ESSAYS ON IMPLEMENTATION AND AUCTIONS

By

Helmuts Āzacis

under the supervision of

Professor Jordi Massó

in partial fulfillment of the requirements for the degree of

Doctor in Economics

at

Universitat Autònoma de Barcelona

Barcelona, September 2005

Contents

Acknowledgments

I am grateful to my thesis advisor Jordi Massó for all the advice and guidance during my work on this dissertation. His support was broader than just research supervision. I am thankful for his patience since probably I was not his easiest student.

I would like to thank Professors Carmen Beviá, David Pérez-Castrillo and Flip Klijn for their helpful comments on Chapter 2, as well as participants at The 2004 Meeting of the Society for Economic Design and The Seventh International Meeting of the Society for Social Choice and Welfare.

Chapter 3 is the result of joint work with Professor Roberto Burguet, and I have benefited from his superior knowledge of the auction field. Professors Michael Creel and Harry J. Paarsch provided suggestions on the early drafts of Chapter 4. I also thank Harry for his hospitality while I was visiting the University of Iowa.

I want to thank the rest of the faculty, the support staff, and the students of IDEA, as well as my friends who made my stay here pleasant and interesting.

I gratefully acknowledge financial support from the European Community's Phare ACE Programme 1998 during academic years 2000/2001 and 2001/2002 and from the Spanish Agency of International Cooperation of the Spanish Ministry of Foreign Affairs for academic year 2004/2005, as well as from the Spanish Ministry of Science and Technology through grant BEC2002-02130 that allowed me to visit the University of Iowa and to assist to the conferences.

Chapter 1

Introduction

My thesis deals with mechanisms to allocate indivisible objects. Any mechanism can be treated as a game form since it defines the actions that the subjects of the mechanism can take. Therefore, we can predict the outcomes of a mechanism using game theoretic tools. However, unlike 'positive' game theory, the designer of the mechanism has the freedom to set the rules of the game so as to achieve the most desirable outcome. For example, social welfare calls for efficient allocation of resources while the objective of an owner of resources is to maximize his revenues. Usually the social planner, auctioneer, or principal lacks the knowledge about the agents' characteristics relevant to achieve the desired objective. The role of a mechanism is to overcome the asymmetric information problem by inducing competition among the agents.

In my thesis I deal with this 'normative' side of game theory, that is, I ask how a mechanism performs with respect to a particular objective and how it can be improved. Thus, in chapter 2, I ask whether the allocation will still be envy-free and efficient with respect to the true preferences when the social planner simply solicits agents to announce their valuations and an envy-free allocation is selected with respect to the announced preferences. The remaining chapters deal with auctions. Chapter 3 compares how various auction formats perform in promoting entry since higher entry leads to stronger competition during bidding and, consequently, to higher revenues of the seller. In the last chapter I test empirically whether the seller can benefit from changing the order of sales in sequential auctions of heterogenous objects.

In Chapter 2 of my thesis I address the implementation of envy-free allocations in the assignment problem. Each agent must be assigned to an indivisible object and pay the price of the object he gets, and prices are required to sum to a given number. An example is the housemate problem where a group of tenants is sharing an apartment and they must decide who gets which room and how much each must pay, subject to the constraint that the sum of their contributions must equal the rent of the apartment. The objective is to select an assignment-price pair that is envy-free with respect to the agents' true preferences. Envy-freeness is a sufficient condition for the stability of the assignment since it guarantees that each agent prefers the object-price pair he has been assigned to any other pair. Moreover, it implies efficiency which makes it an attractive solution concept.

Previous research concentrated on mechanisms that select an envy-free assignment and price pair, assuming that the valuations of objects are known by the social planner. Once we assume that the social planner lacks such perfect knowledge and must instead solicit agents' valuations, a question arises on the scope that agents can manipulate the outcome by misrepresenting their valuations.

I treat the assignment problem as a game and prove that the mechanism, in which agents are simply required to announce their (possibly false) valuations and an envy-free allocation is selected with respect to these announced valuations, will double-implement the set of envy-free allocations both in Nash and strong Nash. This means that, in equilibrium, the selected allocation will be envy-free also with respect to the true preferences. I demonstrate that by choosing an envy-free allocation a social planner does not need to worry about strategic issues, that is, the scope for agents to manipulate the allocations is limited in equilibrium. This result provides a justification on strategic grounds for the use of social choice functions selecting envy-free allocations.

The chapter 3 studies auctions when the presence of a strong buyer deters the entry of other potential bidders, preventing competition in the auction and leading to a low sales price. For example, the issue arose during third-generation mobile communication licenses auctions where incumbent firms could effectively discourage entry by threatening to outbid new entrants.

When attracting entrants is a goal, the allocation mechanism should favor entrants over incumbents. Maskin and Riley [3] have shown that Dutch auction, compared with English auction, tilt the allocation in favor of ex-ante weaker bidders, while sacrificing efficiency. Based on this insight, Paul Klemperer (see Klemperer [2]) have proposed the use of the so-called Anglo-Dutch auction in order to promote entry. It is a mixture of the two types of auctions. It begins with an 'English' phase during which the price rises until all but a number of bidders that exceeds by one the number of objects drop out. At this price the auction switches to a second 'Dutch' phase. In this stage, only the remaining bidders can submit simultaneous, sealed bids and only bids above the price at which the English phase stopped are allowed.

We propose an alternative two-stage English auction, based on Burguet and Sákovics [1], in order to encourage entry. In the first stage there is a reserve price, and if nobody bids at that price, then a second stage is conducted without the reserve price. Potential entrants of the second stage learn that those who entered in the first stage but did not bid have low valuations. This provides incentives for weak bidders to enter in the second stage. Instead of using inefficiencies as the tool to induce entry, what the two-stage English auction uses is the information conveyed by the (absence of) bidding in the first phase.

We analyze a model when valuations can take only two different values, although incumbents have a higher probability of high valuation, and entrants have to incur a cost before learning their valuations. We show that in the one-unit case a two-stage English auction is more efficient than both the English and the Anglo-Dutch auctions. Moreover, we show that the gain in efficiency benefits the seller as well. Indeed, the revenues for the seller are higher in the two-stage

English auction than in the Anglo-Dutch auction.

Subsequently, we extend the model to continuous valuations. Indeed, in the discrete valuations model, the Anglo-Dutch is allocationally efficient, that is, the object is never assigned to a bidder that competes against a bidder with higher valuation, and the only inefficiency comes from inappropriate entry. Instead, in the case of continuous valuations the Anglo-Dutch auction may produce inefficient allocation when a bidder with lower valuation wins the auction. The inefficient allocation of the Anglo-Dutch auction increases seller's revenues by reducing informational rents of ex-ante stronger bidders. To investigate whether we do not underestimate the revenue generation potential of Anglo-Dutch auction we perform numerical computations using uniform distributions. We obtain the same results as in the discrete valuations model.

We also consider multiple units. Here entry decisions in the second stage of the two-stage English auction depends on the number of units that are sold in the first stage. The larger the number of units left unsold the larger the number of entrants in the second stage. In fact, under very extreme values of the parameters, the effect that the number of units available has on entry is very extreme. Only in such cases can Anglo-Dutch auctions dominate in terms of revenues the two-stage English auction. Otherwise, our results for the one unit case hold in the multiple unit case.

Many real life auctions sell more than one object; objects are usually different and often are sold sequentially; hence the questions arise of how the order of sales affects revenue and what the optimal order of sales is. The last chapter addresses these issues. It consists of an empirical analysis of auctions conducted by the state company "Latvia's State Forests" selling rights to harvest timber in state forests. In each auction the company offers several lots of forest, and they are sold sequentially through oral, ascending price auctions.

For the two-object case, theory suggests that the revenues of the seller are highest when objects are sold in the order of decreasing value. I test empirically this hypothesis for the case of many objects and suggest an optimal ordering of sales. Since we do not observe the valuations of lots the bidders have, first I estimate them, using a discrete choice model, based on the assumption that a lot will be sold if and only if its valuation exceeds the reserve price. Next, I test the order of sales effect on the revenues of the seller. I do not reject the hypothesis that the order does not have effect on the revenues of seller.

Bibliography

[1] R. Burguet and J. Sákovics. Reserve prices without commitment. *Games and Economic Behavior*, 15:149–164, 1996.

[2] P. Klemperer. Auctions with almost common values: The wallet game and its applications. *European Economic Review*, 42:757–769, 1998.

[3] E. Maskin and J. Riley. Asymetric auctions. *Review of Economic Studies*, 67:413–438, 2000.

Chapter 2

Double Implementation in a Market for Indivisible Goods with a Price Constraint

2.1 Introduction

I study the problem of assigning a set of indivisible objects to a set of agents. Each agent wants exactly one object, and his preferences are quasilinear in money. Each agent must pay a price corresponding to the object he gets, and prices are required to sum to a given number. A standard example is the housemate problem where a group of tenants is sharing an apartment. The objective is to determine who gets which room and how much each must pay subject to the constraint that the sum of their contributions must equal the rent of the apartment. However, neither the valuations of objects, nor the number that the prices must sum up to, need to be positive. An example with a negative price constraint includes heirs sharing inheritance that besides indivisible objects contains a divisible object - money. Negative valuations imply that the objects are not goods but rather bads or burdens. For example, a central government with fixed budget assigning waste disposal sites and other duties/projects to municipalities.

When deciding on the assignment of agents to objects and the corresponding prices, we may want to meet certain criteria. The usual requirements include efficiency and envy-freeness. Efficiency ensures that the welfare of society as a whole is maximized, while envy-freeness guarantees that each agent prefers the object-price pair he has been assigned. In this sense envy-freeness is a sufficient condition for the stability of the assignment. Moreover, it implies efficiency which makes it an attractive solution concept.

There exists a wide literature that uses the above framework. Shapley and Shubik [13] showed that the set of efficient and envy-free allocations can be found as a solution to a linear programming problem. Subsequent contributions have proposed different algorithms to find a particular envy-free allocation. Examples selecting envy-free allocations when prices are required to sum to a given number include algorithms by Abdulkadiroğlu *et al.* [1] and Haake *et al.* [8]. Since the set of efficient and envy-free allocations is usually not a singleton, these algorithms pick up different solutions, corresponding to different price vectors. In related works, Brams and Kilgour [5] and Chin Sung and Vlach [6] impose the additional constraint that prices must be nonnegative and analyze when the selected allocation is envy-free. The requirement of nonnegative prices is justified when objects are goods and the price constraint is positive, like in the room-sharing problem.

A shortcoming of all of the above studies is that they treat the valuations of objects as known by the social planner. However, in a more realistic setup the social planner lacks such perfect knowledge and instead solicits agents' valuations. Yet agents are interested to maximize their own utility and, in general, have no incentives to reveal their true valuations. Therefore, if we insist on using an algorithm to reach a particular allocation, a question arises on the scope that agents have to manipulate the outcome by misrepresenting the valuations.

Motivated by these algorithms, I consider a mechanism that, given the announced valuations, selects an allocation that is efficient and envy-free with respect to these announced preferences. It is a direct revelation mechanism where agents' actions are messages of the valuations they attach to each object. The particular price vector that the mechanism selects among all possible envy-free prices, coincides with the one that would be selected by the algorithm of Abdulkadiroğlu *et al.* [1]. The advantage of this algorithm is that it provides a formula for the selected price vector in terms of the announced preferences, making it easy to establish whether there is a profitable deviation. In addition, since the algorithm by Abdulkadiroğlu *et al.* [1] does not specify which efficient assignment to select with respect to the announced preferences, I introduce a tie-breaking rule to ensure that the assignment is efficient with respect to the true preferences.

I prove that the proposed mechanism double implements the set of efficient and envy-free allocations both in Nash and strong Nash equilibrium. That is, I show, first, that all envy-free allocations (with respect to the true valuations) are outcomes of some (strong) Nash equilibrium of the game induced by the mechanism and, second, all (strong) Nash equilibrium outcomes of the game are envy-free. Although truth-telling is not an equilibrium strategy, in any (strong) Nash equilibrium the reported preferences will induce an allocation such that agents will be envy-free with respect to the true preferences.

One implication of the result is that by choosing an efficient and envy-free allocation a social planner does not need to worry about strategic issues, that is, the scope for agents to manipulate the allocations is limited in equilibrium. This provides a justification on strategic grounds for the use of social choice functions selecting envy-free allocations. It also gives an answer to the question posed by Abdulkadiroğlu *et al.* [1] on the possible equilibria of the preference manipulation game induced by their algorithm.

When there is no requirement that prices must sum to a certain amount, it has been shown (Leonard [10]) that truth-telling is a dominant strategy for the mechanism which selects an efficient assignment of objects and agents pay the so-called agent-optimal prices. However, when prices are required to sum to a given amount, there is a trade off between strategy-proofness and envy-freeness: if the price that an agent pays depends on his valuations, then he would have incentives to misrepresent them, ruling out truth telling as a dominant strategy. On the other hand, if prices are independent of preferences, then envy-freeness is not guaranteed. Therefore, I use (strong) Nash equilibrium as a solution concept.

Another desirable feature of the proposed mechanism is that it is balanced, unlike the implementation in dominant strategies which requires side-payments to the third party. The latter can be justified in certain cases, for instance, when objects are auctioned and a seller receives the revenue such as in Demange *et al.* [7]. However, there are economic examples where such side-payments are ruled out or their amount is fixed in advance in which case a balanced mechanism is the appropriate one.

Among other studies that address manipulation games the present paper is most closely related to Tadenuma and Thomson [14] and Beviá [4]. Both study the strategic aspects of using envy-freeness as a solution concept to allocate indivisible goods and prove that the set of equilibrium outcomes coincides with the set of envy-free allocations with respect to the true preferences. The work by Tadenuma and Thomson [14] considers allocating only one indivisible object to one of several agents when monetary compensations are available, while Beviá [4] extends the result to the multiple objects case. Beviá's [4] approach is more general in that she works with correspondences while my mechanism selects a single-valued outcome. As a consequence, Beviá [4] must use modified equilibrium concepts appropriate to multi-valued outcomes.

The remaining of the chapter is organized as follows. The following section provides the formal model and some results necessary for the proof. Section 2.3 defines the implementation problem and states the theorem. An example for two-agent two-object case is provided in Section 2.4 before proving the theorem in Section 2.5. Final remarks in Section 2.6 conclude the paper. Some of the proofs are relegated to the Appendix.

2.2 Preliminaries

The set of agents is $I = \{1, ..., n\}$ and generic elements of I will be denoted by i and k. The set of objects is $J = \{1, ..., n\}$ with generic elements of J denoted by j and l. Throughout it is assumed that the number of agents and objects is the same n.[1] It is assumed that each agent wants one and only one object. The matrix of true valuations is $A = [a_{ij}]_{i \in I, j \in J}$ where $a_{ij} \in R$ is the valuation that agent i assigns to object j. The assignment of agents to objects is given by a one-to-one mapping $\mu : I \to J$. I denote a price vector by $p = (p_1, ..., p_n) \in R^n$. Utilities are quasi-linear in prices, namely, the utility of agent i from being assigned to object $\mu(i)$ and paying its price $p_{\mu(i)}$ is $u_i(p_{\mu(i)}) = a_{i\mu(i)} - p_{\mu(i)}$. Let M denote the set of assignments. An allocation is an assignment-price pair $(\mu, p) \in M \times R^n$.

Definition 1 *An assignment $\mu \in M$ is* **efficient** *if $\sum_{i \in I} a_{i\mu(i)} \geq \sum_{i \in I} a_{i\eta(i)}$ for all assignments $\eta \in M$.*

Definition 2 *An allocation $(\mu, p) \in M \times R^n$ is* **envy-free** *if $u_i(p_{\mu(i)}) \geq u_i(p_j)$ for all $i \in I$ and $j \in J$.*

Given an envy-free allocation $(\mu, p) \in M \times R^n$ we will refer to p as an envy-free price. Also denote by M^A the set of efficient assignments relative to the matrix of valuations A. Alkan *et al.* [2] prove that if the allocation (μ, p) is envy-free then the assignment μ is efficient. One can also think of envy-freeness as a sufficient requirement of stability since each individual prefers his object to any other object, given the vector of prices. Therefore, envy-freeness is used as a solution concept in most models dealing with indivisible objects, see for example Alkan *et al.* [2], Aragonés [3], Haake *et al.* [8], and Klijn [9].

