfrak{l}$ we have $X = X_0 + X_1$ where $X_0 \in \mathfrak{l} \cap \mathfrak{r}$ and $X_1 \in \mathfrak{u}$, thus $[\xi, X] = [\xi, X_0] + [\xi, X_1] \in \mathfrak{u} \subset \mathfrak{l}$. □

Let K be a semisimple compact connected real Lie group of odd dimension $2n+1$ and rank $2r+1$ and let $\rho : \mathrm{K} \to \mathrm{G} = \mathrm{K}^{\mathbb{C}}$ be its universal complexification. Choose a maximal torus T of K and a Borel subgroup B of G such that $\mathrm{H} := \rho(\mathrm{T})^{\mathbb{C}} \subset \mathrm{B}$. The subgroup H is isomorphic to $(\mathbb{C}^*)^{2r+1}$ and denoting by U the subgroup of unipotent elements of B we have $\mathrm{B} = \mathrm{H} \cdot \mathrm{U}$.

A Lie groups morphism $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H} \cong (\mathbb{C}^*)^{2r+1}$ is given by the composition of the exponential map with a linear map

$$\Lambda^0 : \mathbb{C}^{r+1} \to \mathbb{C}^{2r+1}$$

$$z := (z_1, \ldots, z_{r+1})^t \mapsto M \cdot z$$

where $\mathrm{M} = (m_i^j)$ is a $(2r+1)\times(r+1)$ complex matrix. Note that $\operatorname{rank}\mathrm{M} = r+1$. We denote by A_Λ the $(2r+1)\times(2r+2)$ real matrix that has the real components of the vectors $\Lambda^0(\mathbf{e}_i)$ as columns, where $\mathbf{e}_1, ..., \mathbf{e}_{2r+2}$ is the canonical basis of $\mathbb{R}^{2r+2} \cong \mathbb{C}^{r+1}$. Namely,

$$\mathrm{A}_\Lambda = \begin{pmatrix} \operatorname{Re} m_1^1 & -\operatorname{Im} m_1^1 & \dots & \operatorname{Re} m_1^{r+1} & -\operatorname{Im} m_1^{r+1} \\ \vdots & \vdots & & \vdots & \vdots \\ \operatorname{Re} m_{2r+1}^1 & -\operatorname{Im} m_{2r+1}^1 & \dots & \operatorname{Re} m_{2r+1}^{r+1} & -\operatorname{Im} m_{2r+1}^{r+1} \end{pmatrix}.$$

Let B_Λ be the $(2r+1)\times 2r$ real matrix obtained by taking the $2r$ last columns of A_Λ.

Lemma 3.1.20. *Let $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H}$ be a Lie group morphism. With the above notation the following conditions are equivalent:*

(a) *The Lie subgroups $\Lambda(\{0\}\times\mathbb{C}^r)$ and $\Lambda(\mathbb{C}^{r+1})$ of H verify*

$$\Lambda(\{0\}\times\mathbb{C}^r)\cap\mathrm{T} = \{0\}, \quad \dim_{\mathbb{R}}\Lambda(\mathbb{C}^{r+1})\cap\mathrm{T} = 1.$$

(b) $\operatorname{rank}\mathrm{A}_\Lambda = 2r+1$ and $\operatorname{rank}\mathrm{B}_\Lambda = 2r$. (I)

Moreover then $\Lambda(\{0\}\times\mathbb{C}^r)$ is a closed subgroup of H.

Remark 3.1.21. Under the hypothesis of the previous lemma the morphism Λ is injective when restricted to $\{0\}\times\mathbb{C}^r$ and $\dim\ker\Lambda = 0$.

Example 3.1.22. The following example shows that in order to assure that **(a)** is fulfilled it is not enough to impose that the complex matrix M has maximal rank. Consider the injective map $\Lambda^0 : \mathbb{C}^2 \to \mathbb{C}^3$ given by

$$(z_1, z_2) \mapsto (z_1 + (1+i)z_2, -z_1 + (1-i)z_2, 0)$$

which induces an injective morphism $\Lambda : \mathbb{C}^2 \to (\mathbb{C}^*)^3$ by the expression

$$(z_1, z_2) \mapsto (\mathrm{e}^{z_1+(1+\mathrm{i})z_2}, \mathrm{e}^{-z_1+(1-\mathrm{i})z_2}, 1).$$

Then with the previous notation

$$\mathrm{M} = \begin{pmatrix} 1 & 1+i \\ -1 & 1-i \\ 0 & 0 \end{pmatrix}; \quad \mathrm{A}_\Lambda = \begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \mathrm{B}_\Lambda = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note that $T = (S^1)^3 \subset (\mathbb{C}^*)^3$. Let us compute $\Lambda(\{0\} \times \mathbb{C}) \cap (S^1)^3$ and $\Lambda(\mathbb{C}^2) \cap (S^1)^3$. In the first case $z_1 = 0$ and $\Lambda((0, z_2)) \in (S^1)^3$ if and only if

$$\mathrm{Re}((1 + \mathrm{i})z_2) = \mathrm{Re}((1 - \mathrm{i})z_2) = 0,$$

that is, if and only if $z_2 = 0$. Therefore $\Lambda(\{0\} \times \mathbb{C}) \cap (S^1)^3 = \{0\}$ and note that this is equivalent to $\operatorname{rank} \mathrm{B}_\Lambda = 2$. On the other hand $\dim_{\mathbb{R}} \Lambda((z_1, z_2)) \cap (S^1)^3 = 1$ if and only if the linear system

$$\begin{cases} \mathrm{Re}(z_1 + (1 + \mathrm{i})z_2) = 0 \\ \mathrm{Re}(-z_1 + (1 - \mathrm{i})z_2) = 0 \end{cases}$$

has rank 3, that is, if $\operatorname{rank} \mathrm{A}_\Lambda = 3$, which is not the case in this example. Note that this shows that it is necessary to impose conditions on both A_Λ and B_Λ, not only on M.

Proof. Notice that $\mathrm{T} = (S^1)^{2r+1} \subset (\mathbb{C}^*)^{2r+1} = \mathrm{H}$ in the usual way. Then

$$\Lambda(z_1, ..., z_{r+1}) = \left(\mathrm{e}^{\sum_{j=1}^{r+1} m_1^j z_j}, ..., \mathrm{e}^{\sum_{j=1}^{r+1} m_{2r+1}^j z_j} \right)$$

and $\Lambda(0, z_2, ..., z_{r+1})$ intersects $(S^1)^{2r+1} \subset (\mathbb{C}^*)^{2r+1}$ if and only if

$$\mathrm{Re}\left(\sum_{j=2}^{r+1} m_1^j z_j \right) = ... = \mathrm{Re}\left(\sum_{j=2}^{r+1} m_{2r+1}^j z_j \right) = 0. \qquad (*)$$

Setting $z_j = x_j + \mathrm{i}y_j$ the condition $(*)$ can be rewritten as

$$\sum_{j=2}^{r+1} \left(\mathrm{Re}\, m_k^j x_j - \mathrm{Im}\, m_k^j y_j \right) = 0 \qquad \forall k = 1, ..., 2r + 1.$$

Note that the coefficients of this homogeneous system are the entries of the matrix B_Λ. Therefore the system $(*)$ admits a unique solution $z_2 = ... = z_{r+1} = 0$ if and only if $\operatorname{rank} \mathrm{B}_\Lambda = 2r$. Finally, an analogous computation shows that the condition $\dim_{\mathbb{R}} \Lambda(\mathbb{C}^{r+1}) \cap \mathrm{T} = 1$ is always verified since $\operatorname{rank} \mathrm{A}_\Lambda = 2r + 1$. We end by verifying that $\Lambda(\{0\} \times \mathbb{C}^r)$ is closed in $\mathrm{H} = (\mathbb{C}^*)^{2r+1}$. It is enough to check that it is closed at a neighborhood of $\Lambda(0) = e = 1$. Note that $\Lambda(\{0\} \times \mathbb{C}^r)$, at a neighborhood of $e = 1$, is isomorphic to an open set of $\mathbb{C}^r$. By the previous calculation $\Lambda(z)$ is close to e if and only if

$$\left| \sum_{j=2}^{r+1} \left(\mathrm{Re}\, m_k^j x_j - \mathrm{Im}\, m_k^j y_j \right) \right| < \epsilon \qquad \forall k = 1, ..., 2r + 1.$$

We conclude that $\Lambda(z)$ is close to e if and only if z is close to 0 in $\mathbb{C}^{r+1}$, thus $\Lambda(\{0\}\times\mathbb{C}^r)$ is closed in H. $\square$

Theorem 3.1.23. *Let* K *be a semisimple compact connected Lie group of odd dimension* $2n+1$ *and rank* $2r+1$ *and let* G *be its universal complexification. Assume that* H *is a Cartan subgroup of* G *and* $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H}$ *a Lie group morphism verifying the transversality condition* (I)*. If* B *is a Borel subgroup of* G *such that* $\mathrm{H} \subset \mathrm{B}$ *and* U *is its subgroup of unipotent elements then the Lie subalgebras* $\mathfrak{l}_\Lambda \subset \mathfrak{l}'_\Lambda$ *of* $\mathfrak{g}$ *associated to the complex Lie subgroups* $\mathrm{L'}_\Lambda = \Lambda(\mathbb{C}^{r+1})\cdot\mathrm{U}$ *and* $\mathrm{L}_\Lambda = \Lambda(\{0\}\times\mathbb{C}^r)\cdot\mathrm{U}$ *of* G *define a left-invariant normal almost-contact structure* K_Λ *on* K*. Moreover, the Lie subgroup* L_Λ *is closed and the CR-structure on* K *determined by* L_Λ *agrees with the one induced by the embedding* $\mathrm{K} \hookrightarrow \mathrm{G}/\mathrm{L}_\Lambda$ *of* K *as a real hypersurface of the complex manifold* $\mathrm{G}/\mathrm{L}_\Lambda$*. Conversely, every left-invariant normal almost contact structure is induced by such a morphism* Λ *from* $\mathbb{C}^{r+1}$ *into a Cartan subgroup* H *of* G*.*

In particular this theorem implies that every semisimple compact connected Lie group admits a left-invariant normal almost contact structure. Considering only $\mathrm{L}_\Lambda = \Lambda(\{0\}\times\mathbb{C}^r)\cdot\mathrm{U}$ we obtain a left-invariant CR-structure of maximal dimension by the same construction, however it is not true that every left-invariant CR-structure of maximal dimension is always of this type, the Lie group SU(2) provides a counterexample as we will see in the last section.

Lemma 3.1.24. *Under the hypothesis of the previous theorem, the connected complex Lie subgroup* $\mathrm{L}_\Lambda = \Lambda(\{0\}\times\mathbb{C}^r)\cdot\mathrm{U}$ *is closed in* G*.*

Proof. Recall that the Borel subgroups and the unipotent subgroups of a complex algebraic group are closed. On the other hand by the Iwasawa decomposition we have a diffeomorphism $\varphi : \mathrm{H}\times\mathrm{U} \to B$ given by $(h,u)\mapsto h\cdot u$. By the lemma 3.1.20 we know that $\Lambda(\{0\}\times\mathbb{C}^r)$ is closed in H. Let us see that this implies that $\mathrm{L}_\Lambda = \Lambda(\{0\}\times\mathbb{C}^r)\cdot\mathrm{U}$ is closed in B and thus in G. Choose $\{h_n\}_{n\in\mathbb{N}}$, $\{u_n\}_{n\in\mathbb{N}}$ such that $h_n \in \Lambda(\{0\}\times\mathbb{C}^r)$ and $u_n \in \mathrm{U}$ and assume that $h_n\cdot u_n \to h\cdot u$ when $n\to\infty$ where $h\in\mathrm{H}$ and $u\in\mathrm{U}$. Then $h_n \to h \in \mathrm{H}$ and $u_n \to u \in \mathrm{U}$ when $n \to \mathbb{N}$ (because φ is a diffeomorphism). As $\Lambda(\{0\}\times\mathbb{C}^r)$ is closed in H we have $h \in \Lambda(\{0\}\times\mathbb{C}^r)$ and $h\cdot u \in \mathrm{L}_\Lambda$. $\square$

Lemma 3.1.25. *The only pairs* $\mathrm{L}\subset\mathrm{L'}$ *of complex Lie subgroups of* $\mathrm{B} = \mathrm{H}\cdot\mathrm{U}$ *of dimensions* n *and* $n+1$ *respectively such that they contain* U, $\mathrm{T}\cap\mathrm{L} = \{e\}$ *and*

$\mathrm{T} \cap \mathrm{L}' = \langle \xi \rangle_{\mathbb{R}}$ *are those of the form* $\mathrm{L}_\Lambda = \Lambda(\{e\} \times \mathbb{C}^r) \cdot \mathrm{U} \subset \mathrm{L'}_\Lambda = \Lambda'(\mathbb{C}^{r+1}) \cdot \mathrm{U}$ *where* $\Lambda : \mathbb{C}^{r+1} \to (\mathbb{C}^*)^{2r+1}$ *is a Lie group morphism verifying* (I).

Proof. It is enough to apply lemma 3.1.13. □

Recall that if we fix a maximal torus T on K and a Borel subgroup B of G with Lie algebras $\mathfrak{t}$ and $\mathfrak{b}$ respectively and such that $\mathrm{T} \subset \mathrm{B}$ the corresponding Iwasawa decomposition states that $\mathrm{G} \cong \mathrm{K} \cdot \mathrm{A} \cdot \mathrm{U}$ where A and U are the simply connected Lie subgroups associated to the Lie algebras $i\mathfrak{t}$ and $\mathfrak{u} = [\mathfrak{b}, \mathfrak{b}]$ respectively. Recall also that $\mathrm{B} = \mathrm{H} \cdot \mathrm{U}$ where H is the Cartan subgroup of G associated to $\mathfrak{r} = \mathfrak{t}^{\mathbb{C}}$ and that $\mathrm{K} \cap \mathrm{B} = \mathrm{T} = (S^1)^{2r+1}$. Then $\mathrm{L}_\Lambda := \Lambda(\mathbb{C}^r \times \{0\}) \cdot \mathrm{U}$ is a closed connected complex Lie subgroup of G of complex dimension n. From the construction of L_Λ and the Iwasawa decomposition of G it follows that the natural inclusion $\mathrm{K} \hookrightarrow \mathrm{G}/\mathrm{L}_\Lambda$ is an embedding of K as a real hypersurface of the complex manifold $\mathrm{G}/\mathrm{L}_\Lambda$. Note that $\mathrm{G}/\mathrm{U} = \mathrm{K} \cdot \mathrm{A}$ where $\mathrm{A} \cong \mathbb{R}^{2r+1}$ and we consider $\mathrm{K} = \mathrm{K} \cdot \{e\} \subset \mathrm{G}/\mathrm{U}$. This inclusion induces a CR-structure on K which is left invariant (since K acts holomorphically on the complex Lie group G by left translations). Notice that if $\mathfrak{l}_\Lambda$ is the Lie subalgebra of $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ corresponding to L_Λ then $\mathfrak{l}_\Lambda$ defines the CR-structure on K induced by the embedding of K in $\mathrm{G}/\mathrm{L}_\Lambda$.

Proof. (Theorem 3.1.23) Let $\mathfrak{l}_\Lambda \subset \mathfrak{l}'_\Lambda$ be the Lie subalgebras of $\mathfrak{g}$ associated to the Lie subgroups L_Λ and $\mathrm{L'}_\Lambda$ respectively. Note that $\dim_{\mathbb{C}} \mathfrak{l}_\Lambda = n$ and $\dim_{\mathbb{C}} \mathfrak{l}'_\Lambda = n+1$. Clearly $\mathfrak{l}_\Lambda \cap \mathfrak{k} = \{e\}$, $\dim_{\mathbb{R}} \mathfrak{l}'_\Lambda \cap \mathfrak{k} = 1$ and $[\mathfrak{l}'_\Lambda, \mathfrak{l}'_\Lambda] \subset [\mathfrak{b}, \mathfrak{b}] = \mathfrak{u} \subset \mathfrak{l}_\Lambda$, therefore they define a left-invariant normal almost contact structure on K^{2n+1}. For the converse we apply proposition 3.1.11 and the previous lemma. □

Let K be a semisimple compact connected Lie group of odd dimension $2n+1$ and rank $2r+1$ and let $\rho : \mathrm{K} \to \mathrm{G} = \mathrm{K}^{\mathbb{C}}$ be the universal complexification of K. We choose two maximal tori T, T′ of K and two Borel subgroups B, B′ of G such that $\mathrm{H} := \rho(\mathrm{T})^{\mathbb{C}} \subset \mathrm{B}$ and $\mathrm{H}' := \rho(\mathrm{T}')^{\mathbb{C}} \subset \mathrm{B}'$ respectively. We denote by U, U′ the unipotent elements subgroups of B and B′ respectively. Let $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H}$ be a morphism of Lie groups verifying (I).

Proposition 3.1.26. *With the above notation, there exists* $k \in \mathrm{K}$ *such that if we denote by* $c_k : \mathrm{K} \to \mathrm{K}$ *the map defined by* $c_k(x) = k \cdot x \cdot k^{-1}$ *then:*

(a) $c_k(\mathrm{B}) = \mathrm{B}'$, $c_k(\mathrm{T}) = \mathrm{T}'$, $c_k(\mathrm{H}) = \mathrm{H}'$ *and* $c_k(\mathrm{U}) = \mathrm{U}'$.

(b) *The Lie group morphism* $\Lambda' : \mathbb{C}^{r+1} \to \mathrm{H}'$ *defined by* $\Lambda' = c_k \circ \Lambda$ *verifies* (I).

(c) *If we denote by* K_Λ *and* $\mathrm{K}_{\Lambda'}$ *the left-invariant normal almost contact structures on* K *obtained as in theorem 3.1.23 then* $c_k : \mathrm{K}_\Lambda \to \mathrm{K}_{\Lambda'}$ *is a left-invariant normal almost contact structure isomorphism.*

In other words, up to conjugacy, the only way to obtain different normal almost contact structures is by making a different choice of the morphism $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H}$.

Proof. **(a)** Since G is a connected algebraic group we know that there exists $g \in \mathrm{G}$ such that $g \cdot \mathrm{B} \cdot g^{-1} = \mathrm{B}'$ and $g \cdot \mathrm{T} \cdot g^{-1} = \mathrm{T}'$ (c.f. [Bor91], p. 156). Moreover, as G is semisimple, from the Iwasawa decomposition and the fact that $N(\mathrm{B}) = \mathrm{B}$ (cf. [Hum75], p.143) we derive that if $g = k \cdot a \cdot u \in \mathrm{K} \cdot \mathrm{A} \cdot \mathrm{U}$ then $\mathrm{B}' = k \cdot \mathrm{B} \cdot k^{-1} = c_k(\mathrm{B})$. Therefore $c_k(\mathrm{T})$ is a maximal torus of K contained in B', but since $\mathrm{K} \cap \mathrm{B}' = \mathrm{T}'$ we conclude that $c_k(\mathrm{T}) = \mathrm{T}'$. Then $c_k(\mathrm{H}) = \mathrm{H}'$ and $c_k(\mathrm{U}) = \mathrm{U}'$.

(b) It is clear.

(c) It is enough to notice that $c_k(\mathrm{G}) = \mathrm{G}$, $c_k(\mathrm{L}_\Lambda) = \mathrm{L}_{\Lambda'}$ and $c_k(\mathrm{L'}_\Lambda) = \mathrm{L'}_{\Lambda'}$.

□

Now we will illustrate the preceding results with some examples of classical Lie groups. We will skip some standard computations (see [MT86] or [Oni90] for details).

• $\mathrm{SU}(2) = \{A \in \mathrm{GL}(2,\mathbb{C}) : A \cdot \bar{A}^t = \mathrm{Id}, \det(A) = 1\}$ is a 3-dimensional compact connected semisimple Lie group. Its Lie algebra is

$$\mathfrak{su}(2) = \{A \in \mathrm{GL}(2,\mathbb{C}) : A + \bar{A}^t = 0, \operatorname{tr}(A) = 0\} = \left\{ \begin{pmatrix} \mathrm{i}y & u \\ -\overline{u} & -\mathrm{i}y \end{pmatrix} : y \in \mathbb{R}, u \in \mathbb{C} \right\}$$

and choosing the basis

$$u_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}; \quad u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad u_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

we see that $[u_1, u_2] = 2u_3$; $[u_2, u_3] = 2u_1$; $[u_3, u_1] = 2u_2$. Indeed,

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} : z, w \in \mathbb{C}, |z|^2 + |w|^2 = 1 \right\} \cong S^3.$$

The universal complexification of SU(2) is the group

$$\mathrm{SU}(2)^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C}) = \{A \in \mathrm{GL}(2,\mathbb{C}) : \det A = 1\}$$

whose Lie algebra is $\mathfrak{sl}(2,\mathbb{C}) = \{A \in M_{2\times 2}(\mathbb{C}) : \mathrm{tr}A = 0\}$. To define a left-invariant normal almost contact structure on SU(2) we will exhibit complex subalgebras $\mathfrak{l} \subset \mathfrak{l}'$ of $\mathfrak{sl}(2,\mathbb{C})$ of complex dimensions 1 and 2 respectively verifying the hypothesis of the section 3.1. Since $\mathrm{rank}\,\mathrm{SU}(2) = 1$ we can choose $\mathfrak{r} = \langle u_1 \rangle_{\mathbb{C}}$, which is unique up to conjugation. There exist two Borel subalgebras containing $\mathfrak{r}$:

$$\mathfrak{b} = \langle u_1, u_2 + \mathrm{i}u_3 \rangle_{\mathbb{C}},\ \mathfrak{u} = \langle u_2 + \mathrm{i}u_3 \rangle_{\mathbb{C}}; \quad \mathfrak{b}' = \langle u_1, u_2 - \mathrm{i}u_3 \rangle_{\mathbb{C}},\ \mathfrak{u}' = \langle u_2 - \mathrm{i}u_3 \rangle_{\mathbb{C}}.$$

Therefore, by the results of the section 3.1 we can conclude the following:

Proposition 3.1.27. *With the above notation:*

(a) *There are only one left-invariant normal almost contact structure on* SU(2) *up to conjugation:* $\mathfrak{u} \subset \mathfrak{b}$.

(b) *There are left-invariant CR-structures on* SU(2) *which cannot be completed to a normal almost contact structure.*

Recall that every 1-dimensional complex subalgebra $\mathfrak{l}$ of $\mathfrak{sl}(2,\mathbb{C})$ such that $\mathfrak{l} \cap \mathfrak{su}(2) = \{0\}$ in $\mathfrak{sl}(2,\mathbb{C})$ defines a left-invariant CR-structure on SU(2). For instance, we can choose $\mathfrak{l} = \langle u_1 + \alpha(u_2 + \mathrm{i}u_3) \rangle_{\mathbb{C}}$ for $\alpha \in \mathbb{R}$, which can not be completed to a normal almost contact structure.

• $\mathrm{SO}(3) = \{A \in \mathrm{GL}(3,\mathbb{R}) : A \cdot A^t = \mathrm{Id},\, \det A = 1\}$ is a 3-dimensional compact connected semisimple Lie group with Lie algebra

$$\mathfrak{so}(3,\mathbb{R}) = \{A \in M_{3\times 3}(\mathbb{R}) :\ A + A^t = 0,\ \mathrm{tr}(A) = 0\}.$$

It admits a basis

$$e_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

such that $[e_1, e_2] = e_3$; $[e_2, e_3] = e_1$ and $[e_3, e_1] = e_2$. It is well known that there exists a covering map $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3)$. Moreover $\mathrm{SO}(3) \cong \mathbb{R}P^3$ and $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$. The universal complexification SO(3) is the group

$$\mathrm{SO}(3,\mathbb{C}) = \{A \in \mathrm{GL}(3,\mathbb{C}) : A \cdot A^t = \mathrm{Id},\ \det A = 1\}$$

with Lie algebra $\mathfrak{so}(3,\mathbb{C}) = \{A \in M_{3\times 3}(\mathbb{C}) :\ A + A^t = 0,\ \mathrm{tr}(A) = 0\}$. Note also that $\mathrm{SO}(3) = \mathrm{SO}(3,\mathbb{C}) \cap \mathrm{SU}(3)$.

As $\operatorname{rank}\mathrm{SO}(3) = 1$ we can choose $\mathfrak{r} = \langle e_1 \rangle_{\mathbb{C}}$ as maximal abelian subalgebra and it is unique up to conjugation. There are two Borel subalgebras containing $\mathfrak{r}$ as a subalgebra: $\mathfrak{b} = \langle e_1, e_2 - \mathrm{i}e_3 \rangle_{\mathbb{C}} \supset \mathfrak{u} = [\mathfrak{b}, \mathfrak{b}] = \langle e_2 - \mathrm{i}e_3 \rangle_{\mathbb{C}}$ and $\mathfrak{b}' = \langle e_1, e_2 + \mathrm{i}e_3 \rangle_{\mathbb{C}} \supset \mathfrak{u}' = [\mathfrak{b}', \mathfrak{b}'] = \langle e_2 + \mathrm{i}e_3 \rangle_{\mathbb{C}}$. As in the previous case, we conclude that:

Proposition 3.1.28. *With the above notation, there is only one left-invariant normal almost contact structure on* SO(3) *up to conjugation:* $\mathfrak{u} \subset \mathfrak{b}$ *.*

Since $\mathfrak{so}(3, \mathbb{R}) \cong \mathfrak{su}(2)$ it is clear that the study of left-invariant normal almost contact structures on SO(3) is the same as in SU(2).

• $\mathrm{SU}(n) = \{A \in M_{n\times n}(\mathbb{C}) :\ A \cdot \bar{A}^t = \mathrm{Id};\ \det A = 1\}$ is a compact connected semisimple real Lie group with Lie algebra

$$\mathfrak{su}(n) = \{A \in M_{n\times n}(\mathbb{C}) :\ A + \bar{A}^t = 0;\ \operatorname{tr} A = 0\}$$

$$= \left\{ \begin{pmatrix} \mathrm{i}\theta_1 & -\bar{z}_1 & & \cdots \\ z_1 & \mathrm{i}\theta_2 & & \cdots \\ \cdots\cdot & & & \\ \cdots\cdot & & \mathrm{i}\theta_{n-1} & -\bar{z}_{n^2-n} \\ \cdots\cdot & & z_{n^2-n} & -\mathrm{i}\sum_{k=1}^{n-1}\theta_k \end{pmatrix} :\ \theta_1, ..., \theta_{n-1} \in \mathbb{R};\ z_j \in \mathbb{C},\ j = 1, ..., n^2 - n \right\}.$$

A small computation shows that $\dim_{\mathbb{R}} \mathrm{SU}(n) = n^2 - 1$ and that $\mathrm{SU}(n)$ is simply connected (this follows from the isomorphism $\mathrm{SU}(n)/\mathrm{SU}(n-1) \cong S^{2n-1}$). Moreover rank $\mathrm{SU}(n) = n - 1$. From now on we assume n is even, so that $\dim_{\mathbb{R}} \mathrm{SU}(n) = n^2 - 1$ is odd.

We can choose a basis of $\mathfrak{su}(n)$ formed by the vectors $\{r_1, ..., r_{n-1}, u_{ij}\}$ where u_{ij} are matrices with a 1 in the position (i, j) for $i \neq j$ and a zero otherwise and r_j are diagonal matrices with 1 in the (j, j) position for $j < n$, -1 in the (n, n) position and zero otherwise. We can easily compute the Lie brackets, we obtain $[r_i, r_j] = 0$ and

$$[u_{ij}, u_{sl}] = \begin{cases} r_i - r_j & j = s,\ i = l \\ u_{il} & j = s,\ i \neq l \\ -u_{sj} & i = l,\ j \neq s \\ 0 & \text{otherwise;} \end{cases}$$

$$[r_k, u_{ij}] = \begin{cases} u_{ij} & i = k,\ j \neq k, 2n \quad \text{o} \quad j = 2n,\ i \neq k, 2n \\ -u_{ij} & i = 2n,\ j \neq k, 2n \quad \text{o} \quad j = k,\ i \neq k, 2n \\ 2u_{ij} & i = k,\ j = 2n \\ -2u_{ij} & i = 2n,\ j = k \\ 0 & \text{otherwise;} \end{cases}$$

The universal complexification of SU(n) is SL(n,$\mathbb{C}$), whose Lie algebra is $\mathfrak{sl}(n,\mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}) : \operatorname{tr}(A) = 0\}$. Up to conjugation we can assume that the maximal abelian subalgebra is $\mathfrak{r} = \langle r_1, ..., r_{n-1}\rangle_{\mathbb{C}}$. As a Borel subalgebra $\mathfrak{b}$ we choose superior diagonal matrices or diagonal inferior matrices. The corresponding subalgebra $\mathfrak{u}$ of unipotent elements is the one generated by $\{u_{ij}\}$ for $i < j$ or $j > i$ respectively.