The assignment problem of indivisible objects was first addressed by Shapley and Shubik [13] who proved that the problem can be translated into a linear programming problem where the efficient assignments are obtained from the primal

[1] See the discussion in Section 2.6 when the number of agents and objects is different. Note, however, that the results stated in this section do not require that $|I| = |J|$.

problem but envy-free prices and the corresponding utilities come from the dual problem as shadow prices. Given the matrix of valuations A, define with the coalitional function $w(A,T,Q)$ the maximal worth that a coalition of agents $T \subseteq I$ can obtain when assigned to a set of objects $Q \subseteq J$. It can be expressed in terms of the following linear programming problem[2]: given the matrix of valuations A and subsets T and Q, choose $(x_{ij})_{i \in T, j \in Q}$ to solve for

$$w(A,T,Q) \equiv \max \sum_{i \in T, j \in Q} a_{ij} x_{ij} \tag{2.1}$$

subject to

$$\begin{aligned}
\sum_{i \in T} x_{ij} &\leq 1 && \text{for any } j \in Q \\
\sum_{j \in Q} x_{ij} &\leq 1 && \text{for any } i \in T \\
x_{ij} &\geq 0 && \text{for any } i \in T, j \in Q \\
\sum_{i \in T, j \in Q} x_{ij} &= min(|T|, |S|).
\end{aligned}$$

This primal problem has a corresponding dual problem where the costs of inputs — agents and objects — are minimized. Shadow prices are prices of objects and utilities of agents. Given the matrix of valuations A and subsets T and Q, choose $(u_i)_{i \in T}$ and $(p_j)_{j \in Q}$ to solve for

$$w(A,T,Q) \equiv \min \sum_{i \in T} u_i + \sum_{j \in Q} p_j \tag{2.2}$$

subject to

$$u_i + p_j \geq a_{ij} \quad \text{for any } i \in T, j \in Q. \tag{2.3}$$

Then the solution of the primal has the property that x_{ij} takes values 0 or 1 for all $i \in T$ and all $j \in Q$. Assume that $T = I$ and $Q = J$. The primal problem solves for an efficient assignment of objects as follows: given the solution

[2]The last condition is needed to ensure that agents will be assigned to objects even if their valuations are negative.

$(x_{ij})_{i\in I, j\in J}$, define the assignment μ by letting $\mu(i) = j$ if and only if $x_{ij} = 1$. The dual problem gives the set of envy-free prices (this follows from constraint (2.3) since $u_i \geq a_{ij} - p_j$ for all $i \in I$ and all $j \in J$) and the corresponding utilities $(u_i \equiv u_i(p_{\mu(i)}))$.

The set of envy-free prices forms a lattice that possess the following property: if p' and p'' are two envy-free price vectors then so are the price vectors $\underline{p}$ and $\overline{p}$ where $\underline{p}_i = \min(p'_i, p''_i)$ and $\overline{p}_i = \max(p'_i, p''_i)$. This property is proven in Shapley and Shubik [13], see also Roth and Sotomayor [12] (chapter 8). The lattice has an agent-optimal price vector $p_* \geq 0$ such that $p \geq p_* \geq 0$ for all envy-free and non-negative prices p.

Given an efficient assignment μ, an agent-optimal price can be calculated[3] (see Leonard [10] or Roth and Sotomayor [12]) using the coalitional function, defined by equation (2.1), as

$$p_{*\mu(i)} = w(A, I\backslash\{i\}, J) - w(A, I\backslash\{i\}, J\backslash\{\mu(i)\}), \tag{2.4}$$

for each $i \in I$. From (2.4) it follows that $p_{*\mu(i)}$ does not depend on the object valuations of agent i. Using this property Leonard [10] proves that the mechanism that selects the agent-optimal prices p_* is strategy-proof. His result is a consequence of the well-known Clark's pivotal mechanism and is a special case of the results proven by Roberts [11] for quasilinear utility functions.

Here I state some additional results that will be useful later in proving the theorem.[4]

Proposition 1 *Given a matrix of valuations A, the coalitional function $w(A, T, Q)$ is continuous and weakly increasing in a_{ij}.*

It is shown in the proof of the Proposition 1 that when $i \in T$ and $j \in Q$ equation (2.1) can be written as

$$w(A, T, Q) = \max(const_1 + a_{ij} \cdot 0, const_2 + a_{ij} \cdot 1) \tag{2.5}$$

[3]For an example how to calculate agent-optimal prices, see Section 2.4.

[4]The proofs are provided in the Appendix.

where $const_2 = w(A, T\backslash\{i\}, Q\backslash\{j\})$ and $const_1$ is a coalitional worth that is obtained by solving the original assignment problem subject to the additional constraint that agent i is not assigned to object j. Since neither $const_1$ nor $const_2$ is affected by the change in a_{ij} they can be considered constants. Function (2.5) is obviously continuous and weakly increasing in a_{ij}.

Proposition 2 *Given a matrix of valuations A, the set of envy-free prices is the same for all efficient assignments of objects.*

Proposition 2 allows us to establish immediately the following result.

Corollary 1 *Fix an envy-free price vector p. Under all efficient assignments of objects each agent gets the same utility:*

$$a_{i\mu_1(i)} - p_{\mu_1(i)} = a_{i\mu_2(i)} - p_{\mu_2(i)}$$

for all $i \in I$ and where μ_1 and μ_2 are any two efficient assignments of objects.

2.3 Implementation Problem

In the implementation problem that I consider I restrict the set of feasible price vectors and require the prices to sum to a given number C: $\sum_{j \in J} p_j = C$. One can think in terms of an economy that consists of the set J of indivisible objects and a quantity C of the divisible object - money - that must be distributed among n agents.

Let $\triangle_C$ denote the set of price vectors that sum to C. Since all feasible price vectors are required to belong to this set, from now on it is understood that an allocation is an assignment-price pair $(\mu, p) \in M \times \triangle_C$. And with an envy-free price vector p I will only refer to price vectors that meet the price constraint: $p \in \triangle_C$. Notice that the set of envy-free prices in $\triangle_C$ is non-empty. For example, if we take the agent-optimal price vector p_* and add a constant c (positive or negative), the envy-freeness is preserved. We can always choose c such that $p_* + c \in \triangle_C$.

Given C and the matrix of valuations A, denote the set of envy-free allocations in $M \times \triangle_C$ with $G(A)$. If a social planner were to choose an allocation (μ, p), arguably, he would prefer to select one from the set of envy-free allocations $(\mu, p) \in G(A)$ since these allocations meet the desirable normative criteria of envy-freeness and hence efficiency. The algorithms proposed by Abdulkadiroğlu *et al.* [1], Aragonés [3], Brams and Kilgour [5], Haake *et al.* [8], and Klijn [9] were designed to select allocations from the set $G(A)$. However, all of them rely on the knowledge of matrix A. If the social planner does not know the true preferences of agents, he will need to solicit them. A question arises whether agents have strategic incentives to reveal their true valuations. That is, an agent can find it profitable to announce valuations of objects different from his true ones.[5] Given this misrepresentation of preferences there is no guarantee anymore that the selected allocation by any of the algorithms will satisfy envy-freeness with respect to the true preferences. However, it will be demonstrated that, with the help of an appropriate tie-breaking rule, selecting an allocation that is envy-free with respect to the announced preferences, not necessarily the true ones, achieves in the equilibrium envy-freeness with respect to the true valuations.

I propose a direct revelation mechanism where each agent is required to announce only his own valuations of objects and the mechanism selects an envy-free allocation with respect to the matrix of announced valuations. Formally, a strategy of agent i is a vector of object valuations $b_i = (b_{i1}, ..., b_{in}) \in R^n$ that he announces. Given the matrix of reported valuations $B = [b_{ij}]_{i \in I, j \in J}$, denote the set of envy-free allocations implied by the matrix B by $G(B)$. A mechanism g is a mapping from the space of valuations into the space of allocations $g : R^{n \times n} \to M \times \triangle_C$. I restrict attention to mechanisms that, for each matrix of valuations B, will select an allocation (μ, p) from the envy-free set $G(B)$. The

[5] Truth-telling is a dominant strategy when the assignment μ is efficient and agent i pays the agent-optimal price $p_{*\mu(i)}$ of the object he gets. However, in general the agent-optimal prices will not meet the price constraint, $p_* \notin \triangle_C$, and therefore, $(\mu, p_*) \notin G(A)$.

justification why to consider such mechanisms was provided before - the envy-freeness is the standard solution concept for such type of problems and there exist works that analyze how to select an allocation from the set $G(B)$ although usually ignoring strategic issues.

In general, the set of envy-free prices is not a singleton. From all envy-free prices the price vector that I select corresponds to the one that would be selected according to the algorithm of Abdulkadiroğlu *et al.* [1] when applied to the matrix B. The advantage of this price vector is its explicit linear relationship with the agent-optimal prices, given by the equation:

$$p_j = p_{*j} + \frac{C - \sum_{m \in J} p_{*m}}{n} \text{ for all } j \in J. \tag{2.6}$$

where p_* is the vector of agent-optimal prices implied by B. According to this formula each agent $i \in I$ pays the agent-optimal price corresponding to the object he gets $p_{*\mu(i)}$ plus the equal share of the difference between the price constraint and the sum of all agent-optimal prices.

The utility of agent i having object $\mu(i)$ and paying price $p_{\mu(i)}$, by applying equation (2.6), is

$$u_i(p_{\mu(i)}) = a_{i\mu(i)} - p_{\mu(i)} = a_{i\mu(i)} - \frac{C}{n} - \frac{n-1}{n} p_{*\mu(i)} + \frac{1}{n} \sum_{l \neq i} p_{*\mu(l)}. \tag{2.7}$$

It follows that the utility of agent i is decreasing in its own agent-optimal price but increasing in each of other agent-optimal prices keeping the assignment μ fixed. We know from equation (2.4) that $p_{*\mu(i)}$ does not depend on the valuations of objects reported by agent i, that is, he cannot affect his own agent-optimal price. However, he can affect the agent-optimal prices of other objects.

If there are several efficient assignments of agents to objects with respect to the reported valuations B then the mechanism g will break ties according to the following rule. Order all objects and all agents, and without loss of generality assume that the order corresponds to the natural one: $\sigma(i) = i$ for all $i \in I$

and $\sigma(j) = j$ for all $j \in J$, and keep these orders fixed. Start with object 1 and proceed iteratively. If all efficient assignments allocate object 1 to the same agent i, then let agent i get it. Otherwise choose among all efficient assignments the one that assigns object 1 to the agent that has announced the smallest valuation for object 1: $\mu(i) = 1$ if $b_{i1} < b_{k1}$ for any k such that there exists an efficient assignment $\nu \in M^B$ under which $\nu(k) = 1$. If $b_{i1} = b_{k1}$ for two or more agents then select the agent from this set who has been assigned the lowest number: $\mu(i) = 1$ if $i < k$ when $b_{i1} = b_{k1}$. In general, assume that objects 1 to $l - 1$ are already assigned. If all remaining efficient assignments allocate object l to the same agent i, then let agent i get it. Otherwise choose among all efficient assignments the one that assigns object l to the agent that has announced the smallest valuation of object l: $\mu(i) = l$ if $b_{il} < b_{kl}$ for any k such that there exists an efficient assignment $\nu \in M^B$ such that $\nu(k) = l$ and $\nu^{-1}(j) = \mu^{-1}(j)$ for already assigned objects $j \in \{1, ..., l - 1\}$. If $b_{il} = b_{kl}$ for two or more agents select the agent from this set who has been assigned the lowest number: $\mu(i) = l$ if $i < k$ when $b_{il} = b_{kl}$. Thus the tie-breaking rule selects a unique assignment among all efficient assignments with respect to B. Thus, the mechanism g defines a game form, and given A, the pair (A, g) is a game in normal form.

Assume that the strategy profile B has been announced. When a set of agents T deviates and announces a different vector of valuations $b'_T \in R^{|T| \times n}$, that leads to another profile $B' = (b'_T, b_{-T})$. When there is only one deviator, $T = \{i\}$, the strategy profile after the deviation is denoted by $B' = (b'_i, b_{-i})$. Denote the allocation induced by the deviation by $g(B') = (\mu', p')$. The solution concept that I use is strong Nash equilibrium.

Definition 3 *A strategy profile* $B \in R^{n \times n}$ *is* **a strong Nash equilibrium** *relative to* (A, g) *if there is no coalition* T *and strategy profile* b'_T *such that* $u_i(p'_{\mu'(i)}) \geq u_i(p_{\mu(i)})$ *for all* $i \in T$ *and* $u_i(p'_{\mu'(i)}) > u_i(p_{\mu(i)})$ *for at least one* $i \in T$.

Denote the set of strong Nash equilibrium outcomes relative to (A, g) by $O^{SNE}_{(A,g)}$, that is $O^{SNE}_{(A,g)} = \{(\mu, p) \in M \times \triangle_C | g(B) = (\mu, p)$ for some pure strategy strong Nash equilibrium B relative to $(A, g)\}$. Similarly, we can define Nash equilibrium if we restrict the set of deviators T to a single agent i and denote the set of Nash equilibrium outcomes relative to (A, g) by $O^{NE}_{(A,g)}$.

Now we are ready to state the main result of the paper:

Theorem *The mechanism g double implements the social choice correspondence G both in Nash and strong Nash equilibrium: $O^{NE}_{(A,g)} = O^{SNE}_{(A,g)} = G(A)$ for all $A \in R^{n \times n}$.*

2.4 An Example

Before providing the proof of the theorem, consider the following numeric two agent-two object example with the price constraint $C = 20$ and the matrix of valuations

$$A = \begin{pmatrix} 15 & 18 \\ 6 & 22 \end{pmatrix}.$$

The efficient assignment is $\mu(1) = 1$ and $\mu(2) = 2$ since $15 + 22 > 6 + 18$. The set of all envy-free prices is delimited by the equations $p_2 = 3 + p_1$ and $p_2 = 16 + p_1$ and shown in Figure 2.1 by the shaded area. To obtain the agent-optimal price of object 1, we find that $w(A, I \backslash \{1\}, J) = 22$ and $w(A, I \backslash \{1\}, J \backslash \{\mu(1)\}) = 22$, and by applying equation (2.4), $p_{*1} = 22 - 22 = 0$. In the same way we can find that $p_{*2} = 18 - 15 = 3$. Thus the agent-optimal prices are $p_* = (0, 3)$.

The prices that sum up to C are represented with the line connecting the points (20,0) and (0,20). The intersection of this line with the shaded region gives the set of envy-free prices that meet the price constraint. In general, there are an infinity of prices that are envy-free and meet the price constraint. The mechanism g that I consider selects, given the announced valuations, envy-free prices obtained from the agent-optimal prices by increasing all of them by the same amount so that the price constraint is met. If the announced valuations are

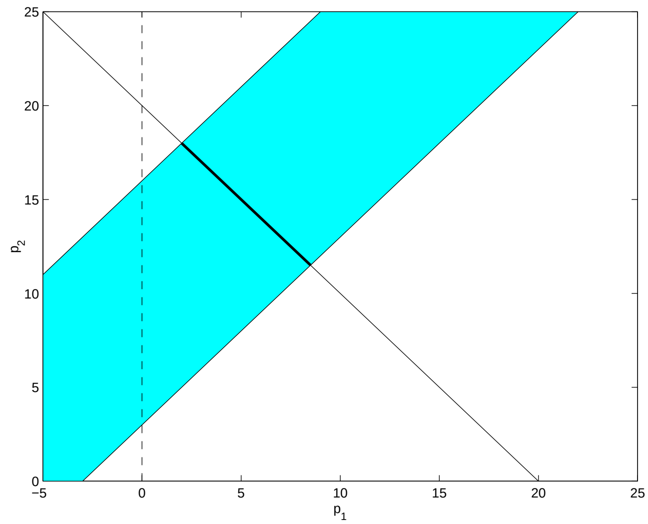

Figure 2.1: The set of envy-free prices

A, then the vector of prices selected by the mechanism is $p = (8.5, 11.5)$ (found by adding 8.5 to the agent-optimal prices p_*).

The algorithm by Abdulkadiroğlu *et al.* [1] would also select the prices $p = (8.5, 11.5)$. Their algorithm finds the first envy-free price when we move from the initial price vector $p^0 = \frac{C}{n}$ along the rent constraint. Thus, the price p obtained by the algorithm is the most 'equal' price among all envy-free prices. In the example we start from $p^0 = (10, 10)$ and reach $p = (8.5, 11.5)$. The proposed mechanism does not ensure neither nonnegative prices nor individual rationality since it depends on the magnitude of C.[6] In the example, if $C < 3$ all envy-free price vectors have at least one negative price and if $C > 37$ then there is no envy-free price that would be individually rational.

In general, agents do not have incentives to announce the true valuations. Agent 1, by announcing the vector $b = (2, 18)$, still gets object 1 according to the tie-breaking rule but pays 2 instead of 8.5. Since, in order to find an efficient assignment, what matters is the relative magnitudes of valuations we can define $\beta_i \equiv b_{i2} - b_{i1}$. Then agent i gets object 2 and agent k gets object

[6]An allocation $(\mu, p) \in M \times \triangle_C$ is **individually rational** if $u_{i\mu(i)}(p_{\mu(i)}) \geq 0$ for all $i \in I$.

1 if $\beta_i > \beta_k$ or if $\beta_i = \beta_k$ and $b_{i1} > b_{k1}$. One can check that when agent i gets object 2 and agent k gets object 1 the agent-optimal prices are given by $p_{*1} = \max(-\beta_i, 0)$ and $p_{*2} = \max(\beta_k, 0)$. It is easy to check that any strategy profile B where $3 \leq \beta_1 = \beta_2 \leq 16$ and $b_{11} \leq b_{21}$ is a Nash equilibrium. When $\beta_1 = \beta_2$ both assignments are efficient with respect to the announced preferences, but by announcing $b_{11} \leq b_{21}$ the tie-breaking rule ensures that the mechanism will select the assignment that is also efficient with respect to the true preferences.

In the proof of the theorem I consider two types of deviations when an agent feels envy. The first occurs when the agent still gets the same object after the deviation but the other agent now must pay a higher price and thus, according to the price constraint, the deviating agent pays a lower price. For example, consider $\beta_2 > \beta_1$, $\beta_2 > 0$ and $\beta_1 < 3$ and matrix A represents the true preferences. Given the announced preferences, $\mu(1) = 1$ and $\mu(2) = 2$ and the agent-optimal prices are $p_{*1} = \max(-\beta_2, 0) = 0$ and $p_{*2} = \max(\beta_1, 0)$. Therefore $p_1 = p_{*1} + (20 - p_{*1} - p_{*2})/2 = (20 - \max(\beta_1, 0))/2 > 8.5$ and $p_2 = 20 - p_1 < 11.5$. Agent 1 feels envy since $15 - p_1 < 18 - p_2$. Agent 1 can deviate and announce $\beta'_1 = \beta_2$ and $b'_{11} < b_{21}$. Then agent 1 still gets object 1 but pays only $p'_1 = (20 - \beta_2)/2 < p_1$ since $p'_{*1} = \max(-\beta_2, 0) = 0$ and $p'_{*2} = \max(\beta'_1, 0) = \beta_2$. Thus he had a profitable deviation.

The second type of deviation occurs when an agent gets the object he envies at the price that the agent who was originally assigned to it paid. For example, if $0 \geq \beta_2 > \beta_1$ and matrix A represents the true preferences then $\mu(1) = 1$ and $\mu(2) = 2$, $p_{*1} = \max(-\beta_2, 0) = -\beta_2$ and $p_{*2} = \max(\beta_1, 0) = 0$. Agent 1 must pay $p_1 = (20 - \beta_2)/2 > 8.5$ and like in the previous case, feels envy. Agent 1 can profitably deviate by announcing $\beta'_1 = \beta_2$ and $b'_{11} > b_{21}$. After the deviation the efficient assignment is $\mu'(1) = 2$ and $\mu'(2) = 1$ with agent-optimal prices $p'_{*1} = \max(-\beta'_1, 0) = -\beta_2$ and $p'_{*2} = \max(\beta_2, 0) = 0$. Agent 1 pays $p'_2 = p_2 = (20 + \beta_2)/2 < 11.5$. A similar profitable deviation exists when $\beta_2 = \beta_1$ and $b_{11} = b_{21}$. Then agent 1 is assigned to object 1 and will feel envy if $\beta_1 < 3$.

Agent 1 is strictly better off by announcing $b'_{11} > b_{21}$ while keeping $\beta'_1 = \beta_2$. If $\beta_1 = 3$ then agent 1 is indifferent between getting object 1 and 2. Observe that the examples discussed cover all the cases when agent 1 could feel envy when he is originally assigned to object 1.

When there are more than two agents, it gets a little bit more complicated to demonstrate the existence of a profitable deviation when an agent feels envy. It may not be anymore possible either to increase the price paid by the agent who is assigned to the object that is envied or to obtain that object at the price that the agent who was originally assigned to it paid. For example, consider the following matrix of announced valuations

$$B = \begin{pmatrix} 5 & 10 & 15 \\ 5 & 10 & 0 \\ 0 & 10 & 20 \end{pmatrix}$$

and $C = 30$. The agent-optimal prices are $p_* = (0, 5, 10)$ and the prices selected that sum to 30 are $p = (5, 10, 15)$. There are two efficient assignments $\mu_1(1) = 1, \mu_1(2) = 2, \mu_1(3) = 3$ and $\mu_2(1) = 2, \mu_2(2) = 1, \mu_2(3) = 3$. The tie-breaking rule selects the first assignment. Suppose that agent 3 envies object 2 at the given prices: $a_{32} - 10 > a_{33} - 15$. Agent 3 can not increase the prices of objects 1 and/or 2 and thus decrease the price of object 3 and still get it. And neither he can obtain object 2 at price $p_2 = 10$. By announcing the vector of valuations $b'_3 = (0, 15 + \epsilon, 20)$ where $\epsilon > 0$ ensures that $\mu'(3) = 2$ and the agent-optimal prices will be $p'_* = (0, 5, 10 - \epsilon)$ and the selected prices $p' = (5 + \epsilon/3, 10 + \epsilon/3, 15 - 2\epsilon/3)$. For ϵ sufficiently small agent 3 will find it advantageous to deviate since $a_{32} - 10 - \epsilon/3 > a_{33} - 15$.

2.5 Proof of The Theorem

Throughout the proof fix a matrix of true valuations A, and assume without loss of generality that the orders of agents and objects needed to define g are both $1, 2, ..., n$.

The set of Nash equilibria contains the set of strong Nash equilibria: $O^{SNE}_{(A,g)} \subseteq O^{NE}_{(A,g)}$. To establish the statement of the theorem, one needs to demonstrate, first, that for every envy-free allocation one can construct a strategy profile B that is a strong Nash equilibrium of the proposed game (A, g) (Lemma 1) implying $G(A) \subseteq O^{SNE}_{(A,g)} \subseteq O^{NE}_{(A,g)}$; second, that a strategy profile B where an agent feels envy at allocation $g(B) = (\mu, p)$ can not be a Nash equilibrium of the game (A, g) (Lemma 2) implying $O^{SNE}_{(A,g)} \subseteq O^{NE}_{(A,g)} \subseteq G(A)$. Combining the results of both Lemmas gives the desired result: $O^{NE}_{(A,g)} = O^{SNE}_{(A,g)} = G(A)$.

Lemma 1 *Let (μ, p) be an envy-free allocation. Then there is a strong Nash equilibrium B of (A, g) such that $g(B) = (\mu, p)$.*

Proof: Take an envy-free allocation $(\mu, p) \in G(A)$. Consider the following strategy profile B: each agent $i \in I$ announces $b_i = p + c_i$ where scalars c_i satisfy the following relationship for any two agents i and k: $c_i < c_k$ if and only if $\mu(i) < \mu(k)$. I claim that the given strategy profile constitutes a strong Nash equilibrium.

Observe that any possible assignment of objects is efficient with respect to B. The only envy-free price vector is p. The way how the scalars c_i for $i = \{1, ..., n\}$ were chosen ensures that the unique assignment, selected according to the tie-breaking rule, will be μ: an agent i who announced the smallest b_{i1} among all agents will be assigned to object 1 and by construction it was agent $\mu^{-1}(1)$. Among the remaining $n - 1$ agents, agent $\mu^{-1}(2)$ announced the smallest b_{i2} therefore he is assigned to object 2, and so forth.

Assume on the contrary that there exists a profitable deviation by a group of agents T. Given the strategy profile after deviation $B' = (b'_T, b_{-T})$, the mechanism g selects an allocation (ν, p'). Since before deviation all agents $i \in T$ preferred their object to any other object and for a deviation to be profitable it must be that

$$a_{i\nu(i)} - p'_{\nu(i)} \geq a_{i\mu(i)} - p_{\mu(i)} \geq a_{i\nu(i)} - p_{\nu(i)} \tag{2.8}$$

with the first inequality strict for at least one agent $i \in T$. It follows that for all $i \in T$

$$p'_{\nu(i)} \leq p_{\nu(i)} \tag{2.9}$$

with at least one inequality strict. Thus there exits an object j whose price has strictly decreased: $p'_j < p_j$. Observe that if $T = I$ it follows immediately that the new price vector does not sum to C, a contradiction.

If $T \subsetneq I$ choose one of the objects j whose price has decreased the most. Since after the deviation the selected allocation $g(B') = (\nu, p')$ is envy-free with respect to the matrix B' then for each non-deviating agent $i \in I \backslash T$ we have an inequality

$$b_{i\nu(i)} - p'_{\nu(i)} \geq b_{ij} - p'_j. \tag{2.10}$$

Using the fact that before the deviation $b_{i\nu(i)} - p_{\nu(i)} = b_{ij} - p_j$ since $b_i = p + c_i$ we obtain for each agent $i \in I \backslash T$ that

$$0 > p'_j - p_j \geq p'_{\nu(i)} - p_{\nu(i)} \tag{2.11}$$

for the assignment ν. Thus it follows that $p'_{\nu(i)} < p_{\nu(i)}$ for all $i \in I \backslash T$. Combining it with (2.9) and summing up over all objects gives

$$\sum_{j=1}^{n} p'_j < \sum_{j=1}^{n} p_j,$$

a contradiction since both price vectors must sum to C. ■

Lemma 1 says that an envy-free allocation can be supported as a strong Nash equilibrium of (A, g). Therefore, if $(\mu, p) \in G(A)$ then $(\mu, p) \in O^{SNE}_{(A,g)}$. Note also that the proof does not depend on any particular way the prices are determined as long as they are envy-free with respect to the announced matrix B.

Lemma 2 *Let B be a strategy profile such that $g(B) = (\mu, p) \notin G(A)$. Then B is not a Nash equilibrium of (A, g).*

Proof: Let $g(B) = (\mu, p)$ be given and assume that agent i envies object j:

$$u_i(p_j) > u_i(p_{\mu(i)}). \tag{2.12}$$

I will construct a profitable deviation in two steps. In the first step consider a possible deviation b_i' where agent i announces

$$b_{ij}' = w(B, I, J) - w(B, I\backslash\{i\}, J\backslash\{j\}) \geq b_{ij}$$

and $b_{ik}' = b_{ik}$ for all $k \neq j$. According to (2.5) we can distinguish between two cases before the deviation. First, there was an efficient assignment $\nu \in M^B$ such that $\nu(i) = j$. Then we have $const_1 \leq w(B, I\backslash\{i\}, J\backslash\{j\}) + b_{ij} = w(B, I, J)$. Then by the construction of the deviation $b_{ij}' = b_{ij}$ and $w(B', I, J) = w(B, I, J)$. Second, there was no efficient assignment $\nu \in M^B$ such that $\nu(i) = j$. It implies that $w(B, I\backslash\{i\}, J\backslash\{j\}) + b_{ij} < const_1 = w(B, I, J)$. By substituting this result for $const_1$ in equation (2.5) but applied to calculate $w(B', I, J)$, it again follows that $w(B', I, J) = w(B, I, J)$. Since after the deviation every assignment that achieves the coalitional worth equal to $w(B', I, J)$ is efficient, it follows that all assignments that were efficient before the deviation remain efficient after. That is, the deviation b_i' was constructed in such a way that no assignment that was efficient is destroyed by the deviation and if the deviation adds an additional efficient assignment, it must assign agent i to object j: if $\nu \in M^B$ then $\nu \in M^{B'}$, and if $\nu \in M^{B'}$ but $\nu \notin M^B$ then $\nu(i) = j$. It follows that $\mu \in M^{B'}$. Therefore we can take the assignment μ to find the agent-optimal price of any object $l \in J$ after the deviation according to (2.4):

$$p_{*l}' = w(B', I\backslash\{\mu^{-1}(l)\}, J) - w(B', I\backslash\{\mu^{-1}(l)\}, J\backslash\{l\}). \tag{2.13}$$

Since $w(B', I, J) = w(B, I, J)$ and $b_{\mu^{-1}(l)l}' = b_{\mu^{-1}(l)l}$ for all $l \in J$ because the only valuation to change was b_{ij} but $\mu(i) \neq j$, therefore the second term of (2.13) does not change:

$$w(B', I\backslash\{\mu^{-1}(l)\}, J\backslash\{l\}) = w(B', I, J) - b_{\mu^{-1}(l)l}' = w(B, I, J) - b_{\mu^{-1}(l)l}.$$

By Proposition 1 the first term is weakly increasing in b_{ij}:

$$w(B', I\backslash\{\mu^{-1}(l)\}, J) \geq w(B, I\backslash\{\mu^{-1}(l)\}, J).$$

Therefore none of the agent-optimal prices can decrease as a result of the deviation.

In the continuation I analyze the following two cases:

Case 1 *The agent-optimal price of object j strictly increases: $p'_{*j} > p_{*j}$.*

It means that

$$w(B', I\backslash\{\mu^{-1}(j)\}, J) > w(B, I\backslash\{\mu^{-1}(j)\}, J).$$

This can only happen if $b'_{ij} > b_{ij}$, which means that there was no efficient assignment $\nu \in M^B$ that would allocate agent i to object j. Applying (2.5) we obtain that[7]

$$\begin{aligned} w(B', I\backslash\{\mu^{-1}(j)\}, J) &= const_2 + b'_{ij} > \\ w(B, I\backslash\{\mu^{-1}(j)\}, J) &= \max(const_1, const_2 + b_{ij}). \end{aligned} \tag{2.14}$$

Now consider a deviation where agent i announces, given a sufficiently small $\epsilon > 0$,

$$b''_{ij} = b'_{ij} - \epsilon > b_{ij}$$

and $b''_{ik} = b_{ik}$ for all $k \neq j$. In what follows I compare the strategy profile after the deviation $B'' = (b''_i, b_{-i})$ with the initial strategy profile $B = (b_i, b_{-i})$. First, after the deviation the set of efficient assignments does not change $M^B = M^{B''}$, and so does the selected assignment: $\mu \in M^B$. Second, using the same argument as when discussing the deviation $B' = (b'_i, b_{-i})$, none of the agent-optimal prices can decrease as a result of the deviation. Third, from (2.14) it follows that

$$w(B', I\backslash\{\mu^{-1}(j)\}, J) > w(B'', I\backslash\{\mu^{-1}(j)\}, J) > w(B, I\backslash\{\mu^{-1}(j)\}, J),$$

[7]Observe that the values of constants $const_1$ and $const_2$ change depending on the sets $T \subseteq I$ and $Q \subseteq J$ but not on the value of b_{ij} while keeping the rest b_{kl}, for $k \neq i$ and $l \neq j$, fixed.

and, as a result, $p''_{*j} = p'_{*j} - \epsilon > p_{*j}$. Fourth, the agent-optimal price of object $\mu(i)$ does not change $p_{\mu(i)} = p''_{\mu(i)}$ since by (2.4) it does not depend on the valuations of agent i. Then, according to (2.7), agent i is strictly better off after the deviation b''_i, thus strategy profile B was not an equilibrium.

Case 2 *The agent-optimal price of object j remains the same: $p'_{*j} = p_{*j}$.*

First I argue that this case implies that none of the agent-optimal prices will change due to the deviation b'_i, namely, $p'_{*k} = p_{*k}$ for all $k \in J$. From Proposition 2, in order to check whether a price vector is envy-free, it is sufficient to consider any efficient assignment. Choose $\mu \in M^B$ since by the construction of B' the assignment $\mu \in M^{B'}$. Clearly, the only agent who could feel envy under the price vector p_*, given the matrix of valuations B', is agent i and only with respect to object j, that is, $b_{i\mu(i)} - p_{*\mu(i)} < b'_{ij} - p_{*j}$. However, the deviation was constructed to ensure that there exists an assignment $\nu \in M^{B'}$ such that $\nu(i) = j$. Corollary 1 says that $b_{i\mu(i)} - p'_{*\mu(i)} = b'_{ij} - p_{*j}$. Agent i cannot affect the agent-optimal price of the object he is assigned to under some efficient assignment rule, therefore $p'_{*\mu(i)} = p_{*\mu(i)}$. Thus nobody feels envy relative to B' under price vector p_*. And it was argued before that as a result of the deviation b'_i, the agent-optimal prices cannot decrease, therefore p_* must be the vector of agent-optimal prices after the deviation.