Proposition 3.1.29. *With the above notation, fixed the previous Cartan subalgebra* $\mathfrak{r}$ *and Borel subalgebra* $\mathfrak{b}$*, the left invariant normal almost contact structures on* SU(n) *described in theorem 3.1.23 are the pairs of complex subalgebras* $\mathfrak{l} \subset \mathfrak{l}' \subset \mathfrak{sl}(n,\mathbb{C})$ *of the form*

$$\mathfrak{l} = \mathfrak{u} \oplus \left\langle \sum_{j=1}^{n-1} a_k^j r_j \right\rangle_{k=1}^{n/2-1} \quad ; \qquad \mathfrak{l}' = \mathfrak{l} \oplus \left\langle \sum_{j=1}^{n-1} b^j r_j \right\rangle .$$

where $a_k^j, b^j \in \mathbb{C}$ *and* $\operatorname{rank}(\operatorname{Re} a_k^j, \operatorname{Im} a_k^j) = n-2$. *Moreover these are the only left-invariant normal almost contact structures up to conjugation.*

We end this section by proving a last corollary of theorem 3.1.23:

Corollary 3.1.30. *Let* K *be a compact connected Lie group of odd dimension. Then* K *admits at least one left-invariant normal almost contact structure (and in particular a left-invariant CR-structure of maximal dimension).*

Remarks 3.1.31. **(a)** By Samelson-Wang theorem we know that every semisimple compact connected Lie group of even dimension admits a left-invariant complex structure.

(b) Let K be a semisimple compact connected real Lie group of even dimension endowed with a left-invariant complex structure. Then the odd-dimensional compact connected Lie group K$\times S^1$ admits a natural left-invariant normal almost contact structure. Indeed, if $\mathfrak{l}$ is a complex subalgebra of $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ defining the distribution of vector fields of type $(0,1)$ and $\frac{\partial}{\partial t}$ is a tangent vector field of S^1 inducing the S^1-action then the subalgebras $\mathfrak{l} \subset \mathfrak{l}' = \mathfrak{l} \oplus \langle \frac{\partial}{\partial t}\rangle^{\mathbb{C}}$ of $\mathfrak{l} \oplus \mathbb{C}$ define a normal almost contact structure on K $\times S^1$.

(c) Let M and N be compact connected Lie groups. If M and N carry respectively a left-invariant normal almost contact structure and left-invariant complex structure then the product M$\times$N carries a natural left-invariant normal almost contact structure (by an argument analogous to the previous one).

Proof. If K is semisimple we apply theorem 3.1.23. Otherwise we know that up to a finite covering K is isomorphic to $\mathrm{K}' \times (S^1)^s$ where K' is a semisimple compact Lie group (see section 1.7.5). If s is even we apply theorem 3.1.23 to obtain a left-invariant normal almost contact structure on K' and complexify $(S^1)^s$ in a natural way. If s is odd we fix a left-invariant complex structure on K', then $\mathrm{K}' \times S^1$ admits a left-invariant normal almost contact structure, and we complexify $(S^1)^{s-1}$ in a natural way. To conclude recall that the finite quotient on $\mathrm{K}' \times (S^1)^s$ which yields K is given by left-translations of elements of $Z(\mathrm{K}') \times \{e\}$. Since the normal almost contact structure that we have defined on $\mathrm{K}' \times (S^1)^s$ is left-invariant it induces a well defined normal almost contact structure on the quotient K. □

Remark 3.1.32. Since we are considering left-invariant structures all the previous operations could be described in terms of the Lie algebra, however this argument has the advantage that generalizes easily to the non-invariant case.

3.2 Non-invariant normal almost contact structures

Let K be a semisimple compact connected real Lie group of odd dimension $2n+1$ and rank $2r+1$ with Lie algebra $\mathfrak{k}$ and let $\rho : \mathrm{K} \to \mathrm{G} = \mathrm{K}^{\mathbb{C}}$ be the universal complexification of K.

Definition 3.2.1. A normal almost contact structure on a compact connected Lie group K of odd dimension $2n+1$ is determined by a complex subbundle V of $T^{\mathbb{C}}\mathrm{K}$ of rank n and a real vector field ξ on K such that $V' := V \oplus \langle \xi \rangle_{\mathbb{C}}$ is a complex subbundle of $T^{\mathbb{C}}\mathrm{K}$ of rank $n+1$ fulfilling:

(a) $V \cap T_p\mathrm{K} = \{e\}$;

(b) $[V, V] \subset V$;

(c) $[\xi, V] \subset V$;

for every $p \in \mathrm{K}$.

Remark 3.2.2. By convention V corresponds to the distribution of vector fields of $T_p^{\mathbb{C}}\mathrm{K}$ of type $(0, 1)$. In the left-invariant case it was enough to fix V_e and ξ_e so that we obtained Lie subalgebras of $\mathfrak{k}^{\mathbb{C}}$.

We fix a maximal torus $\mathrm{T} \subset \mathrm{K}$ and a Borel subgroup B of G such that $\mathrm{H} := \rho(\mathrm{T})^{\mathbb{C}} \subset \mathrm{B}$, with Lie algebras $\mathfrak{t}$ and $\mathfrak{b}$ respectively. The corresponding Iwasawa decomposition induces a diffeomorphism $\mathrm{G} \cong \mathrm{K} \cdot \mathrm{A} \cdot \mathrm{U}$ where A and U are the simply connected Lie subgroups associated to the Lie algebras $i\mathfrak{t}$ and $\mathfrak{u} := [\mathfrak{b}, \mathfrak{b}]$ respectively. Since $\mathrm{G}/\mathrm{U} \cong \mathrm{K} \cdot \mathrm{A}$ there is a projection $\pi : \mathrm{G}/\mathrm{U} \to \mathrm{A} \cong \mathbb{R}^{2r+1}$. We will denote by $\mathrm{K} \cdot a$ its fibers and by $\mathrm{K} := \mathrm{K} \cdot e$. Then the inclusion $\mathrm{K} \hookrightarrow \mathrm{G}/\mathrm{U}$ is an embedding.

Lemma 3.2.3. *The action of* $(\mathbb{C}^*)^{4r+2} \cong \mathrm{H} \times \mathrm{H} = \mathrm{B}/\mathrm{U} \times \mathrm{B}/\mathrm{U}$ *on the homogeneous space* G/U *given by*

$$\mathrm{H} \times \mathrm{H} \times \mathrm{G}/\mathrm{U} \to \mathrm{G}/\mathrm{U}$$

$$(h_1, h_2, [g]) \mapsto [h_1 \cdot g \cdot h_2]$$

is well defined.

Proof. The right action of H on G/U is well defined for $N(\mathrm{U}) = \mathrm{B}$ (cf. [Hum75], p.144). The left action is the restriction of the action of G on G/U. Finally since both actions commute the action of H×H is well-defined. □

Given a locally free holomorphic $\mathbb{C}^{r+1}$-action $\varphi : \mathbb{C}^{r+1} \times \mathrm{G}/\mathrm{U} \to \mathrm{G}/\mathrm{U}$ and $x \in \mathrm{G}$ we define

$$F_x := \varphi(\{0\} \times \mathbb{C}^r, [x]); \qquad F'_x := \varphi(\mathbb{C}^{r+1}, [x]),$$

that is, the leaves through $[x] \in \mathrm{G}/\mathrm{U}$ of the foliations defined by the actions of $\mathbb{C}^r \cong \{0\} \times \mathbb{C}^r$ and $\mathbb{C}^{r+1}$ respectively. We denote by $z_0, z_1, \ldots, z_r$ the linear coordinates of $\mathbb{C}^{r+1}$.

Lemma 3.2.4. *A locally free holomorphic* $\mathbb{C}^{r+1}$*-action* $\varphi : \mathbb{C}^{r+1} \times \mathrm{G}/\mathrm{U} \to \mathrm{G}/\mathrm{U}$ *fulfilling:*

- **(i)** $\dim_{\mathbb{R}}(F_p \cap \mathrm{K} \cdot a) = 0$*, for* $a = e$ *and each* $p \in \mathrm{K} \cdot e = \mathrm{K}$*,*
- **(ii)** $\dim_{\mathbb{R}}(F'_p \cap \mathrm{K} \cdot a) = 1$*, for* $a = e$ *and each* $p \in \mathrm{K} \cdot e = \mathrm{K}$*,*
- **(iii)** *there exists* $\lambda \in \mathbb{C}$ *such that* $\xi = d\varphi\big(\mathrm{Re}\,\big(\lambda \frac{\partial}{\partial z_0}\big)\big)$ *is tangent to* $\mathrm{K} \cdot a$*, for* $a = e$ *and each* $p \in \mathrm{K} \cdot e = \mathrm{K}$*,*

induces a normal almost contact structure on K*. We refer to the previous conditions as the transversality hypothesis* (II)*.*

Proof. Let $\tilde{\pi} : \mathrm{G} \to \mathrm{G}/\mathrm{U}$ be the natural projection. We define

$$V_p := \tilde{\pi}^*(TF_p)_{|\mathrm{K}} \subset T_p\mathrm{G}.$$

One can easily see that V_p is well defined and depends analytically on p. By definition $T_p\mathrm{U} \subset V_p$. Clearly all the conditions in the above proposition hold since the action is locally free by hypothesis and the fundamental vector fields of the action commute. □

Remark 3.2.5. Unlike the case of complex structures (see lemma 3.0.8) a $\mathbb{C}^{r+1}$-action on a complex manifold X containing K as a real submanifold and such that the intersection of the leaves of the associated foliation $\mathcal{F}$ with K has real dimension 1 does not determine a normal almost contact structure for there is no natural choice of a distribution transverse to the vector field. Therefore the previous lemma should be considered as analogous of lemma 3.0.8 for normal almost contact structures.

• Let us consider non-invariant normal almost contact structures on SU(2). We use the same notation as in the example of the non-invariant case. The Iwasawa decomposition associated to $\mathfrak{r} = \langle u_1 \rangle_{\mathbb{C}}$ and $\mathfrak{u} = \langle u_2 - \mathrm{i} u_3 \rangle_{\mathbb{C}}$ is

$$\mathrm{SL}(2, \mathbb{C}) = \mathrm{SU}(2) \cdot \mathrm{A} \cdot \mathrm{U}$$

where $\mathrm{A} = \left\{ \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix} \right) : \lambda \in \mathbb{R}^+ \right\}$, $\mathrm{H} = \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix} \right) : \alpha \in \mathbb{C}^* \right\}$, $\mathrm{U} = \{ \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix} \right) : a \in \mathbb{C} \}$. Therefore

$$\mathrm{SU}(2) \hookrightarrow \mathrm{SL}(2, \mathbb{C})/\mathrm{U} \cong \mathbb{C}^2 \backslash \{0\}$$

where the identification of $\mathrm{SL}(2, \mathbb{C})/\mathrm{U}$ with $\mathbb{C}^2 \backslash \{0\}$ is induced by the transitive action of $\mathrm{SL}(2, \mathbb{C})$ over $\mathbb{C}^2 \backslash \{0\}$ defined by

$$\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^2 \backslash \{0\} \to \mathbb{C}^2 \backslash \{0\}$$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), (z, w) \mapsto (az + bw, cz + dw)$$

(note that the isotropy group of $(1, 0)$ is U). With this identification SU(2)$\cong S^3$ is embedded in the usual way in $\mathbb{C}^2 \backslash \{0\}$. The action of $\mathrm{H} \times \mathrm{H} \cong (\mathbb{C}^*)^2$ on $\mathbb{C}^2 \backslash \{0\}$ defined by the product on the right and on the left corresponds to

$$(\mathrm{H} \times \mathrm{H}) \times \mathbb{C}^2 \backslash \{0\} \to \mathbb{C}^2 \backslash \{0\}$$

$$(\alpha, \beta), (z, w) \mapsto (\alpha\beta z, \alpha^{-1}\beta w).$$

A morphism $\Lambda : \mathbb{C} \to (\mathbb{C}^*)^2$ is of the form $\Lambda(t) = (e^{at}, e^{bt})$ for $a, b \in \mathbb{C}$. Therefore the locally free $\mathbb{C}$-action on $\mathbb{C}^2\backslash\{0\}$ induced by the action of $\mathrm{H} \times \mathrm{H}$ is

$$\mathbb{C} \times \mathbb{C}^2\backslash\{0\} \to \mathbb{C}^2\backslash\{0\}$$

$$(t, (z, w)) \mapsto (e^{(a+b)t}z, e^{(b-a)t}w)$$

where $a, b \in \mathbb{C}$ are parameters. We will now study for which choices of a and b the previous action verifies the transversality hypothesis (II). Notice that the action φ_Λ is the induced by the linear vector field

$$\eta = (a + b)z\frac{\partial}{\partial z} + (b - a)w\frac{\partial}{\partial w},$$

which intersects S^3 in a 1-dimensional orbit if and only if there does not exist $\mu \in \mathbb{R}^-$ such that $a + b = \mu(b - a)$ or $b - a = \mu(a + b)$. Therefore the condition $\dim_{\mathbb{R}} \varphi_\Lambda(\mathbb{C}, p) \cap \mathrm{K} = 1$ for all $p \in \mathrm{SU}(2)$ is verified if and only if there does not exist $\mu \in \mathbb{R}^-$ such that $a + b = \mu(b - a)$ or $b - a = \mu(a + b)$. On the other hand $\varphi(t, (z, w)) \in S^3$ if and only if

$$e^{2\,\mathrm{Re}((a+b)t)}|z|^2 + e^{2\,\mathrm{Re}((b-a)t)}|w|^2 = 1 \qquad (*).$$

If the condition **(iii)** of the transversality hypothesis is verified then

$$\begin{cases} \mathrm{Re}((a + b)t) = 0 \\ \mathrm{Re}((b - a)t) = 0 \end{cases}$$

because the $t \in \mathbb{C}$ that verify $(*)$ must be independent of the point $(z, w) \in S^3$. We decompose $t = t_1 + it_2$, $a = a_1 + ia_2$, $b = b_1 + ib_2$ where $t_1, t_2, a_1, a_2, b_1, b_2 \in \mathbb{R}$ and the previous system can be rewritten as

$$\begin{cases} (a_1 + b_1)t_1 - (a_2 + b_2)t_2 = 0 \\ (b_1 - a_1)t_1 - (b_2 - a_2)t_2 = 0 \end{cases}$$

and we must impose that the vector subspace of solutions has real dimension 1, that is,

$$(a_1 + b_1)(a_2 - b_2) + (a_2 + b_2)(b_1 - a_1) = \mathrm{Im}((a + b)\overline{(b - a)}) = 0,$$

or equivalently, that there exists $\mu \in \mathbb{R}$ such that $a + b = \mu(b - a)$ or $b - a = \mu(a + b)$. Note that if $\mu \in \mathbb{R}^-$ then $\dim_{\mathbb{R}}(F'_p \cap S^3) = 2$ at some point $p \in S^3$. To sum up, we have obtained that a morphism $\Lambda : \mathbb{C} \to (\mathbb{C}^*)^2$ verifies the transversality hypothesis (II) for SU(2) if and only if there exists $\mu \in \mathbb{R}^+$ such that $a+b=\mu(b-a)$ or $b-a=\mu(a+b)$. By the previous lemma we conclude that each such morphism Λ induces a normal almost contact structure on SU(2). Note that the CR-structure is always left-invariant (and corresponds to the Lie subalgebra $\mathfrak{u}$). On the other hand the vector field

$$\xi = \mathrm{Re}\left(\lambda \cdot \left((a+b)z\frac{\partial}{\partial z} + (b-a)w\frac{\partial}{\partial w}\right)\right) = \mathrm{Re}\left(\mathrm{i}\left(z\frac{\partial}{\partial z} + \mu w\frac{\partial}{\partial w}\right)\right)$$

where $\mu \in \mathbb{R}^+$ is left-invariant if and only if $\mu = 1$. Indeed, one can verify the last statement explicitly taking into account that the action of

$$S^3 \cong \mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\}$$

on the left on $\mathbb{C}^2 \backslash \{0\} \cong \mathrm{SL}(2,\mathbb{C})/\mathrm{U}$ by the previous identification is given by

$$(\alpha,\beta),(z,w) \mapsto (\alpha z - \overline{\beta} w, \beta z + \overline{\alpha} w).$$

Proposition 3.2.6. *There is a family of non-invariant normal almost contact structures on* SU(2).

• Let us consider non-invariant structures on SO(3,$\mathbb{R}$), like before we use the same notation as in the left-invariant case. Note that the difference with SU(2) consists only in a finite quotient. As every normal almost contact structure on SO(3,$\mathbb{R}$) admits a lift to a normal almost contact structure on SU(2) it is enough to study which of the structures on SU(2) are invariant by the map $\nu : \mathbb{C}^2\backslash\{0\} \to \mathbb{C}^2\backslash\{0\}$ defined by $\nu(z,w) = (-z,-w)$ (with the same notation as before). Since all the CR-structures are left-invariant and $\nu^*\eta = \eta$ there is no restriction in the choice of the morphism Λ. Therefore every normal almost contact structure on SU(2) induces a normal almost contact structure on SO(3,$\mathbb{R}$). This conclusion is not surprising since we know that finite coverings of Lie groups are obtained by quotients of discrete subgroups of the center.

Proposition 3.2.7. *There is a bijection between normal almost contact structures on* SU(2) *and on* $\mathrm{SO}(3,\mathbb{R})$. *In particular there is a family of non-invariant normal almost contact structures on* $\mathrm{SO}(3,\mathbb{R})$.

Theorem 3.2.8. *Let* K *be a semisimple compact connected Lie group of odd dimension* $2n+1$ *and rank* $2r+1$ *and let* G *be its universal complexification.*

Assume that H $\subset$ B *are a Cartan subgroup and a Borel subgroup of* G *respectively. Then every morphism of Lie groups* $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H} \times \mathrm{H}$ *inducing a locally free holomorphic action* $\varphi_\Lambda : \mathbb{C}^{r+1} \times \mathrm{G/U} \to \mathrm{G/U}$ *verifying* (II) *determines a normal almost contact structure in a natural way by lemma 3.2.4. Moreover, such a normal almost contact structure is left-invariant if and only if* $\Lambda = (e, \Lambda_2)$ *where* $\Lambda_2 : \mathbb{C}^{r+1} \to \mathrm{H}$ *is a morphism verifying the transversality hypothesis* (I). *In particular, there exist small deformations of the previous ones obtained by deforming* Λ *which induce suitable* $\mathbb{C}^{r+1}$*-actions defining normal almost contact structures on* K *generically non-invariant.*

Remark 3.2.9. Note that when rank K > 1, that is, $r > 0$, it is clear that there exist suitable deformations of the actions inducing left-invariant normal almost contact structures. Indeed, it is enough to fix the action on the first variable (so that the condition **(iii)** of the transversality hypothesis (II) holds) and deform slightly the action on the others variables. Then one obtains a left-invariant vector field and a generically non-invariant CR-structure. On the other hand the only compact connected Lie groups of rank 1 are S^1, SO(3) and SU(2) (see [BtD85], p.185). In the first case a normal almost contact structure is simply a non-vanishing vector field. For the other two cases we have already proved directly that there are non-invariant normal almost contact structures. Notice that for these groups the CR-structure turned out to be always left-invariant whereas the vector field were not.

• Let us apply the previous theorem to construct non-invariant normal almost contact structures on SU(n). We recover the notation of the left-invariant case. Fix the following Iwasawa decomposition associated to $\mathfrak{r} = \langle r_1, ..., r_{n-1}\rangle_{\mathbb{C}}$:

$$\mathrm{SL}(n, \mathbb{C}) = \mathrm{SU}(n) \cdot \mathrm{A} \cdot \mathrm{U}$$

where U is the subgroup of diagonal inferior matrices with 1's on the diagonal, A is the subgroup of diagonal matrices with entries $\lambda_1, ..., \lambda_n \in \mathbb{R}^+$ such that $\lambda_1 \cdot \lambda_2 \cdot ... \cdot \lambda_n = 1$ and H is the subgroup of diagonal matrices with entries $\alpha_1, ..., \alpha_n \in \mathbb{C}$ such that $\alpha_1 \cdot ... \cdot \alpha_n = 1$. Denote $n = 2r + 2$ and

$$\Lambda : \mathbb{C}^{r+1} \longrightarrow \mathrm{H} \times \mathrm{H} \cong (\mathbb{C}^*)^{2n-2}$$

$$(t_1, ..., t_{r+1}) \mapsto \left(\mathrm{e}^{\sum_{j=1}^{r+1} c_1^j t_j}, ..., \mathrm{e}^{\sum_{j=1}^{r+1} c_{n-1}^j t_j}, \mathrm{e}^{\sum_{j=1}^{r+1} d_1^j t_j}, ..., \mathrm{e}^{\sum_{j=1}^{r+1} d_{n-1}^j t_j}\right)$$

for $c_k^j, d_k^j \in \mathbb{C}$. If condition (II) holds we obtain a normal almost contact structure on SU(n). If $\mathrm{rank}(\mathrm{Re}\, d_k^j, -\mathrm{Im}\, d_k^j)_{k=1}^{n-1} = n - 1$, $\mathrm{rank}(\mathrm{Re}\, d_k^j, -\mathrm{Im}\, d_k^j)_{k=2}^{n-1} = n - 2$,

$c_k^1 = 0$ for $k = 1, ..., n-1$ and c_k^j are small enough for $k = 2, ..., n-1$ and $j = 1, ..., r+1$ we are under the hypothesis of theorem 3.2.8 and we obtain normal almost contact structures on $\mathrm{SU}(n)$ that are not left-invariant unless $c_k^j = 0$ for every j, k (notice though that the vector field of the normal almost contact structure *is* left-invariant).

Proposition 3.2.10. *There is a family of non-invariant normal almost contact structures on* $\mathrm{SU}(n)$.

The proof of the above theorem is straightforward once we prove the following two propositions. The first one assures us that a small deformation of a morphism Λ under the above hypothesis induce a locally free action on $\mathrm{G/U}$ and the second one characterizes when a normal almost contact structure so obtained is left-invariant. We will hereinafter consider that a Lie group morphism $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H} \times \mathrm{H}$ induces a holomorphic action $\varphi_\Lambda : \mathbb{C}^{r+1} \times \mathrm{G/U} \to \mathrm{G/U}$ by means of the action of $\mathrm{H} \times \mathrm{H}$ on $\mathrm{G/U}$. We denote such a morphism $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H} \times \mathrm{H}$ by $\Lambda = (\Lambda_1, \Lambda_2)$ where $\Lambda_i : \mathbb{C}^{r+1} \to \mathrm{H}$ are Lie group morphisms.

Lemma 3.2.11. *Let* $\varphi_\Lambda : \mathbb{C}^{r+1} \times \mathrm{G/U} \to \mathrm{G/U}$ *be a holomorphic action induced by a Lie group morphism* $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H} \times \mathrm{H}$*. Then for every* $x \in \mathrm{K}$*,* $c \in \mathbb{C}^{r+1}$ *and* $a \in \mathrm{A}$ *we have*

$$\varphi_\Lambda(c, [x \cdot a]) = \varphi_\Lambda(c, [x]) \cdot a.$$

Proof. Note that $\mathrm{A} \subset \mathrm{H} \subset N(\mathrm{U}) = \mathrm{B}$. Then

$$\varphi_\Lambda(c, [x \cdot a]) = [\Lambda_1(c) \cdot x \cdot a \cdot \Lambda_2(c)] = [\Lambda_1(c) \cdot x \cdot \Lambda_2(c) \cdot a] = \varphi_\Lambda(c, [x]) \cdot a.$$

□

Proposition 3.2.12. *Let* $\varphi_\Lambda : \mathbb{C}^{r+1} \times \mathrm{G/U} \to \mathrm{G/U}$ *be a holomorphic action induced by a Lie group morphism* $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H} \times \mathrm{H}$ *such that the action* φ_Λ *is locally free and verifies* (II) *for* $a = e$ *and for each* p *in* K*. Then the action* φ_Λ *verifies* (II) *for each* $a \in \mathrm{A}$ *and for each* $p \in \mathrm{K} \cdot a$.

Proof. Choose $[y] \in \mathrm{K} \cdot a$. There exist $x \in \mathrm{K}$ and $v \in \mathrm{U}$ such that $y = x \cdot a \cdot v$ and by lemma 3.2.11 we know that

$$\varphi_\Lambda(c, [y]) = \varphi_\Lambda(c, [x]) \cdot a.$$

We derive easily that if $\dim_{\mathbb{R}} F_p \cap \mathrm{K} \cdot e = 0$ and $\dim_{\mathbb{R}} F'_p \cap \mathrm{K} \cdot e = 1$ hold for every $p \in \mathrm{K} \cdot e$ the previous equalities are also true for $p \in \mathrm{K} \cdot a$ substituting K by $\mathrm{K} \cdot a$. Analogously one proves **(iii)**. □

Proposition 3.2.13. *Let $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H} \times \mathrm{H}$ be a Lie group morphism inducing a locally free holomorphic action $\varphi_\Lambda : \mathbb{C}^{r+1} \times \mathrm{G/U} \to \mathrm{G/U}$ verifying* (II). *With the same notation as in theorem 3.2.8 the following conditions are equivalent:*

(a) *The normal almost contact structure K_Λ on K induced by Λ is left-invariant.*

(b) *Given $x, k \in \mathrm{K}$ we have $k \cdot F_x = F_{k \cdot x}$ and $k \cdot F'_x = F'_{k \cdot x}$.*

(c) *Given $y, g \in \mathrm{G}$ we have $g \cdot F_y = F_{g \cdot y}$ and $g \cdot F'_y = F'_{g \cdot y}$.*

(d) *The morphism Λ is of the form $\Lambda = (e, \Lambda_2)$ where $\Lambda_2 : \mathbb{C}^{r+1} \to \mathrm{H}$ is a Lie group morphism verifying* (I).

Proof. **(a)⇔(b):** The normal almost contact structure K_Λ obtained by lemma 3.2.4 is left-invariant if and only if for every $x, k \in \mathrm{K}$ we have

$$k \cdot \mathrm{d}\varphi_\Lambda(\{0\} \times \mathbb{C}^r, [x]) = \mathrm{d}\varphi_\Lambda(\{0\} \times \mathbb{C}^r, [k \cdot x]), \;\; k \cdot \mathrm{d}\varphi_\Lambda(\mathbb{C}^{r+1}, [x]) = \mathrm{d}\varphi_\Lambda(\mathbb{C}^{r+1}, [k \cdot x]),$$

(by abuse of notation we denote here by $k \cdot$ the differential of this map). These conditions say that the distributions of $\mathbb{C}^r$ and $\mathbb{C}^{r+1}$-planes tangent to the leaves F_x and F'_x of the foliations associated to the actions $\varphi_\Lambda(\{0\} \times \mathbb{C}^r, \cdot)$ and φ_Λ respectively are left-invariant by elements of K when restricted to K. From a differential version of lemma 3.2.11 we conclude that these distributions are left-invariant by elements of K when restricted to K if and only if they are left-invariant over every fibre $\mathrm{K} \cdot a$ of $\pi : \mathrm{G/U} \to \mathrm{A} \cong \mathbb{C}^{2r+1}$ (and consequently over G/U). Integrating the distributions we obtain the leaves of the foliations associated to the actions $\varphi_\Lambda(\{0\} \times \mathbb{C}^r, \cdot)$ and φ_Λ. It follows that the normal almost contact structure is left-invariant if and only if

$$k \cdot \varphi_\Lambda(\{0\} \times \mathbb{C}^r, [x]) = \varphi_\Lambda(\{0\} \times \mathbb{C}^r, [k \cdot x]), \quad k \cdot \varphi_\Lambda(\mathbb{C}^{r+1}, [x]) = \varphi_\Lambda(\mathbb{C}^{r+1}, [k \cdot x]),$$

that is, if and only if $k \cdot F_x = F_{k \cdot x}$ and $k \cdot F'_x = F'_{k \cdot x}$ for every $x, k \in \mathrm{K}$.
(b)⇔(c): We will see first that it is enough to prove that for every $y \in \mathrm{G}$ and $k \in \mathrm{K}$ we have $k \cdot F_y = F_{k \cdot y}$ and $k \cdot F'_y = F'_{k \cdot y}$. We will denote these two conditions by $(*)$. Indeed, if $(*)$ hold we define

$$\widetilde{\mathrm{G}} = \{g \in \mathrm{G} : \; g \cdot F_y = F_{g \cdot y}, \; g \cdot F'_y = F'_{g \cdot y}, \; \forall y \in \mathrm{G}\}.$$

Note that $\widetilde{G}$ is a complex Lie subgroup G because $e \in \widetilde{G}$ and $g \cdot F_{g^{-1}\cdot y} = F_y$, so $F_{g^{-1}\cdot y} = g^{-1} \cdot F_y$. As K is totally real in $G = K^{\mathbb{C}}$ and $(*)$ implies $K \subset \widetilde{G}$ we conclude that $G = \widetilde{G}$. Let us then prove the second equality in $(*)$ (for the first one we would proceed analogously). Choose $k \in K$, $c \in \mathbb{C}^{r+1}$ and $y \in G$. We want to prove that there exists $d \in \mathbb{C}^{r+1}$ such that

$$k \cdot \varphi_\Lambda(c, [y]) = \varphi_\Lambda(d, [k \cdot y]) \qquad (\clubsuit).$$

There exist unique $x \in K$, $a \in A$ and $v \in U$ such that $y = x \cdot a \cdot v$. The equation $(\clubsuit)$ is equivalent to see that

$$k \cdot \Lambda_1(c) \cdot x \cdot a \cdot v \cdot \Lambda_2(c) = \Lambda_1(d) \cdot k \cdot x \cdot a \cdot v \cdot \Lambda_2(d) \cdot u$$

for some $u \in U$. Assume that such a solution exists. As $N(U) = B$ there exist $v', v'' \in U$ such that

$$v \cdot \Lambda_2(c) = \Lambda_2(c) \cdot v' \quad \text{and} \quad v \cdot \Lambda_2(d) = \Lambda_2(d) \cdot v'' \qquad (\spadesuit).$$

Moreover a commutes with $\Lambda_2(c)$ and $\Lambda_2(d)$, thus we obtain

$$k \cdot \Lambda_1(c) \cdot x \cdot \Lambda_2(c) = \Lambda_1(d) \cdot k \cdot x \cdot \Lambda_2(d) \cdot a \cdot (v'' \cdot u \cdot v'^{-1}) \cdot a^{-1}.$$

Note that $a \cdot (v'' \cdot u \cdot v'^{-1}) \cdot a^{-1} \in U$. By hypothesis (see **(b)**) we know that there exist $d \in \mathbb{C}^{r+1}$ and $\tilde{u} \in U$ such that

$$k \cdot \Lambda_1(c) \cdot x \cdot \Lambda_2(c) = \Lambda_1(d) \cdot k \cdot x \cdot \Lambda_2(d) \cdot \tilde{u}.$$

Setting $u = v''^{-1} \cdot (a^{-1} \cdot \tilde{u} \cdot a) \cdot v' \in U$ where v', v'' are those that verify $(\spadesuit)$ for this $d \in \mathbb{C}^{r+1}$ we obtain

$$k \cdot \Lambda_1(c) \cdot y \cdot \Lambda_2(c) = \Lambda_1(d) \cdot k \cdot y \cdot \Lambda_2(d) \cdot u$$

and have therefore concluded.
(c)⇒(d): Assume that there exists $c \in \mathbb{C}^{r+1}$ such that $\Lambda_1(c) \neq e$. If $\mathfrak{r} = \oplus_{\alpha \in \widetilde{\Phi}} \mathfrak{h}_\alpha$ and $\mathfrak{b} = \mathfrak{r} \oplus_{\alpha \in \widetilde{\Phi}} \mathfrak{g}_\alpha$ (the Lie subalgebras associated to H and B respectively) set $\mathfrak{b}' := \mathfrak{r} \oplus_{\alpha \in \widetilde{\Phi}} \mathfrak{g}_{-\alpha}$. Let B' and U' be the connected Lie subgroups associated to $\mathfrak{b}'$ and $\mathfrak{u}' = [\mathfrak{b}', \mathfrak{b}']$ respectively. Note that $B' = H \cdot U'$ and $U \cap U' = \{e\}$. Choose $g \in U'$, by hypothesis there exists $d \in \mathbb{C}^{r+1}$ such that

$$g^{-1} \cdot \varphi_\Lambda(c, [g]) = \varphi_\Lambda(d, [e]).$$

Equivalently there exists $u \in \mathrm{U}$ such that

$$g \cdot \Lambda_1(d) \cdot \Lambda_2(d) = \Lambda_1(c) \cdot g \cdot \Lambda_2(c) \cdot u \qquad (\diamondsuit).$$

Note that $g \in \mathrm{U}'$ and $\Lambda_1(c), \Lambda_2(c), \Lambda_1(d), \Lambda_2(d) \in \mathrm{H}$, therefore $g \cdot \Lambda_1(d) \cdot \Lambda_2(d)$, $\Lambda_1(c) \cdot g \cdot \Lambda_2(c) \in \mathrm{B}'$. As $\mathrm{U} \cap \mathrm{B}' = \{e\}$ the equation $(\diamondsuit)$ is equivalent to

$$g \cdot \Lambda_1(d) \cdot \Lambda_2(d) = \Lambda_1(c) \cdot g \cdot \Lambda_2(c) \qquad (\heartsuit)$$

which can be rewritten as

$$\Lambda_1(d)^{-1} \cdot g \cdot \Lambda_1(d) = \Lambda_1(c-d) \cdot \Lambda_2(c-d) \cdot \Lambda_2^{-1}(c-d) \cdot g \cdot \Lambda_2(c-d)$$

where $\Lambda_1(d)^{-1} \cdot g \cdot \Lambda_1(d), \Lambda_2^{-1}(c-d) \cdot g \cdot \Lambda_2(c-d) \in \mathrm{U}'$ and $\Lambda_1(c-d) \cdot \Lambda_2(c-d) \in \mathrm{H}$. Since $\mathrm{H} \cap \mathrm{U}' = \{e\}$ this implies $\Lambda_1(c-d) = \Lambda_2(d-c)$ or equivalently

$$\Lambda_1(c) \cdot \Lambda_2(c) = \Lambda_1(d) \cdot \Lambda_2(d).$$

Combining the last equation with $(\heartsuit)$ we obtain

$$g = \Lambda_1(c)^{-1} \cdot g \cdot \Lambda_1(c).$$

Therefore $\Lambda_1(c)$ belongs to the center of B' which is equal to the center of $\mathrm{G} = \mathrm{K}^{\mathbb{C}}$ (for G is a connected algebraic group, cf. [Hum75], p.140) that is discrete for G is semisimple. If $\Lambda_1(c) \neq e$ by continuity there exists a small neighborhood B_c of c in $\mathbb{C}^{r+1}$ such that for every $\tilde{c} \in B_c$ we have $\Lambda_1(\tilde{c}) \neq e$. This is a contradiction unless for every $\tilde{c} \in B_c$ we have $\Lambda_1(\tilde{c}) = \Lambda_1(c)$. In this case, since Λ_1 is a holomorphic map it should be constant and since it is a Lie group morphism it would follow that $\Lambda_1 \equiv \Lambda_1(0) = e$, which contradicts our first assumption.
(d)⇒(a): Clear. □

Proposition 3.2.14. *Let* $\Lambda : \mathbb{C}^{r+1} \to \mathrm{H} \times \mathrm{H}$ *be a morphism of Lie groups inducing a locally free holomorphic action* φ_Λ *which verifies* (II). *With the same notation as in proposition 3.2.8 the normal almost contact structure* K_Λ *on* K *induced by* Λ *is left-invariant by the action of* $Z(\mathrm{K})$.

Proof. It is enough to apply the same arguments as in proposition 3.2.13 taking into account that $Z(\mathrm{K}) = Z(\mathrm{G})$. □

Corollary 3.2.15. *Let* K *be a compact connected Lie group of odd dimension. Then* K *admits a non-invariant normal almost contact structure (and in particular a non-invariant CR-structure of maximal dimension).*

The proof is analogous to the invariant case (see theorem 3.1.30) thanks to the previous proposition.

Chapter 4

Three constructions of complex structures

In this chapter we construct complex structures on some classes of smooth manifolds obtained by geometrical constructions from manifolds in the class $\mathcal{T}$. We begin by considering the problem of describing an integrable almost-complex structure on a particular class of transversely holomorphic foliations given by a smooth action of $\mathbb{R}^2$. Next, we discuss with detail the three cases that we announced at the introduction and we verify that we can apply the previous study on the complexification of foliations to produce complex structures on these manifolds.

4.1 Complexification of 2-foliations.

Let $\mathcal{F}$ be a transversely holomorphic foliation on a compact manifold M induced by a locally free action of $\mathbb{R}^2$, i.e. there are two global linearly independent vector fields T_1 and T_2 on M such that $T\mathcal{F} = \langle T_1, T_2 \rangle$ and $[T_1, T_2] = 0$. Let $\mathcal{D}$ be a real distribution such that $T\mathrm{M} = \mathcal{D} \oplus T\mathcal{F}$.

Set $\tau \in \mathbb{C}\backslash\mathbb{R}$. We define a complex vector field $v = T_1 - \tau T_2$ and a complex-valued 1-form χ on M imposing that $\ker(\chi) = \mathcal{D}$, $\chi(v) = 1$ and $\chi(\bar{v}) = 0$. Under the above assumptions we can define an almost-complex structure K on $T\mathrm{M}$ imposing that K is compatible with the transverse holomorphic structure for $\mathcal{F}$ and that χ is of type $(1,0)$.

Proposition 4.1.1. *With the above notation and hypothesis K is integrable if and only if $d\chi^{0,2} = 0$. We will denote by (M, χ), or simply by M, the compact complex manifold so obtained.*

Remark 4.1.2. The integrability condition $d\chi^{0,2} = 0$ must be understood in terms of the almost-complex structure K.

Notice that the complex structure transverse to $\mathcal{F}$ induces an almost complex structure J on $\mathcal{D}$. Let us denote by $T^{1,0}$ the subbundle of vectors in $\mathcal{D} \otimes \mathbb{C}$ of type $(1,0)$ with respect to J. An equivalent way to define K is to require that $Q^{1,0} = T^{1,0} \oplus \langle v \rangle$ is the subbundle of vector fields of type $(1,0)$ of $T^{\mathbb{C}}\mathrm{M} = T\mathrm{M} \otimes \mathbb{C}$. We could also have imposed that $\langle \chi \rangle \oplus \Phi^{1,0}$ are the forms of type $(1,0)$ where

$$\Phi^{1,0} = \{\alpha \in \Omega^1(\mathrm{M}) \otimes \mathbb{C} : \ \alpha(X) = 0, \text{ for } X \in \langle T_1, T_2 \rangle \oplus T^{0,1}\}.$$

We can define $\Phi^{p,q}$ in an analogous way. Finally it is also equivalent to define K by the conditions $K_{|\mathcal{D}} = J$ and

$$K(T_2) = \frac{|\tau|^2 T_1 - (\mathrm{Re}\,\tau) T_2}{\mathrm{Im}\,\tau}; \qquad K(T_1) = \frac{(\mathrm{Re}\,\tau) T_1 - T_2}{\mathrm{Im}\,\tau}.$$

Proof. With the above notation, from the fact that $\mathcal{F}$ is an transversely holomorphic foliation, we obtain

(a) $[v, T^{1,0}] \subset T^{1,0} \oplus \langle v, \overline{v} \rangle$,

(b) $[T^{1,0}, T^{1,0}] \subset T^{1,0} \oplus \langle v, \overline{v} \rangle$.

Thus the almost complex structure K is integrable if and only if $[v, T^{1,0}] \subset Q^{1,0}$ and $[T^{1,0}, T^{1,0}] \subset Q^{1,0}$. These conditions are equivalent to $d\overline{\chi}(v, X) = d\overline{\chi}(X, Y) = 0$ for all $X, Y \in T^{1,0}$, which can be rewritten as $d\chi^{0,2} = 0$. □

Proposition 4.1.3. *With the above notation, if K is integrable:*

(a) *The vector field v is holomorphic on (M, χ) if and only if the 1-form $L_v d\chi$ is of type $(1,0)$.*

(b) *The form χ is holomorphic if and only if $d\chi = d\chi^{2,0}$.*

Proof. For the proof of **(a)** we will use that a vector field Z of type $(1,0)$ is holomorphic if and only if $[Z, Q^{0,1}] \subset Q^{0,1}$ where $Q^{0,1} = \overline{Q^{1,0}} = T^{0,1} \oplus \langle \overline{v} \rangle$. Since $[v, \overline{v}] = 0$ and $[v, T^{0,1}] \subset T^{0,1} \oplus \langle v, \overline{v} \rangle$ by hypothesis, it is enough to verify that $[v, T^{0,1}] \subset T^{0,1} \oplus \langle \overline{v} \rangle$. By the same argument as in lemma 4.1.1 this last condition can be written as $d\chi(v, T^{0,1}) = 0$. The assertion in **(b)** is clear. □

Corollary 4.1.4. *If $d\chi$ is a basic form the vector field v is holomorphic.*

That will be the case when the distribution $\mathcal{D}$ is preserved by the action, i.e. $[T_i, \mathcal{D}] \subset \mathcal{D}$ for $i = 1, 2$.

Proposition 4.1.5. *Given* M *and $\mathcal{F}$ as before, the distribution $\mathcal{D}$ is preserved by the action if and only if $d\chi$ is basic.*

Proof. It follows from the equality

$$2i_{T_i} d\chi(X) = 2d\chi(T_i, X) = X\chi(T_i) - T_i\chi(X) - \chi[T_i, X].$$

The direct implication is straightforward. For the reciprocal apply the same equality to $X \in \mathcal{D}$, it shows that $\chi[T_i, X] = 0$ for $i, j \in \{1, 2\}$, thus $[T_i, \mathcal{D}] \subset \mathcal{D}$. □

We will hereinafter restrict ourselves to the case of a distribution $\mathcal{D}$ preserved by the action, in particular the vector field v is holomorphic. Recall that the existence of a holomorphic vector field without zeros on a compact complex manifold implies that the Chern classes of M must vanish (c.f. [Kob72], p.121). When $d\chi$ is basic $d\chi^{0,2}$ denotes the component of type $(0, 2)$ of $d\chi$ with respect to the transversely holomorphic structure of the departing foliation $\mathcal{F}$. Therefore, when $d\chi$ is basic the integrability condition does not depend on how we have defined the almost-complex structure on the tangent space of $\mathcal{F}$, so it is much easier to control. Furthermore, as $\mathcal{D}$ is preserved by T_1 and T_2, we have $[T_i, T^{1,0}] \subset \mathcal{D}^{\mathbb{C}}$. The fact that the foliation is transversely holomorphic implies then $[v, T^{1,0}] \subset T^{1,0}$. If moreover $\dim_{\mathbb{R}} \mathrm{M} = 4$ then $d\chi = 0$, since $d\chi$ is a basic 2-form and the transverse complex dimension is 1, therefore the integrability condition for K is empty.

Definition 4.1.6. With the above notation, assume that the distribution $\mathcal{D}$ is invariant with respect to the vector fields T_1 and T_2. We define the *Euler classes of $\mathcal{F}_1$ and $\mathcal{F}_2$* as the cohomology classes in $H^2(\mathrm{M}/\mathcal{F}, \mathbb{R})$ given by

$$e_{\mathcal{F}_1}(\mathrm{M}) = [d\omega_1], \qquad e_{\mathcal{F}_2}(\mathrm{M}) = [d\omega_2],$$

where ω_1, ω_2 are 1-forms on M such that $\omega_i(T_j) = \delta_{ij}$ and $i_{T_i} d\omega_j = 0$ for $i, j = 1, 2$.

Note that the Euler classes $e_{\mathcal{F}_1}(\mathrm{M})$ and $e_{\mathcal{F}_2}(\mathrm{M})$ do not depend on the distribution $\mathcal{D}$, as long as it is invariant with respect to T_1 and T_2.

The case of an invariant distribution $\mathcal{D}$ presents another interesting property. We denote by $\Omega^*(\mathrm{M}/\mathcal{F})$ the space of basic holomorphic forms for the holomorphic flow $\mathcal{F}$, i.e. $\alpha \in \Omega^*(\mathrm{M})$ such that $i_v\alpha = i_v d\alpha = 0$. Let $\nu = T\mathrm{M}/T\mathcal{F}$ be the normal bundle, $K(\mathrm{M}/\mathcal{F}) = \bigwedge^{n-1}\nu^*$ the transverse canonical line bundle and $K(\mathrm{M}) = \bigwedge^n T\mathrm{M}^*$ the canonical line bundle.

Proposition 4.1.7. *With the above notation, if $d\chi$ is a basic form and $K(\mathrm{M}/\mathcal{F})$ is trivial then $K(\mathrm{M})$ is also trivial.*

Proof. By hypothesis there exists a holomorphic $(n-1)$-form ω without zeros such that $i_v\omega = 0$. Then $\chi \wedge \omega$ is a form of type $(n, 0)$ and it clearly has no zeros. It suffices to show that $\chi \wedge \omega$ is holomorphic on (M, χ), i.e. $\bar{\partial}(\chi \wedge \omega) = 0$. As M has complex dimension n it is equivalent to prove that $\chi \wedge \omega$ is closed. Moreover ω is a form of type $(n-1, 0)$ such that $\bar{\partial}\omega = 0$, hence $d\omega$ is a form of type $(n, 0)$. Recall that $d(\chi \wedge \omega) = \chi \wedge d\omega - d\chi \wedge \omega$. As $d\chi$ is a basic form with no component of type $(0, 2)$ the form $d\chi \wedge \omega$ is transverse and only has components of type $(n+1, 0)$ and $(n, 1)$. Since the complex dimension of the transverse part is $n-1$ we can conclude that $d\chi \wedge \omega = 0$. Finally $\chi \wedge d\omega$ is of type $(n+1, 0)$, so it must also be 0. ☐

Proposition 4.1.8. *Let* M *be a compact complex manifold with a holomorphic vector field v without zeros. The complex structure on* M *can be constructed by means of proposition 4.1.1. Moreover if* M *is Kähler we can assume $d\chi = 0$, in particular $\mathcal{D}$ is a Levi-flat distribution invariant by the action.*

Proof. We can construct a 1-form χ of type $(1, 0)$ on M such that $\chi(v) = 1$ and since M is a complex manifold $d\chi^{0,2} = 0$. We decompose $v = T_1 - iT_2$ so that T_1, T_2 are real vector fields. As v is holomorphic we have $[T_1, T_2] = 0$. The distribution $\mathcal{D}$ is given by $\ker(\chi)$. We can thus conclude that every complex structure on a compact manifold that admits a non-vanishing holomorphic vector field is under the hypothesis of this section and can be obtained by the preceding method. If M is Kähler we can apply a result by Carrell-Lieberman (see [CL73]) to conclude that there exists a holomorphic 1-form χ such that $\chi(v) \neq 0$. As M is compact we can assume that $\chi(v) = 1$ and as M is Kähler χ is closed. ☐

Although we will only apply this discussion to three particular cases the previous result gives us an idea of how general the situation we have considered is. For instance, all complex structures on the product of spheres $S^{2n-1} \times S^{2m-1}$ described in [LN96], which are not Kähler, can be recovered by proposition 4.1.1 for a distribution $\mathcal{D}$ invariant by the action given by the product of the usual contact distribution on a odd-dimensional sphere, even though they do not fall into any of the three cases that we will study in the next sections.

Let us recall the Loeb-Nicolau construction of complex structures on $S^{2n-1} \times S^{2m-1}$ (see [LN96]). We consider

$$S^{2n-1} \times S^{2m-1} = \{(z_1, \dots, z_n, w_1, \dots, w_m) \in \mathbb{C}^{n+m} : \sum_{i=1}^{n} |z_i|^2 = 1, \sum_{j=1}^{m} |w_j|^2 = 1\}$$

and the linear holomorphic vector field on $\mathbb{C}^{n+m}$ given by

$$\xi = \sum_{i=1}^{n} \lambda_i z_i \frac{\partial}{\partial z_i} + \sum_{j=1}^{m} \mu_j w_j \frac{\partial}{\partial w_j}$$

where $(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m) \in \mathbb{C}^{n+m}$ belongs to the Poincare domain, i.e. the convex hull of the points $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m \in \mathbb{C}$ does not contain $0 \in \mathbb{C}$, and

$$0 = \arg(\lambda_1) \leq \dots \leq \arg(\lambda_n) < \arg(\mu_1) \leq \dots \leq \arg(\mu_m) < \pi.$$

Under this conditions each leaf of $\mathcal{F}_\xi$ meets transversely the product of spheres $S^{2n-1} \times S^{2m-1}$. Therefore ξ induces a complex structure on this product, we denote by $\Sigma^{n,m}$ the complex manifold that results. The vector field

$$\eta = \sum_{i=1}^{n} \mathrm{Re}(\lambda_i) z_i \frac{\partial}{\partial z_i} + \sum_{j=1}^{m} \mathrm{Re}(\mu_j) w_j \frac{\partial}{\partial w_j}$$

induces a holomorphic vector field v without zeros on $\Sigma^{n,m}$, which can be written on $S^{2n-1} \times S^{2m-1}$ as $v = \mathrm{Re}(\eta) - \mathrm{Re}(\xi) + \mathrm{i}\,\mathrm{Im}(\eta)$. Furthermore

$$Z_{ij} = \overline{z}_i \frac{\partial}{\partial z_j} - \overline{z}_j \frac{\partial}{\partial z_i}, \quad W_{ij} = \overline{w}_i \frac{\partial}{\partial w_j} - \overline{w}_j \frac{\partial}{\partial w_i}$$

are vector fields of type $(1,0)$ on $\mathbb{C}^{n+m}$ and $\Phi^{1,0} = \langle Z_{ij}, W_{ij} \rangle$ is a distribution of vector fields of type $(1,0)$ on $\Sigma^{n,m}$ transverse to the vector field v. We set $\mathcal{D}$ as the distribution given by $\mathcal{D} = \langle \mathrm{Re}(Z_{ij}), \mathrm{Im}(Z_{ij}), \mathrm{Re}(W_{ij}), \mathrm{Im}(W_{ij}) \rangle$ and $v = T_1 - iT_2$ where T_1 and T_2 are real vector fields on $S^{2n-1} \times S^{2m-1}$. An explicit calculation allows us to obtain the equalities

$$[T_1, Z_{ij}] = \frac{1}{2\mathrm{i}} \big(\mathrm{Im}(\lambda_i) + \mathrm{Im}(\lambda_j) \big) Z_{ij}, \quad [T_2, W_{ij}] = \frac{1}{2\mathrm{i}} \big(\mathrm{Re}(\mu_i) + \mathrm{Re}(\mu_j) \big) W_{ij}.$$

Therefore the distribution $\mathcal{D}$ is preserved by the vector fields T_1 and T_2. Thus we are exactly under the hypothesis of this section for a distribution $\mathcal{D}$ preserved by T_1 and T_2, the integrability being assured by the fact that we depart from a complex structure. For the choice of the parameters $(1, ..., 1, \mu, ..., \mu)$ we obtain Calabi-Eckmann complex structures, which are the total space of an elliptic principal bundle over $\mathbb{P}^n \times \mathbb{P}^m$. However it must be noted that not all the complex structures described by Loeb-Nicolau are elliptic principal bundles. Furthermore generically these complex structures cannot be recovered by any of the particular constructions that we describe next. Note that no complex structure on $S^{2n-1} \times S^{2m-1}$ can admit a Kähler metric unless $n = m = 1$, since $H^2(S^{2n-1} \times S^{2m-1}) = 0$. This example also shows that even if we restrict ourselves to the case of the distribution $\mathcal{D}$ being preserved by the vector fields T_1 and T_2 the construction of proposition 4.1.1 still provides interesting examples of complex manifolds which are not Kähler.

Next we discuss how to use the manifolds in the class $\mathcal{T}$ to produce explicit examples of smooth manifolds which are under the hypothesis of the previous section. In the three cases the distribution $\mathcal{D}$ considered is invariant by the $\mathbb{R}^2$-action.

4.2 Products of two manifolds in the class $\mathcal{T}$ (case A)

Proposition 4.2.1. *Let* M_1 *and* M_2 *be two manifolds in the class* $\mathcal{T}$*. There exists a 1-parametric family of integrable almost complex structures* K_τ *on the product* $\mathrm{M}_1 \times \mathrm{M}_2$ *for* $\tau \in \mathbb{C}\backslash\mathbb{R}$ *so that the complex manifold* $\mathrm{M}_1 \times \mathrm{M}_2$ *admits a non-vanishing holomorphic vector field* v*.*

Proof. Let us denote by (T_1, ω_1) and (T_2, ω_2) the vector fields and the 1-forms of the CR-structures and the CR-actions on M_1 and M_2 respectively. The distribution $\mathcal{D}$ is given by $\mathcal{D} = \ker\omega_1 \oplus \ker\omega_2$ and the 2-foliation $\mathcal{F}$ generated by T_1 and T_2 is transversely holomorphic. Clearly $[T_1, T_2] = 0$. Moreover $\chi = \frac{i}{2\,\mathrm{Im}\,\tau}(\bar{\tau}\omega_1 + \omega_2)$. Therefore $d\chi$ is basic and we are under the hypothesis of proposition 4.1.1. The integrability condition is verified for every $\tau \in \mathbb{C}\backslash\mathbb{R}$ since $d\chi$ is of type $(1, 1)$. □

Example 4.2.2. Recall that in chapter 2 we have discussed some examples of compact 3-manifolds in the class $\mathcal{T}$. If we apply the previous propostion to a pair of Seifert fibrations M_1 and M_2 we obtain a complex structure on the product $\mathrm{M}_1 \times \mathrm{M}_2$ which is a principal Seifert elliptic fibration over an orbifold of complex dimension 2. If M_1 and M_2 are linear foliations of $\mathbb{T}^3$ the above construction yields a complex torus of dimension 3. When M_1 and M_2 are S^3 or lens spaces one obtains Calabi-Eckmann complex structures on the product $S^3 \times S^3$ or finite quotients of those. Analogously, Calabi-Eckmann complex structures on the product of spheres $S^{2n-1} \times S^{2m-1}$ can be obtained by the preceding construction. When $\mathrm{M}_1 = S^3$ and $\mathrm{M}_2 = S^2 \times S^1$ with the flow induced by the suspension of an irrational rotation of S^2 we obtain a complex structure on $S^2 \times S^1 \times S^3$ which is a topologically trivial analytic fibre bundle with fibre $\mathbb{P}^1$ over a primary Hopf surface.

Remark 4.2.3. Let M_1 and M_2 be two manifolds in the class $\mathcal{T}$ with normal almost contact structures $(\varphi_j, T_j, \omega_j)$ on M_j for $j = 1, 2$. Morimoto defines an almost complex structure on $\mathrm{M}_1 \times \mathrm{M}_2$ by

$$K(X_1, X_2) = (\varphi_1(X_1) - \omega_2(X_2)T_1, \varphi_2(X_2) + \omega_1(X_1)T_2),$$

which corresponds to K_{-i} in the preceding proposition (see [Mor63]).

Example 4.2.4. Let K be a compact connected real Lie group of odd dimension. The previous proposition describes a complex structure on the product $\mathrm{K} \times S^1$. Note that as $\mathrm{K} \times S^1$ is also a Lie group this can be seen as a particular case of a result by Samelson (cf. [Sam53]). Moreover, it is known that X cannot be Kähler. Indeed, up to a finite covering, we can assume that $\mathrm{K} \cong \mathrm{K}' \times (S^1)^r$, where K' is a compact connected semisimple real Lie group. Since $b_1(\mathrm{K}') = b_2(\mathrm{K}') = 0$ (see section 1.7.5) by Kunneth's formula we conclude that $H^2(\mathrm{K} \times S^1) \cong H^2((S^1)^{r+1})$ so there cannot exist a Kähler form $[\omega] \in H^2(\mathrm{K} \times S^1)$ such that $[\omega]^{2s} \neq 0$ in $H^{2s}(\mathrm{K} \times S^1)$ for $s = \dim_{\mathbb{C}}(\mathrm{K} \times S^1)$. We have therefore proved that there exists a finite analytic covering of $\mathrm{K} \times S^1$ that is not Kähler, which implies that $\mathrm{K} \times S^1$ cannot be Kähler.

4.3 S^1-principal bundles over a manifold in the class $\mathcal{T}$ (case B)

Proposition 4.3.1. *Let* M *be a manifold in the class* $\mathcal{T}$*. Denote by* T *the vector field inducing the CR-action and by* $\mathcal{F}_T$ *the transversely holomorphic flow induced by* T*. Let* $\pi : \mathrm{X} \to \mathrm{M}$ *be a* S^1*-principal bundle over* M *with Chern class* $[d\beta]$*, where* β *is a 1-form on* X *such that* $d\beta \in \pi^*\Omega^{1,1}(M/\mathcal{F}_T)$*, that is,* $d\beta$ *is the* *pull-back of a closed* $(1,1)$*-form on* M*. Then there exists a 1-parametric family of integrable almost complex structures* K_τ *on* X *for* $\tau \in \mathbb{C}\backslash\mathbb{R}$ *so that the complex manifold* X *admits a non-vanishing holomorphic vector field* v*.*

Proof. Let ω be the 1-form associated to the CR-structure on M and the vector field T. We denote by $\widetilde{T}$ the vector field on X contained in $\ker\beta$ such that $\pi_*(\widetilde{T}) = T$ and define the 1-form $\widetilde{\omega} = \pi^*\omega$. Let R denote the fundamental vector field of the action corresponding to the S^1-fibration $\pi : \mathrm{X} \to \mathrm{M}$ such that $\beta(R) = 1$. The compact manifold X is under the hypothesis of proposition 4.1.1 with an invariant distribution for the vector fields $\widetilde{T}$ and R, the distribution $\mathcal{D} = \ker\beta \cap \ker\widetilde{\omega}$, and the transverse holomorphic structure for $\mathcal{F} = \langle\widetilde{T}, R\rangle$ induced by the CR-structure of M. The holomorphic vector field v is $\widetilde{T} - \tau R$ for $\tau \in \mathbb{C}\backslash\mathbb{R}$. We define a complex-valued 1-form χ by imposing $\ker\chi = \mathcal{D}$, $\chi(v) = 1$ and $\chi(\overline{v}) = 0$. The hypothesis $d\beta \in \pi^*\Omega^{1,1}(\mathrm{M}/\mathcal{F}_T)$ and $d\omega \in \Omega^{1,1}(\mathrm{M}/\mathcal{F}_T)$ imply that $d\chi$ is of type $(1,1)$, thus the complex structure defined on section 4.1 is integrable. □

When M is the total space of a S^1-principal bundle over a complex manifold and the normal almost contact structure on M is obtained as we described in corollary 2.2.1 then the resulting complex manifold X is an elliptic principal bundle. Conversely, every complex manifold which is the total space of an analytic elliptic principal bundle can be constructed in this way. The Iwasawa manifold is an example of this situation.