Now consider a deviation b''_i where agent i announces, for sufficiently small ϵ,

$$b''_{ij} = b'_{ij} + \epsilon$$

and $b''_{ik} = b_{ik}$ for all $k \neq j$. After the deviation all efficient assignments will allocate object j to agent i: $\nu(i) = j$ for all $\nu \in M^{B''}$ and $B'' = (b''_i, b_{-i})$. In what follows I compare the situation when the strategy profile $B' = (b'_i, b_{-i})$ was used with the strategy profile $B'' = (b''_i, b_{-i})$. Take any efficient assignment after the deviation b''_i: $\nu \in M^{B''}$. This assignment was efficient before the deviation: $\nu \in M^{B'}$. Again, agent i cannot affect his own agent-optimal price, here, the price of object j. According to (2.4), before the deviation the price of any object $l \neq j$ is equal to

$$p_{*l} = w(B', I \backslash \{\nu^{-1}(l)\}, J) - w(B', I \backslash \{\nu^{-1}(l)\}, J \backslash \{l\}), \tag{2.15}$$

where

$$w(B', I \backslash \{\nu^{-1}(l)\}, J \backslash \{l\}) = w(B', I \backslash \{\nu^{-1}(l), i\}, J \backslash \{l, j\}) + b'_{ij}$$

since $\nu(i) = j$. After the deviation b''_{ij} the second term of (2.15) has increased by ϵ, that is,

$$w(B'', I \backslash \{\nu^{-1}(l)\}, J \backslash \{l\}) = w(B', I \backslash \{\nu^{-1}(l)\}, J \backslash \{l\}) + \epsilon$$

for all $l \neq j$. The first term of (2.15) before the deviation is

$$w(B', I \backslash \{\nu^{-1}(l)\}, J) = \max(const_1, const_2 + b'_{ij}).$$

Therefore, after the deviation b''_{ij}, it belongs to the interval:

$$w(B', I \backslash \{\nu^{-1}(l)\}, J) + \epsilon \geq w(B'', I \backslash \{\nu^{-1}(l)\}, J) \geq w(B', I \backslash \{\nu^{-1}(l)\}, J).$$

It follows that the agent-optimal prices of objects other than j cannot increase and each of them can decrease at most by ϵ: $p_{*l} - \epsilon \leq p''_{*l} \leq p_{*l}$ for all $l \neq j$. Since, according to (2.7), the utility of agent i is increasing in the agent-optimal prices paid by other agents, consider the worst case: $p''_{*l} = p_{*l} - \epsilon$ for all $l \neq j$. Then the utility of the agent i after the deviation is $u_i(p''_j) = u_i(p_j) - \frac{n-1}{n}\epsilon$. By (2.12), for sufficiently small ϵ,

$$u_i(p''_j) = u_i(p_j) - \frac{n-1}{n}\epsilon > u_i(p_{\mu(i)}).$$

Thus, for sufficiently small ϵ, announcing

$$b''_{ij} = w(B, I, J) - w(B, I \backslash \{i\}, J \backslash \{j\}) + \epsilon$$

is a profitable deviation for agent i and the matrix B could not form a profile of Nash equilibrium strategies. ■

From Lemma 2 it follows that if B is a Nash equilibrium of (A, g) it must be envy-free, that is, if $g(B) \in O^{NE}_{(A,g)}$ then $g(B) \in G(A)$. Lemma 1 already established the converse inclusion $G(A) \subseteq O^{SNE}_{(A,g)} \subseteq O^{NE}_{(A,g)}$. Therefore $G(A) = O^{SNE}_{(A,g)} = O^{NE}_{(A,g)}$. Thus I have proven that the simple and natural mechanism double implements the set of efficient and envy-free allocation both in Nash and strong Nash equilibrium. That is, the sets of Nash and strong Nash equilibrium outcomes and envy-free allocations coincide.

2.6 Concluding Remarks

Given the announced preference profile, the mechanism selects a particular envy-free price vector although there may exist other envy-free prices. Abdulkadiroğlu *et al.* [1] point to some advantages of the price selected by the algorithm. In particular, if there exists an envy-free price vector that is nonnegative, then the price selected by their algorithm must also be nonnegative. In any case, the use of the given price selection rule entails no loss of generality since with the same rule one can achieve any envy-free price with respect to the true preferences as an equilibrium outcome of the game.

A feature of the mechanism is that an equilibrium strategy profile B will usually imply multiple efficient assignments with respect to the announced valuations. Therefore the mechanism always needs to rely on the tie-breaking rule to select the right assignment. However, the set of possible equilibria is not affected by the particular tie-breaking rule. One could substitute the present tie-breaking rule with any other rule that selects correctly the efficient assignment with respect to the true valuations. For example, a valid tie-breaking rule could be that additionally to their valuations agents announce the object they prefer. In equilibrium, each agent would announce the object that would be assigned to him if the true valuations were known. An advantage of the proposed tie-breaking rule however is that it requires that agents announce only their own valuations.

Notice that the mechanism does not ensure individual rationality, that is, agents may get lower utility by playing the game than by choosing not to participate. Thus it implicitly assumes that agents are forced to participate in the game. The reason is that the model imposes an exogenous price constraint which if big enough, rules out the existence of individually rational and envy-free allocations. On the other hand, if the constraint is variable so that it accommodates individual rationality, one could simply apply the results of Leonard [10] and Demange *et al.* [7] to implement in dominant strategies.

The model explicitly assumes that the number of agents and objects is the same. If the number of agents exceeded the number of objects one could introduce fictitious objects and the previous analysis would still apply. However, when the number of objects exceeds the number of agents, the introduction of fictitious agents does not work since it implies that some fictitious agent would need to pay a price or receive a transfer of the object he is assigned to. As a result the actually paid prices would not meet the price constraint.

2.7 Appendix: Proofs of Propositions 1 and 2

Proposition 1 *Given a matrix of valuations A, the coalitional function $w(A,T,Q)$ is continuous and weakly increasing in a_{ij}.*

Proof: If either $i \notin T$ or $j \notin Q$ then $w(A,T,Q)$ does not depend on a_{ij} and can be treated as constant - obviously continuous and weakly increasing in a_{ij}. Assume that $i \in T$ and $j \in Q$. Given a solution $(x_{ij})_{(i,j)\in T\times Q}$ to the primal problem, we can write equation (2.1) in the following form:

$$w(A,T,Q) = \sum_{(k,l)\in T\times Q\setminus\{(i,j)\}} a_{kl}x_{kl} + a_{ij}x_{ij}. \tag{2.16}$$

First, I claim that if as a result of the change from a_{ij} to a'_{ij}, keeping the rest of valuations fixed, there is no change in $x_{ij} = x'_{ij}$, then there is no change in the solution $x_{kl} = x'_{kl}$ for all $(k,l) \in T \times Q$. Assume, on the contrary, that $x_{kl} \neq x'_{kl}$ for some $(k,l) \in T \times Q \backslash \{(i,j)\}$, and that x'_{kl} for all $(k,l) \in T \times Q$ was not another solution of the original problem. Then we can write the system of equations:

$$\begin{aligned} \sum_{(k,l) \in T \times Q \backslash \{(i,j)\}} a_{kl} x'_{kl} + a_{ij} x_{ij} &< \sum_{(k,l) \in T \times Q \backslash \{(i,j)\}} a_{kl} x_{kl} + a_{ij} x_{ij} \\ \sum_{(k,l) \in T \times Q \backslash \{(i,j)\}} a_{kl} x'_{kl} + a'_{ij} x_{ij} &\geq \sum_{(k,l) \in T \times Q \backslash \{(i,j)\}} a_{kl} x_{kl} + a'_{ij} x_{ij}, \end{aligned}$$

where the first inequality holds under original valuations and the second holds after the change in a_{ij}. Thus we obtain a contradiction. Given this result, we can write equation (2.1) as

$$w(A,T,Q) = \max(const_1 + a_{ij} \cdot 0, const_2 + a_{ij} \cdot 1). \tag{2.17}$$

The function in (2.17) is obviously continuous and weakly increasing in a_{ij}. Note that $const_2 = w(A, T \backslash \{i\}, Q \backslash \{j\})$ since agent i has been assigned to object j and each agent can be assigned to at most one object and vice versa. ■

Proposition 2 *Given a matrix of valuations A, the set of envy-free prices is the same for all efficient assignments of objects.*

Proof: Take any two efficient assignments μ_1 and μ_2. Assume, on the contrary, that the price vector p is envy-free for the assignment μ_1 but it is not envy-free for the assignment μ_2. Envy-freeness of μ_1 implies that

$$a_{i\mu_1(i)} - p_{\mu_1(i)} \geq a_{i\mu_2(i)} - p_{\mu_2(i)} \tag{2.18}$$

for all $i \in I$. Assume without loss of generality that agent 1 envies object j under assignment μ_2:

$$a_{1\mu_1(1)} - p_{\mu_1(1)} \geq a_{ij} - p_j > a_{1\mu_2(1)} - p_{\mu_2(1)}. \tag{2.19}$$

Summing up equation (2.18) across all agents and using equation (2.19) we obtain

$$\sum_{i \in I} a_{i\mu_1(i)} - \sum_{i \in I} p_{\mu_1(i)} > \sum_{i \in I} a_{i\mu_2(i)} - \sum_{i \in I} p_{\mu_2(i)},$$

contradicting the assumption that μ_2 was an efficient assignment. ■

Bibliography

[1] A. Abdulkadiroğlu, T. Sönmez, and M.U. Ünver. Room assignment - rent division: A market approach. *Social Choice and Welfare*, 22:515–538, 2004.

[2] A. Alkan, G. Demange, and D. Gale. Fair allocation of indivisible objects and criteria of justice. *Econometrica*, 59:1023–1039, 1991.

[3] E. Aragonés. A derivation of the money rawlsian solution. *Social Choice and Welfare*, 12:267–276, 1995.

[4] C. Beviá. Manipulation games in economies with indivisible goods. May 2001.

[5] S.J. Brams and D.M. Kilgour. Competitive fair division. *Journal of Political Economy*, 109:418–443, 2001.

[6] S. Chin Sung and M. Vlach. Competitive envy-free division. *Social Choice and Welfare*, 23:103–111, 2004.

[7] G. Demange, D. Gale, and M. Sotomayor. Multi-item auctions. *Journal of Political Economy*, 94:863–872, 1986.

[8] C.-J. Haake, M.G. Raith, and F.E. Su. Bidding for envy-freeness: A procedural approach to n-player fair-division problems. *Social Choice and Welfare*, 17:201–215, 2002.

[9] F. Klijn. An algorithm for envy-free allocations in an economy with indivisible objects and money. *Social Choice and Welfare*, 17:201–215, 2000.

[10] H.B. Leonard. Elicitation of honest preferences for the assignment of individuals to positions. *Journal of Political Economy*, 91:461–479, 1983.

[11] K. Roberts. The characterization of implementable choice rules. In Laffont J.J., editor, *Aggregation and Revelation of Preferences*, chapter 18, pages 321–349. North-Holland, Amsterdam, 1979.

[12] A.E. Roth and M.A. Oliveira Sotomayor. *Two-sided matching: A study in game-theoretic modeling and analysis.* Cambridge University Press, 1990.

[13] L. S. Shapley and M. Shubik. The assignment game 1: The core. *International Journal of Game Theory*, 1:111–130, 1972.

[14] K. Tadenuma and W. Thomson. Games of fair division. *Games and Economic Behavior*, 9:191–204, 1995.

Chapter 3

Incumbency and Entry in License Auctions: The Anglo-Dutch Auction Meets Other Simple Alternatives

3.1 Introduction

One of the salient features of the recent wave of spectrum license auctions has been the disparity of prices across experiences. As an example, the Dutch auction (meaning, the auction in the Netherlands) of licenses for the new UMTS fetched less than a third than the corresponding British auction in per capita terms. One of the explanations proposed for such disparity is the disparities across countries in the ratio of incumbents to licenses, and the effects of this ratio on entry and competition for licenses. (See, for instance, Klemperer [4] and Milgrom [7] .)

Certainly, insufficient entry may limit competition during the bidding process, leading to a low price. The entry decision of a firm depends on a comparison between its costs of participation and its expected benefits. For a potential bidder the costs may include resources needed to assess the value of the license, interest foregone on deposits, etc. These are sunk costs that a firm incurs before it knows whether it wins a license. Therefore, the firm will decide to participate only if it believes that its odds to win the auction are sufficient.

But, why is the ratio of incumbents to licenses important in this regard? First, incumbents are strong competitors. Indeed, they may have a client base and lower expected network roll-out cost. That is, their expected valuation of a license is higher. Also, their cost of market prospective may be significantly lower. If there are at least as many incumbents as licenses, then entrants will probably assess as too likely that all licenses will end up in the hands of incumbents, and then entry is not likely to occur. That needs not be either inefficient or problematic in terms of expected license revenue if there are more incumbents than licenses. Yet, if the number of incumbents and licenses coincide, the lack of entry will destroy all sources of competition, and that will have an enormous effect on revenue.

When attracting entrants is a goal, the allocation mechanism should favor entrants over incumbents. For instance, Dutch (or first-price) auctions tilt the allocation in favor of ex-ante weaker bidders, and thus weaker bidders prefer Dutch auctions to efficient, English auctions (see Maskin and Riley [5]). Based on this fundamental insight, Paul Klemperer (see Klemperer [3]) and others have proposed the use of the so-called Anglo-Dutch auction when a number of identical objects (licenses) are to be allocated and an identical number of ex-ante stronger incumbents are potential buyers. An Anglo-Dutch auction is a mixture of the two types of auction. It begins with an 'English' phase during which the price rises until all but a number of bidders that exceeds by one the number of objects drop out. At this moment (and price), the auction switches to a second 'Dutch' phase. In this stage, only the remaining bidders can submit (simultaneous, sealed) bids and only bids above the price at which the English phase stopped are allowed.

The first goal of this paper is to show in a very simple model how this auction indeed improves the expected revenues of the seller at the cost of sacrificing efficiency. But our main goal is to investigate other simple alternatives that dominate, both in terms of efficiency and revenues, the Anglo-Dutch auction. We propose what we could term Anglo-Anglo auction: a two-stage, English auction. The design is inspired by Burguet and Sákovics [2], and consists of two

English phases, the first one run with a (relatively high) reserve price. Instead of using inefficiencies or allocation preference as the tool to induce entry, what the two-stage, English auction uses is the information conveyed by the (absence of) bidding in the first phase. Indeed, if some of the participants in the first phase (incumbents included) are unwilling to bid above the reserve price, they will be perceived as "weaker than expected" bidders. Thus, potential entrants that did not venture to enter in the first round may now consider doing so for the second round. We show that a two-stage, English auction is more efficient than both the English and the Anglo-Dutch auctions. By allowing entry conditional on some private information (entry conditional on bidding behavior), the two-stage entry auction improves upon the most efficient one-stage entry auction, namely, the English auction. Moreover, we show that the gain in efficiency (entry) benefits the seller as well. Indeed, the revenues for the seller are higher in the two-stage, English auction than in the Anglo-Dutch auction.

The analysis is carried out in an extremely simple model, presented in Section 3.2, where entrants have to incur a cost before learning their valuations. Moreover, valuations can take only two different values, although incumbents have a higher probability of high valuation. We simplify further by assuming only one unit for sale and one incumbent. The analysis of this model and the results are presented in Section 3.3. Subsequently, we extend the model in two main directions: multiple units and continuous valuations.

The extension to continuous types is relevant. Indeed, in the two-types model, the Anglo-Dutch is allocationally efficient, in the sense that the object is never assigned to a bidder that competes against a bidder with higher valuation. That is, the only inefficiency may come from inappropriate (excessive) entry. The allocation is still tilted in favor of the ex-ante weaker bidder: when two bidders have the same type, an event with positive probability, the ex-ante weaker bidder will win the auction (will bid higher) with higher probability. Yet one can suspect that by not considering the allocation inefficiency (a tool to reduce informational rents of ex-ante stronger bidders) of Dutch auctions, their revenue generation potential is underestimated. The analysis of the continuous valuations case necessarily relies on numerical computations, since there is no analytical solution to asymmetric Dutch auctions (the second stage of an Anglo-Dutch auction) and the bidding behavior (when to bid) in a two-stage English auction has no simple closed form. Using numerical methods for a family of simple continuous distributions, we obtain the same results as in the base model.

Then we consider multiple units (and the same number of incumbents). Here entry decisions in the second stage of the two-stage, English auction depends on the number of units that are sold in the first stage. The larger the number of units left unsold the larger the number of entrants in the second stage. In fact, under very extreme values of the parameters, the effect that the number of units available has on entry is very extreme. Only in such cases can Anglo-Dutch auctions dominate in terms of revenues the two-stage, English auction. Otherwise, our results for the one unit case hold in the multiple unit case.

3.2 Rules of the auctions

There are q identical units available for sale and the same number of incumbents. Besides there is a sufficiently large number of potential entrants. In order to learn his valuation and prepare his bid, an entrant has to incur a cost c. To simplify the analysis, we assume that incumbents already know their valuations and incur no further cost of participation. Each bidder has demand only for one unit. Valuations are private and independently distributed. The valuations of incumbents and entrants are drawn from distribution functions $F_1(v)$ and $F_2(v)$, respectively.

Rules of the Anglo-Dutch auction: Before bidding starts entrants decide whether or not to incur cost c and to learn their valuations.[1] The auction starts as an English auction where bidders continuously raise their bids. We use the clock modelling, so that once a bidder drops he cannot reenter the auction. When $q+1$ bidders are left, the auction switches to the Dutch auction, or more precisely, to the discriminatory auction, which is a generalization of first-price auction for multiple units. Thus, surviving bidders simultaneously bid in this stage, and the q bidders with the highest bids will win one license each. Winners pay their bids. In this Dutch stage, the price at which the last bidder dropped out in the English stage is set as a reserve price or minimum acceptable bid.

In the discrete valuation case, several bidders may drop out from the English auction at a given price, leaving less than $q+1$ bidders active. In that case, we will assume that some of these simultaneously dropping bidders are randomly selected to participate in the Dutch stage so that $q+1$ bidders are still present.

Rules of the two-stage English auction: In the first stage the seller sets a reserve price r, common to all units, and then entrants decide whether to enter or not. Then the oral auction starts. Units are awarded to, at most, the q last bidders to drop. The price is the maximum of r and the price at which only q bidders stay. If less than q bidders offer the reserve price, so that some units fail to sell in the first stage, these remaining units are auctioned in the second stage with the reserve price now set equal to zero. Before this second stage starts, entrants who did not enter in the first stage have a new chance to enter and compete for the remaining units against incumbents and first-stage entrants who abstained from bidding in the first stage. Again, if q' units were left unsold after the first stage, they are awarded to the q' bidders at the price at which only these many bidders are left.

[1] We will consider pure strategy equilibria in entry. That allows us to leave the total number of potential entrants unspecified. Thus, we will assume at the outset some exogenous order of entry that entrants would consider natural.

3.3 Discrete valuations, one unit case

Each bidder's valuation may be $\overline{v}$ or $\underline{v}$ with $\overline{v} > \underline{v}$. Bidder 1, the incumbent, has probability μ_1 of having the high valuation $\overline{v}$, and all other bidders (entrants) have probability μ_2 of having high valuation. We assume that $\mu_1 > \mu_2$.

3.3.1 Anglo-Dutch auction

Assume that, apart from the incumbent, bidder 1, n bidders enter the auction. The English stage will stop at the price at which only two bidders remain in the auction. Also, in this stage bidders' weakly dominant strategy is to stay in the auction until the price reaches their valuations. Thus, if there are more than two bidders with valuation $\overline{v}$, the price will continue increasing until it reaches this value. In such case, the English stage stops at price $\overline{v}$, two bidders are randomly selected to bid in the Dutch part, where they bid $\overline{v}$. On the other hand, if two or less bidders have valuation $\overline{v}$, the English stage stops at price $\underline{v}$. In this case, bidders with high valuation, if any, stay for the Dutch stage. If there are less than two bidders with high valuation, then bidders among dropping (low valuation) bidders are randomly selected so that two bidders take part in the Dutch stage. We assume that bidders recognize each other, so that the identity of bidder 1 known. However, when the English stage stops at price $\underline{v}$, the participants in the Dutch part cannot be sure whether their opponent has been randomly selected among the dropping bidders (i.e., has a low valuation) or not (i.e., has a high valuation).

Bidding in the Dutch stage

We only need to analyze subgames in which the clock stops at price $\underline{v}$ in the English stage. Assume bidder 1 and an entrant play the Dutch stage. It is easy to see that there could be no pure strategy equilibrium in the bidding game. Also, in equilibrium bidders bid $\underline{v}$, the reserve price set by the English stage, if that is their valuation. For valuation $\overline{v}$ bidders, we consider only bidding in an interval. It is easy to rule out bidding (with positive probability) strictly above the supremum of the set of bids that the rival use with positive probability. That is, this supremum needs being common to both bidders. Also, it is equally easy to rule out mass points at such supremum, as is easy to rule out mass points (or "holes") anywhere in the interior of the intervals of bids used in equilibrium. Finally, equilibrium where the infimum of the intervals are different, or different from $\underline{v}$ can be ruled out as well: there is no point in bidding in the interior of an open interval with zero probability of containing a rival bid. Finally, we can rule out that both bidders' strategies contain a mass point at $\underline{v}$, but we cannot rule out that one of the bidders' strategy has a mass point there.

To avoid open-set problems, we assume that in case two bidders with different valuations tie in the Dutch auction then the winner is the one with higher valuation. We will comment on this tie-breaking rule later.

Thus, let us characterize an equilibrium where bidders bid $\underline{v}$ when their valuation is $\underline{v}$, and bid on $[\underline{v}, \overline{b}]$ when their valuation is $\overline{v}$, for some $\overline{b}$. The entrant bids according to a distribution H_2, and the incumbent, bidder 1, bids according to a distribution H_1. Clearly, bidders with valuation $\underline{v}$ expect zero profits. Since $\overline{b}$ is common and H_i, for $i = 1, 2$, has no atoms at $\overline{b}$ then the expected profits of either an entrant or the incumbent that has valuation $\overline{v}$ when the clock stops at price $\underline{v}$ in the English stage equals $[\overline{v} - \overline{b}]$. Indeed, by bidding $\overline{b}$ either bidder expects to win with probability 1. Since expected profits should be independent on the pure strategy played in a mixed strategy equilibrium, we conclude that the entrant and the incumbent will expect the same equilibrium profit upon entry if they have the same valuation.

This is in fact the way the Anglo-Dutch auction is expected to foster entry. Indeed, notice that entrants with high valuation expect lower profits than incumbents in a standard English auction, since the rival has higher expected valuation. The Dutch stage tilts competition in favor of entrants, and in the case of discrete valuations, this is enough to perfectly level the field.

In order to characterize $\overline{b}$ and H_i, for $i = 1, 2$, we need to consider the posterior beliefs for each type of bidder about the rival bidder. Conditioning of being one of the two bidders in the Dutch auction when the English auction stopped at $\underline{v}$, from the point of view of a rival with high type, an entrant's type is $\overline{v}$ with probability

$$\gamma_2 = \frac{\mu_2}{\mu_2 + \frac{(1-\mu_2)}{n}}.$$

Indeed, given that stopping price (i.e., conditioning on no more than 1 rival bidder having high valuation), the rival knows that the entrant would be one of the participants in the Dutch stage for sure if his valuation is high, and with probability $\frac{1}{n}$ in case his valuation is (as everybody else's) low. Similarly, the rival with high valuation updates his beliefs about the incumbent having high valuation assigning this event a probability

$$\gamma_1 = \frac{\mu_1}{\mu_1 + \frac{(1-\mu_1)}{n}}.$$

Observe that $\gamma_1 > \gamma_2$. With these posteriors, the expected profit for bidder 1 is, for all b on $(\underline{v}, \overline{b}]$,

$$\pi_1(b) = [(1 - \gamma_2) + \gamma_2 H_2(b)] (\overline{v} - b). \tag{3.1}$$

Similarly, for bidder 2, and for all b in $(\underline{v}, \overline{b}]$

$$\pi_2(b) = [(1 - \gamma_1) + \gamma_1 H_1(b)] (\overline{v} - b). \tag{3.2}$$

Thus, given $\overline{b}$, (3.1) and (3.2) characterize H_i, for $i = 1, 2$. Notice that since $\pi_1(b) = \pi_2(b)$ but $\gamma_1 > \gamma_2$, $H_2(b) < H_1(b)$ for all b. That is, as is usually the case in the Dutch auction, the ex-ante weaker bidder bids more aggressively. Now we apply the condition that the infimum of the intervals of both mixed strategies should be $\underline{v}$. Since only one bidder can have a mass point at that infimum, and $H_2(b) < H_1(b)$ for all b, we conclude that $H_2(\underline{v}) = 0$. That is, from (3.1)

$$\pi_1 = (1 - \gamma_2) (\overline{v} - \underline{v}) = \pi_2. \tag{3.3}$$

Substituting in (3.1) and (3.2), we obtain

$$H_2(b) = \frac{1-\gamma_2}{\gamma_2}\frac{b-\underline{v}}{\overline{v}-b}.$$

and

$$H_1(b) = \frac{1-\gamma_1}{\gamma_1}\frac{b-\underline{v}}{\overline{v}-b} + \frac{\gamma_1-\gamma_2}{\gamma_1}\frac{\overline{v}-\underline{v}}{\overline{v}-b},$$

so that

$$H_1(\underline{v}) = \frac{\gamma_1-\gamma_2}{\gamma_1}.$$

Also, using $H_2(\overline{b}) = 1$ we obtain

$$\overline{b} = \gamma_2\overline{v} + (1-\gamma_2)\,\underline{v}.$$

The equilibrium when two entrants play the Dutch auction is even simpler. Here both bidders are symmetric and update according to γ_2 the probability that the rival has high type when he himself does. Then, their expected profits are given by (3.1), and using $H_2(\underline{v}) = 0$ we obtain that these profits equal (3.3). This implies that $H_2(b)$ obtained above still describes the equilibrium, except that now this is the common bidding strategy for both rivals.

We can summarize all that has been obtained in the following

Lemma 3 *If selected to play the Dutch stage when the English stage stops at price* $\underline{v}$*,*

i) entrants expect zero profits if their valuation is $\underline{v}$ *and profits* $(1-\gamma_2)\,(\overline{v}-\underline{v})$ *if their valuation is* $\overline{v}$ *independently of the identity of the rival,*

ii) entrants bid independently of the identity of the rival, but more aggressively than the incumbent.

We should comment now on the role of our tie-breaking assumption. We assume that in case two bidders bid $\underline{v}$, which occurs with positive probability when the incumbent has valuation $\overline{v}$ and the entrant has valuation $\underline{v}$, the bidder with high valuation wins. This allows the latter to bid $\underline{v}$ and still expect to win with probability γ_2. That is, this allows the incumbent to identify the lowest bid that still allows him to defeat the bid of a rival with type $\underline{v}$. Therefore, the tie-breaking assumption comes to play the role of a smallest unit of money. Notice, however, that we do not need assuming that the incumbent has preference when bidding against an entrant with type $\overline{v}$. That is, an entrant that may defeat the incumbent.

We now turn to the analysis of entry and the English stage.

Entry and bidding in the English stage

First we compute the profits that an entrant expects net of the entry cost. Notice that the entrant expects positive profits only when his type is $\overline{v}$ and no more than one other bidder has this type. His profits in this case are independent of the pure strategy (among the ones that belong to the support of his equilibrium mixed strategy) that he chooses to play. One of these strategies is to bid $b = \underline{v}$, in which case he wins $(\overline{v} - \underline{v})$ when all rivals have $\underline{v}$ or when the rival is the incumbent and bids $\underline{v}$. This event has probability

$$[(1-\mu_1) + \mu_1 H_1(\underline{v})]\,(1-\mu_2)^{n-1}.$$

Thus, substituting for $H_1(\underline{v})$, the expected profits of an entrant are

$$\Pi^a(n) = \mu_2\,(1-\mu_2)^{n-1}\left[\frac{(\mu_1-\mu_2)}{(n-1)\mu_2+1} + (1-\mu_1)\right](\overline{v} - \underline{v}) - c. \tag{3.4}$$

This is a decreasing function of n. Entry occurs to the point where the above expression is non negative, and the same expression is negative for $n+1$. That is, treating n as a continuous variable, the number of entrants in the Anglo-Dutch auction, n^a, satisfies $\Pi^a(n^a) = 0$.

Compare this entry decision with the entry decision in a standard English auction. Again, treating n as a continuous variable, the number of entrants in a standard English auction, n^e, solves

$$\mu_2\,(1-\mu_1)\,(1-\mu_2)^{n^e-1}\,(\overline{v} - \underline{v}) - c = 0. \tag{3.5}$$

Since $\frac{(\mu_1-\mu_2)}{(n-1)\mu_2+1} > 0$, and $\Pi^a(n)$ is decreasing in n, we conclude that

Lemma 4 *The Anglo-Dutch auction promotes entry beyond what is obtained in the standard English auction:* $n^a \geq n^e$.

To conclude with the Anglo-Dutch auction, we can compute the profits expected by the incumbent. Again, planning to bid $\underline{v}$ in the Dutch stage when his type is $\overline{v}$ and the clock stops at $\underline{v}$ at the English stage, the incumbent's expects profits are

$$\mu_1 (1-\mu_2)^n (\overline{v} - \underline{v}). \tag{3.6}$$

3.3.2 Two-stage English auction

Assume that, besides the incumbent, k new entrants enter and learn their valuations in the first stage and, if nobody bids, that is, if all bidders drop before the reserve price r is called, then some additional l bidders enter in the second stage. We look for a separating equilibrium where high valuation bidders of the first stage prefer to bid (i.e., prefer to stay past the reserve price r) while low valuation bidders abstain from bidding in the first stage.[2]

Thus, assume all bidders (entrants and incumbent) with valuation $\overline{v}$ stay past the reserve price $r \in (\underline{v}, \overline{v})$. Then it is weakly dominant to drop only when the price in the first stage reaches $\overline{v}$, since dropping before implies zero profits. Bidding in the second stage for all $k+l+1$ participants is simple: again it is weakly dominant to drop at a price equal to valuation.

Apart from entry, the only other important choice for a bidder with valuation $\overline{v}$ present at the first stage is between participating in this first stage and waiting in the hope that there is a second one. The incumbent will prefer to bid in this first stage if

$$(1-\mu_2)^k(\overline{v} - r) \geq (1-\mu_2)^{k+l}(\overline{v} - \underline{v}).$$

[2]Entrants that would not "bid" in the first stage even if their valuation is high would not enter. Thus, the only pooling equilibrium having no "bids" in the first stage is one where no entrants enter and all wait until the second stage. We will consider this case below.

Indeed, in either case he will earn positive profits, $(\overline{v}-\underline{v})$ in the second stage or $(\overline{v}-r)$ in the first, only if no other participant has valuation $\overline{v}$. Similarly, first stage entrants with high type will prefer to bid if:

$$(1-\mu_1)(1-\mu_2)^{k-1}(\overline{v}-r) \geq (1-\mu_1)(1-\mu_2)^{k+l-1}(\overline{v}-\underline{v}).$$

In both cases, the restriction can be written as:

$$r \leq \overline{v} - (1-\mu_2)^l(\overline{v}-\underline{v}). \tag{3.7}$$

We now turn to the entry decisions in each stage. If the second stage of the auction takes place, potential new entrants learn that the incumbent and the k entrants in the first stage all have low valuations. Thus, treating l as a continuous variable, it satisfies[3]

$$\mu_2(1-\mu_2)^{l-1}(\overline{v}-\underline{v}) = c. \tag{3.8}$$

Notice that l is independent of r. Similarly, the zero profit (entry) condition in the first stage is

$$\mu_2(1-\mu_1)(1-\mu_2)^{k-1}(\overline{v}-r) = c. \tag{3.9}$$

Certainly k and l are integers, and thus in general neither of the conditions above are satisfied with equality. That is, entrants expect "some" positive profits. Again, if there is a "natural" order for potential entrants to enter, as we are assuming, and expected profits in the first stage are no lower than expected profits in the second this creates no additional coordination problems. Also, the seller could consider a small entry in the second stage to keep entrants at their indifference level.

If we compare (3.8) with (3.4), we can conclude that $l \geq n^a$. That is, the number of new entrants in the second stage is at least equal to the number of entrants in the Anglo-Dutch auction.