Example 4.3.2. Let $H = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} : z_j \in \mathbb{C} \right\}$. The *Iwasawa manifold* is the compact homogeneous space $\mathrm{M} = \Gamma\backslash H$ where $\Gamma = \left\langle \begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$ for $n_j \in \mathbb{Z} + i\mathbb{Z}$ is a cocompact discrete group. It admits an structure of elliptic fibre bundle over a complex torus $\mathbb{T}^2$ considering the projection:

$$\begin{array}{rccc} \pi : & \Gamma/H & \longrightarrow & \mathbb{T}^2 = \mathbb{C}^2/(\mathbb{Z}+i\mathbb{Z})^2 \\ & \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} & \longrightarrow & (z_1, z_2). \end{array}$$

The fiber is $\{z_3 \in \mathbb{C} : z_3 \sim z_3 + n_3\}$ where $n_3 \in \mathbb{Z} + i\mathbb{Z}$, thus it is an elliptic curve. The vector field of the action is $\frac{\partial}{\partial z_3}$ and $\chi = dz_3 - z_1 dz_2$ is a connection of type $(1,0)$ with $d\chi = -dz_1 \wedge dz_2$, which is a non-exact form on $\mathbb{T}^2$. It is not difficult to verify that the complex structure on the Iwasawa manifold can be recovered by our procedure. Also note that Γ/H is parallisable but not abelian, therefore Wang's theorem (see corollary 4.4.10) implies that it is not Kähler.

4.4 Suspensions of manifolds in the class $\mathcal{T}$ (case C)

Definition 4.4.1. Let M^{2n+1} be a compact manifold with CR-structure $\Phi^{1,0}$ of dimension n and a vector field T inducing a CR-action. We define

$$\mathrm{Aut}_{\mathcal{T}}(\mathrm{M}) = \{f \in \mathrm{Aut}_{\mathrm{CR}}(\mathrm{M}) : f_*T = T\}.$$

Proposition 4.4.2. *Let* M^{2n+1} *be a manifold in the class* $\mathcal{T}$ *with a CR-structure* $\Phi^{1,0}$ *and a vector field* T *inducing a transverse CR-action. Given* $f \in \mathrm{Aut}_{\mathcal{T}}(\mathrm{M})$ *the suspension* X *of* M *by* f *admits a 1-parametric family of integrable almost complex structures* K_τ *for* $\tau \in \mathbb{C}\backslash\mathbb{R}$ *so that the complex manifold* X *admits a non-vanishing holomorphic vector field* v *induced by* $T - \tau\frac{\partial}{\partial s}$.

The proof is straightforward, the distribution $\mathcal{D}$ is induced by the CR-structure $\Phi^{1,0}$ on M. There is a fibration $\pi : \mathrm{X} \to S^1$ with fibre M so that the flow defined on X by the vector field induced by $\frac{\partial}{\partial s}$ is transverse to this fibration. Note also that if the CR-structure on M is Levi-flat the manifold X admits a holomorphic foliation transverse to v. For $f = \mathrm{id}$ we obtain $\mathrm{X} = \mathrm{M} \times S^1$ and the complex structure on X corresponds to the one described in proposition 4.2.1.

Lemma 4.4.3. *The inclusion* $i : \{g \in \mathrm{Aut}_{\mathcal{T}}(\mathrm{M}) : g \circ f = f \circ g\} \to \mathrm{Aut}_{\mathbb{C}}(\mathrm{X})$ *induced by* $g \mapsto (g, \mathrm{id})$ *is an injective homomorphism.*

Proof. It is enough to observe that if $g \in \mathrm{Aut}_\tau(\mathrm{M})$ and $f \circ g = g \circ f$ then (g, id) is well-defined and it preserves the vector fields of type $(1,0)$ on $\mathrm{M} \times \mathbb{R}$ with respect to K_τ. □

Remark 4.4.4. Recall that in lemma 2.4.2 we remarked that a suspension of a complex manifold N by an automorphism $g \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$ gives an example of normal almost contact structure. Applying proposition 4.4.2 to such a manifold in the class $\mathcal{T}$ is equivalent to consider the quotient X of $\mathrm{N} \times \mathbb{C}$ by $F(x,z) = (f(x), z+1)$ and $G(x,z) = (g(x), z+\tau)$, where $f \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$ so that $f \circ g = g \circ f$, which we will call double suspension of a compact complex manifold. Furthermore there is a holomorphic fibration $\pi : \mathrm{X} \to E_\tau = \mathbb{C}/\langle 1, \tau\rangle$ such that the vector field $\frac{\partial}{\partial z}$ is transverse to the fibers.

Theorem 4.4.5. *Every compact Kähler manifold* X *admitting a non-vanishing holomorphic vector field v can be obtained by the construction of proposition 4.4.2.*

Proof. By a result by Carell-Lieberman (see [CL73]) there exists a holomorphic 1-form χ over X such that $\chi(v) = 1$. Denote $b_1(\mathrm{X}) = 2k$, let $\gamma_1, ..., \gamma_{2k}$ be closed paths giving a basis of $H_1(\mathrm{X}, \mathbb{Z})$ modulus torsion and let $\xi_1, ..., \xi_{2k}$ be the dual basis of closed 1-forms. Fix a basis $\omega_1, ..., \omega_k$ of $H^0(\mathrm{X}, \Omega^1)$. By Hodge's decomposition theorem we have

$$\xi_i = a_i^1\omega_1 + ... + a_i^k\omega_k + b_i^1\overline{\omega}_1 + ... + b_i^k\overline{\omega}_k + df_i = \eta_i + df_i$$

where f_i is a differentiable function, for $i = 1, ..., 2k$, and $a_i^j, b_i^j \in \mathbb{C}$. By Stokes theorem the two sets of 1-forms $\{\xi_i\}$ and $\{\eta_i\}$ have the same periods. In particular $\{\eta_1, ..., \eta_{2k}\}$ is a basis of $H^1(\mathrm{X}, \mathbb{C})$ dual of $\{\gamma_1, ..., \gamma_{2k}\}$. Since $\mathbb{Q} + \mathrm{i}\mathbb{Q}$ is a dense subset in $\mathbb{C}$ we can choose $a_i \in \mathbb{C}$ for $i = 1, ..., 2k$ arbitrarily small so that $\eta = \chi + \sum a_i\eta_i$ is a closed 1-form and $\int_{\gamma_j} \eta \in \mathbb{Q} + \mathrm{i}\mathbb{Q}$ for $j = 1, ..., 2k$. Moreover by construction the 1-form η is of the form:

$$\eta = c_1\omega_1 + ... + c_k\omega_k + d_1\overline{\omega}_1 + ... + d_k\overline{\omega}_k$$

with $c_i, d_i \in \mathbb{C}$ for $i = 1, ..., k$. It follows that $\eta(v)$ is constant and close to 1 by construction, set $\eta(v) = \delta$. In an analogous way $\eta(\bar{v})$ is constant and close to 0, set $\eta(\bar{v}) = \epsilon$. Therefore $\Gamma = \left\{ \int_\gamma \eta : \gamma \in H_1(\mathrm{X}, \mathbb{Z})\right\}$ is finitely generated and it is contained in $\mathbb{Q} + \mathrm{i}\mathbb{Q}$, thus $\Gamma \cong \mathbb{Z} + \mathrm{i}\mathbb{Z}$. Fixing a base point p_0 the differentiable map

$$\begin{aligned} \pi_1 : \mathrm{X} &\to \mathbb{C}/\Gamma \\ p &\mapsto \int_{p_0}^{p} \eta \bmod \Gamma \end{aligned}$$

over the elliptic curve $\mathbb{C}/\Gamma$ is well defined. Furthermore π_1 is a proper submersion and thus a fibration. The real vector fields $v + \bar{v}$ and $\mathrm{i}(v - \bar{v})$ are transverse to the fibres of π_1 and preserve the fibration (because $\eta(v + \bar{v}) = \epsilon + \delta$, $\eta(\mathrm{i}(v - \bar{v})) = \mathrm{i}(\delta - \epsilon)$ are constants close to 1 and i respectively and η is closed). Since $\eta(v+\bar{v}) = \delta+\epsilon \sim 1$ we can find a linear map $h : \mathbb{C} \to \mathbb{R}$ such that $h(\eta(v+\bar{v})) > 0$ and $h(\Gamma) \subset \mathbb{Z}$. Let $\bar{h} : \mathbb{C}/\Gamma \to \mathbb{R}/\mathbb{Z}$ be the induced fibration. The composition $\pi_2 = \bar{h} \circ \pi_1 : \mathrm{X} \to \mathbb{R}/\mathbb{Z}$ is a fibration over the circle. The fibres of π_2, denoted by $\mathrm{M} = \pi_2^{-1}(p)$, admit a CR-structure induced by the complex structure on X.

There exists $a \in \mathbb{C}\backslash\mathbb{R}$ such that the real vector field $v_1 = \mathrm{Re}(av)$ is tangent to M. As the flow associated to v is holomorphic the vector field v_1 preserves the CR-structure of M and induces a transverse CR-action. On the other hand there exists $b \in \mathbb{R}^+$ such that the vector field $v_2 = \mathrm{Re}(bv)$ projects over the vector field $\frac{\partial}{\partial t}$ on S^1. The flow of v_2 preserves the CR-structure over M and clearly $[v_1, v_2] = 0$. Finally setting $\tau = \bar{a} \cdot b^{-1}$ we obtain $v_1 - \tau v_2 = \mathrm{Re}(av) - \tau\,\mathrm{Re}(bv) = \mu \cdot v$, where $\mu \in \mathbb{C}$. Taking the automorphism f over M induced by the flow of v_2 for time 1 the complexification of 4.4.2 for the preceding τ gives rise to the original complex structure. □

Theorem 4.4.6. *Every compact Kähler manifold* X *admitting a holomorphic vector field v without zeros admits a complex structure on the underlying smooth manifold* X *arbitrarily close to the original one that can be obtained by the construction of remark 4.4.4.*

Remark 4.4.7. The statement that the new complex structure on the underlying smooth manifold X is arbitrarily close to the original one can be stated more precisely in the following way. Let $\Omega^{1,0}(\mathrm{X})$ be the subspace of smooth (1,0)-forms on X and $\{\alpha, \alpha_1, ..., \alpha_k\}$ a family of $(1,0)$-forms which span $\Omega^{1,0}(\mathrm{X})$ as a $\mathcal{C}^\infty(\mathrm{X})$-module such that $\alpha(v) = 1$ and $\alpha_j(v) = 0$ for $j = 1, ..., k$. Then there exists a closed 1-form β of type (0,1) arbitrarily small such that $\beta(\bar{v}) = 0$ and for the new complex structure on X the set of forms $\{\alpha + \beta, \alpha_1, ..., \alpha_k\}$ is a family of $(1,0)$-forms which span $\Omega^{1,0}(\mathrm{X})$ as a $\mathcal{C}^\infty(\mathrm{X})$-module.

Proof. There exists a holomorphic 1-form α on X such that $\alpha(v) = 1$ and $d\alpha = 0$. We proceed as in theorem 4.4.5 to obtain a closed 1-form η with group of periods $\Gamma \cong \mathbb{Z} + \mathrm{i}\mathbb{Z}$ and a smooth fibration $\pi_1 : \mathrm{M} \to \mathbb{C}/\Gamma$ given by $x \mapsto \int_{x_0}^{x} \eta$. Every fibre N of π_1 is transverse to the foliation $\mathcal{F}_v$ generated by v. Therefore N admits a complex structure. Note that $\eta = \pi_1^*(dz)$. Consider the universal covering $p : \mathbb{C} \to \mathbb{C}/\Gamma_\tau$ and the pullback $\pi_2 : \mathrm{N} \times \mathbb{C} \to \mathbb{C}$ of the fibration π_1 by the map p. There exists a map $q : \mathrm{N} \times \mathbb{C} \to \mathrm{X}$ such that $\pi_1 \circ q = p \circ \pi_2$. The holomorphic vector field v is transverse to the leaves of π_1 and it preserves the complex structure on N. We recall that $\eta(v) = \delta \sim 1$ and that $\eta(\bar{v}) = \epsilon \sim 0$. Fix $\tau \in \mathbb{C}\backslash\mathbb{R}$. We decompose $\delta^{-1}v = v_1 - \tau v_2$, where v_1 and v_2 are real vector fields. Then v_1 and v_2 are transverse to the fibers of π_1, they preserve the fibration and the complex structure on N and $[v_1, v_2] = 0$ (for $\delta^{-1}v$ is holomorphic). Finally, they project over $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ on $\mathbb{C}/\Gamma$ respectively where $z = \frac{\mathrm{i}}{2\,\mathrm{Im}\,\tau}(ds + \bar{\tau}dt)$. We set $f, g \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$ as the flows v_1 and v_2 for time 1 and -1 respectively. Thus X is diffeomorphic to the suspension $\mathrm{N} \times \mathbb{C}/\langle F, G\rangle$ where $F(x,z) = (f(x), z+1)$, $G(x,z) = (g(x), z+\tau)$ and $f \circ g = g \circ f$. The construction of the remark 4.4.4 gives a complex structure on X which is arbitrarily close to the original complex structure on X. Note that with the new complex structure on X the fibration $\pi_1 : \mathrm{X} \to \mathbb{C}/\Gamma$ is holomorphic and the fibres N are analytic submanifolds. Choosing η close enough to the starting holomorphic 1-form α we obtain a complex structure as close to the original as we wish. □

There is a natural generalization of the construction of the remark 4.4.4 to a suspension of a compact complex manifold N by a commutative subgroup $\Gamma = \langle f_1, ..., f_s, g_1, ..., g_s\rangle$ of $\mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$. The resulting complex manifold has complex dimension $\dim_{\mathbb{C}} \mathrm{N} + s$ and fibers over the torus $\mathbb{T}^s$. Then one can prove, with the same arguments as in theorem 4.4.6, the result below. Recall that $\mathfrak{h}$ is the Lie algebra of holomorphic vector fields on X and $\mathfrak{h}_0$ the Lie algebra of holomorphic vector field with zeros.

Theorem 4.4.8. *Let* X *be a compact Kähler manifold such that* $\mathfrak{h}$ *admits an abelian subalgebra* $\widetilde{\mathfrak{h}}$ *of holomorphic vector fields without zeros such that* $\dim_{\mathbb{C}} \widetilde{\mathfrak{h}} = s > 0$*. The underlying smooth manifold* X *admits a complex structure arbitrarily close to the original one, obtained as a suspension over the complex torus* $\mathbb{T}^s$*.*

Remark 4.4.9. Recall that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}_0$. Therefore if $\dim_{\mathbb{C}} \mathfrak{h} = s > 0$ and $\mathfrak{h}_0 = 0$ the hypothesis of the above theorem holds. Any complex manifold which is a product $\mathbb{T}^s \times \mathrm{N}$ where N is a Kähler compact manifold is also a trivial example of a manifold under the above hypothesis.

The limit case, i.e. when $\dim_{\mathbb{C}} \widetilde{\mathfrak{h}} = \dim_{\mathbb{C}} \mathrm{X}$, is a classical result by Wang's:

Corollary 4.4.10. *Let* X *be a complex parallisable compact Kähler manifold, then* X *is a complex torus.*

Proof. Since X is parallisable $\mathfrak{h}_0 = 0$ and $dim_{\mathbb{C}}\mathfrak{h} = \dim_{\mathbb{C}} \mathrm{X} = s > 0$. We now apply the preceding theorem for $n = 0$. We obtain X as the suspension over a compact complex manifold N of dimension 0, that is, a point. Since the obstruction in the previous theorem to obtain the original complex structure was due to the fact that N was not a complex submanifold of X, we can conclude. □

4.5 Complexifications of manifolds in the class $\mathcal{T}$.

Let M be a manifold in the class $\mathcal{T}$. We denote by T the vector field defining the CR-action and by $\mathcal{F}$ the transversely holomorphic flow induced by T.

Definition 4.5.1. With the above notation, we say that a compact complex manifold X endowed with a non-singular holomorphic vector field v is a *complexification* of the pair (M, T) if:

- **(i)** M is a real submanifold of X.
- **(ii)** The CR-structure of M is compatible with the complex structure of X.
- **(iii)** There exists $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda v) = T$.

Remark 4.5.2. Both the constructions of case A and case C produce complex manifolds that are complexifications in the previous sense of the departing manifolds in the class $\mathcal{T}$. Indeed, if $v = T_1 - \tau T_2$ for $\tau \in \mathbb{C}\backslash\mathbb{R}$ then $T_1 = \mathrm{Re}\left(\frac{\mathrm{i}\bar{\tau}}{\mathrm{Im}\,\tau} v\right)$ and $T_2 = \mathrm{Re}\left(\frac{\mathrm{i}}{\mathrm{Im}\,\tau} v\right)$.

4.6 A remark on deformations.

Let (M, χ) be a compact complex manifold with a holomorphic foliation $\mathcal{F}$ obtained by proposition 4.1.1 with two vector fields T_1 and T_2 and $\chi = \chi_1 + \tau\chi_2$. The flow $\mathcal{F}$ is a transversely holomorphic foliation when we forget part the tangent part of the complex structure of M, we will denote this foliation by $\mathcal{F}^{tr}$. We call *f-deformations* to those deformations of the holomorphic flow $\mathcal{F}$ which keep fixed its transversal type. For specific definitions, details and proofs we refer the reader to [GN89]. Families of f-deformations can be viewed as families of complex structures on M for which $\mathcal{F}^{tr}$ becomes a holomorphic foliation, therefore it is the natural context to study the effect of changing the parameters of our construction (the two 1-forms χ_1 and χ_2 and the complex number $\tau \in \mathbb{C}\backslash\mathbb{R}$ or equivalently $\chi = \chi_1 + \tau\chi_2$) on the resulting complex structure.

Let $\Theta_{\mathcal{F}}$ be the sheaf of germs of holomorphic vector fields over M preserving $\mathcal{F}$ and $\Theta^f_{\mathcal{F}}$ the subsheaf of $\Theta_{\mathcal{F}}$ consisting of those elements of $\Theta_{\mathcal{F}}$ tangent to the leaves of $\mathcal{F}$. Note that in our case $\Theta^f_{\mathcal{F}}$ is isomorphic to the sheaf of germs of holomorphic functions $\mathcal{O}_{\mathrm{M}}$, for there is a non-vanishing holomorphic vector field v tangent to $\mathcal{F}$.

Let $\mathcal{S}$ be an analytic space with a distinguished point 0. A family of f-deformations of $\mathcal{F}$ parameterized by $\mathcal{S}$ is an analytic space X with a proper projection $p : \mathrm{X} \to \mathcal{S}$ inducing on each fibre $\mathrm{M}_s = p^{-1}(s)$ a complex structure together with a holomorphic foliation $\mathcal{F}_s$, which depend analytically on s, and a smooth trivialization $\mathrm{X} \cong \mathrm{M} \times \mathcal{S}$ compatible with p such that $\mathcal{F}_s = \mathcal{F}^{tr}$ on M_s as transversely holomorphic foliations. We suppose also that there is given an isomorphism $i : (\mathrm{M}, \mathcal{F}) \to (\mathrm{M}_0, \mathcal{F}_0)$. We will only deal with germs of families, that is, we will only consider the behavior near the fibre M_0. Thus a family will be parameterized by a germ $(\mathcal{S}, 0)$ of $\mathcal{S}$ at 0.

A family of f-deformations is called *versal* if for any other family of f-deformations of $\mathcal{F}$ there is a morphism of germs of analytic spaces $\varphi : (\mathcal{S}', 0) \to (\mathcal{S}, 0)$ inducing an isomorphism of families of f-deformations and such that the tangent map of $d_0\varphi$ of φ at 0 is unique. If a versal family $(\mathcal{S}, 0)$ exists then it is unique up to isomorphism and $\mathcal{S}$ is called the *versal space*.

Theorem 4.6.1 ([GN89])**.** *With the above notation, there is a germ of analytic space* $(K^f, 0)$ *parameterizing a family* $p : Z^f \to K^f$ *of* f*-deformations of* $\mathcal{F}$ *which is versal with respect to* f*-deformations. More precisely, there is an open neighborhood* V *of* 0 *in* $H^1(\mathrm{M}, \Theta^f_{\mathcal{F}})$ *and an analytic map* $\zeta_f : V \to H^1(\mathrm{M}, \Theta^f_{\mathcal{F}})$ *such that* $(K^f, 0)$ *is isomorphic to the germ at* 0 *of* $\zeta_f^{-1}(0)$*.*

The Kodaira-Spencer map

$$\rho : T_0\mathcal{S} \to H^1(\mathrm{M}, \Theta_{\mathcal{F}f})$$

is defined in the following way. Let $T^{0,1}$ and $T^{0,1}_s$ denote the complex subbundles of vector fields of type $(0,1)$ of $T^{\mathbb{C}}\mathrm{M}$ and $T^{\mathbb{C}}\mathrm{M}_s$ respectively. There is an analytic family of 1-forms φ_s on M defined imposing that $(\varphi_s + \mathrm{Id})(T^{0,1}) = T^{0,1}_s$ and from the integrability condition on $T^{0,1}_s$ it follows that

$$\overline{\partial}\varphi_s - \frac{1}{2}[\varphi_s, \varphi_s] = 0.$$

Then we define $\rho(\frac{\partial}{\partial s}|_{s=0}) = \frac{\partial \varphi_s}{\partial s}|_{s=0} \in H^1(\mathrm{M}, \Theta^f_{\mathcal{F}})$. The Kodaira-Spencer map is an isomorphism from T_0K^f to $H^1(\mathrm{M}, \Theta^f_{\mathcal{F}})$. From the previous theorem it follows that if $(\mathcal{S}, 0)$ is a germ of an analytic space $\mathcal{S}$ at 0 parameterizing a family of f-deformations such that the Kodaira-Spencer map $\rho : T_0\mathcal{S} \to H^1(\mathrm{M}, \Theta^f_{\mathcal{F}})$ is not identically zero then the family is not trivial.

Assume that (M, χ') is a complex structure on M obtained by proposition 4.1.1 with the vector fields T_1 and T_2 and $\chi' = \chi'_1 + \tau'\chi'_2$. The Kodaira-Spencer map

$$\rho : T_0\mathcal{S} \to H^1(\mathrm{M}, \Theta^f_{\mathcal{F}}) \cong H^1(\mathrm{M}, \mathcal{O}_{\mathrm{M}})$$

can be easily computed in this situation when $\mathcal{S}$ is an analytic space parameterizing the 1-forms χ' close to χ. If we denote by $T^{0,1}$ and $T'^{0,1}$ the subbundles of vectors of type (0,1) in (M, χ) and (M, χ') respectively and set

$$\varphi = \frac{\tau' - \bar{\tau}'}{\tau - \bar{\tau}'}(\chi')^{0,1} \otimes v,$$

then $(\mathrm{Id} + \varphi)(T^{0,1}) = T'^{0,1}$ (it is an easy computation, just note that $\chi'(v) = \frac{\bar{\tau}' - \tau}{\tau' - \bar{\tau}'}$). Note that $\overline{\partial}\alpha = 0$. Thus $\rho(\chi') = \frac{\tau' - \bar{\tau}'}{\tau - \bar{\tau}'}(\chi')^{0,1}$.

Assume now that $\tau = \tau'$, we can consider a family of complex structures on M defined by $\chi_s = (1 - s)\chi + s\chi'$. Then $\varphi_s = s(\chi')^{0,1} \otimes v$ so $\rho(\chi') = (\chi')^{0,1}$. We conclude that if there exists χ' close enough to χ and such that $\rho(\chi')$ represents a non-zero cohomology class in $H^1(\mathrm{M}, \mathcal{O}_{\mathrm{M}})$ we can obtain, changing the parameters of our construction, complex structures on M for which there does not exist a diffeomorphism tangent to the leaves of $\mathcal{F}$, close to the identity and preserving the transversally holomorphic foliations $\mathcal{F}^{tr}$ sending one into another. A similar computation can be done when $\chi'_1 = \chi_1$ and $\chi'_2 = \chi_2$ changing the complex parameter τ and yields a similar result.

Chapter 5

Criteria of Kählerianity

In this chapter we discuss some criteria to determine when the complex manifolds obtained by the constructions of the previous chapter are Kählerian. We first prove that in order to obtain a Kähler manifold, in the three cases we have discussed (cases A, B and C), the Euler class of the flows associated to the departing normal almost contact structures must be zero. When the flows are isometric we give sufficient and necessary conditions for the resulting complex manifold to be Kählerian. Finally, in the context of suspensions (case C) we exhibit more necessary conditions for the complexification to be Kählerian which in some cases, for instance a double suspension, are also sufficient.

5.1 The Euler class

The next result is an obstruction for the complex manifolds obtained as in the section 4.1 to admit a Kähler metric.

Theorem 5.1.1. *Let* $(\mathrm{X}, \mathcal{F})$ *be a compact complex manifold with a holomorphic flow whose complex structure has been obtained as in proposition 4.1.1 with a distribution* $\mathcal{D}$ *invariant with respect to the vector fields* T_1 *and* T_2*. If* X *is Kählerian then* $e_{\mathcal{F}_1}(\mathrm{X}) = e_{\mathcal{F}_2}(\mathrm{X}) = 0$ *(where* $\mathcal{F}_i$ *is the flow defined by* T_i *for* $i = 1, 2$*).*

Proof. Since X is a compact Kähler manifold with a holomorphic vector field v without zeros by Carrell-Liebermann's theorem there exists a holomorphic 1-form α such that $\alpha(v) \neq 0$. As X is compact we can assume that $\alpha(v) = 1$ and as X is Kählerian the form α is closed. We decompose $\alpha = \frac{i}{2\,\mathrm{Im}\,\tau}(\alpha_2 + \bar{\tau}\alpha_1)$ where

α_1 and α_2 are real closed 1-forms. Using that $\alpha(v) = 1$ and $\alpha(\overline{v}) = 0$, a direct computation shows that $\alpha_i(T_j) = \delta_{ij}$ for $i, j = 1, 2$. Thus $e_{\mathcal{F}_1}(\mathrm{X}) = e_{\mathcal{F}_2}(\mathrm{X}) = 0$. □

Let M be a manifold in the class $\mathcal{T}$. We denote by T the vector field defining the CR-action and by $\mathcal{F}$ the transversely holomorphic flow induced by T.

Proposition 5.1.2. *With the above notation, assume that the compact complex manifold* X *is a complexification of* (M, T) *(cf. definition 4.5.1). If* X *is Kählerian then* $e_{\mathcal{F}}(\mathrm{M}) = 0$.

Corollary 5.1.3. *With the above notation, if* $e_{\mathcal{F}}(\mathrm{M}) \neq 0$ *no complexification obtained by proposition 4.2.1 (case A) or proposition 4.4.2 (case C) can admit a Kähler structure.*

Corollary 5.1.4. *With the above notation, if* M *admits a normal contact structure compatible with the CR-action induced by* T *then* (M, T) *admits no Kähler complexification.*

It follows from corollary 2.2.4.

Corollary 5.1.5. *With the above notation, if* $b_1(\mathrm{M}) = 0$*, in particular if* M *is simply connected, then* (M, T) *admits no Kähler complexification.*

It is a consequence of corollary 2.1.6.

Corollary 5.1.6. *With the above notation, let* M *be a compact connected semisimple real Lie group of odd dimension endowed with a normal almost contact structure. Then* (M, T) *admits no Kähler complexification.*

It is enough to recall that for any such group $b_1(\mathrm{M}) = 0$.