[3]In fact, it would be given by

$$l = \max\{m|\mu_2(1-\mu_2)^{m-1}(\overline{v}-\underline{v}) \geq c\}.$$

Let us define $r(k)$, for $k = 1, 2...$ as the solution in r of (3.9) above. Notice that $r(k)$ is decreasing in k. Also, denote by $r^{\times}(0)$ the solution to (3.7) with equality. This is the highest reserve price compatible with the incumbent bidding in the first stage. Notice that $r^{\times}(0) > r(1)$. Indeed, using (3.8) we have

$$r^{\times}(0) = \overline{v} - \frac{1 - \mu_2}{\mu_2} c,$$

whereas $r(1)$, substituting in (3.9) for $k = 1$, is

$$r(1) = \overline{v} - \frac{1}{\mu_2(1 - \mu_1)} c < r(0).$$

Finally, define the reserve price $r(0)$ as

$$r(0) = \overline{v} - \frac{1 - \mu_2}{\mu_2(1 - \mu_1)} c.$$

At $r(0)$, an incumbent with high valuation expects the same profits bidding in the first stage of the two-stage English auction as in a standard English auction. Notice that $r^{\times}(0) > r(0) > r(1)$.

We should note that for any reserve price in $[r(0), r^{\times}(0)]$ there exist two equilibria. In one of them, the incumbent is expected not to participate in the first stage no matter what valuation he has, and therefore the first stage never sells the license. Thus, the second stage (and the whole two-stage English auction) becomes a regular English auction. In the other equilibrium the incumbent is expected to bid in the first round if his valuation is high, so that in the second stage l new entrants enter. Given this, the incumbent indeed prefers bidding in the first stage when his valuation is high. What is important is that the first equilibrium does not exist for $r \in (\underline{v}, r(0))$, and then we will only consider reserve prices in this range.

3.3.3 Comparing total surplus

One feature of both the Anglo-Dutch auction and the two-stage English auction in this setting is that the license is assigned to the user that values it most among the ones present at the round in which it is assigned. In a more general, continuous type model this is true for the two-stage English auction, but not for the Anglo-Dutch auction. Nevertheless, in our setting efficiency comparisons depend only on the entry decisions.

The standard English auction maximizes the gains from trade among the mechanisms at which entry occurs only at one point in time. Indeed, given n entrants, a new entrant adds surplus only if the $n-1$ previous entrants and the incumbent all have valuation $\underline{v}$ and the new entrant has valuation $\overline{v}$. This event has probability $\mu_2(1-\mu_1)(1-\mu_2)^{n-1}$ and the increase in surplus is $(\overline{v}-\underline{v})$ in this case. Trading this increased expected surplus with the cost of entry c results in (3.5), the entry condition in an English auction. In this sense, the Anglo-Dutch auction fosters entry beyond what is efficient.

In a two-stage English auction, entry takes place at more than one point in time. If the license is not assigned in the first stage, then new entrants will enter to take part in a final, English auction. This second-stage entry conditional on all first-stage participants having low type $\underline{v}$ is also (conditionally) efficient. Indeed, the expected surplus, given that there is a second stage, is

$$\underline{v} + \left[1 - (1-\mu_2)^l\right](\overline{v} - \underline{v}) - lc$$

where l is given by (3.8). Now,

$$\left[1 - (1-\mu_2)^l\right](\overline{v} - \underline{v}) = (\overline{v} - \underline{v}) \sum_{m=1}^{l} \mu_2(1-\mu_2)^{m-1}.$$

Notice that there are l terms on the right hand side, and they are decreasing in m. Thus, the smallest one is $\mu_2(1-\mu_2)^{l-1}(\overline{v} - \underline{v})$. But this term is equal to c, if (3.8) is satisfied.

It should not come as a surprise that a two-stage entry mechanism may result in a higher surplus than even the most efficient one-shot entry. In fact, this is so for any reserve price choices.

Proposition 3 *The expected surplus in any two-stage English auction is higher than in a standard English auction and therefore also higher than in an Anglo-Dutch auction.*

Proof: We have already established that the second stage entry l is conditionally efficient, i.e., that l minimizes $(1-\mu_2)^l(\overline{v}-\underline{v})+lc$, also that $l \geq n^e$, and that l is independent of the reserve price, and therefore independent of first stage entry k. Notice that for $r=\underline{v}+\epsilon$ and ϵ small, (3.9) is (virtually) the entry condition in the English auction, therefore $k \leq n^e$ for any r. Now, the total net surplus from a two-stage English auction given entry decisions $k \leq n^e$ is

$$\begin{aligned}
& \overline{v}-(1-\mu_1)(1-\mu_2)^k\left[(1-\mu_2)^l(\overline{v}-\underline{v})+lc\right]-kc \\
> \; & \overline{v}-(1-\mu_1)(1-\mu_2)^k\left[(1-\mu_2)^{n^e-k}(\overline{v}-\underline{v})+(n^e-k)c\right]-kc \\
= \; & \overline{v}-(1-\mu_1)(1-\mu_2)^{n^e}(\overline{v}-\underline{v})-(1-\mu_1)(1-\mu_2)^k(n^e-k)c-kc \\
> \; & \overline{v}-(1-\mu_1)(1-\mu_2)^{n^e}(\overline{v}-\underline{v})-n^e c,
\end{aligned}$$

where the last line is the expected surplus in the standard English auction. QED

■

According to the above proposition, any reserve price, including $r=\underline{v}$ $(+\epsilon)$, the one that maximizes entry in the first round and still allows a positive probability of new entry in the second round, induces more efficient entry than the most efficient one-shot entry auction. But what is the efficient level of first-stage entry in the two-stage English auction? The answer to this question will also be relevant when discussing revenues. When there is a second opportunity to experiment, i.e., to obtain valuation draws, assigning the license in the first round has an opportunity cost above $\underline{v}$. Then efficient entry in the first stage needs not be the highest compatible with screening low valuation types. Indeed,

Lemma 5 *Maximizing surplus in a two-stage English auction requires limiting entry. In particular, efficient entry k^* satisfies*

$$r(k^*)=\underline{v}+(\overline{v}-\underline{v})\left[1-(1-\mu_2)^{l-1}(1+(l-1)\mu_2)\right]. \tag{3.10}$$

Proof: The marginal contribution of a new entrant in the first stage of the two stage English auction is

$$\mu_2(1-\mu_1)(1-\mu_2)^{k-1}[(1-\mu_2)^l(\overline{v}-\underline{v})+lc]-c. \tag{3.11}$$

Indeed, $\mu_2(1-\mu_1)(1-\mu_2)^{k-1}$ is the probability that the new entrant has a high valuation $\overline{v}$, and the rest of entrants have low valuation $\underline{v}$. In this case, we would have higher (gross) surplus with the additional entrant in the first stage if future entrants were to have low type as well, which has probability $(1-\mu_2)^l$. The additional (gross) surplus would be $(\overline{v}-\underline{v})$, but entry of the second period entrants would also imply a cost lc, which a good realization of a first stage entrant would save. Now, treating k and l as continuous variables and substituting equation (3.8), entry in the first stage should take place until the point where

$$\mu_2(1-\mu_1)(1-\mu_2)^{k-1}(1-\mu_2)^{l-1}(\overline{v}-\underline{v})(1+(l-1)\mu_2)=c.$$

It follows that efficient first stage entry $k^*<n^e$. Now, $\mu_2(1-\mu_1)(1-\mu_2)^{k-1}(\overline{v}-r)=c$. Substituting in (3.9), we obtain (3.10).QED ■

3.3.4 Comparing revenues

Revenues and efficiency are intimately related. Indeed, disregarding the integer problem, entrants expect zero profits (net of the entry cost) both in a standard English, an Anglo-Dutch, and a two-stage English auctions. Therefore, we need only looking at total surplus (net of entry costs) and the profits of the incumbent when comparing the seller's revenues in both auctions. The incumbent's profits are $\mu_1(1-\mu_2)^n(\overline{v}-\underline{v})$ both in the standard English and in the Anglo-Dutch auctions, except that n may differ in both. Thus, the revenues for the seller in each case are

$$R(n^i)=\overline{v}-(1-\mu_2)^{n^i}(\overline{v}-\underline{v})-n^ic,$$

for $i = e, a$. We can compute $R(n) - R(n-1)$ for any n to obtain

$$R(n) - R(n-1) = \mu_2 (1 - \mu_2)^{n-1} (\overline{v} - \underline{v}) - c.$$

This is decreasing in n. Thus, $R(n)$ is maximized when this expression is zero, i.e., when $n = l$ (again, treating n as a continuous variable), increasing for $n < l$, and decreasing for $n > l$. Compare this with entry decisions in Anglo-Dutch and English auctions, obtained respectively from (3.4) and (3.5) above. Since $l > n^a > n^e$,

Proposition 4 *The revenues of the seller are higher in the Anglo-Dutch auction than in the standard English auction.*

As conjectured, the Anglo-Dutch auction increases the revenues of the seller by increasing entry.

Consider now the two-stage English auction. When maximizing revenues, the seller needs only consider the maximum of the reserve prices compatible with any amount of entry, k, that is, $r(k)$ defined above. Indeed, any two values for the reserve price that induce the same first period entry (and also the same second period entry) also induce the same total (gross) surplus and the same cost of entry, whereas both the profits of first period entrants and incumbents are lower for the highest of the two reserve prices.

Then, let us compute the expected profit of the incumbent, i.e., $\mu_1(1-\mu_2)^k(\overline{v} - r(k))$ for different values of k. Substituting (3.9), we can write this expected profits as

$$\frac{\mu_1}{1-\mu_1}\frac{1-\mu_2}{\mu_2}c.$$

Thus,

Lemma 6 *The expected profits of the incumbent evaluated at $r(k)$, are independent of k.*

In other words, from the point of view of the seller's revenues, the reduction in reserve price that is necessary to attract one more entrant in the first round exactly compensates the increase of competition obtained in this way, starting from any level of entry $k \geq 1$. Thus, we have

Corollary 2 *From the point of view of the seller, the optimal reserve price is $r(k^*)$, which also maximizes total surplus.*

Remark: If we select the equilibrium where the incumbent with high valuation bids in the first stage, in the range $[r(0), r^{\times}(0)]$, then $r^{\times}(0)$ may result in higher revenues for the seller. [4]

We are now ready to compare seller's revenues in the Anglo-Dutch auction and in the two-stage English auction.

Proposition 5 *A two-stage English auction with appropriate reserve price $r(k^*)$ results in higher revenues for the seller than the Anglo-Dutch auction.*

Proof: The revenues for the seller in an Anglo-Dutch auction are

$$R^{AD} = R(n^a) = \overline{v} - (1-\mu_2)^{n^a}(\overline{v} - \underline{v}) - n^a c,$$

whereas the revenues of the seller in a two-stage English auction with $r = r(k^*)$, using the definition of $r(k^*)$, can be written as

$$\begin{aligned} R^{2S}(r(k^*)) &= \overline{v} - (1-\mu_1)(1-\mu_2)^{k^*+l}(\overline{v} - \underline{v}) - k^* c - \\ &\quad (1-\mu_1)(1-\mu_2)^{k^*} lc - \frac{\mu_1}{1-\mu_1}\frac{1-\mu_2}{\mu_2}c, \end{aligned}$$

[4] Notice that this amounts to make a take it or leave it offer to the incumbent. If the seller has the ability to exclude bidders from future stages, this may be optimal. In fact, McAfee and McMillan [6] have shown in a model that could be reduced to our model except for its symmetry, that the optimal mechanism for a seller with this ability when the number of potential buyers is unbounded is to induce one by one entry and offer each entrant a (constant across periods) price. With a finite number of potential entrants, this reserve price would have to be decreasing (see Burguet [1]), so that buyers could buy in future periods even if they reject the offer at the time they enter. The mechanism, however, would have to be complemented with asymmetric subsidies even when buyers are symmetric.

where the last term represents the profits of the incumbent. Then, since at k^*, (3.11) equals zero, substituting this expression for c in that last term, we have

$$R^{2S}(r(k^*)) = \overline{v} - (1-\mu_2)^{k^*}\left[(1-\mu_2)^l(\overline{v}-\underline{v}) + lc\right] - k^*c.$$

Notice that, for all $m \leq l$,

$$\begin{aligned}(1-\mu_2)^m(\overline{v}-\underline{v}) + mc &= \left[(1-\mu_2)^{m-1} - \mu_2(1-\mu_2)^{m-1}\right](\overline{v}-\underline{v}) + mc \leq \\ &\quad (1-\mu_2)^{m-1}(\overline{v}-\underline{v}) + (m-1)c.\end{aligned}$$

Thus, repeating this for $m = l, l-1, ..., n^a - k^*$,

$$\begin{aligned}R^{2S}(r(k^*)) &\geq \overline{v} - (1-\mu_2)^{k^*}\left[(1-\mu_2)^{n^a-k^*}(\overline{v}-\underline{v}) + (n^a-k^*)c\right] - k^*c = \\ &\quad \overline{v} - (1-\mu_2)^{n^a}(\overline{v}-\underline{v}) - \left[(1-\mu_2)^{k^*}(n^a-k^*) + k^*\right]c \geq R^{AD}.\end{aligned}$$

QED ■

In an Anglo-Dutch auction, the seller sacrifices surplus to foster entry and obtain higher revenues. A two-stage English auction increases the revenues of the seller by improving the efficiency of entry decisions. As a result, both the revenues of the seller and the efficiency of the allocation are higher than what they are in a Anglo-Dutch auction.

3.4 Remarks on generalizations

In the previous sections we have analyzed a very stylized model of competition, where buyers' valuations could take one of two specific values. This was enough to illustrate the main insights behind the proposal to use Anglo-Dutch auctions to foster entry in the presence of a strong incumbent. Indeed, the incumbent bids less aggressively in the Dutch part, so that the probability that an entrant obtains the good (license) at a profit is enhanced. This fosters entry and enhances the revenues for the seller at an efficiency cost: excessive entry. The stylized model also shows that a two-stage English auction may be more appropriate to attain the goal of high revenues with no cost of (and even enhancing of) efficiency.

Yet, there is one aspect of Anglo-Dutch auctions that the discrete case does not reflect: the Dutch stage may introduce inefficiency that goes beyond excessive entry, i.e., allocation inefficiency. Indeed, in asymmetric settings, a Dutch auction may assign the good or license to a buyer different from the one that has the highest willingness to pay. This inefficiency is in general to the advantage of both the entrant and the seller (see Maskin and Riley [5]).

3.4.1 Continuous distributions

Thus, the first generalization we may consider is to assume continuous distributions of types, where this allocational inefficiency appears. We assume now that v_i, the valuation of a buyer, is a (independent) realization of a continuous random variable. The incumbent draws his type from a distribution $F_1(v)$, whereas all entrants draw their types from distribution $F_2(v)$. We further assume that F_1 stochastically dominates F_2.

In the Anglo-Dutch auction, it is still weakly dominant for bidders to stay in the auction up to the moment when the price reaches their respective valuations. If the clock stops at price ρ, with two entrants as the remaining bidders, then these bidders participate in a symmetric Dutch auction with reserve price ρ, the one with highest valuation will win, and the revenues for the seller will be equal to the expected value of the second largest valuation. When one of the two remaining bidders is the incumbent, however, bidders participate in an asymmetric Dutch auction. In general, bidding strategies in this case, and expected revenues for the seller, can only be obtained through numerical methods.

With respect to the two-stage English auction, and for any given reserve price set by the seller, r, both the incumbent's and entrants' optimal behavior is to drop at a price equal to their willingness to pay, if they do participate in any of the stages. Thus, we only need analyzing participation decisions. We can conjecture that these will be characterized by two cut-off values, w_1, w_2, such that the incumbent decides to participate in the first period if $v_1 \geq w_1$, and any first period entrant i participates if $v_i \geq w_2$. Both of these values will depend on the number of entrants in the first period and how many are expected in the second. Treating entry as a continuous variable, then zero profits for entrants in both stages and indifference between participating and waiting both for entrants and the incumbent at their respective cut-off valuations are the four equations that the solution (w_1, w_2, k, l) solves.