Proof. (Proposition 5.1.2) We denote by v the holomorphic vector field on X and by λ the complex number such that $T = \mathrm{Re}(\lambda v)$ on M. Since X is a compact Kähler manifold with a non-singular vector field v by the same argument as in the previous theorem there exists a holomorphic closed 1-form α such that $\alpha(v) = \lambda^{-1}$. We can decompose $(\lambda \cdot v)_{|\mathrm{M}} = T - iS$ where S is a real vector field. Set $\alpha = \frac{1}{2}(\alpha_1 + i\alpha_2)$ where α_1, α_2 are real 1-forms on X. Then α_1 and α_2 are closed and $\alpha_1(T) = 1$ on M. The closed real 1-form $\omega := \alpha_{1|\mathrm{M}}$ verifies $\omega(T) = 1$ (because $\alpha(v) = 1$ and $\alpha(\overline{v}) = 0$), therefore $e_{\mathcal{F}}(\mathrm{M}) = 0$. □

Proposition 5.1.7. *With the above notation, assume that* X *is a compact complex manifold constructed as in proposition* 4.3.1 *(case B) from a manifold* M *in the class* $\mathcal{T}$*. If* X *is Kählerian then* $e_{\mathcal{F}}(\mathrm{M}) = 0$ *and the* S^1*-principal bundle* $\pi : \mathrm{X} \to \mathrm{M}$ *is flat. In particular, if* X *is Kähler and* $H^2(\mathrm{M}, \mathbb{Z})$ *has no torsion then the* S^1*-principal bundle is topologically trivial. Moreover, if* α *is a connection 1-form on* X *such that* $d\alpha \in \pi^*\Omega^{1,1}(\mathrm{M}/\mathcal{F})$ *then* $[d\alpha] = 0$ *in* $H^2(\mathrm{M}/\mathcal{F})$*.*

Proof. With the notation of proposition 4.3.1, by the same argument as before if v is the holomorphic vector field of the complexification there exists a closed holomorphic 1-form α on X such that $\alpha(v) = 1$. The connected group S^1 acts holomorphically on X (as the group of the action of the S^1-principal bundle), therefore the forms α and $\bar{\alpha}$ are invariant by the action of S^1. Notice that $v = \widetilde{T} - \tau R$ where R is the vector field of the action and, $\widetilde{T}$ is the vector field contained in $\ker \beta$ such that $\pi_*(\widetilde{T}) = T$ (recall that T is the vector field inducing the CR-action on M) and $\tau \in \mathbb{C} \backslash \mathbb{R}$. We decompose $\alpha = \frac{i}{2 \operatorname{Im} \tau}(\alpha_2 + \bar{\tau}\alpha_1)$ where α_1, α_2 are real 1-forms. Then α_1 and α_2 are closed 1-forms invariant by the action of S^1 (for they are a linear combination of α, $\bar{\alpha}$ with constant coefficients) such that $\alpha_1(\widetilde{T}) = \alpha_2(R) = 1$ and $\alpha_1(R) = \alpha_2(\widetilde{T}) = 0$ (because $\alpha(v) = 1$ and $\alpha(\overline{v}) = 0$). Since α_1 is a closed real basic S^1-invariant 1-form it induces a closed 1-form ω on M such that $\omega(T) = 1$, thus $e_{\mathcal{F}}(\mathrm{M}) = 0$. Finally, α_2 is a closed connection 1-form for the S^1-principal bundle $\pi : \mathrm{X} \to \mathrm{M}$, so it is flat. When $H^2(\mathrm{M}, \mathbb{Z})$ has no torsion all flat bundles are topologically trivial. Moreover, if α is a connection 1-form on X such that $d\alpha \in \pi^*\Omega^{1,1}(\mathrm{M}/\mathcal{F})$ then $[d\alpha] = [d\alpha_2] = 0$ in $H^2(\mathrm{M}/\mathcal{F})$. □

5.2 Criteria for isometric flows

Theorem 5.2.1. *Let* $(\mathrm{X}, \mathcal{F})$ *be a compact complex manifold with a holomorphic flow whose complex structure has been obtained as in the previous chapter. Assume that the real foliation* $\mathcal{F} = \langle T_1, T_2 \rangle$ *is Riemannian. The manifold* X *is Kähler if and only if the flows* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are isometric,* $e_{\mathcal{F}_1}(\mathrm{X}) = e_{\mathcal{F}_2}(\mathrm{X}) = 0$ *and* $\mathcal{F}$ *is transversely Kählerian.*

Proof. $\Rightarrow$) : The same argument as in theorem 5.1.1 shows that there are two closed real 1-forms α_1 and α_2 on M such that $\alpha_i(T_j) = \delta_{ij}$ for $i, j = 1, 2$. In particular the flows are isometric (see section 1.5), α_1 and α_2 are characteristic forms for $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively and $e_{\mathcal{F}_1}(\mathrm{X}) = e_{\mathcal{F}_2}(\mathrm{X}) = 0$. We denote by $(\varphi_1)_t$

and $(\varphi_2)_t$ the 1-parametric groups associated to T_1 and T_2 respectively and by H the closure in Isom(X) of the abelian group generated by $(\varphi_1)_t$ and $(\varphi_2)_t$. If Φ is the Kähler form on X then the transverse part $\Psi(\cdot,\cdot)$ with respect to $\mathcal{F}$ of

$$\int_{\mathrm{H}} \Phi(\sigma_*\cdot, \sigma_*\cdot),$$

where we integrate with respect to the Haar mesure on H, is a transverse Kähler form.

$\Leftarrow$) : With the notation of the section 4.1, we denote by ω_1 and ω_2 the 1-forms defined by $\omega_i(T_j) = \delta_{ij}$ and $\omega_{i|\mathcal{D}} = 0$ for $i, j = 1, 2$. Since $e_{\mathcal{F}_1}(\mathrm{X}) = e_{\mathcal{F}_2}(\mathrm{X}) = 0$ there exist $\beta_1, \beta_2 \in \Omega^1(\mathrm{X}/\mathcal{F})$ such that $d\beta_i = d\omega_i$ for $i = 1, 2$. We denote by β the basic form $\beta = \frac{i}{2\,\mathrm{Im}\,\tau}(\beta_2 + \overline{\tau}\beta_1)$. It follows that $d\beta = d\chi$. We begin by showing that it is enough to find $\alpha \in \Omega^1(\mathrm{X}/\mathcal{F}, \mathbb{C})$ of type $(1,0)$ such that $d\alpha = d\chi$. Indeed, if α exists the form $\Phi = (\chi - \alpha) \wedge (\overline{\chi} - \overline{\alpha})$ is closed and of type $(1,1)$. Adding to Φ a positive multiple of the transverse Kähler form of $\mathcal{F}$ we obtain a Kähler form on X and the proof is complete. We will now show that such a form exists. Since $d\beta^{0,2} = d\chi^{0,2} = 0$ we have $d\beta = d(\beta^{1,0}) + \partial(\beta^{0,1})$, i.e. $\overline{\partial}(\beta^{0,1}) = 0$, and

$$d(\partial\beta^{0,1}) = (\partial + \overline{\partial})(\partial\beta^{0,1}) = -\partial\overline{\partial}\beta^{0,1} = 0$$

so $\partial(\beta^{0,1})$ is a $(1,1)$-form which is ∂-exact as a basic form and d-closed. Applying the basic $\partial\overline{\partial}$-lemma (see section 1.6) to $\partial(\beta^{0,1})$ we obtain a basic function f such that

$$\partial(\beta^{0,1}) = \partial\overline{\partial} f = \overline{\partial}\partial(-f).$$

Then $-\partial f$ is a basic form of type $(1,0)$ such that $d(-\partial f) = -\overline{\partial}\partial f = \partial(\beta^{0,1})$. The form $\alpha = \beta^{1,0} - \partial f$ is basic, of type $(1,0)$ and $d\alpha = d\beta = d\chi$ so the conclusion follows. □

Remark 5.2.2. Every Riemannian holomorphic flow $\mathcal{F}$ in a compact complex surface S is transversely Kählerian. Therefore with the notation and hypothesis of the above theorem, when the complex manifold X has dimension 2, it is Kähler if and only if the flows $\mathcal{F}_1$ and $\mathcal{F}_2$ are isometric and $e_{\mathcal{F}_1}(\mathrm{X}) = e_{\mathcal{F}_2}(\mathrm{X}) = 0$.

Corollary 5.2.3. *Let* X *be a complex manifold obtained by proposition 4.2.1 (case A) from two manifolds* M_1 *and* M_2 *in the class* $\mathcal{T}$ *such that the flows* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *in* M_1 *and* M_2 *respectively induced by the vector fields of the normal* *almost contact structures are Riemannian. Then* X *is Kählerian if and only if* $e_{\mathcal{F}_1}(\mathrm{M}_1) = e_{\mathcal{F}_2}(\mathrm{M}_2) = 0$ *and the flows* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are isometric and transversely Kählerian (on* M_1 *and* M_2 *respectively).*

Corollary 5.2.4. *Let* X *be a complex manifold obtained by proposition 4.3.1 (case B) from a manifold* M *in the class* $\mathcal{T}$ *such that the flow* $\mathcal{F}_T$ *on* M *is Riemannian. Then* X *is Kähler if and only if the* S^1*-principal bundle* $\pi : \mathrm{X} \to \mathrm{M}$ *is flat,* $e_{\mathcal{F}_T}(\mathrm{M}) = 0$ *and the flow* $\mathcal{F}_T$ *is isometric and transversely Kählerian on* M*.*

Remark 5.2.5. Recall that if $H^2(\mathrm{M}, \mathbb{Z})$ has no torsion then the S^1-principal bundle $\pi : \mathrm{X} \to \mathrm{M}$ is flat if and only if it is topologically trivial.

The previous theorem also allows us to derive some results for elliptic principal bundles, $\mathbb{C}$-principal bundles and $\mathbb{C}^*$-principal bundles.

Corollary 5.2.6. *Let* B *be a compact complex manifold and* $\pi : \mathrm{X} \to \mathrm{B}$ *an elliptic principal bundle. Then* X *is Kähler if and only if* B *is Kähler and the fibre bundle* $\pi : \mathrm{X} \to \mathrm{B}$ *is flat. If* $H^2(\mathrm{B}, \mathbb{Z})$ *has no torsion then* X *is Kähler if and only if* B *is Kähler and the fibre bundle* $\pi : \mathrm{X} \to \mathrm{B}$ *is topologically trivial. Moreover, if* X *is Kähler then* $\pi : \mathrm{X} \to \mathrm{B}$ *admits an holomorphic connection form.*

It follows directly from the preceding theorem. Recall that an elliptic principal bundle $\pi : \mathrm{X} \to \mathrm{B}$ over a compact manifold B with fibre $\mathrm{E} = \mathbb{C}/\langle 1, \tau\rangle$ for $\tau \in \mathbb{C}\backslash\mathbb{R}$ is topologically trivial if and only if the Chern class $[\Omega_1 + \tau\Omega_2] \in H^2(\mathrm{B}, \mathbb{Z}\oplus\tau\mathbb{Z})$ is zero or equivalently if there exists a $\mathbb{C}$-principal bundle $p : \widetilde{\mathrm{X}} \to \mathrm{B}$ and a group Γ acting properly discontinuously on $\widetilde{\mathrm{X}}$ so that $\widetilde{\mathrm{X}}/\Gamma \cong \mathrm{X}$ and if we denote by $q : \widetilde{\mathrm{X}} \to \widetilde{\mathrm{X}}/\Gamma \cong \mathrm{X}$ then $\mathrm{id} \circ p = \pi \circ q$.

Remark 5.2.7. In particular this criterium shows that the Iwasawa manifold cannot be Kähler, since it is an elliptic principal bundle over a torus $\mathbb{T}^2$ (note that $H^2(\mathbb{T}^2, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$) which is not topologically trivial. Note also that the Iwasawa manifold admits a holomorphic connection, therefore this condition is not equivalent to the other ones.

Corollary 5.2.8. *If* $\pi : \mathrm{X} \to \mathrm{B}$ *is a* $\mathbb{C}$*-principal bundle over a Kähler manifold* B *then* X *is Kähler and* $\pi : \mathrm{X} \to \mathrm{B}$ *admits a holomorphic connection form.*

Proof. We cannot directly apply theorem 5.2.1 because the total space of a $\mathbb{C}$-principal bundle is not compact. Nevertheless every $\mathbb{C}$-principal bundle is topologically trivial. Given any lattice $\Gamma_\tau = \mathbb{Z} + \tau\mathbb{Z}$ in $\mathbb{C}$ with $\operatorname{Im}\tau \neq 0$, it induces a topologically trivial $\mathrm{E} = \mathbb{C}/\langle 1, \tau\rangle$-principal bundle $p : \mathrm{M} \to \mathrm{B}$, thus the total space M is Kähler. We consider the commutative diagram of fibre bundles:

$$\begin{array}{ccc} \mathbb{C} & & \mathbb{C}/\Gamma_\tau \\ \downarrow & & \downarrow \\ \mathrm{X} & \xrightarrow{q} & \mathrm{M} \cong \mathrm{X}/\Gamma_\tau \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ \mathrm{B} & \xrightarrow{\mathrm{id}} & \mathrm{B}. \end{array}$$

Since Γ_τ acts properly discontinuously the projection q is a covering map and we can conclude that X is Kähler. Taking the pull-back by q of the holomorphic connection form on M we obtain a holomorphic connection form on X. □

A similar argument yields the following:

Corollary 5.2.9. *Let* B *be a compact complex manifold and* $\pi : \mathrm{X} \to \mathrm{B}$ *a* $\mathbb{C}^*$*-principal bundle. Then the following statements are equivalent:*

(a) B *is Kähler and* $\pi : \mathrm{X} \to \mathrm{B}$ *is topologically trivial.*

(b) B *is Kähler and* $\pi : \mathrm{X} \to \mathrm{B}$ *is obtained as a quotient of a* $\mathbb{C}$*-principal bundle.*

If they hold then X *is Kähler and* $\pi : \mathrm{X} \to \mathrm{B}$ *admits a holomorphic connection form.*

5.3 Criteria for suspensions

5.3.1 Double suspensions

Let N be a compact Kähler manifold and $f, g \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$ such that $f \circ g = g \circ f$. We consider the compact complex manifold X obtained as the suspension

$$\mathrm{X} = \mathrm{N} \times \mathbb{C}/\langle F, G\rangle$$

where $F(x, z) = (f(x), z + 1)$, $G(x, z) = (g(x), z + \tau)$ and $\operatorname{Im}(\tau) \neq 0$. There is a holomorphic submersion over the elliptic curve $\mathrm{E} = \mathbb{C}/\langle 1, \tau\rangle$ given by

$$p : \mathrm{X} \to \mathrm{E}; \qquad (x, z) \mapsto z$$

and the holomorphic vector field without zeros v induced by $\frac{\partial}{\partial z}$ projects over the vector field $\frac{\partial}{\partial z}$ on E. We denote by $\mathrm{Aut}_0(\mathrm{N})$ the connected component of the identity in the group of holomorphic transformations $\mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$ of N.

Lemma 5.3.1. *Let* M *be a compact manifold,* $f \in \mathrm{Aut}(\mathrm{M})$ *and* $\mathrm{X} = \mathrm{M} \times_f \mathbb{R}$. *Then the de Rham cohomology groups* $H^r(\mathrm{X})$ *is isomorphic to*

$$\{[\sigma] \in H^r(\mathrm{M}) : \, f^*[\sigma] = [\sigma]\} \oplus \left(\frac{H^{r-1}(\mathrm{M})}{\{[\sigma - f^*\sigma] : \, [\sigma] \in H^{r-1}(\mathrm{M})\}} \right) \wedge [ds].$$

In particular, if $f^* = \mathrm{id}$ *acting on* $H^*(\mathrm{M})$, *then*

$$H^r(\mathrm{X}) \cong H^r(\mathrm{M}) \oplus H^{r-1}(\mathrm{M}) \wedge [ds].$$

Remark 5.3.2. With the above notation, if M is connected then

$$H^1(\mathrm{X}) \cong \{[\sigma] \in H^1(\mathrm{M}) : \, f^*[\sigma] = [\sigma]\} \oplus \langle [ds] \rangle.$$

Proof. For the proof of the lemma it is enough to use Mayer-Vietoris sequence for the De Rham cohomology groups (see [BT82] for details). Let X be a compact manifold and assume $\mathrm{X} = U \cup V$ where U, V are open subsets of X. There is a short exact sequence given by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^*(\mathrm{X}) & \longrightarrow & \Omega^*(U) \oplus \Omega^*(V) & \longrightarrow & \Omega^*(U \cap V) & \longrightarrow & 0 \\ & & \alpha & \longrightarrow & (\alpha_{|U}, \alpha_{|V}) & & & & \\ & & & & (\omega, \tau) & \longrightarrow & \omega_{|U\cap V} - \tau_{|U \cap V} & & \end{array}$$

which induces an exact sequence for the De Rham cohomology groups:

$$H^{q-1}(U \cap V) \xrightarrow{d^*} H^q(\mathrm{X}) \longrightarrow H^q(U) \oplus H^q(V) \xrightarrow{F_q} H^q(U \cap U)$$

The coboundary operator is given by

$$d^*[\omega] = \begin{cases} [-d(\rho_V \omega)] & \text{on } U \\ [d(\rho_U \omega)] & \text{on } V \end{cases}$$

where $\{\rho_U, \rho_V\}$ is a partition of unity associated to the covering $\{U, V\}$. Now recall that a suspension admits a fibration $p : \mathrm{X} \to S^1$ with fibre M. We choose as open sets $U = p^{-1}(S^1 \backslash \{1\}) \cong (0,1) \times \mathrm{M}$ and $V = p^{-1}(S^1 \backslash \{-1\}) \cong (0,1) \times \mathrm{M}$. Therefore $H^*(U) \cong H^*(V) \cong H^*(\mathrm{M})$ and $H^*(U \cap V) \cong H^*(\mathrm{M}) \oplus H^*(\mathrm{M})$. The map $F_q : H^q(U) \oplus H^q(V) \to H^q(U \cap V)$ corresponds to

$$\begin{array}{ccc} H^q(\mathrm{M}) \oplus H^q(\mathrm{M}) & \xrightarrow{F_q} & H^q(\mathrm{M}) \oplus H^q(\mathrm{M}) \\ ([\alpha], [\beta]) & \longrightarrow & ([\alpha - \beta], [\alpha - f^*\beta]) \end{array}$$

so $\mathrm{Ker}(F_q) \cong \{[\sigma] \in H^q(\mathrm{M}) : f^*[\sigma] = [\sigma]\}$ and $\mathrm{Im}(F_q) \cong [ds] \wedge (H^{q-1}(\mathrm{M}) \oplus \{[\sigma - f^*\sigma] : [\sigma] \in H^{q-1}(\mathrm{M})\})$. Finally, considering the short exact sequence

$$0 \longrightarrow H^q(\mathrm{M}) \oplus H^q(\mathrm{M})/\mathrm{Im}(F_{q-1}) \longrightarrow H^q(\mathrm{X}) \longrightarrow \mathrm{Ker}(F_q) \longrightarrow 0$$

we can conclude. □

Corollary 5.3.3. *Let* K *be a non-abelian compact connected real Lie group of odd dimension endowed with a normal almost contact structure,* $f \in \mathrm{Aut}_{\mathcal{T}}(\mathrm{K})$ *and* $\mathrm{X} = \mathrm{K} \times_f \mathbb{R}$ *endowed with the complex structure described in proposition 4.4.2. Then* X *cannot admit a Kähler metric.*

Proof. We know that $\mathrm{X} = \mathrm{K} \times_f \mathbb{R}$ admits a finite covering $\widetilde{\mathrm{X}} = \mathrm{M} \times_{\tilde{f}} \mathbb{R}$ such that $\mathrm{M} \cong \mathrm{K}' \times (S^1)^r$ where K' is a compact connected semisimple real Lie group, $0 \leq r < \dim_{\mathbb{R}} \mathrm{K}$ and $\widetilde{f}$ the lift of f to $\widetilde{\mathrm{X}}$. Using Kunneth's formula and taking into account that $b_1(\mathrm{K}') = b_2(\mathrm{K}') = 0$ (see section 1.7.5) we conclude that $H^2(\mathrm{M}) \cong H^2((S^1)^r)$ and $H^1(\mathrm{M}) \cong H^1((S^1)^r)$. Now, the previous lemma says that $H^2(\widetilde{\mathrm{X}})$ is isomorphic to

$$\{[\sigma] \in H^2(\mathrm{M}) : f^*[\sigma] = [\sigma]\} \oplus \left(\frac{H^1(\mathrm{M})}{\{[\sigma - f^*\sigma] : [\sigma] \in H^1(\mathrm{M})\}}\right) \wedge [ds].$$

If $\widetilde{\mathrm{X}}$ were Kähler then there would be a class $[\omega] \in H^2(\widetilde{\mathrm{X}})$ such that $[\omega^s] \neq 0$ in $H^{2s}(\widetilde{\mathrm{X}})$, where $s = \dim_{\mathbb{C}} \widetilde{\mathrm{X}}$, which is impossible. Therefore $\widetilde{\mathrm{X}}$ is not Kähler and it follows that X cannot be Kähler. □

Corollary 5.3.4. *Let* N *be a compact complex manifold and* $f, g \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$ *such that* $f \circ g = g \circ f$ *and let* X *be the suspension* $\mathrm{N} \times \mathbb{C}/\langle F, G\rangle$ *where* $F(x,z) = (f(x), z+1)$, $G(x,z) = (g(x), z+\tau)$ *and* $\mathrm{Im}(\tau) \neq 0$. *If* f^*, g^* *act as the identity on* $H^1(\mathrm{N}, \mathbb{R})$ *then* $b_1(\mathrm{X}) = b_1(\mathrm{N}) + 2$.

Theorem 5.3.5. *Let* N *be a compact complex manifold and* $f, g \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$ *such that* $f \circ g = g \circ f$ *and let* X *be the suspension* $\mathrm{N} \times \mathbb{C}/\langle F, G\rangle$ *where* $F(x,z) = (f(x), z+1)$, $G(x,z) = (g(x), z+\tau)$ *and* $\mathrm{Im}(\tau) \neq 0$. *Then the following conditions are equivalent:*

- **(i)** X *is Kähler.*
- **(ii)** *There is a Kähler form* ω *on* N *such that* $[f^*\omega] = [g^*\omega] = [\omega]$.
- **(iii)** N *is Kähler and there are integers* $n, m > 0$ *such that* $f^n, g^m \in \mathrm{Aut}_0(\mathrm{N})$.

Recall that if h_1, h_2 are homotopic endomorphisms of a manifold (or topological space) X then $h_1^* = h_2^*$ on $H^*(\mathrm{X}, \mathbb{R})$. In particular if $f \in \mathrm{Aut}_0(\mathrm{N})$ then $f^* = \mathrm{id}$ in $H^*(\mathrm{N})$.

Corollary 5.3.6. *Let* N *be a compact Kähler manifold, if* $b_2(\mathrm{N}) = 1$ *any compact complex manifold* X *obtained by a double suspension on* N *is Kähler.*

Example 5.3.7. If $\dim_{\mathbb{C}} \mathrm{N} = 1$ then $b_1(\mathrm{N}) = 1$ and we conclude that any compact complex manifold X obtained by a double suspension on N is Kähler. We can see that any surface obtained as in the remark 4.4.4 is Kähler using the fact that a compact surface is Kähler if and only if it has even first Betti number (cf. [Buc99]). Note that when $f, g \in \mathrm{Aut}_0(\mathrm{N})$ we have $f^* = g^* = \mathrm{id}$ on $H^1(\mathrm{N})$, then by the previous corollary $b_1(\mathrm{X}) = b_1(\mathrm{N}) + 2$. If $\dim_{\mathbb{C}} \mathrm{N} = 1$ then $b_1(\mathrm{N})$ is even (for N is Kähler), then it follows that $b_1(\mathrm{X})$ is even.

Proof. (Corollary 5.3.6) Applying Hodge decomposition theorem to N one concludes that $H^2(\mathrm{N}, \mathbb{C}) \cong H^{1,1}(\mathrm{N}, \mathbb{C}) \cong \mathbb{C}$. Then if ω is a Kähler form on N and $f \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$ we have $[f^*\omega] = [\lambda\omega]$, where $\lambda \in \mathbb{R}^+$. Then $f^*\omega = \lambda\omega + d\alpha$ where α is a 1-form. Moreover we know that $[\lambda^n \omega^n] = f^*[\omega^n] = [\omega^n]$, therefore $\lambda = \pm 1$. Assume that $\lambda = -1$, then the Kähler form $\omega + f^*\omega = d\alpha$ is exact, which is a contradiction. Thus $\lambda = 1$ and we conclude that the hypothesis **(ii)** in the theorem 5.3.5 is trivially fulfilled. □

Example 5.3.8. If N is a compact Kähler manifold and $\mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$ is finite then any complex manifold X thus obtained from N must be Kähler. Among the examples of manifolds N under the previous assumptions we have compact Kähler manifolds with negative definite Ricci tensor, quotients D/Γ of a bounded domain of $\mathbb{C}^n$ by a properly discontinuous group Γ of holomorphic transformations acting freely on D and any non-singular hypersurface of degree d in $\mathbb{P}^n$ for $n > 4$ and $d \geq 3$ (see [Kob72], p.86-88).

Lemma 5.3.9. *Let* $\widetilde{\mathrm{X}}$ *be the finite covering of* $n \cdot m$ *sheets of* X *obtained as the suspension*

$$\widetilde{\mathrm{X}} = \mathrm{N} \times \mathbb{C}/\langle F^n, G^m \rangle$$

where $F^n(x, z) = (f^n(x), z + n)$ *and* $G^m(x, z) = (g^m(x), z + m\tau)$. *Then* $\widetilde{\mathrm{X}}$ *is Kähler if and only if* X *is Kähler.*

Remark 5.3.10. With the above notation, $\widetilde{X}$ is isomorphic to the suspension $\mathrm{N} \times \mathbb{C}/\langle \widetilde{F}, \widetilde{G} \rangle$, for $\widetilde{F}(x, z) = (f^n(x), z + 1)$ and $\widetilde{G}(x, z) = (g^m(x), z + \frac{m}{n}\tau)$.

Proof. Clearly, if X is Kähler the pull-back of a Kähler form on X is a Kähler form on $\widetilde{\mathrm{X}}$. Assume now that Φ is a Kähler form on $\widetilde{\mathrm{X}}$, which we can represent by a Kähler form on $\mathrm{N} \times \mathbb{C}$ invariant by F^n and G^m. Then the form Ψ on $\mathrm{N} \times \mathbb{C}$ defined by:

$$\Psi = \Sigma_{i=0,..,n-1;j=0,...m-1}(F^i)^* \circ (G^j)^*(\Phi)$$

is a Kähler form on $\mathrm{N} \times \mathbb{C}$ invariant by F and G, thus it represents a Kähler form on X. □

During the proof of the theorem we will make use of the following result by A.Blanchard (cf. [Bla56], p.192):

Theorem 5.3.11 (Blanchard)**.** *Let* X *be a fibred complex compact analytic space with base* B *and fibre* F*. Let us assume that* $\pi_1(\mathrm{B})$ *acts trivially on* $H^1(\mathrm{F}, \mathbb{R})$*. Then* X *is Kähler if and only if the following conditions hold:*

- **(i)** *There is a Kähler form on* F *which represents a cohomology class invariant by* $\pi_1(\mathrm{B})$*.*
- **(ii)** B *is Kähler.*
- **(iii)** $b_1(\mathrm{X}) = b_1(\mathrm{B}) + b_1(\mathrm{F})$*.*

The basic tools to prove the preceding result are the following two versions of the $\partial\overline{\partial}$-lemma with parameters. Their proof can also be found in [Bla56].

Lemma 5.3.12. *Let* M *be a compact Kähler manifold and let* ω *be an exact form of type* (p, q) *on* M *such that* $\omega = d_{\mathrm{M}}\alpha$ *and* ω *and* α *depend smoothly on a complex parameter* $z \in \mathbb{C}$*. Then there exist a* $(p-1, q)$*-form* μ *and a* $(p, q-1)$*-form* ν *that depend smoothly on* $z \in \mathbb{C}$ *and such that* $\omega = d_{\mathrm{M}}\mu = d_{\mathrm{M}}\nu$*.*

Lemma 5.3.13. *Let* M *be a compact Kähler manifold and let* ω *be an exact form on* M *such that* $\omega = d_{\mathrm{M}}\alpha$ *and* ω *and* α *depend smoothly on a complex parameter* $z \in \mathbb{C}$*. Assume that* $\omega = \omega_1 + \omega_2$ *where* ω_1 *is of type* $(p+1, q)$ *and* ω_2 *of type* $(p, q+1)$*, both depending smoothly on* $z \in \mathbb{C}$*. Then there exists a* (p, q)*-form* ν *which depends smoothly on* $z \in \mathbb{C}$ *and such that* $\omega = d_{\mathrm{M}}\nu$*.*

Remark 5.3.14. Since the total space X of a suspension is locally a product of N and an elliptic curve $\mathrm{E} = \mathbb{C}/\langle 1, \tau \rangle$ the exterior derivative d on X can be decomposed as the sum $d = d_{\mathrm{N}} + d_{\mathrm{E}}$. Moreover $d_{\mathrm{N}} = \partial_{\mathrm{N}} + \overline{\partial}_{\mathrm{N}}$; $d_{\mathrm{E}} = \partial_{\mathrm{E}} + \overline{\partial}_{\mathrm{E}}$ and $\partial_{\mathrm{E}}, \overline{\partial}_{\mathrm{E}}$ and $\partial_{\mathrm{N}}, \overline{\partial}_{\mathrm{N}}$ are the usual well-defined operators on N and E respectively.

Note also that any of these differential operators of E anti-commute with any of these operators on N, for instance $d_{\mathrm{E}} \circ d_{\mathrm{N}} = -d_{\mathrm{N}} \circ d_{\mathrm{E}}$. This is also true if one considers the contractions $i_{\frac{\partial}{\partial z}}$ and $i_{\frac{\partial}{\partial \bar{z}}}$ and any of the previous operators on N.

Proof. (Theorem 5.3.5) (**i**) $\Longrightarrow$ (**ii**): Let Ψ be a Kähler form on X. Its pull-back $\Phi = \pi^*\Psi$ by the covering map $\pi : \mathrm{N} \times \mathbb{C} \to \mathrm{X}$ is a Kähler form on $\mathrm{N} \times \mathbb{C}$. Let us choose $z_0 \in \mathbb{C}$. If we denote by $\mathrm{N}_z = \mathrm{N} \times \{z\}$ and by $\omega_z(x) = \Phi(x,z)_{|\mathrm{N}_z}$ then ω_{z_0} is a Kähler form on N_{z_0}. It suffices to show that $[f^*\omega_{z_0}] = [\omega_{z_0}]$ (for g the argument is analogous). By construction of Φ we know that $F^*\Phi = \Phi$ where $F(x,z) = (f(x), z+1)$, therefore $f^*\omega_{z_0} = \omega_{z_0-1}$. Recall that if $[\Phi] = [\Phi'] \in H^2(\mathrm{N} \times \mathbb{C}, \mathbb{R})$ then $[\Phi(x,z_0)_{|\mathrm{N}}] = [\Phi'(x,z_0)_{|\mathrm{N}}]$ in $H^2(\mathrm{N}, \mathbb{R})$. Therefore it is enough to see that $[\Phi(x,z)] = [\Phi(x, z+a_0)]$ in $H^2(\mathrm{N} \times \mathbb{C}, \mathbb{R})$ for all $a_0 \in \mathbb{C}$. Since the map $(x,z) \mapsto (x, z+a_0)$ is homotopic to the identity on $\mathrm{N} \times \mathbb{C}$ this condition is verified and we can conclude.

(**ii**) $\Longrightarrow$ (**iii**): It is an immediate consequence of a result by D.Lieberman (see [Lie78]) that asserts the following: if M is a compact Kähler manifold, ω a Kähler form and we denote by $\mathrm{Aut}_\omega(\mathrm{M})$ the group of automorphisms of M preserving the Kähler class $[\omega]$ then $\mathrm{Aut}_\omega(\mathrm{M})/\mathrm{Aut}_0(\mathrm{M})$ is a finite group. Indeed, if we consider $\{f, f^2, ...\}$ there must exist $n_1 > n_2 > 0$ such that $f^{n_1} = f^{n_2} \cdot h$ with $h \in \mathrm{Aut}_0(\mathrm{N})$. Therefore for $n = n_1 - n_2$ we have $f^n \in \mathrm{Aut}_0(\mathrm{N})$ (and we would proceed identically for g).

(**iii**) $\Longrightarrow$ (**ii**): Let $\widetilde{\omega}$ be a Kähler form on N whose Kähler class is preserved both by f^n and g^m. Considering the Kähler form

$$\omega = \sum_{i=0,..,n-1; j=0,...,m-1} (f^i)^* \circ (g^j)^* \widetilde{\omega},$$

the class $[\omega]$ is preserved both by f and g.

(**iii**) $\Longrightarrow$ (**i**): It is enough to show that a double suspension of a compact Kähler manifold N by $f, g \in \mathrm{Aut}_0(\mathrm{N})$ is Kähler (see lemma 5.3.9 and remark 5.3.10). This follows from the theorem by A.Blanchard stated above: note that since $f, g \in \mathrm{Aut}_0(\mathrm{N})$ the fundamental group $\pi_1(\mathrm{E})$ acts trivially on $H^1(\mathrm{N}, \mathbb{R})$ and consequently $b_1(\mathrm{X}) = b_1(\mathrm{N}) + 2$.

For the sake of clarity we discuss the argument under our hypothesis. The Kähler form on X is constructed in several steps. We begin by fixing a Kähler form ω on N. We can choose an open covering $\{U_i\}$ of E so that the fibration $p : \mathrm{X} \to \mathrm{E}$ is trivial over U_i. Then ω induces a well-defined form ω_i on

$p^{-1}(U_i) \cong U_i \times \mathrm{N}$. Let $\{\rho_i\}$ be a partition of unity associated to $\{U_i\}$, then $\Phi_0 = \sum_i \rho_i(z,\bar{z})\omega_i$ is a real global $(1,1)$-form on X so that $\Phi_{0|\mathrm{N}}$ is a Kähler form on N representing a fixed cohomology class (as f^* and g^* preserve $H^*(\mathrm{N})$ all the forms ω_i belong to the same cohomology class in $H^2(\mathrm{N})$). Moreover $d\Phi_0 = d_{\mathrm{E}}\Phi_0$. Now we want to obtain a closed real valued $(1,1)$-form Φ on X such that $\Phi_{|\mathrm{N}} = \Phi_0$. If $\{x_i\}$ are local coordinates on an open set U of N and $\{z\}$ is the usual complex coordinate on E then Φ will have the local form:

$$\Phi = \Sigma_{jk}a_{jk}dx_j \wedge d\bar{x}_k + \Sigma_j h_j dx_j \wedge d\bar{z} + \Sigma_j \bar{h}_j d\bar{x}_j \wedge dz + \mathrm{i} f dz \wedge d\bar{z}.$$

where $a, h_j, \bar{h}_j, f$ are functions on X depending on $x, \bar{x}, z, \bar{z}$ and f is real-valued. Equivalently Φ can be written as

$$\Phi = \Phi_0 + H \wedge d\bar{z} + \overline{H} \wedge dz + \mathrm{i} f dz \wedge d\bar{z},$$

where H is a $(1,0)$-form on X and f a real valued function on X. The hypothesis $d\Phi = 0$ is equivalent to the following equations

$$\begin{cases} \partial_{\mathrm{E}}\Phi_0 + \partial_{\mathrm{N}}\overline{H} \wedge dz = 0 & \text{(I)} \\ \partial_{\mathrm{N}}H = 0 & \text{(II)} \\ \partial_{\mathrm{E}}H + \mathrm{i}\partial_{\mathrm{N}}f \wedge dz = 0 & \text{(III)} \end{cases}$$

Obtaining the form Φ is equivalent to finding H and f that solve the previous system. Roughly speaking, we will first solve (I) and (II) so that we determine H and then define f as the solution of (III). The equations (I) and (II) are equivalent to

$$d_{\mathrm{N}}H = i_{\frac{\partial}{\partial \bar{z}}}\bar{\partial}_{\mathrm{E}}\Phi_0 \qquad \text{(IV)}.$$

Note that the form $\omega = i_{\frac{\partial}{\partial \bar{z}}}\bar{\partial}_{\mathrm{E}}\Phi_0$ can be seen locally as a $(1,1)$-form on N which depends smoothly on the complex parameter z. To apply lemma 5.3.12 to obtain a local $(1,0)$-form H_i which solves (IV) we must assure that ω is d_{N}-exact. Note that

$$d_{\mathrm{N}}\omega = i_{\frac{\partial}{\partial \bar{z}}}\bar{\partial}_{\mathrm{E}}d_{\mathrm{N}}\Phi_0 = 0.$$

To prove the exactness it is enough to see that

$$\int_C \omega = i_{\frac{\partial}{\partial \bar{z}}}\bar{\partial}_{\mathrm{E}}\int_C \Phi_0 = 0$$

for every cycle C on N. This holds because Φ_0 represents the same cohomology class on every fibre so $\int_C \Phi_0$ does not depend on z or $\bar{z}$. Therefore there exist local solutions $\{H_i\}$ to (IV) which depend smoothly on z. Using a partition of unity as above we obtain a global solution $H_0 = \sum_i \rho(z,\bar{z})H_i$. Now, to proceed with our plan, we should define f as the solution of (III) for the previous H_0. The equation (III) is equivalent to

$$d_{\mathrm{N}}f = \mathrm{i}\big(i_{\frac{\partial}{\partial \bar{z}}}\bar{\partial}_{\mathrm{E}}\overline{H} - i_{\frac{\partial}{\partial z}}\partial_{\mathrm{E}}H\big) =: \nu \qquad \text{(V)}.$$

Note that the term on the right ν is a real form which is the sum of a form of type $(1,0)$ and a form of type $(0,1)$, therefore we can try to apply lemma 5.3.13. Moreover ν is d_{N}-closed, for

$$\begin{aligned} d_{\mathrm{N}}\nu &= \mathrm{i}\big(i_{\frac{\partial}{\partial \bar{z}}}\bar{\partial}_{\mathrm{E}}d_{\mathrm{N}}\overline{H} - i_{\frac{\partial}{\partial z}}\partial_{\mathrm{E}}d_{\mathrm{N}}H\big) \\ &= \mathrm{i}\big(i_{\frac{\partial}{\partial z}} \circ i_{\frac{\partial}{\partial \bar{z}}}\bar{\partial}_{\mathrm{E}}\partial_{\mathrm{E}}\Phi_0 - i_{\frac{\partial}{\partial \bar{z}}} \circ i_{\frac{\partial}{\partial z}}\partial_{\mathrm{E}}\bar{\partial}_{\mathrm{E}}\Phi_0\big) = 0 \end{aligned}$$

Nevertheless, here we encounter a difficulty, since we cannot prove that ν_0, for the previous solution H_0, is d_{N}-exact. To overcome it we will modify H_0 to obtain a new solution of (IV) for which the corresponding ν in (V) is d_{N}-exact. Fix a basis $\{\alpha_1,...,\alpha_k,\overline{\alpha}_1,...,\overline{\alpha}_k\}$ of $H^0(\mathrm{N},\Omega^1) \oplus \overline{H^0(\mathrm{N},\Omega^1)}$. Note that since α_i are holomorphic forms on a compact Kähler manifold N they are fixed by $f,g \in \mathrm{Aut}_0(\mathrm{N})$, therefore we can assume that they are well-defined holomorphic forms on X. Indeed, as f,g preserve the cohomology classes, $f^*\omega_i - \omega_i$ and $g^*\omega_i - \omega_i$ are holomorphic exact 1-forms for $i = 1,...,k$, thus zero. In other words $f^*\omega_i = g^*\omega_i = \omega_i$ for $i = 1,...,k$ and $\omega_1,...,\omega_k$ are well-defined forms in $H^0(\mathrm{X},\Omega^1)$ defining a basis of $H^0(\mathrm{N},\Omega^1)$. By construction they are also d_{E}-closed. Let $\{\gamma_1,...,\gamma_k,\overline{\gamma}_1,...,\overline{\gamma}_k\}$ be the dual basis of $H_1(\mathrm{N},\mathbb{C})$. Note that if α is a 1-form $\overline{\gamma}(\alpha) := \overline{\gamma(\overline{\alpha})}$. We define

$$u_j = \mathrm{i}\int_{\gamma_j}(\bar{\partial}_{\mathrm{E}}\overline{H}_0 \wedge dz + \partial_{\mathrm{E}}H_0 \wedge d\bar{z}) = \int_{\gamma_j}\nu_0 \wedge dz \wedge d\bar{z}$$

for $1 \leq j \leq k$. It is not difficult to see that they are d_{E}-closed $(1,1)$ forms on E. Moreover

$$u_j = d_{\mathrm{E}}\int_{\gamma_j}\mathrm{i}(\overline{H}_0 \wedge dz + H_0 \wedge d\bar{z})$$

so they are also d_{E}-exact. Applying lemma 5.3.12 to E and u_j we obtain a family $\{v_1,...,v_k\}$ of $(0,1)$-forms on E so that $d_{\mathrm{E}}v_j = u_j$. We can define now a new solution H of (IV) by the formula

$$H \wedge d\bar{z} = H_0 \wedge d\bar{z} + \mathrm{i}\sum_{j=1}^{k}\alpha_j \wedge v_j.$$

We will next verify that the integral of $\nu dz \wedge d\bar{z} = \mathrm{id}_{\mathrm{E}}(\overline{H} \wedge dz + H \wedge d\bar{z})$ is zero for any cycle C on N, for it is equivalent to $\int_C \nu = 0$. Note that since we saw that ν is d_{N}-closed when H is a solution of (IV) it is enough to check that $\int_{\gamma_j} \nu dz \wedge d\bar{z} = \int_{\bar{\gamma}_j} \nu dz \wedge d\bar{z} = 0$. Indeed,

$$\begin{aligned}
\int_{\gamma_j} \nu dz \wedge d\bar{z} &= \int_{\gamma_j} \nu_0 dz \wedge d\bar{z} - d_{\mathrm{E}} v_j = u_j - u_j = 0 \\
\int_{\bar{\gamma}_j} \nu dz \wedge d\bar{z} &= \int_{\bar{\gamma}_j} \nu_0 dz \wedge d\bar{z} + d_{\mathrm{E}} \bar{v}_j = -\bar{u}_j + \bar{u}_j = 0.
\end{aligned}$$

Therefore, we are left to solve $d_{\mathrm{N}} f = \nu$ with ν satisfying all the hypothesis in lemma 5.3.13 to obtain local real functions f_i so that $d_{\mathrm{N}} f_i = \nu$. We define thus f by means of a partition of unity, $f = \sum_i \rho_i f_i$. To finish the proof it is enough to add to Φ the pullback of a Kähler form on E positive enough so that we obtain a positive closed $(1,1)$-form on X. □

Let now N be a compact Kähler manifold. We denote by Alb(N) its Albanese torus, by J the Jacobi map and by $f!$ the automorphism of Alb(N) induced by a given $f \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$ (see section 1.3). We want to determine when the suspension X of a compact projective manifold N by $f, g \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$ such that $f \circ g = g \circ f$ is projective. Since the condition is expressed in terms of the Albanese torus of X we will begin by computing it.

Proposition 5.3.15. *Let* N *be a compact projective manifold and* $f, g \in \mathrm{Aut}_0(\mathrm{N})$ *such that* $f \circ g = g \circ f$*. Then* Alb(X) *is isomorphic to the suspension* $\mathrm{Alb}(\mathrm{N}) \times \mathbb{C}/\langle F!, G! \rangle$ *where* $f!, g!$ *are the translations of* Alb(N) *induced by* f, g *respectively,* $F!(y,z) = (f!(y), z+1)$ *and* $G!(y,z) = (g!(y), z+\tau)$*.*

Proof. Since $H_1(\mathrm{X}, \mathbb{Z}) \cong H_1(\mathrm{N}, \mathbb{Z}) \oplus H_1(\mathrm{E}, \mathbb{Z})$ (because X is homeomorphic to $\mathrm{N} \times \mathrm{E}$) we can obtain a basis of $H_1(\mathrm{X}, \mathbb{Z})$ taking a basis $\gamma_1, ...\gamma_{2k}$ of $H_1(\mathrm{N}, \mathbb{Z})$ and σ_1, σ_2 which are defined as the composition of ρ_1 and a path on the fibre N from x_0 to $f(x_0)$ and of ρ_2 and a path on the fibre N from x_0 to $g(x_0)$ respectively where ρ_1 and ρ_2 are respectively the projection on X of the paths

$$\begin{array}{ccccccc}
[0,1] & \longrightarrow & \mathrm{N} \times \mathbb{C} & ; & [0,1] & \longrightarrow & \mathrm{N} \times \mathbb{C} \\
t & \longrightarrow & (x_0, t) & & t & \longrightarrow & (x_0, \tau t).
\end{array}$$

Assume $\omega_1, ..., \omega_k$ are a basis of $H^0(\mathrm{N}, \Omega^1)$. As f, g preserve the cohomology classes, $f^*\omega_i - \omega_i$ and $g^*\omega_i - \omega_i$ are holomorphic exact 1-forms for $i = 1, ..., k$, thus zero. In other words $f^*\omega_i = g^*\omega_i = \omega_i$ for $i = 1, ..., k$ and $\omega_1, ..., \omega_k$ are well-defined forms in $H^0(\mathrm{X}, \Omega^1_{\mathrm{X}})$ defining a basis of $H^0(\mathrm{N}, \Omega^1_{\mathrm{N}})$. We complete to a basis of $H^0(\mathrm{X}, \Omega^1_{\mathrm{X}})$ by adding $\sigma = p^*(dz)$ where $p : \mathrm{X} \to \mathrm{E}$ is the natural projection. Therefore if $\Delta_{\mathrm{N}} = \left\{ \left(\int_\gamma \omega_1, ..., \int_\gamma \omega_k \right) \,:\, \gamma \in H_1(\mathrm{N}, \mathbb{Z}) \right\}$, the Albanese torus is isomorphic to $\mathrm{Alb(N)} = \mathbb{C}^k/\Delta_{\mathrm{N}}$ and the Albanese map is

$$\begin{aligned} J : \mathrm{N} &\longrightarrow \quad \mathrm{Alb(N)} \\ x &\longrightarrow \left(\int_{x_0}^{x} \omega_1, ..., \int_{x_0}^{x} \omega_k \right). \end{aligned}$$

Then a basis of Δ_{X}, where $\mathrm{Alb(X)} = \mathbb{C}^{k+2}/\Delta_{\mathrm{X}}$, can be given by the rows of the matrix

$$\begin{pmatrix} \int_{\gamma_1} \omega_1 & \dots & \int_{\gamma_1} \omega_k & 0 \\ & \dots & & 0 \\ \int_{\gamma_{2k}} \omega_1 & \dots & \int_{\gamma_{2k}} \omega_k & 0 \\ \int_{x_0}^{f(x_0)} \omega_1 & \dots & \int_{x_0}^{f(x_0)} \omega_k & 1 \\ \int_{x_0}^{g(x_0)} \omega_1 & \dots & \int_{x_0}^{g(x_0)} \omega_k & \tau \end{pmatrix} = \begin{pmatrix} \Omega_{\mathrm{M}} & 0 \\ J(f(x_0)) & 1 \\ J(g(x_0)) & \tau \end{pmatrix}.$$

□

The result is the following:

Theorem 5.3.16. *Let* N *be a projective compact manifold and* $f, g \in Aut_{\mathbb{C}}(\mathrm{N})$ *such that* $f \circ g = g \circ f$ *and let* X *be the suspension* $\mathrm{N} \times \mathbb{C}/\langle F, G \rangle$ *where* $F(x, z) = (f(x), z + 1)$, $G(x, z) = (g(x), z + \tau)$ *and* $\mathrm{Im}(\tau) \neq 0$. *Assume that* X *is Kähler. Then the following conditions are equivalent:*

- **(i)** X *is projective.*
- **(ii)** Alb(X) *is projective.*

Corollary 5.3.17. *Let* N *be a projective compact manifold and* $f, g \in Aut_{\mathbb{C}}(\mathrm{N})$ *such that* $f \circ g = g \circ f$ *and let* X *be the suspension* $\mathrm{N} \times \mathbb{C}/\langle F, G \rangle$ *where* $F(x, z) = (f(x), z + 1)$, $G(x, z) = (g(x), z + \tau)$ *and* $\mathrm{Im}(\tau) \neq 0$. *Assume that* X *is Kähler and that there exists* $x_0 \in \mathrm{N}$ *such that* $x_0 = f(x_0) = g(x_0)$. *Then* X *is projective.*

Proof. It is enough to use the computation in the previous proposition with base point x_0 to deduce that $\mathrm{Alb(X)} = \mathrm{Alb(N)} \times \mathrm{E}$, thus it is projective. □

Example 5.3.18. If $\mathrm{M} = \mathbb{P}^1$ then $f, g \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^1)$ commute if and only if they have a common fixed point $x_0 \in \mathbb{P}^1$, therefore every suspension of $\mathbb{P}^1$ is projective.

The previous theorem is based on the following result by A.Blanchard, also in [Bla56], and on the lemma below:

Theorem 5.3.19 (Blanchard)**.** *Let* X *be a fibred complex compact analytic space with base* B*, connected fibre* F *and connected structural group. Assume that* X *is Kähler. Then* X *is projective if and only if the following conditions hold:*

(i) F *and* B *are projective.*

(ii) Alb(X) *is projective.*

Note that we cannot apply Blanchard's theorem directly to our situation because our structural group is connected if and only if $f, g \in \mathrm{Aut}_0(\mathrm{N})$. The following lemma is well-known but we include the proof for the sake of clarity.

Lemma 5.3.20. *Let* N *be a projective compact manifold and* $f, g \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{N})$ *so that there exist* $n, m \geq 0$ *such that* $f^n, g^m \in \mathrm{Aut}_0(\mathrm{N})$*. Let* $\widetilde{\mathrm{X}}$ *be the finite covering of* $n \cdot m$ *leaves of* X *obtained as the suspension*

$$\widetilde{\mathrm{X}} = \mathrm{N} \times \mathbb{C}/\langle F^n, G^m \rangle$$

where $F^n(x, z) = (f^n(x), z + n)$ *and* $G^m(x, z) = (g^m(x), z + m\tau)$*. Then* X *is projective if and only if* $\widetilde{\mathrm{X}}$ *is projective.*

Proof. We will use Kodaira embedding theorem for compact Hodge manifolds. It is also worth recalling that if $p : \widetilde{\mathrm{X}} \to \mathrm{X}$ is a finite covering map of $n \cdot m$ sheets then there is a transfer map $p! : H^*(\widetilde{\mathrm{X}}) \to H^*(\mathrm{X})$ such that $p! \circ p^* = n \cdot m \cdot \mathrm{id}$ and $p^* \circ p! = n \cdot m \cdot \mathrm{id}$ on $\mathrm{Im}(p^*)$ defined by

$$p![\omega] := \sum_{i=0,\ldots,n-1; j=0,\ldots,m-1} [(f^i)^* \circ (g^j)^* \omega],$$

see [Bre97]. The direct implication of the lemma is clear since the pull-back of a Kähler integer form on X is an integer cohomology class. If Φ is a Kähler form in $\widetilde{\mathrm{X}}$ such that $[\Phi] \in H^{1,1}(\widetilde{\mathrm{X}}) \cap H^2(\widetilde{\mathrm{X}}, \mathbb{Z})$ it induces a Kähler form Ψ on $\mathrm{N} \times \mathbb{C}$ in the same way as in lemma 5.3.9, namely $[\Psi] = p![\Phi]$, and by construction $[\Psi] \in H^{1,1}(\mathrm{N} \times \mathbb{C}) \cap H^2(\mathrm{N} \times \mathbb{C}, \mathbb{Z})$. Then the class that Ψ induces, up to a integer positive constant, is in $H^2(\mathrm{X}, \mathbb{Z})$. □

Proof. (Theorem 5.3.16) (**i**) $\Longrightarrow$ (**ii**): Since X is projective it is also Kähler. Then there exists $n, m > 0$ such that $f^n, g^m \in \mathrm{Aut}_0(\mathrm{N})$. The suspension $\widetilde{\mathrm{X}}$ of N by

f^n, g^m is projective if and only if the suspension X of N by f, g is projective, by the lemma above. Then by Blanchard's theorem applied to $\widetilde{\mathrm{X}}$ we conclude that $\mathrm{Alb}(\widetilde{\mathrm{X}})$ is projective. Finally, by the above lemma we can also see that Alb(X) is projective if and only if $\mathrm{Alb}(\widetilde{\mathrm{X}})$ is projective.

(ii) $\Longrightarrow$ **(i)**: It is enough to show that a double suspension of a compact projective manifold N by $f, g \in \mathrm{Aut}_0(\mathrm{N})$ is projective, by the above lemma. This follows from the theorem by A.Blanchard stated above. □

5.3.2 Suspensions of manifolds in the class $\mathcal{T}$

Let M^{2n+1} be a compact manifold in the class $\mathcal{T}$ and let $\mathcal{F}$ be the transversely holomorphic flow induced by the vector field T. We denote by ω the 1-form such that $\omega(T) = 1$ and $i_T d\omega = 0$ corresponding to the normal almost contact structure on M. Fixed $\tau \in \mathbb{C}\backslash\mathbb{R}$ and given $f \in \mathrm{Aut}_{\mathcal{T}}(\mathrm{M})$ we denote by X the suspension $\mathrm{M} \times_f \mathbb{R}$ endowed with the complex structure described in proposition 4.4.2. Let v be the holomorphic vector field on X induced by $T - \tau\frac{\partial}{\partial s}$ and $\mathcal{F}_v$ the holomorphic flow on X defined by v.

Let $\Omega^p_f(\mathrm{M}/\mathcal{F})$ be the vector space of holomorphic closed basic p-forms β on M with respect to the transversal holomorphic structure such that $f^*\beta = \beta$. Let $H^{p,q}_f(\mathrm{M}/\mathcal{F})$ be the Dolbeaut cohomology groups of basic forms fixed by f. We denote by $\Omega^p(\mathrm{X}/\mathcal{F}_v)$ the holomorphic p-forms on X basic with respect to $\mathcal{F}_v$.

Theorem 5.3.21. *With the above notation, if* X *is Kähler then:*

(i) $e_{\mathcal{F}}(\mathrm{M}) = 0$. *Moreover* $[d\omega] = 0$ *in* $H^{1,1}_f(\mathrm{M}/\mathcal{F})$.

(ii) *There exists a basic* $(1,0)$*-form* ν *on* M *such that* $d\omega = d\nu$ *and* $f^*\nu = \nu$. *Moreover*

$$H^1(\mathrm{X}, \mathbb{C}) \cong \Omega^1_f(\mathrm{M}/\mathcal{F}) \oplus \overline{\Omega^1_f(\mathrm{M}/\mathcal{F})} \oplus \langle[\omega - \nu]\rangle \oplus \langle[ds]\rangle.$$

(iii) *There exists a closed positive form* Φ *on* M *such that* $\Phi_{|\ker\omega}$ *is of type* $(1,1)$ *with respect to the holomorphic transverse structure and* $[f^*\Phi] = [\Phi]$. *In particular, if* $d\omega = 0$ *then* $\Phi_{|\ker\omega}$ *is a Kähler form on the leaves* V *of the foliation defined by* $\ker\omega$ *on* M.

Proof. **(i),(ii)** Since v is a holomorphic vector field without zeros there is a holomorphic 1-form α such that $\alpha(v) = 1$ and $d\alpha = 0$. In local coordinates the form α has an expression of the type

$$\alpha = \frac{\mathrm{i}}{2\operatorname{Im}\tau}(h(x,s)ds + \bar{\tau}\widetilde{\omega}) = \frac{\mathrm{i}}{2\operatorname{Im}\tau}(h(x,s)ds + \bar{\tau}\sum a_i(x,s)dx_i)$$

where x are real coordinates on M and $a_i(x,s)$ and $h(x,s)$ are complex-valued functions. Since $\alpha(v) = 1$ and $\alpha(\bar{v}) = 0$ it follows that $h = \widetilde{\omega}(T) = 1$. Therefore $\widetilde{\omega}$ is closed and it does not depend on s (because $d\alpha = 0$), i.e. $\widetilde{\omega}$ is a closed form on M and $f^*\widetilde{\omega} = \widetilde{\omega}$. Set $\chi = \frac{i}{2\operatorname{Im}\tau}(ds + \bar{\tau}\omega)$, since χ is of type $(1,0)$ there exists a $(1,0)$-form μ in X such that $\alpha = \chi + \mu$, therefore $\widetilde{\omega} = \omega - \frac{2\mathrm{i}\operatorname{Im}\tau}{\bar{\tau}}\mu$. The form $\nu = \frac{2\mathrm{i}\operatorname{Im}\tau}{\bar{\tau}}\mu$ is a basic $(1,0)$-form such that $d\nu = d\omega$ and $f^*\nu = \nu$. By Hodge decomposition theorem we know that

$$H^1(\mathrm{X},\mathbb{C}) \cong H^0(\mathrm{X},\Omega^1) \oplus \overline{H^0(\mathrm{X},\Omega^1)}.$$

We can assume that there is a basis of $H^0(\mathrm{X},\Omega^1)$ of closed holomorphic forms $\alpha, \alpha_1, ..., \alpha_k$ such that $\alpha_j(v) = 0$ for $j = 1...k$. One can easily prove that the forms $\alpha_1, ..., \alpha_k$ can be taken as forms on M basic with respect to the flow $\mathcal{F}$ and such that $f^*\alpha_j = \alpha_j$ for $j = 1, ..., k$ (the last assertion follows from the fact that (f, id) induces a holomorphic automorphism on X). It follows that

$$H^1(\mathrm{X},\mathbb{C}) \cong \Omega^1_f(\mathrm{M}/\mathcal{F}) \oplus \overline{\Omega^1_f(\mathrm{M}/\mathcal{F})} \oplus \langle [ds] \rangle \oplus \langle [\omega - \nu] \rangle.$$

(iii) The same argument than in the case of the double suspension yields the result. Choose a Kähler form Ψ_0 on X and the pullback Ψ to $\mathrm{M} \times \mathbb{R}$. The restriction $\Phi = \Psi(x, s_0)_{|\mathrm{M}}$ is the desired form. □

Corollary 5.3.22. *If* X *is Kähler then*

$$\alpha := \frac{\mathrm{i}}{2\operatorname{Im}\tau}\big(ds + \bar{\tau}(\omega - \nu)\big)$$

is a closed holomorphic 1-form on X *such that* $\alpha(v) = 1$.