Using numerical computations, we have solved for the continuous model with asymmetric, uniform distributions, $F_i(v) = \frac{v}{\overline{v}_i}$, $i = 1, 2$, with $1 = \overline{v}_2 \leq \overline{v}_1$. The details are shown in the Appendix. In all cases analyzed, the total surplus is higher under a two-stage English auction than under an Anglo-Dutch auction. Revenues also follow this ranking except for one case: when $\overline{v}_1 = 1.2$, and $c = 0.072$. This cost was chosen (as in all examples) so that entrants in the Anglo-Dutch auction break even, the most favorable case from the point of view of the seller. In this particular case, 2 entrants enter the Anglo-Dutch auction. For these same values, the optimal reserve price in a two-stage English auction is $r = \frac{2}{3}$, which generates zero entry in the first stage of the two-stage English auction, and 2 entrants in the second stage. ($r = \frac{2}{3}$ is the maximum reserve price compatible with the incumbent "bidding" in the first stage with positive probability, since the expected highest bid from the two second period entrants is $\frac{2}{3}$. Consequently, $w_1 = 1$.) This obviously results in lower revenues for the seller (0.528, instead of 0.531). Why is entry not higher in the two-stage English auction? The answer has to do with the integer nature of entry. Indeed, with $l = 2$, entrants expect positive profits, equal to 0.0116. (Recall that entrants expect zero profits in the Anglo-Dutch auction with these values of the parameters.) Yet with $l = 3$ they would expect negative profits. The seller needs only setting an entry fee in the second stage of the two-stage English auction above 0.0016 (much lower than 0.0116), which entrants would be willing to pay, to improve upon the Anglo-Dutch auction.

3.4.2 More than one unit

We could also ask whether our results extend to the case where the seller has more than one unit for sale, and the equality between the number of units and the number of incumbents still holds. In the Appendix we consider the case where 2 units are to be sold and there are 2 incumbents. Other things are as in the model analyzed above. The main novelty here is that, in the second stage of a two-stage English auction, there is now a chance that one of the units for sale is assigned in the first stage, but the other unit is still available in the second stage. Let l_1 be the number of second-stage entrants in case 1 unit is left and l_2 the number of second-stage entrants when 2 units are still available. These are defined by

$$\mu_2(1-\mu_2)^{l_1-1}(\overline{v}-\underline{v})=c, \tag{3.12}$$

and

$$\mu_2[(1-\mu_2)^{l_2-1}+(l_2-1)\mu_2(1-\mu_2)^{l_2-2}](\overline{v}-\underline{v})=c. \tag{3.13}$$

Notice that the second stage is independent of the identity of the winner of the first, given our assumptions. In the Appendix we show that the socially efficient entry and allocation in one-stage auctions is achieved using, for instance, an English auction. However, a two-stage English auction always results in higher surplus than any one-stage auction. Thus, our results on efficiency carry over to this multiple-unit case.

With respect to the seller's revenues a sufficient condition for the ranking to be the same is $l_2 \leq 2(l_1+1)$. In fact, this leaves a lot of slack. We have obtained cases where the Anglo-Dutch auction performs better than the two-stage English auction. However, these cases involve both extremely low values of μ_2 and c so that even though no entry would take place in an English auction, a large number of firms would enter in an Anglo-Dutch or a second stage of the two-stage English auction.

3.5 Concluding remarks

We have offered theoretical support to the claim that an Anglo-Dutch auction results in higher revenues than an ascending auctions when there are as many licenses as incumbents. By favoring ex-ante weaker entrants, the Anglo-Dutch auction fosters entry and this results in higher prices of licenses at the cost of efficiency. However, we have also proposed another simple alternative to this Anglo-Dutch auction: a two-stage English auction. What we could term an Anglo-Anglo auction. Instead of relying on inefficient allocations to induce entry, the two-stage English auction relies on information revealed through bidding, an information on which entrants can condition their entry decisions. As a result, entry is more efficient and surplus is higher, which works to the advantage of the seller. Thus, this simple design not only increases revenues but also the gains from trade.

There is one aspect in which the two-stage English auction is more complex than the Anglo-Dutch auction: its intrinsic multistage nature. Indeed, the Anglo-Dutch auction requires two stages, but no lag of time is needed between them. On the contrary, in a two-stage English auction potential new entrants should be given enough time to "enter " (find about their valuation, prepare a bidding strategy) before the second stage takes place. In cases where this waiting is costly, this waiting cost would have to be weighed against the gains that we have discussed. In any case, the conclusion of this work is that the learning that takes place in dynamic designs may be a better alternative than inefficiencies when fostering competition through entry is the goal.

3.6 Appendix

3.6.1 Continuous valuations

Consider the Anglo-Dutch auction. There are initially n entrants plus the incumbent who participate in the English stage. A weakly dominant strategy for bidders is to stay in the auction up to the moment when the price reaches their respective valuations and then to drop out. Suppose that the bidder with third highest valuation drops out at the valuation ρ. Then the remaining two bidders participate in the Dutch stage with reserve price ρ. Suppose that two bidders with the highest valuations are entrants. Then the expected revenue for seller from the Dutch stage is

$$\int_{\rho}^{\overline{v}_2} v(1 - F_2(v))f_2(v)dv,$$

which is the expected value of the second highest valuation given that it exceeds ρ. The density (on $[0, \rho)$) of the highest valuation ρ (reserve price) among the remaining $n - 1$ bidders is

$$f_{n-1:n-1}(\rho) = F_2(\rho)^{n-2}f_1(\rho) + (n-2)F_2(\rho)^{n-3}F_1(\rho)f_2(\rho).$$

There are $n(n-1)$ permutations when two bidders with the highest valuations are entrants. Thus, the revenue for seller (times the probability of the event) accruing when two entrants play the Dutch part is

$$\begin{aligned} R_{w,w} &= n(n-1)\int_0^{\overline{v}_2}\left(\int_{\rho}^{\overline{v}_2} v(1-F_2(v))f_2(v)dv\right)dF_2(\rho)^{n-2}F_1(\rho) \quad (3.14) \\ &= n(n-1)\int_0^{\overline{v}_2} v(1-F_2(v))F_2(v)^{n-2}F_1(v)f_2(v)dv. \end{aligned}$$

Suppose now that one of the two bidders with the highest valuations is the incumbent. Define the truncated distributions

$$\begin{aligned} G_2(v) &\equiv \frac{F_2(v) - F_2(\rho)}{1 - F_2(\rho)} \\ G_1(v) &\equiv \frac{F_1(v) - F_1(\rho)}{1 - F_1(\rho)} \end{aligned}$$

Let $v = \phi_2(b)$ and $v = \phi_1(b)$ be inverse bid functions, respectively, for entrant and incumbent, defined on $[\rho, b^*]$, such that $\phi_i(\rho) = \rho$ and $\phi_i(b^*) = \overline{v}_i$ for $i = 1, 2$. When an entrant bids b he wins with probability $G_1(\phi_1(b))$ and pays his bid. Similarly, the distribution of an entrant's bid is $G_2(\phi_2(b))$. Therefore the expected revenue for the seller when accruing from an entrant-winner is

$$R_2 = \int_{\rho}^{b^*} bG_1(\phi_1(b))dG_2(\phi_2(b)),$$

and the expected revenue accruing from an incumbent-winner is

$$R_1 = \int_{\rho}^{b^*} bG_2(\phi_2(b))dG_1(\phi_1(b)).$$

The distribution of ρ is

$$f_{n-1:n-1}(\rho) = (n-1)F_2(\rho)^{n-2}f_2(\rho).$$

Then, the expected revenue accruing to the seller from an entrant when the incumbent is one of the two bidders with the highest valuations is

$$R_{w,s} = n\int_0^{\overline{v}_2} R_2(1 - F_2(q))(1 - F_1(q))(n-1)F_2(q)^{n-2}f_2(q)dq, \tag{3.15}$$

and the expected revenue accruing to the seller from the incumbent is

$$R_{s,w} = n\int_0^{\overline{v}_2} R_1(1 - F_2(\rho))(1 - F_1(\rho))(n-1)F_2(\rho)^{n-2}f_2(\rho)d\rho. \tag{3.16}$$

Thus, the total revenues of the seller are $R_{w,w} + R_{w,s} + R_{s,w}$. Notice that R_1 and R_2 depend on the (inverse) bidding functions $\phi_i(b)$. These have to be computed using numerical methods.

Now we turn to a two-stage English auction. Again assume that, besides the incumbent, k entrants enter in the first stage and, if nobody bids, some additional l bidders enter in the second stage. Since first stage entrants must bid at least the reserve price r, they will decide to participate in the first stage bidding if their valuations will exceed a cut-off value w_2. Similarly, the incumbent will decide to participate in the bidding if his valuation is above a cut-off value w_1.

Define truncated distributions for $i = 1, 2$ as

$$H_i(v) \equiv \frac{F_i(v)}{F_i(w_i)}.$$

We can distinguish three cases: (1) $\overline{v}_2 \geq w_2 \geq w_1 \geq 0$, (2) $\overline{v}_2 \geq w_1 \geq w_2 \geq 0$, (3) $\overline{v}_1 \geq w_1 \geq \overline{v}_2 \geq w_2 \geq 0$. (And two more separate cases when $k = 0$ since then w_2 does not exist: $\overline{v}_2 \geq w_1 \geq 0$ and $\overline{v}_1 \geq w_1 \geq \overline{v}_2$.) Here we present only derivations of cut-off points for the first case. Note that once bidders decide to bid (both in the first and second stages) it is weakly dominant for them to bid their true valuations. The cut-off point w_1 for the incumbent is found when he is indifferent between obtaining the object in stage 1 at reserve price r or waiting till stage 2 and obtaining it at the highest valuation among $k + l$ entrants. Thus,

$$(w_1 - r)F_2(w_2)^k = \int_0^{w_1} (w_1 - v)dF_2(v)^l F_2(v)^k = \int_0^{w_1} F_2(v)^l F_2(v)^k dv, \quad (3.17)$$

and the cut-off point w_2 for each of the k entrants satisfies

$$\begin{aligned} & (w_2 - r)F_2(w_2)^{k-1}F_1(w_1) + \int_{w_1}^{w_2} (w_2 - v)F_2(w_2)^{k-1}dF_1(v) \quad (3.18) \\ = & \int_{w_1}^{w_2} (w_2 - v)F_1(w_1)dF_2(v)^l F_2(v)^{k-1} + \int_0^{w_1} (w_2 - v)F_1(v)dF_2(v)^l F_2(v)^{k-1}. \end{aligned}$$

Here we have an extra term since an entrant with valuation w_2 will win against an incumbent whose valuation takes value in (w_1, w_2). Rearranging,

$$\begin{aligned} & (w_1 - r)F_2(w_2)^{k-1}F_1(w_1) + \int_{w_1}^{w_2} F_2(w_2)^{k-1}F_1(v)dv \\ = & \int_0^{w_1} F_2(v)^l F_2(v)^{k-1}F_1(v)dv + \int_{w_1}^{w_2} F_2(v)^l F_2(v)^{k-1}F_1(w_1)dv. \end{aligned}$$

We can express both conditions using truncated distributions

$$w_1 - r = \int_0^{w_1} F_2(v)^l H_2(v)^k dv$$

$$\begin{aligned} & w_1 - r + F_1(w_1)^{-1} \int_{w_1}^{w_2} F_1(v)dv \\ = & \int_0^{w_1} F_2(v)^l H_2(v)^{k-1}H_1(v)dv + \int_{w_1}^{w_2} F_2(v)^l H_2(v)^{k-1}dv. \end{aligned}$$

Combining both equations gives

$$\int_0^{w_1} F_2(v)^l H_2(v)^k dv + F_1(w_1)^{-1} \int_{w_1}^{w_2} F_1(v) dv \tag{3.19}$$
$$= \int_0^{w_1} F_2(v)^l H_2(v)^{k-1} H_1(v) dv + \int_{w_1}^{w_2} F_2(v)^l H_2(v)^{k-1} dv.$$

Let us define the revenue of the seller from the incumbent by R_s, from each of first-stage entrants by R_k, and from each of second-stage entrants by R_l. The total revenue to the seller then is $R_s + kR_k + lR_l F_2(w_2)^k F_1(w_1)$. Also define the expected profit of each of first-stage entrants by P_k, and of each of second-stage entrants by P_l.

For fixed entry (k, l) we solve the following maximization problem:

$$\max_{w_1, w_2} R_s(w_1, w_1) + kR_k(w_1, w_2) + lR_l(w_1, w_2) F_1(w_1) F_2(w_2)^k \tag{3.20}$$

subject to constraint (3.19), and inequalities $\overline{v}_2 \geq w_2 \geq w_1 \geq 0$, $P_k(w_2, w_1) \geq c$ and $P_l(w_2, w_1) \geq c$. After solving for this, we can find the reserve price r from either equation (3.17) or (3.18). Next we present expressions for revenues R_s, R_k, R_l, and profits P_k, P_l.

The expected revenue from the incumbent is

$$R_s = \int_{\overline{v}_2}^{\overline{v}_1} J_1(v) f_1(v) dv + \int_{w_2}^{\overline{v}_2} J_1(v) F_2(v)^k f_1(v) dv +$$
$$\int_{w_1}^{w_2} J_1(v) F_2(w_2)^k f_1(v) dv + \int_0^{w_1} J_1(v) F_2(v)^{k+l} f_1(v) dv. \tag{3.21}$$

The expected revenue from each of the k first stage entrants is

$$R_k = \int_{w_2}^{\overline{v}_2} J_2(v) F_1(v) F_2(v)^{k-1} f_2(v) dv + \int_{w_1}^{w_2} J_2(v) F_1(w_1) F_2(v)^{k+l-1} f_2(v) dv +$$
$$\int_0^{w_1} J_2(v) F_1(v) F_2(v)^{k+l-1} f_2(v) dv, \tag{3.22}$$

and the expected revenue from each of the l second stage entrants is

$$R_l = \int_{w_2}^{\overline{v}_2} J_2(v) f_2(v) dv + \int_{w_1}^{w_2} J_2(v) H_2(v)^k F_2(v)^{l-1} f_2(v) dv +$$
$$\int_0^{w_1} J_2(v) H_1(v) H_2(v)^k F_2(v)^{l-1} f_2(v) dv. \tag{3.23}$$

The expected profit of each of the k first stage entrants is

$$
\begin{aligned}
P_k &= \int_{w_2}^{\overline{v}_2}(1-F_2(v))F_1(v)F_2(v)^{k-1}dv + \int_{w_1}^{w_2}(1-F_2(v))F_1(w_1)F_2(v)^{k+l-1}dv + \\
&\quad \int_0^{w_1}(1-F_2(v))F_1(v)F_2(v)^{k+l-1}dv \geq c, \qquad (3.24)
\end{aligned}
$$

and the expected profit of each of the l second stage entrants is

$$
\begin{aligned}
P_l &= \int_{w_2}^{\overline{v}_2}(1-F_2(v))F_2(v)^{l-1}dv + \int_{w_1}^{w_2}(1-F_2(v))H_2(v)^k F_2(v)^{l-1}dv + \\
&\quad \int_0^{w_1}(1-F_2(v))H_1(v)H_2(v)^k F_2(v)^{l-1}dv \geq c. \qquad (3.25)
\end{aligned}
$$

For numerical simulations we assume that valuations come from the uniform distributions on $[0, \overline{v}_i]$

$$
F_i(v) = \frac{v}{\overline{v}_i}, \qquad (3.26)
$$

with $\overline{v}_2 \leq \overline{v}_1$. The results from numerical simulations are summarized in Table 3.1. We have fixed $\overline{v}_2 = 1$ for all simulations. Table 3.1 illustrates results when $\overline{v}_1$ varies. The entry cost c was chosen to ensure that n entrants in Anglo-Dutch auction earn exactly zero net profits. With the uniform distributions, and when first-stage entry is positive, one can show that $w_1 = w_2$ satisfies the equations for cut-off points. Among the results presented in the Table 3.1 only in one case the revenues of seller are lower in two-stage English auction than in Anglo-Dutch auction, namely, in the auction that does not induce strictly larger (overall) number of entrants than Anglo-Dutch auction: $\overline{v}_1 = 1.2$, $c = 0.072$, where R^{AD} is equal to 0.531, and $R^{2S} = 0.528$. Yet, this is due to an integer problem. Indeed, two entrants expect substantial positive profits in the second stage of the two-stage English auction (0.0116 net of entry cost, in this case), yet a third one would expect negative profits. If we keep the profits of entrants in the second stage to zero, for instance, charging these entrants an entry fee of 0.0116, which would be paid in case the incumbent did not bid (an event with probability 1/1.2), then $R^{2S} = 0.547$ and then the revenues for the seller would again be larger in the two-stage auction.