Recall that the form $\chi = \frac{\mathrm{i}}{2\operatorname{Im}\tau}\big(ds + \bar{\tau}\omega\big)$ is always of type $(1,0)$ but it might be neither closed nor holomorphic.

Proposition 5.3.23. *If* X *is Kähler then*

- **(i)** $H^{p,0}(\mathrm{X}) \cong \Omega^p_f(\mathrm{M}/\mathcal{F}) \oplus \Omega^{p-1}_f(\mathrm{M}/\mathcal{F}) \wedge [\alpha]$.
- **(ii)** $\Omega^p(\mathrm{X}/\mathcal{F}_v) \cong \Omega^p_f(\mathrm{M}/\mathcal{F})$.

Proof. Let ζ be a closed $(p,0)$-form on X, in particular ζ is holomorphic. Since $i_v\zeta$ is a holomorphic $(p-1)$-form on X we have $di_v\zeta = 0$. It follows that $\zeta = \alpha \wedge i_v\zeta + \zeta_b$ where ζ_b is a closed $(p,0)$-form basic with respect to the real flows induced by T and $\frac{\partial}{\partial s}$ (since we also have $i_{\bar{v}}\zeta = 0$). Moreover, since $\zeta_b, i_v\zeta$ are closed and basic they do not depend on s, thus $\zeta_b, i_v\zeta \in \Omega^{*,0}(\mathrm{M}/\mathcal{F})$. Since a holomorphic form on a compact Kähler manifold is closed and never exact it follows that

$$H^{p,0}(\mathrm{X}) \cong \Omega^p_f(\mathrm{M}/\mathcal{F}) \oplus \Omega^{p-1}_f(\mathrm{M}/\mathcal{F}) \wedge [\alpha].$$

There is a natural inclusion of $\Omega^p_f(\mathrm{M}/\mathcal{F})$ into $\Omega^p(\mathrm{X}/\mathcal{F}_v)$. On the other hand, given $\zeta \in \Omega^p(\mathrm{X}/\mathcal{F}_v)$ it can be regarded as a holomorphic p-form on X, thus $(f, \mathrm{id})^*\zeta = f^*\zeta = \zeta$. The same argument as before proves that $\zeta \in \Omega^{*,0}(\mathrm{M}/\mathcal{F})$, thus we can conclude. □

When the flow $\mathcal{F}$ on the compact manifold M is isometric we can give more specific conditions for the suspension X to be Kähler:

Proposition 5.3.24. *Assume that the flow $\mathcal{F}$ induced by the vector field T on* M *is isometric. If* X *is Kähler then the following conditions hold:*

- **(i)** *The Euler class $e_{\mathcal{F}}(\mathrm{M})$ is zero.*
- **(ii)** *The flow $\mathcal{F}$ is transverselly Kähler and there exists a basic Kähler form Φ such that $[f^*\Phi] = [\Phi] \in H^{1,1}(\mathrm{M}/\mathcal{F})$.*

Proof. Applying theorem 5.1.1 we conclude that $e_{\mathcal{F}}(\mathrm{M}) = 0$. Choose now a Kähler form Ψ_0 on X and define Ψ as the pullback to $\mathrm{M} \times \mathbb{R}$. We define $\Phi_0 = \Psi(x, s_0)_{|\mathrm{M}}$ so that $[f_*\Phi_0] = [\Phi_0]$ and Φ_0 is a closed form of type $(1,1)$ with respect to the holomorphic transverse structure. We denote by φ_t the 1-parameter group on M associated to T and by H the closure in Isom(M) of the abelian group generated by φ_t. The form Φ defined as the basic part of

$$\int_{\mathrm{H}} \Phi_0(\sigma_*\cdot, \sigma_*\cdot),$$

where we integrate with respect to the Haar mesure on H, is a basic Kähler form on M. Moreover since $f^*\Phi_0 = \Phi_0 + d\alpha$ where α is a 1-form on M, the form $f^*\Phi$ is the basic part of

$$\begin{aligned} \int_{\mathrm{H}} f^*(\Phi_0(\sigma_*\cdot, \sigma_*\cdot)) &= \int_{\mathrm{H}} (f^*\Phi_0)(\sigma_*\cdot, \sigma_*\cdot) = \int_{\mathrm{H}} \Phi_0(\sigma_*\cdot, \sigma_*\cdot) + \int_{\mathrm{H}} d\alpha(\sigma_*\cdot, \sigma_*\cdot) \\ &= \int_{\mathrm{H}} \Phi_0(\sigma_*\cdot, \sigma_*\cdot) + d\int_{\mathrm{H}} \alpha(\sigma_*\cdot) \end{aligned}$$

(see [GHV72] and [GHV73] and note that $f_*T = T$ so $f \circ \sigma = \sigma \circ f$). Therefore $[f^*\Phi] = [\Phi]$. $\square$

Let M^{2n+1} be a compact manifold in the class $\mathcal{T}$ and $\mathcal{F}$ the transversely holomorphic flow induced by the vector field T. We denote by ω the 1-form such that $\omega(T) = 1$ and $i_T d\omega = 0$ corresponding to the normal almost contact structure on M.

Definition 5.3.25. We say that a compact connected manifold M with a fixed normal almost contact structure belongs to the *class* $\mathcal{T}_0$ if and only if its CR-structure is Levi-flat.

Lemma 5.3.26. *A compact connected manifold* M *with a fixed normal almost contact structure belongs to the class* $\mathcal{T}_0$ *if and only if* $d\omega = 0$.

Proof. Recall that a CR-structure is Levi-flat if and only if $\omega \wedge d\omega = 0$. Since $d\omega$ is basic for the foliation induced by T on M this condition is equivalent to $d\omega = 0$. Therefore the manifolds in class $\mathcal{T}_0$ are exactly the manifolds in class $\mathcal{T}$ with a Levi-flat CR-structure $\Phi^{1,0}$. $\square$

If M is a compact connected manifold in the class $\mathcal{T}_0$ and X is the suspension of M by $f \in \mathrm{Aut}_{\mathcal{T}}(\mathrm{M})$ the $(1,0)$-form χ induced by $\frac{\mathrm{i}}{2Im\tau}(ds + \bar{\tau}\omega)$ on X is closed, in particular χ is holomorphic. Moreover if M belongs to the class $\mathcal{T}_0$ then $\ker\omega$ defines a foliation $\widetilde{\mathcal{F}}$ on X transverse to the vector field T such that every leaf is biholomorphic to the same complex (possibly non compact) manifold V. The biholomorphism between the leaves is given by the 1-parameter flow φ_t associated to T, since $\varphi_t \in \mathrm{Aut}_{CR}(\mathrm{M})$.

The *suspension* M of a compact complex manifold N by $g \in Aut_{\mathbb{C}}(\mathrm{N})$ with the normal almost contact structure described in proposition 2.4.2 belongs to the class $\mathcal{T}_0$. Conversely, one has the following result:

Theorem 5.3.27. *Let* M^{2n+1} *be a manifold in class* $\mathcal{T}_0$ *with CR-structure* $\Phi^{1,0}$ *and vector field* T. *If* ω *has group of periods* $\Gamma_\omega \cong \mathbb{Z}$ *then the CR-structure* $\Phi^{1,0}$ *and the vector field* T *can be obtained as in lemma 2.4.2 as a suspension of the compact manifold* V *by* $g \in \mathrm{Aut}(\mathrm{V})$.

Proof. Assume that ω has periods group $\Gamma_\omega \cong \mathbb{Z}$, then there is a well defined fibration
$$\pi : \mathrm{M} \to S^1 \cong \mathbb{R}/\Gamma_\omega, \quad x \mapsto \int_{x_0}^{x} \omega$$

with compact fibers isomorphic to V which are the leaves of the foliation induced by $\ker\omega$. The compact leaf V carries a complex structure induced by the CR-structure $\Phi^{1,0}$ on M. The automorphism $g \in \mathrm{Aut}(\mathrm{V})$ corresponding to the suspension giving rise to M is induced by φ_{t_0}, where t_0 verifies $\varphi_t(x_0) \notin V$ for $0 < t < t_0$ and $\varphi_{t_0}(x_0) \in V$ for every $x_0 \in V$. The vector field T is transverse to the leaves of the fibration and $\omega = \pi^*(dt)$, where t denotes the real coordinate on S^1. The choice of g and the hypothesis $\omega(T) = 1$ imply that T is the vector field induced by $\frac{\partial}{\partial t}$. □

Theorem 5.3.28. *Let* M^{2n+1} *be a manifold in the class* $\mathcal{T}_0$ *with CR-structure* $\Phi^{1,0}$ *and vector field* T*. Assume that the flow* $\mathcal{F}$ *induced by the vector field* T *on* M *is isometric and that* $f^* = \mathrm{id}$ *acting on* $H^1(\mathrm{M}, \mathbb{C})$*. Then* X *is Kähler if and only there exists a basic Kähler form* Φ *for the flow* $\mathcal{F}$ *such that* $[f^*\Phi] = [\Phi] \in H^{1,1}(\mathrm{M}/\mathcal{F})$*.*

Lemma 5.3.29. *Let* M *a compact manifold with a transversely Kählerian flow* $\mathcal{F}$ *depending smoothly on a real parameter* $s \in \mathbb{R}$*. Let* ω *be an exact basic form on* M *of type* (p, q) *such that* $\omega = d_{\mathrm{M}}\alpha$ *for a basic form* α *depending smoothly on* s*. Then there exists a basic* $(p-1, q)$*-form* μ *and a basic* $(p, q-1)$*-form* ν *that depend smoothly on* s *and such that* $\omega = d_{\mathrm{M}}\mu = d_{\mathrm{M}}\nu$*.*

Lemma 5.3.30. *Let* M *a compact manifold with a transversely Kählerian flow* $\mathcal{F}$ *depending smoothly on a real parameter* $s \in \mathbb{R}$*. Let* ω *be an exact basic form on* M *such that* $\omega = d_{\mathrm{M}}\alpha$ *for a basic form* α *depending smoothly on* s*. Assume that* $\omega = \omega_1 + \omega_2$ *where* ω_1 *is of type* $(p+1, q)$ *and* ω_2 *is of type* $(p, q+1)$*, both basic and depending smoothly on* s*. There exists a basic* (p, q)*-form* ν *such that* $\omega = d_{\mathrm{M}}\nu$ *and it depends smoothly on* s*.*

The proofs of the previous lemmas are analogous to the ones of the lemmas 5.3.12 and 5.3.13 using the fact that on a transversely Kähler isometric flow there exists a transversal Hodge theory regarding basic forms and in particular a basic $\partial\bar{\partial}$-lemma (see section 1.6). Note that locally every suspension is a product, thus we can consider the exterior derivative d_s with respect to the local coordinate s and then $d = d_{\mathrm{M}} + d_s$.

Proof. $\Longrightarrow$) : It follows from proposition 5.3.24.
$\Longleftarrow$) : We define α as the (1,0)-form induced by $\frac{\mathrm{i}}{2\,\mathrm{Im}\,\tau}(ds + \bar{\tau}\omega)$, which is a closed holomorphic 1-form on X such that $\alpha(v) = 1$. Besides, from Gysin's

exact sequence it follows that

$$H^1(\mathrm{M}) \cong H^1(\mathrm{M}/\mathcal{F}) \oplus \langle[\omega]\rangle.$$

From the hypothesis $f^* = \mathrm{id}$ acting on $H^1(\mathrm{X})$ and lemma 5.3.1, it follows that $H^1(\mathrm{X}) \cong H^1(\mathrm{M}) \oplus \langle[ds]\rangle$. Since the flow $\mathcal{F}$ is transversely Kählerian on M by Hodge theory

$$H^1(\mathrm{M}/\mathcal{F}) \cong H^{1,0}(\mathrm{M}/\mathcal{F}) \oplus H^{0,1}(\mathrm{M}/\mathcal{F})$$

and there exists a basis $\alpha_1, ..., \alpha_k$ of $H^{1,0}(\mathrm{M}/\mathcal{F})$ of closed (1,0)-forms (and therefore holomorphic) on M such that $f^*\alpha_j = \alpha_j$ for $j = 1, ..., k$ (for $f^* = \mathrm{id}$ on $H^1(\mathrm{M})$ and the representatives of cohomology classes of type (1,0) are unique as a consequence of the transversal Hodge theory). Note also that $H^{1,0}(\mathrm{X}/\mathcal{F}_v) \cong H^{1,0}(\mathrm{M}/\mathcal{F})$ since a basic 1-form on X can not depend on s and therefore lives on M. Fix the transverse Kähler form Φ on M. We can choose an open covering $\{U_i\}$ of S^1 so that the fibration $p : \mathrm{X} \to S^1$ is trivial over U_i. Then Φ induces a well-defined form Φ_i on $p^{-1}(U_i) \cong U_i \times \mathrm{M}$. Let $\{\rho_i\}$ be a unit partition associated to $\{U_i\}$. Then $\Phi_0 = \sum_i \rho_i(s)\Phi_i$ is a real global $(1,1)$-form on X so that $\Phi_{0|\mathrm{M}}$ is a transverse Kähler form on M representing a fixed cohomology class, since $[f^*\Phi] = [\Phi]$. Moreover $d\Phi_0 = d_s\Phi_0$. Now we want to obtain a closed real valued $(1,1)$-form $\widetilde{\Phi}$ on X such that $\widetilde{\Phi}_{|\mathrm{M}} = \Phi_0$. We search $\widetilde{\Phi}$ of the type:

$$\widetilde{\Phi} = \Phi_0 + H \wedge \bar{\alpha} + \overline{H} \wedge \alpha + \mathrm{i}F\alpha \wedge \bar{\alpha},$$

where H is a basic $(1,0)$-form on X and F a real valued function on X. The hypothesis $d\widetilde{\Phi} = 0$ is then equivalent to the following equations:

$$\left\{\begin{array}{rcll} d_s\Phi_0 - \frac{1}{2\mathrm{i}\,\mathrm{Im}\,\tau}(-d_{\mathrm{M}}H \wedge ds + d_{\mathrm{M}}\overline{H} \wedge ds) & = & 0 & \text{(I)} \\ -\tau d_{\mathrm{M}}H \wedge \omega + \bar{\tau} d_{\mathrm{M}}\overline{H} \wedge \omega & = & 0 & \text{(II)} \\ \frac{1}{2\mathrm{i}\,\mathrm{Im}\,\tau}(-\tau d_sH \wedge \omega + \bar{\tau} d_s\overline{H} \wedge \omega) - idF\alpha \wedge \bar{\alpha} & = & 0. & \text{(III)} \end{array}\right.$$

Note that

$$\mathrm{i}dF\alpha \wedge \bar{\alpha} = \frac{-dF}{2\,\mathrm{Im}\,\tau} ds \wedge \omega.$$

Obtaining the form $\widetilde{\Phi}$ is equivalent to finding H and F that solve the previous system. Roughly speaking, we will first solve (I) and (II) so that we determine H and then define F as the solution of (III). The equation (I) is equivalent to

$$2\mathrm{i}\,\mathrm{Im}\,\tau\, i_{\frac{\partial}{\partial s}} d_s\Phi_0 = d_{\mathrm{M}}\overline{H} - d_{\mathrm{M}}H = d_{\mathrm{M}}(\overline{H} - H)$$

which is fulfilled if

$$d_{\mathrm{M}}H = \bar{\tau} i_{\frac{\partial}{\partial s}} d_s \Phi_0 \qquad \text{(IV)}.$$

The form $\gamma := \bar{\tau} i_{\frac{\partial}{\partial s}} d_s \Phi_0$ can be seen locally as a basic form on M of type (1,1) that depend smoothly on the real parameter s. To apply lemma 5.3.29 to obtain local basic $(1,0)$-forms H_i on $p^{-1}(U_i) \cong U_i \times \mathrm{M}$ which solve (IV) we must assure that γ is d_{M}-exact. Note that

$$d_{\mathrm{M}}\gamma = \bar{\tau} i_{\frac{\partial}{\partial s}} d_s d_{\mathrm{M}} \Phi_0 = 0.$$

To prove the exactness it is enough to see that

$$\int_C \gamma = \bar{\tau} i_{\frac{\partial}{\partial s}} d_s \int_C \Phi_0 = 0$$

for every cycle C on M. This holds because Φ_0 represents the same cohomology class on every fibre so $\int_C \Phi_0$ does not depend on s. Therefore there exist 1-forms $\{\beta_i\}$ on M depending smoothly on s such that $\gamma = d_{\mathrm{M}}\beta_j$. We denote by φ_t the 1-parameter group associated to T and by L the closure of the abelian group generated by φ_t. Define $\tilde{\beta}_j$ as the transverse part of $\int_{\mathrm{L}} \beta_j(\sigma\cdot)$ (integrating with respect to the Haar measure on L). Then $\tilde{\beta}_j$ is a transverse 1-form on X which depends smoothly on s and such that

$$d_{\mathrm{M}}\tilde{\beta}_j = d_{\mathrm{M}} \int_{\mathrm{L}} \sigma^* \beta_j = \int_{\mathrm{L}} \sigma^* d_{\mathrm{M}} \beta_j = \int_{\mathrm{L}} \sigma^* \gamma = \gamma$$

(recall that γ is basic, so $\int_{\mathrm{L}} \sigma^* \gamma = \gamma$). Therefore there exist basic local solutions $\{H_i\}$ to (IV) which depends smoothly on s. Using a unit partition as above we obtain a global solution $H_0 = \sum_i \rho(s) H_i$. Note that by construction of H it verifies $\tau d_{\mathrm{M}} H = \bar{\tau} d_{\mathrm{M}} \overline{H}$, therefore it is also a solution of (II). Now, to proceed with our plan we should define F as the solution of (III) for the previous H_0. Instead of finding a solution of (III) we will solve the following equation:

$$d_{\mathrm{M}}F = \mathrm{i}\big(\tau i_{\frac{\partial}{\partial s}} d_s H - \bar{\tau} i_{\frac{\partial}{\partial s}} d_s \overline{H}\big) = \nu \qquad \text{(V)}.$$

Note that the term on the right ν is a basic real form on M depending smoothly on s which is the sum of a form of type $(1,0)$ and a form of type $(0,1)$, therefore we can try to apply lemma 5.3.30. Moreover ν is d_{M}-closed, for

$$d_{\mathrm{M}}\nu = \mathrm{i} \cdot i_{\frac{\partial}{\partial s}} d_s (\tau d_{\mathrm{M}} H - \bar{\tau} d_{\mathrm{M}} \overline{H}) = 0$$

Nevertheless, here we encounter a difficulty, since we cannot prove that ν_0, for the previous solution H_0, is d_{M}-exact. To overcome it we will modify H_0 to obtain a new solution of (IV) for which the corresponding ν in (V) is d_{M}-exact. Consider the basis $\{\alpha_1, ..., \alpha_k, \overline{\alpha}_1, ..., \overline{\alpha}_k\}$ of $H^{1,0}(\mathrm{M}/\mathcal{F}) \oplus H^{0,1}(\mathrm{M}/\mathcal{F})$ of forms fixed by f so that they are well-defined closed forms on X (note that $\alpha_1, ..., \alpha_k$ are holomorphic). Let $\{\gamma_1, ..., \gamma_{k+1}, \overline{\gamma}_1, ..., \overline{\gamma}_{k+1}\}$ be the basis of $H_1(\mathrm{M}, \mathbb{C})$ dual to $\{\alpha_1, ..., \alpha_k, \alpha, \overline{\alpha}_1, ..., \overline{\alpha}_k, \overline{\alpha}\}$. We define

$$u_j = \mathrm{i} \int_{\gamma_j} \big(\tau d_s H - \bar{\tau} d_s \overline{H}\big) = \mathrm{i} \cdot d_s \int_{\gamma_j} (\tau H - \bar{\tau} \overline{H})$$

for $1 \leq j \leq k$. Notice that

$$\bar{u}_j = \mathrm{i} \cdot d_s \int_{\overline{\gamma}_j} (\tau H - \bar{\tau} \overline{H})$$

It is not difficult to see that they are d_s-closed and exact real 1-forms on S^1 (the base of the natural fibration). There exists a family $\{v_1, ..., v_k\}$ of real functions on S^1 (that we extend by pullback to X) such that $d_s v_j = u_j$, namely

$$v_j = \mathrm{i} \int_{\gamma_j} (\tau H - \bar{\tau} \overline{H}).$$

We can define now a new solution H of (IV), and therefore a solution of (I) and (II), by the formula

$$H = H_0 + \mathrm{i} \cdot \tau^{-1} \sum_{j=1}^{k} \alpha_j \cdot v_j$$

so that

$$\nu ds = \mathrm{i}\big(\tau d_s H_0 - \bar{\tau} d_s \overline{H}_0\big) + \sum_{j=1}^{k} u_j \wedge \alpha_j - \sum_{j=1}^{k} \bar{u}_j \wedge \overline{\alpha}_j.$$

We will next verify that the integral of $\nu ds = \mathrm{i}\big(\tau d_s H - \bar{\tau} d_s \overline{H}\big)$ is zero for any cycle C on M, for it is equivalent to $\int_C \nu = 0$. Note that since we saw that ν is d_{M}-closed when H is a solution of (IV) it is enough to check that $\int_{\gamma_j} \nu ds = \int_{\bar{\gamma}_j} \nu ds = 0$ (note that if $\nu = d_{\mathrm{M}} G$ the function G must be basic). Indeed,

$$\begin{aligned} \int_{\gamma_j} \nu ds &= \int_{\gamma_j} \nu_0 ds - d_s v_j = u_j - u_j = 0 \\ \int_{\bar{\gamma}_j} \nu ds &= \int_{\bar{\gamma}_j} \nu_0 ds - d_s \bar{v}_j = \bar{u}_j - \bar{u}_j = 0. \end{aligned}$$

Therefore, we are left to solve $d_{\mathrm{M}}F = \nu$ with ν satisfying all the hypothesis in lemma 5.3.29 to obtain local real functions F_i such that $d_{\mathrm{M}}F_i = \nu$. We define thus F by means of a unit partition, $F = \sum_i \rho_i F_i$. To finish the proof it is enough to add a positive constant to F so that we obtain a positive closed $(1,1)$-form on X. □

Let β be a closed 1-form on M such that $\beta(T) = 1$ and $f^*\beta = \beta$. Consider on M the operator $d_{\mathcal{T},\beta}$ acting on the complex

$$\Omega^*_{\mathcal{T},f}(\mathrm{M},\mathbb{C}) = \{\sigma \in \Omega^*(\mathrm{M}) : \, i_T\sigma = 0, \, f^*\sigma = \sigma\}$$

by the following formula:

$$d_{\mathcal{T},\beta} : \Omega^p_{\mathcal{T},f}(\mathrm{M},\mathbb{C}) \to \Omega^{p+1}_{\mathcal{T},f}(\mathrm{M},\mathbb{C})$$
$$\alpha \mapsto d\alpha - \beta \wedge (i_T d\alpha).$$

Choose $\alpha \in \Omega^p_f(\mathrm{M},\mathbb{C})$, then

$$d^2_{\mathcal{T},\beta}(\alpha) = d_{\mathcal{T},\beta}(d\alpha - \beta \wedge (i_T d\alpha)) = \beta \wedge d \circ i_T \circ d\alpha - \beta \wedge d \circ i_T \circ d\alpha = 0.$$

Consequently we can consider the cohomology groups corresponding to $d_{\mathcal{T},\beta}$, that we will denote by $H^*_{\mathcal{T},\beta}(\mathrm{M},\mathbb{C})$. If one considers the complex $\Omega^k_f(\mathrm{M})$ of k-forms on M fixed by f the maps

$$i : \Omega^k_{\mathcal{T},f}(\mathrm{M},\mathbb{C}) \to \Omega^k_f(\mathrm{M})/\langle\beta\rangle; \qquad \chi : \Omega^k_f(\mathrm{M})/\langle\beta\rangle \to \Omega^*_{\mathcal{T},f}(\mathrm{M},\mathbb{C}),$$

where i is the natural projection and $\chi[\sigma] = \sigma - \beta \wedge i_T\sigma$, induce an isomorphism between $H^*_{\mathcal{T},\beta}(\mathrm{M},\mathbb{C})$ and the cohomology of the complex $\Omega^k_f(\mathrm{M})/\langle\beta\rangle$ together with the restriction of the usual differential operator on $\Omega^*(\mathrm{M})$.

Proposition 5.3.31. *Let* M^{2n+1} *be a manifold in the class* $\mathcal{T}_0$ *with CR-structure* $\Phi^{1,0}$ *on* M. *With the usual notation assume* $f \in \mathrm{Isom}(\mathrm{M})$ *for some Riemannian metric on* M. *Then* $\mathrm{X} = \mathrm{M} \times_f \mathbb{R}$ *is Kahler if and only if there exists a closed form* Φ *on* M *such that* $\Phi_{|\ker\omega}$ *is of type* $(1,1)$ *with respect to the holomorphic transverse structure and positive,* $f^*\Phi = \Phi$ *and the form* $\mu = \tau^{-1}(i_T\Phi)^{1,0} + \bar{\tau}^{-1}(i_T\Phi)^{0,1}$ *verifies* $d_{\mathrm{M}}\mu \wedge \omega = 0$ *and* $0 = [L_T\mu] \in H^1_{\mathcal{T},\omega}(\mathrm{M},\mathbb{C})$.

Remark 5.3.32. Under the above hypothesis the flow induced by $\frac{\partial}{\partial s}$ is an isometric flow with zero Euler class. Recall also that if M is a compact Riemannian manifold then Isom(M) is compact (see [Kob72]).

Proof. $\Rightarrow$) : We choose a Kähler form Ψ_0 on X. We denote by φ_s the 1-parameter group associated to $\frac{\partial}{\partial s}$ and by L the closure in Isom(X) of the abelian group generated by φ_s which is compact for $\frac{\partial}{\partial s}$ generates an isometric flow on X. We define

$$\Psi_1(\cdot,\cdot) = \int_{\mathrm{L}} \Psi_0(\sigma^*\cdot, \sigma^*\cdot)$$

where we integrate with respect to the Haar measure on L. The $(1,1)$-form Ψ_1 is closed and positive on X, therefore it is a Kähler form on X. Moreover, when expressed in local coordinates its coefficients do not depend on s. Set now $\Psi := \pi^*\Psi_1$ where $\pi : \mathrm{M}\times\mathbb{R} \to \mathrm{X}$ is the natural projection and $\Phi := \Psi_{|\mathrm{M}}$. Note that we need not fix $s_0 \in \mathbb{R}$ since $\Psi_{|\mathrm{M}}$ does not depend on s. The form Φ is closed and $\Phi_{|\ker\omega}$ is of type $(1,1)$ and positive on the leaves V of the foliation associated to $\ker\omega$. Moreover if $(z,\bar z,t)$ are local coordinates on M such that $T = \frac{\partial}{\partial t}$, $z,\bar z$ are coordinates on V, that is, $\mathcal{F} = \{z = \mathit{const}, \bar z = \mathit{const}\}$ and $V = \{t = \mathit{const}\}$, and s is the coordinate on $\mathbb{R}$, the form Ψ has an expression of the type:

$$\begin{aligned}\Psi \quad = \quad & \sum_{j,k=1}^{n} A_{jk}(z,\bar z,t)dz_j\wedge d\bar z_k + \sum_{j=1}^{n} H_j(z,\bar z,t)dz_j \wedge (ds+\tau dt) + \\ & \sum_{j=1}^{n} \overline{H}_j(z,\bar z,t)d\bar z_j\wedge(ds+\bar\tau dt) + F(z,\bar z,t)dt\wedge ds.\end{aligned}$$

We set $A := \sum_{j,k=1}^{n} A_{jk}(z,\bar z,t)dz_j\wedge d\bar z_k$ and $H := \sum_{j=1}^{n} H_j(z,\bar z,t)dz_j\wedge(ds+\tau\omega)$ so that

$$\Psi = A + H\wedge(ds+\tau\omega) + \overline{H}\wedge(ds+\bar\tau\omega) + F(z,\bar z,t)\omega\wedge ds. \qquad (*)$$

Note that $H = \tau^{-1}(i_T\Phi)^{1,0}$ and $\overline{H} = \bar\tau^{-1}(i_T\Phi)^{0,1}$. Note that $d_{\mathrm{M}}A = d_tA$ and $\partial_{\mathrm{M}}H = \bar\partial_{\mathrm{M}}\overline{H} = 0$. Moreover $i_TA = i_{\frac{\partial}{\partial s}}A = 0$. The form Ψ is closed if and only if the following two equations hold:

$$\begin{cases} d_{\mathrm{M}}A + (\tau d_{\mathrm{M}}H + \bar\tau d_{\mathrm{M}}\overline{H})\wedge\omega = 0 & \text{(I)}\\ d_{\mathrm{M}}H + d_{\mathrm{M}}\overline{H} + d_{\mathrm{M}}F\wedge\omega = 0 & \text{(II)}\end{cases}$$

which imply $d_{\mathrm{M}}A + (\tau-\bar\tau)d_{\mathrm{M}}H\wedge\omega = 0$. The form Φ has local expression

$$\Phi = A + (\tau H + \bar\tau\overline{H})\wedge\omega.$$

Since $F^*\Psi = \Psi$ where $F : \mathrm{M}\times\mathbb{R}\to\mathrm{M}\times\mathbb{R}$ is defined as $F(x,s) = (f(x), s+1)$ and the coefficients of Ψ do not depend on s we have $f^*\Phi = \Phi$. The form

$\mu = \tau^{-1}(i_T\Phi)^{1,0}+\bar{\tau}^{-1}(i_T\Phi)^{0,1} = H+\overline{H}$ verifies $d_{\mathrm{M}}\mu = d_{\mathrm{M}}H+d_{\mathrm{M}}\overline{H} = -d_{\mathrm{M}}F\wedge\omega$ so $d_{\mathrm{M}}\mu\wedge\omega = 0$ and $L_T\mu = d_{\mathrm{M}}F - i_T d_{\mathrm{M}}F\wedge\omega = d_{\mathcal{T},\omega}(F)$. Finally, from $F^*\Psi = \Psi$, $f^*\omega = \omega$ and the uniqueness of the decomposition on $(*)$ it follows that $f^*F = F$.
$\Leftarrow)$: The form Φ is a well defined closed form on $\mathrm{X} = \mathrm{M}\times_f \mathbb{R}$. We begin by modifying it so that we obtain a well-defined real $(1,1)$-form on X. Set

$$\Phi_0 := \Phi + \tau^{-1}(i_T\Phi)^{1,0}\wedge ds + \bar{\tau}^{-1}(i_T\Phi)^{0,1}\wedge ds = \Phi + \mu\wedge ds.$$

Recall that $v = T - \tau\frac{\partial}{\partial s}$ is a holomorphic vector field. Moreover we can complete v to a local basis of vector fields of type (1,0) by adding $\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}$ where $z_1, ..., z_n$ are holomorphic coordinates on $\mathrm{V} = \ker\omega$ as before. Then

$$i_{\frac{\partial}{\partial z_j}}\Phi_0 = i_{\frac{\partial}{\partial z_j}}\Phi - \tau^{-1}(i_T \circ i_{\frac{\partial}{\partial z_j}}\Phi)\wedge ds$$

so $i_{\frac{\partial}{\partial z_k}} \circ i_{\frac{\partial}{\partial z_j}}\Phi_0 = 0$ (for Φ is of type (1,1) on V) and

$$i_v \circ i_{\frac{\partial}{\partial z_j}}\Phi_0 = i_T \circ i_{\frac{\partial}{\partial z_j}}\Phi - i_T \circ i_{\frac{\partial}{\partial z_j}}\Phi = 0.$$

The same arguments show that Φ_0 has no part of type $(0,2)$ with respect to the complex structure on X. We would like to complete Φ_0 to a closed $(1,1)$-form $\widetilde{\Phi}$ on X of the form

$$\widetilde{\Phi} := \Phi_0 + iF\alpha\wedge\bar{\alpha}$$

where F is a function on M such that $f^*F = F$. The condition $d\widetilde{\Phi} = 0$ is equivalent to

$$d_{\mathrm{M}}\mu + d\left(\frac{F}{2\,\mathrm{Im}\,\tau}\right)\wedge\omega = 0.$$

Since $d_{\mathrm{M}}\mu\wedge\omega = 0$ by Cartan's lemma $d_{\mathrm{M}}\mu = \beta\wedge\omega$. Therefore $L_T\mu = i_T\beta\wedge\omega - \beta = dG - i_T dG\wedge\omega$ for some function G on M such that $f^*G = G$. Since $\beta + dG = (i_T\beta + i_T dG)\wedge\omega$ we conclude $d_{\mathrm{M}}\mu + dG\wedge\omega = 0$ and it is enough to choose $F := (2\,\mathrm{Im}\,\tau)G + K$ where K is a positive constant. □

Essentially the same argument proves the following:

Proposition 5.3.33. *Let* M^{2n+1} *be a manifold in the class* $\mathcal{T}$ *with CR-structure* $\Phi^{1,0}$ *on* M. *With the usual notation assume* $f \in \mathrm{Isom}(\mathrm{M})$ *for some Riemannian metric on* M. *Then* $\mathrm{X} = \mathrm{M}\times_f\mathbb{R}$ *is Kahler if and only if there exists a closed positive form* Φ *on* M *such that* $\Phi_{|\ker\omega}$ *is of type* $(1,1)$ *with respect to the holomorphic transverse structure,* $f^*\Phi = \Phi$ *and the form* $\mu = \tau^{-1}(i_T\Phi)^{1,0} + \bar{\tau}^{-1}(i_T\Phi)^{0,1}$ *verifies* $d_{\mathrm{M}}\mu = d_{\mathrm{M}}G\wedge\omega + G\wedge d_{\mathrm{M}}\omega$ *for some function* G *on* M *such that* $f^*G = G$.

Chapter 6

Examples of compact complex surfaces

In this chapter we discuss some examples of compact complex surfaces that can be obtained by means of the constructions in chapter 4. Let M^3 be a compact connected manifold in the class $\mathcal{T}$ and let S be a compact complex surface obtained as in proposition 4.3.1 on the total space of a S^1-principal bundle over M^3 (case (B), note that $\mathrm{S} \cong \mathrm{M}^3 \times S^1$, i.e. case (A), is included in this one) or as in proposition 4.4.2, i.e. case (C), on a suspension of M^3 by $f \in \mathrm{Aut}_{\mathcal{T}}(\mathrm{M}^3)$. Since S admits a holomorphic vector field v without zeros, it is well known that S must be minimal, we will include here the proof for the sake of completeness.

Lemma 6.1. *Let* S *be a compact complex surface and* v *a holomorphic vector field on* S *without zeros. Then* S *is minimal.*

Proof. Assume that there exists a holomorphic curve $\mathrm{C} \cong S^2$ such that $(\mathrm{C}, \mathrm{C}) = -1$. Since C does not admit any deformation the vector field v must be tangent to C at every point. Therefore v must have a singularity, which contradicts the hypothesis. □

We recall now the classification of minimal compact complex surfaces admitting a non-trivial holomorphic vector field (see [DOT00] and [DOT01]):

Theorem 6.2. *A compact minimal surface* S *admits a non-trivial holomorphic vector field if and only if it belongs to the following list:*

1. $kod(\mathrm{S}) \geq 0$.

- **(a)** *Complex tori.*
- **(b)** *Principal Seifert fibre bundles over a Riemann surface with fibre an elliptic curve.*

2. *kod*(S) $= -\infty$ *and* S *is Kählerian.*

- **(a)** *Holomorphic fibrations with fibre* $\mathbb{P}^1$ *and structural group* $\mathbb{C}^*$ *over a Riemann surface of genus* $g \geq 1$, $\mathbb{P}^2$ *and the Hizerbruch surfaces* F_n *with* $n = 0, 2, ...$
- **(b)** *Holomorphic fibrations with fibre* $\mathbb{P}^1$ *and solvable connected structural group over a Riemann surface of genus* $g \geq 1$ *such that the associated line bundle has a non-trivial holomorphic section.*

3. *kod*(S) $= -\infty$ *and* S *non-Kählerian.*

- **(a)** *Almost-homogeneous Hopf-surfaces.*
- **(b)** *Inoue surfaces of type* $\mathrm{S}^+_{N,p,q,r,t}$.
- **(c)** *Some particular surfaces of class* VII_0^+.

With the exception of the case **3(c)** the last result has been known for a long time, see also [Miz78] and [CHK73]. A classical argument by Blanchard (see [Ghy96a]) shows that every holomorphic vector field v on a ruled surface preserves the fibration. Thus, either v is tangent to the fibre $\mathbb{P}^1$ or projects over a non trivial vector field of the base. Since $\mathbb{P}^2$ and Riemann surfaces of genus $g \geq 2$ do not admit vector fields without zeros it follows that the only surfaces in case **2** which can admit vector fields without zeros are some ruled surfaces over an elliptic curve. On the other hand it is known that surfaces in cases **3(b)** and **3(c)** only admit vector fields with zeros. Thus, taking into account that v has no zeros, we can conclude that a priori we have the following list of possibilities for S:

- **(I)** Complex tori.
- **(II)** Principal Seifert fibre bundles over a Riemann surface of genus $g \geq 1$ with fiber an elliptic curve (this includes hyperelliptic, Kodaira and properly elliptic surfaces).
- **(III)** Ruled surfaces over an elliptic curve.
- **(IV)** Almost-homogeneous Hopf-surfaces.

Our next goal is to determine which of the previous surfaces are obtained by our constructions for each possible choice of the manifold M^3 in the class $\mathcal{T}$. We will make use essentially of two criteria: whether the resulting complex surface S is Kählerian or not and a study of the analytic (or differential) universal covering of S. Recall that surfaces in the classes **(I)** and **(III)** are always Kählerian whereas those in the class **(IV)** are non-Kählerian. Note also that an elliptic fibre bundle over $\mathbb{P}^1$ is either a product or a Hopf surface (cf. [BPVdV84], p.146), therefore we can impose genus $g \geq 1$ in case **(II)** without any loss of generality. For Seifert principal fibrations over a Riemann surface N_g with $g \geq 1$, by theorem 5.2.1, we know that the total space is Kähler if and only if the Euler classes are zero.

Finally we will make use of the following classical argument:

Lemma 6.3. *The analytic universal covering of a Seifert fibre bundle* S *over a Riemann surface* N *of genus* $g(N) \geq 1$ *with fibre* F *is either* $\mathbb{D} \times \widetilde{F}$ *or* $\mathbb{C} \times \widetilde{F}$ *where* $\widetilde{F}$ *is the analytic universal covering of* F*.*

Proof. Taking a suitable ramified covering $\widetilde{N}$ of N we obtain an analytic fibre bundle $\widetilde{S} \to \widetilde{N}$ with fibre F such that $\widetilde{S}$ is a smooth covering space of S. From the inequality $g(\widetilde{N}) \geq g(N) = 1$ we deduce that the universal covering of $\widetilde{N}$ is $\mathbb{D}$ or $\mathbb{C}$. By pull-back we obtain a fibration $\pi : X \to \mathbb{D}$ or $\pi : X \to \mathbb{C}$ that is analytically trivial since $\mathbb{D}$ and $\mathbb{C}$ are contractible and Stein (by Oka's principle, see [Gra58]), hence $X \cong \mathbb{D} \times \widetilde{F}$ or $X \cong \mathbb{C} \times \widetilde{F}$. Finally, it is clear that X is the analytic universal covering of $\widetilde{S}$, and consequently of S. □

Thus we conclude that the analytic universal covers of the previous surfaces are: $\mathbb{C}^2$ for complex tori, $\mathbb{C}^2$ or $\mathbb{D} \times \mathbb{C}$ for case **(II)**, $\mathbb{P}^1 \times \mathbb{C}$ or $\mathbb{P}^1 \times \mathbb{D}$ for case **(III)** and $\mathbb{C}^2 \backslash \{0\}$ for Hopf surfaces. The previous lemma will also allow us to classify all connected compact 3-manifolds in the class $\mathcal{T}$. We can sum up the information we have gathered up to now in the following table :

Surface	*Kähler*	*Universal covering*
Complex tori	Yes	$\mathbb{C}^2$
Principal elliptic Seifert fibre bundles over a Riemann surface N_g, $g \geq 1$	Iff the Euler classes are zero	$\mathbb{C}^2$ or $\mathbb{D} \times \mathbb{C}$
Ruled surfaces over an elliptic curve	Yes	$\mathbb{P}^1 \times \mathbb{C}$ or $\mathbb{P}^1 \times \mathbb{D}$
Hopf surfaces	No	$\mathbb{C}^2 \backslash \{0\}$

Let M^3 be a compact connected 3-manifold in the class $\mathcal{T}$ for a CR-structure $\Phi^{1,0}$, a CR-action induced by a vector field T and a 1-form ω. We denote by $\mathcal{F}$ the flow induced by the vector field T. Recall that in order to apply the construction of proposition 4.3.1, case (B), we must choose a S^1-fibre bundle over M and a connection 1-form α such that $d\alpha \in \Omega^{1,1}(\mathrm{M}/\mathcal{F})$.

Assume that $\mathcal{F}$ is an isometric flow induced by a Killing vector field T and that ω is a characteristic 1-form of $\mathcal{F}$ on M defining a normal almost contact structure. If the flow $\mathcal{F}$ is isometric then $H^2(\mathrm{M}/\mathcal{F}) \cong \mathbb{C}$ (see section 1.5) and in particular the flow $\mathcal{F}$ is homologically orientable. Furthermore the transverse part of $\mathcal{F}$ has complex dimension 1 and the flow is transversely Kählerian, hence by the Hodge decomposition theorem for transversely holomorphic flows (see section 1.6) one has $H^2(\mathrm{M}/\mathcal{F}) \cong H^{1,1}(\mathrm{M}/\mathcal{F}) \cong \mathbb{C}$. Applying Gysin's exact sequence for isometric flows (see [RP01]) it follows that if $[d\omega] \neq 0$ in $H^2(\mathrm{M}/\mathcal{F})$ then the map $H^2(\mathrm{M}/\mathcal{F}) \to H^2(\mathrm{M})$ is identically trivial. Indeed, one has

$$\ldots H^0(\mathrm{M}/\mathcal{F}) \xrightarrow{\ \delta\ } H^2(\mathrm{M}/\mathcal{F}) \xrightarrow{\ i^*\ } H^2(\mathrm{M}) \xrightarrow{\ \int^*\ } H^1(\mathrm{M}/\mathcal{F}) \ldots$$

where i^* is the map induced by the inclusion, $\int^*$ the integral along the leaves and $\delta[c] = [c \wedge d\omega]$. Since $H^2(\mathrm{M}/\mathcal{F}) \cong \mathbb{C}$, if $[d\omega] \neq 0$ in $H^2(\mathrm{M}/\mathcal{F})$ then $\operatorname{Im}\delta = H^2(\mathrm{M}/\mathcal{F}) = \operatorname{Ker} i^*$. Therefore if the Euler class of $\mathcal{F}$ is not zero, that is, if $[d\omega] \neq 0$ in $H^2(\mathrm{M}/\mathcal{F})$, then the only S^1-principal bundles $p : \mathrm{X} \to \mathrm{M}$ which admit a complex structure by the construction of proposition 4.3.1 are the flat ones. Recall that to apply proposition 4.3.1 the S^1-principal bundle $p : \mathrm{X} \to \mathrm{M}$ must admit a connection form α such that $d\alpha \in p^*\Omega^{1,1}(\mathrm{M}/\mathcal{F})$. Then if $[d\omega] \neq 0$ in $H^2(\mathrm{M}/\mathcal{F})$ the previous discussion shows that $[d\alpha] = 0$ in $H^2(\mathrm{M})$, that is, the fibre bundle $p : \mathrm{X} \to \mathrm{M}$ is flat. If $H^2(\mathrm{M}, \mathbb{Z})$ has no torsion then the fibre bundle is topologically trivial.

Proposition 6.4. *With the notation above, if* S *is a complex surface obtained from* M^3 *as in proposition 4.3.1, case* (B)*, then:*

(a) S *is parallisable.*

(b) *If* $[d\alpha] \neq 0$ *or* $[d\omega] \neq 0$ *in* $H^2(\mathrm{M}/\mathcal{F})$ *then* S *is not Kählerian.*

Proof. **(a)** Recall that every compact orientable 3-manifold is parallisable. Using the connection form α we take a horizontal elevation to S of a basis of global vector fields of M^3. The vector fields that result together with the fundamental

vector field of the S^1-action are a parallism for S. In particular the tangent bundle is topologically trivial.
(b) As the flow associated to a S^1-fibre bundle is always isometric, this is a consequence of proposition 5.1.7. □

On the other hand the classification of transversely holomorphic flows on a compact connected 3-manifold (see [Bru96] and [Ghy96b]) together with the condition of the existence of a CR-structure and a transverse CR-action give a short list of possibilities for M^3 in the class $\mathcal{T}$. Indeed we prove the following:

Theorem 6.5. *Let* M^3 *be a compact connected manifold of dimension 3 in the class* $\mathcal{T}$. *Then, up to diffeomorphism, the manifold* M^3 *and the vector field inducing the CR-action belong to the following list:*

- **(i)** *Seifert fibrations over a Riemann surface with a vector field tangent to the fibres such that the isometric flow of the action admits a characteristic 1-form* ω *such that* $d\omega$ *is of type* $(1,1)$.
- **(ii)** *Linear vector fields in* $\mathbb{T}^3$.
- **(iii)** *Foliations on* S^3 *induced by a singularity of a holomorphic vector field in* $\mathbb{C}^2$ *in the Poincaré domain and their finite quotients, i.e. foliations on the lens spaces* $L_{p,q}$.
- **(iv)** *Suspensions of a holomorphic automorphism of* $\mathbb{P}^1$ *with a vector field tangent to the flow associated to the suspension.*

Moreover, all the previous manifolds admit a normal almost contact structure such that the CR-action is the one induced by the corresponding vector field.

In the first case we described a family of CR-structures and transverse CR-actions in chapter §2. For $\mathbb{T}^3$ a suitable closed linear form provides a CR-structure adapted to the vector field. In **(iii)** we can choose the canonical contact structure on S^3 as the CR-structure. Finally, for suspensions of $\mathbb{P}^1$ the tangent space of $\mathbb{P}^1$ defines a Levi-flat CR-structure which is compatible with the vector field. In the classification of transversely holomorphic flows there are two more cases that are not in the class $\mathcal{T}$:

- **(v)** Strong stable foliations associated to suspensions of hyperbolic diffeomorphisms of $\mathbb{T}^2$.

(vi) $\mathbb{C} \times \mathbb{R} \backslash \{(0,0)\} / \sim$ where $(z,t) \sim (\lambda z, 2t)$ for $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$ with the flow induced by the vertical vector field $\frac{\partial}{\partial t}$.

To prove the theorem it is enough to rule out these two cases. The flows in **(v)** are examples of non-isometric Riemannian flows (see [Car84]), note that if they admitted an invariant CR-structure together with a transverse CR-action they would be isometric (see section 1.5), therefore they cannot be in the class $\mathcal{T}$. The manifolds in **(vi)** are excluded by the following proposition:

Proposition 6.6. *Let* M^3 *be* $\mathbb{C} \times \mathbb{R} \backslash \{(0,0)\} / \sim$ *where* $(z,t) \sim (\lambda z, 2t)$ *for* $\lambda \in \mathbb{C}$ *such that* $|\lambda| > 1$ *with the flow* $\mathcal{F}$ *induced by the vertical vector field* $\frac{\partial}{\partial t}$. *The manifold* M^3 *cannot admit a normal almost contact structure with a CR-structure transverse to* $\mathcal{F}$ *and a CR-action tangent to* $\mathcal{F}$.

Proof. Suppose that M^3 admits a CR-structure $\Phi^{1,0}$ transverse to $\mathcal{F}$ and a vector field T tangent to $\mathcal{F}$ inducing a CR-action, i.e. M^3 is in the class $\mathcal{T}$. Let us define a complex structures on $\mathrm{X} = \mathrm{M}^3 \times S^1$ as in proposition 4.2.1. Since X is homeomorphic to $S^2 \times S^1 \times S^1$ the complex surface X must be a ruled surface over an elliptic curve (if it were an elliptic fibre bundle with base $\mathbb{P}^1$ it would be trivial, thus a product, which is also a ruled surface). However we will see that this is a contradiction. Recall that the universal covering of a ruled surface over an elliptic curve is either $\mathbb{D} \times \mathbb{P}^1$ or $\mathbb{C} \times \mathbb{P}^1$. By construction of the complex structure on $\mathrm{X} = \mathrm{M} \times S^1$ we have that $t + \tau s$ is a holomorphic coordinate for some $\tau \in \mathbb{C} \backslash \mathbb{R}$, therefore the analytic universal covering $\widetilde{\mathrm{X}} = \mathbb{C} \times \mathbb{R} \backslash \{(0,0)\} \times \mathbb{R}$ of X admits a holomorphic projection $p : \widetilde{\mathrm{X}} \to \mathbb{C}$ defined by $p(z,t,s) = z$ with fiber an open subset of $\mathbb{C}$. As $\mathbb{P}^1$ is compact by the maximum principle it is immersed in the fibers of p, which is a contradiction. □

Recall that when one complexifies M^3 by means of the construction in proposition 4.3.1, case (B), we are able to decide if the resulting complex surface is Kählerian only when the flow is isometric. This is always the case for the flows in **(i)** and **(ii)**, the flows in **(iii)** are isometric if and only the flow is given by $\varphi_t[z_1, z_2] = [e^{i\mu_1 t} z_1, e^{i\mu_2 t} z_2]$ where $\mu_1/\mu_2 \in \mathbb{R}$ and the flows in **(iv)** are isometric if and only if the automorphism is a rotation (cf. [Car84]).

Theorem 6.7. *Let* S *be a compact complex surface obtained from a manifold* M^3 *in the class* $\mathcal{T}$ *by means of the constructions of cases (A), (B) or (C). Then:*

(i) *Assume that* M^3 *is a Seifert fibration. In case (B) the surface* S *is an elliptic Seifert principal fibre bundle over a Riemann surface* N_g *with* $g \geq 1$*. Moreover* S *is Kählerian if and only if* $e_{\mathcal{F}}(\mathrm{M}) = 0$ *and the* S^1*-principal bundle* $\pi : \mathrm{S} \to \mathrm{M}$ *is topologically trivial.*

(ii) *Assume* $\mathrm{M}^3 \cong \mathbb{T}^3$*. In case (B), when* $\pi : \mathrm{S} \to \mathrm{M}$ *is a topologically trivial fibration the surface* S *is a complex torus, otherwise* S *is a non-Kählerian elliptic Seifert principal fibre bundle over a Riemann surface* N_g *with* $g \geq 1$*. In case (C) we can obtain a complex torus or a Seifert elliptic principal fibre bundle over a Riemann surface* N_g *with* $g \geq 1$*.*

(iii) *If* $\mathrm{M}^3 \cong S^3$ *or* $\mathrm{M}^3 \cong L_{p,q}$ *then* S *is a Hopf surface.*

(iv) *Assume that* M^3 *is the suspension of an automorphism of* $\mathbb{P}^1$*. If* $\mathrm{S} \to \mathrm{M}^3$ *is a* S^1*-bundle which is not topologically trivial then* S *is a Hopf surface. Otherwise it is a ruled surface over an elliptic curve.*

Remark 6.8. The 3-manifolds in the previous list might admit different CR-structures for which the above vector fields define transverse CR-actions. Nevertheless the statements in the theorem are true for the corresponding complexifications independently of the normal almost contact structure.

We can always regard case (A) as a particular case of case (B) for the trivial fibre bundle. In the case **(i)** we will partially discuss the surfaces that can be obtained by the construction of case (C) in the next proposition, when M^3 is the total space of a S^1-principal bundle instead of a Seifert fibration. Finally, note that for all the previous 3-manifolds M^3 in cases **(ii)**, **(iii)** and **(iv)** the cohomology group $H^2(\mathrm{M}, \mathbb{Z})$ has no torsion, therefore a S^1-principal bundle over M is flat if and only if it is topologically trivial.

Proof. **(i)** The first statement is clear and the second one follows directly from theorem 5.2.1, for the second cohomology group of a Riemann surface has no torsion.
(ii) Assume that we have a topologically trivial S^1-principal bundle $\pi : \mathrm{S} \to \mathbb{T}^3$. We obtain a Kählerian surface S homeomorphic to $(S^1)^4$. If we assume that $d\omega = 0$, where ω denotes the 1-form of the normal almost contact structure, and $\mathrm{S} = \mathbb{T}^3 \times S^1$ we obtain a complex torus. Since every deformation of a complex torus is a complex torus (cf. [Cat02]) we conclude that when the S^1-principal bundle $\pi : \mathrm{S} \to \mathbb{T}^3$ is topologically trivial S is a complex torus. Otherwise, that is, if the S^1-principal bundle $\pi : \mathrm{S} \to \mathbb{T}^3$ is not topologically trivial, the surface S

cannot be Kähler (by theorem 5.2.1) and its universal covering is $\mathbb{R}^4$, therefore it must be a non-Kählerian principal elliptic Seifert fibre bundles over a Riemann surface N_g with $g \geq 1$. In case (C), since the universal covering is $\mathbb{R}^4$ we must be in case **(I)** or **(II)**.

(iii) It is enough to consider the case $\mathrm{M}^3 = S^3$. As $H^2(S^3, \mathbb{Z}) = 0$ every S^1-principal bundle over S^3 is topologically trivial so in case (B) the surface S is homeomorphic to $S^3 \times S^1$. In particular S is not Kählerian and we obtain a primary Hopf surface. For $L_{p,q}$ we obtain secondary Hopf surfaces. On the other hand, since $\mathrm{Diff}^+(S^3)$ is connected (by Cerf's theorem, c.f. [Cer68]) in case (C) the surface S is also homeomorphic to $S^3 \times S^1$ and the same arguments apply.

(iv) Assume first that $p : \mathrm{S} \to \mathrm{M}$ is a S^1-principal bundle which is not topologically trivial, that is, $[d\alpha] \neq 0$. Note that in this case S is not Kählerian. On the other hand the universal covering of M^3 is $p : S^2 \times \mathbb{R} \to \mathrm{M}^3$ so by pull-back of the S^1-principal bundle $\pi : \mathrm{S} \to \mathrm{M}^3$ we obtain a fibre bundle with total space a product of a finite quotient of S^3 times $\mathbb{R}$. It follows that $S^3 \times \mathbb{R}$ is the universal covering of S and we can conclude. If $\mathrm{S} \to \mathrm{M}^3$ is topologically trivial the universal covering is $S^2 \times \mathbb{R}^2$. Note that since $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^1) = \mathrm{PSL}(2, \mathbb{C})$ is connected $\mathrm{M}^3 \cong S^2 \times S^1$. Moreover the universal covering of S is $S^2 \times \mathbb{R}^2$. Therefore S is a ruled surface over an elliptic curve. □

Proposition 6.9. *Let* M^3 *be a* S^1-principal bundle *over a Riemann surface* B *endowed with a normal almost contact structure and let* S *be a compact complex surface obtained from* M^3 *by means of the construction of case (C). Then:*

- **(a)** *When* B *is a Riemann surface* N_g *of genus* $g \geq 2$*, the surface* S *is a Seifert fibration over* N_g *with fibre an elliptic curve.*
- **(b)** *When* $\mathrm{B} = \mathbb{P}^1$ *the surface* S *is a ruled surface over an elliptic curve when* $e_{\mathcal{F}}(\mathrm{M}) = 0$ *and a Hopf surface when* $e_{\mathcal{F}}(\mathrm{M}) \neq 0$*.*
- **(c)** *When* B *is an elliptic curve the surface* S *is either a complex torus or a principal elliptic Seifert bundle over an elliptic curve.*

Moreover if $e_{\mathcal{F}}(\mathrm{M}) \neq 0$ *the surface* S *is not Kählerian*

Proof. **(a)** Assume now that we are in case (C) and let S be the suspension of M by $f \in \mathrm{Aut}_{\mathcal{T}}(\mathrm{M})$. As f preserves the vector field of the CR-action it preserves the fibration of M and projects over an automorphism $\widetilde{f}$ of N_g. Up to a finite covering we can assume that $\widetilde{f} = \mathrm{id}$, for $\mathrm{Aut}_{\mathbb{C}}(\mathrm{N}_g)$ is a finite group when $g \geq 2$. It is not difficult to see that an automorphism preserving such a normal almost contact structure and projecting over the identity can only be φ_{t_0}, where φ_t is the 1-parameter group corresponding to the vector field T of the action and $t_0 \in \mathbb{R}^+$ (it is enough to impose the conditions $f_*T = T$ and $f^*\omega = \omega$). It follows that, up to a finite covering, S is an elliptic fibration over N_g, which is Kähler if and only $e_{\mathcal{F}}(\mathrm{M}) = 0$. Thus S is a Seifert principal elliptic fibre bundle over N_g with fibre an elliptic curve. Finally notice that there is a finite analytic covering of S which is Kähler if and only if $e_{\mathcal{F}}(\mathrm{M}) = 0$ (as the total space of the finite covering is the product $\mathrm{M} \times S^1$ the criterium of theorem 5.2.1 can be applied). To conclude it is enough to notice that a finite analytic covering of S is Kählerian if and only if S is Kählerian.

(b) Note that $e_{\mathcal{F}}(\mathrm{M}) = 0$ if and only if the S^1-principal bundle is topologically trivial. The S has universal covering $S^2 \times \mathbb{R}^2$ if $e_{\mathcal{F}}(\mathrm{M}) = 0$ and $S^3 \times \mathbb{R}$ if $e_{\mathcal{F}}(\mathrm{M}) \neq 0$, which allows us to conclude.

(c) It enough to notice that the universal cover must be $\mathbb{C}^2$, therefore we can obtain a complex torus or a principal elliptic Seifert bundle over an elliptic curve. Moreover if $e_{\mathcal{F}}(\mathrm{M}) \neq 0$ the surface S is not Kählerian, therefore it cannot be a complex torus. □

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Index