Table 3.1: Numerical simulations of auctions for uniform distributions with $\bar{v}_2 = 1$

$\overline{v}_1$	c	n	S^{AD}	R^{AD}	k	l	w_1	r	S^{2S}	R^{2S}
1	0.083	2	0.583	0.500	1	2	0.552	0.510	0.630	0.529
1	0.050	3	0.650	0.600	1	3	0.700	0.652	0.692	0.631
1	0.024	5	0.738	0.715	2	5	0.771	0.745	0.770	0.745
1.2	0.072	2	0.663	0.531	0	2	1.000	0.667	0.689	0.528
1.2	0.043	3	0.720	0.628	2	4	0.555	0.547	0.754	0.647
1.2	0.020	5	0.795	0.737	3	5	0.711	0.697	0.826	0.751
1.5	0.059	2	0.791	0.570	0	3	0.836	0.714	0.850	0.635
1.5	0.035	3	0.839	0.663	0	4	0.954	0.796	0.883	0.710
1.5	0.016	5	0.901	0.766	0	6	1.000	0.857	0.934	0.786
2	0.045	2	1.018	0.620	0	3	1.000	0.750	1.082	0.675
2	0.027	3	1.057	0.708	0	4	1.000	0.800	1.113	0.733
2	0.012	5	1.107	0.802	0	7	1.000	0.875	1.151	0.826

3.6.2 The two unit case

Again we show that the two-stage English auction is more efficient than both the standard English auction and the Anglo-Dutch auction for all reserve prices r. The expected surplus for any one-stage auction is given by

$$S^1 = 2\overline{v} - \{2\mu_1(1-\mu_1)(1-\mu_2)^n + (1-\mu_1)^2[2(1-\mu_2)^n + n\mu_2(1-\mu_2)^{n-1}]\}(\overline{v} - \underline{v}) - nc,$$

which is maximized by n such that

$$\mu_2[(1-\mu_1)^2\{(1-\mu_2)^{n-1} + (n-1)\mu_2(1-\mu_2)^{n-2}\} + 2\mu_1(1-\mu_1)(1-\mu_2)^{n-1}](\overline{v} - \underline{v}) = c,$$

and is achieved using an English auction. Thus, necessarily expected surplus from Anglo-Dutch auction is at most equal to the expected surplus from the standard English auction. The expected surplus for the two-stage English auction is given by

$$\begin{aligned} S^2 &= 2\overline{v} - \{2\mu_1(1-\mu_1)(1-\mu_2)^k + (1-\mu_1)^2 k\mu_2(1-\mu_2)^{k-1}\}[(1-\mu_2)^{l_1}(\overline{v} - \underline{v}) + l_1 c] \\ &\quad -(1-\mu_1)^2(1-\mu_2)^k[\{2(1-\mu_2)^{l_2} + l_2\mu_2(1-\mu_2)^{l_2-1}\}(\overline{v} - \underline{v}) + l_2 c] - kc. \quad (3.27) \end{aligned}$$

First we observe that $k \leq n$ since the highest first-stage entry is achieved by setting the reserve price $r = \underline{v} + \varepsilon$. Define $l = n - k$ and rewrite expression for

S^1 as follows:

$$\begin{aligned} S^1 &= 2\overline{v} - \{2\mu_1(1-\mu_1)(1-\mu_2)^k + (1-\mu_1)^2 k\mu_2(1-\mu_2)^{k-1}\}[(1-\mu_2)^l(\overline{v} - \underline{v}) + lc] \\ &\quad -(1-\mu_1)^2(1-\mu_2)^k[\{2(1-\mu_2)^l + l\mu_2(1-\mu_2)^{l-1}\}(\overline{v} - \underline{v}) + lc] - kc \qquad (3.28) \\ &\quad -(1 - \{2\mu_1(1-\mu_1)(1-\mu_2)^k + (1-\mu_1)^2[(1-\mu_2)^k + k\mu_2(1-\mu_2)^{k-1}]\})lc. \end{aligned}$$

Since l_1 and l_2 maximize expected surplus of the second stage when one and two units, respectively, are available, it follows that

$$\begin{aligned} (1-\mu_2)^{l_1}(\overline{v} - \underline{v}) + l_1 c &\leq (1-\mu_2)^l(\overline{v} - \underline{v}) + lc \\ \{2(1-\mu_2)^{l_2} + l_2\mu_2(1-\mu_2)^{l_2-1}\}(\overline{v} - \underline{v}) + l_2 c &\leq \{2(1-\mu_2)^l + l\mu_2(1-\mu_2)^{l-1}\}(\overline{v} - \underline{v}) + lc. \end{aligned}$$

Comparing (3.28) with (3.27) implies that $S^2 > S^1$. We summarize the result in the following lemma.

Lemma 7 *The expected surplus in a two-stage English auction is higher than in a standard English auction and an Anglo-Dutch auction.*

Observe that the previous argument easily extends to more than two units.

The revenues of the seller are the difference between net surplus and the profits of incumbents. We have already demonstrated that the surplus is higher in two-stage English auction than in the Anglo-Dutch auction. For the Anglo-Dutch auction to yield higher revenues it must be that the part of expected social surplus received by incumbents is smaller in the Anglo-Dutch auction than in the two-stage second price auction.

It can be shown, like in the case of one unit, that the revenues in a two-stage auction are maximized either when there is no entry in the first stage and we set the highest reserve price that induces incumbents with high valuations to bid in the first stage, or when the entry level in the first stage is (almost) socially efficient. (It will not be exactly socially efficient entry level because now profits of incumbents are not independent of entry k.) We provide partial results on revenue rankings in both auctions. Consider the case when $k = 0$ and

$$r = \overline{v} - [(1-\mu_1)\{(1-\mu_2)^{l_2} + l_2\mu_2(1-\mu_2)^{l_2-1}\} + \mu_1(1-\mu_2)^{l_1}](\overline{v} - \underline{v}).$$

Then the seller's revenues in the two-stage English auction are

$$\begin{aligned} R^2 \quad = \quad & 2\overline{v} - 2\mu_1(1-\mu_1)[(1-\mu_2)^{l_1}(\overline{v} - \underline{v}) + l_1 c] \\ & -(1-\mu_1)^2[\{2(1-\mu_2)^{l_2} + l_2\mu_2(1-\mu_2)^{l_2-1}\}(\overline{v} - \underline{v}) + l_2 c] \\ & -2\mu_1[\mu_1(1-\mu_2)^{l_1} + (1-\mu_1)\{(1-\mu_2)^{l_2} + l_2\mu_2(1-\mu_2)^{l_2-1}\}](\overline{v} - \underline{v}). \end{aligned} \tag{3.29}$$

It can be shown that the expected utility of incumbent in the Anglo-Dutch auction take the same expression as in (one-stage) English auction

$$\mu_1[(1-\mu_2)^n + (1-\mu_1)n\mu_2(1-\mu_2)^{n-1}](\overline{v} - \underline{v}). \tag{3.30}$$

Thus the expected revenue of the seller in the Anglo-Dutch auction can be written as

$$\begin{aligned} R^1 \quad = \quad & 2\overline{v} - 2\mu_1(1-\mu_1)[(1-\mu_2)^n(\overline{v} - \underline{v}) + nc] \\ & -(1-\mu_1)^2[\{2(1-\mu_2)^n + n\mu_2(1-\mu_2)^{n-1}\}(\overline{v} - \underline{v}) + nc] \\ & -2\mu_1[\mu_1(1-\mu_2)^n + (1-\mu_1)\{(1-\mu_2)^n + n\mu_2(1-\mu_2)^{n-1}\}](\overline{v} - \underline{v}) - \mu_1^2 nc \end{aligned} \tag{3.31}$$

It can be shown that the number of entrants in the Anglo-Dutch auction is given by

$$
\begin{aligned}
n = \max \quad & \{m|\mu_2 \left[\left\{ \mu_1^2(1-\mu_2)^{m-1} + \frac{1}{m} 2\mu_1(1-\mu_1)(1-\mu_2)^{m-1} \right.\right. \\
& \left. + \frac{2}{m(m+1)}(1-\mu_1)^2(1-\mu_2)^{m-1} \right\} \\
& \times \frac{\mu_1(1-\mu_2) + \frac{2}{m+1}(1-\mu_1)(1-\mu_2) + (1-\mu_1)\mu_2}{\mu_1(1-\mu_2) + \mu_1 m \mu_2 + \frac{2}{m+1}(1-\mu_1)(1-\mu_2) + (1-\mu_1)\mu_2} \\
& + \left\{ \frac{m-1}{m} 2\mu_1(1-\mu_1)(1-\mu_2)^{m-1} + 2\mu_1(1-\mu_1)(n-1)\mu_2(1-\mu_2)^{m-2} \right. \\
& \left. + \frac{4(m-1)}{m(m+1)}(1-\mu_1)^2(1-\mu_2)^{m-1} + \frac{2}{m}(1-\mu_1)^2(n-1)\mu_2(1-\mu_2)^{m-2} \right\} \\
& \times \frac{\frac{1}{m+1}(1-\mu_2)^2 + \mu_2(1-\mu_2)}{\frac{1}{m+1}(1-\mu_2)^2 + \mu_2(1-\mu_2) + \frac{m}{2}\mu_2^2} \\
& + \frac{(m-1)(m-2)}{(m+1)m}(1-\mu_1)^2(1-\mu_2)^{m-1} \\
& \left. + \frac{m-2}{m}(1-\mu_1)^2(m-1)\mu_2(1-\mu_2)^{m-2} \right] (\overline{v} - \underline{v}) \geq c\}.
\end{aligned} \tag{3.32}
$$

We want to know when the revenue of the seller is higher in the two-stage English auction (3.29), where entry is given by conditions (3.12) and (3.13), than in the Anglo-Dutch auction where entry is given by the condition (3.32). First, observe that when $\mu_1 = 0$, n is given by

$$
\mu_2[(1-\mu_2)^{n+1} + (n+1)\mu_2(1-\mu_2)^n](\overline{v} - \underline{v}) = c
$$

implying $n = l_2 - 2$. Differentiating the expression in square brackets of (3.32) with respect to μ_1 we obtain that the derivative is negative. The expression in square brackets of (3.32) is also declining with respect to n. Therefore we may conclude that higher probability μ_1 leads to lower entry n, and it is, at most, $l_2 - 2$. Assuming that $n = l_2 - 2$ for all μ_1 holds, when comparing (3.29) and (3.31), we obtain that $R^{2S} \geq R^{1S}$ if

$$
\begin{aligned}
& 2\mu_1(1-\mu_1)[(1-\mu_2)^{l_1}(\overline{v} - \underline{v}) + l_1 c] + 2\mu_1^2(1-\mu_2)^{l_1}(\overline{v} - \underline{v}) \\
\leq \; & 2\mu_1(1-\mu_1)[(1-\mu_2)^{l_2-2}(\overline{v} - \underline{v}) + (l_2-2)c] + 2\mu_1^2(1-\mu_2)^{l_2-2}(\overline{v} - \underline{v}) + \mu_1^2(l_2-2)c
\end{aligned}
$$

or

$$(1-\mu_2)^{l_1}(\overline{v}-\underline{v})+l_1c-\mu_1 l_1 c \leq (1-\mu_2)^{l_2-2}(\overline{v}-\underline{v})+(l_2-2)c-\frac{\mu_1}{2}(l_2-2)c.$$

The inequality will hold if $l_2 \leq 2(l_1+1)$, since $(1-\mu_2)^{l_1}(\overline{v}-\underline{v})+l_1c \leq (1-\mu_2)^{l_2-2}(\overline{v}-\underline{v})+(l_2-2)c$. (Because l_1 was chosen to minimize $(1-\mu_2)^{l_1}(\overline{v}-\underline{v})+l_1c$.)

Bibliography

[1] R. Burguet. Optimal repeated purchases when sellers are learning about costs. *Journal of Economic Theory*, 68:440–455, February 1996.

[2] R. Burguet and J. Sákovics. Reserve prices without commitment. *Games and Economic Behavior*, 15:149–164, 1996.

[3] P. Klemperer. Auctions with almost common values: The wallet game and its applications. *European Economic Review*, 42:757–769, 1998.

[4] P. Klemperer. How (not) to run auctions: The european 3g telecom auctions. *European Economic Review*, 46:829–845, 1998.

[5] E. Maskin and J. Riley. Asymetric auctions. *Review of Economic Studies*, 67:413–438, 2000.

[6] R.P. McAfee and J. McMillan. Search mechanisms. *Journal of Economic Theory*, 44:99–123, February 1988.

[7] P. Milgrom. *Putting Auction Theory to Work*. Cambridge University Press, 2004.

Chapter 4

Estimating the Order of Sales Effect in Sequential Auctions

4.1 Introduction

Using data from timber auctions in Latvia, I study the empirical evidence on whether the order of sales affects the revenues of seller. State company "Latvia's State Forests" sells lots of standing trees sequentially through oral, ascending price auctions. The lots are highly heterogenous because they differ in size, composition and in other characteristics. Since they are so heterogenous we want to know whether the order in which they are sold matters for the revenues of the seller. And if it does, what is the optimal sequence in which seller should sell the lots.

The literature on auctions suggests that the order of sales becomes relevant if, on the one hand, auctioned lots are heterogenous and, on the other hand, firms face capacity and/or budget constraints and therefore are not able to compete for all the available lots. Intuitively, once buyers exhaust their budgets or capacities, they will not participate in the bidding for the later units. Thus, competition is less intense for the last lots in the auction. In the homogenous goods' case the bidders when bidding for the initial lots will take it into account and accordingly will bid less aggressively. As a result, in equilibrium, a bidder will be indifferent between acquiring a lot now or dropping from bidding and acquiring another lot later. When goods are heterogenous, bidders may not be able to equalize the profits from obtaining different objects since their bids are constrained to be nonnegative and not to exceed budgets. When more valuable lots are sold first, the competition for them is strong. If they are sold last, the competition is weak both for the initial less valuable lots, and also for later more valuable lots.

In order to test whether the order of sales affects the revenues, I construct an econometric model. In general, we do not observe the valuations of the lots the bidders have. Therefore, in the first stage I estimate them, using a discrete choice model, based on the assumption that a lot will be sold if and only if its valuation exceeds the reserve price. In the second stage I regress the revenues of the seller on the obtained valuations allowing for different bidding strategies depending on whether more valuable lot was sold first or last. I apply this model to the sequential auctions of timber sold by "Latvia's State Forests". The results from the first stage, where I obtain estimates of the valuations, agree with other indirect evidence on the valuations like the prices the seller uses to construct the reserve price and bids from related roundwood auctions. In the second stage I can not reject the hypothesis that the order does not affect the revenues of the seller.

The literature on multi-object auctions in the presence of constraints has been mainly restricted to the two-object case. When objects are sold sequentially, the analysis is usually done using English or second price auctions. The use of these auction formats ensures the existence of dominant strategies in the second round, irrespective of the information revealed during the first round. Additionally, other simplifying assumptions have been employed to derive equilibrium strategies; for instance, only two bidders, perfect correlation of valuations of both objects, or complete information.

Elmaghraby [6] considers procurement auctions under private information where each bidder can complete only one job. Costs of providing both jobs are different but depend on the bidder's type. The author allows for quite a general cost structure subject to the following restrictions - it is cheaper to fulfill the first task than the second for all types, and the cost of the first task is strictly monotonically decreasing and/or the cost of the second task is strictly monotonically increasing in bidders type. He shows the existence of an efficient ordering and derives equilibrium bids under this ordering. He has only partial results on the revenues but they tend to indicate that the revenue of the seller is higher under the efficient ordering.

Gale and Hausch [7] restrict the analysis to two bidders, each willing to obtain only one object, but they allow the valuations of both objects to be imperfectly correlated, although independent between bidders. They prove that the auction where the seller sets the order does not guarantee efficiency while the right-to-choose auction does it. They also establish an important link between the revenues of the seller and the decline in prices of subsequently sold objects - the revenues are higher in the efficient right-to-choose auction if and only if the expected price declines in the standard auction.

Beggs and Graddy [3] construct a theoretical model for auctioning heterogeneous objects where each bidder wants only one object, and they apply it to art auctions. They restrict attention to auctions that order items by declining valuation, and prove that this ordering achieves efficiency, maximizes seller's revenues, and implies declining prices over the auction.

The research that considers budget constraints, rather than capacity, has been mainly restricted to the complete information setup. Pitchik and Schotter [10] conduct an experimental study of sequential auctions. There are two bidders in their model, they value the objects differently and have different budget constraints. Valuations and budgets are common information. They conclude that the trembling-hand perfect equilibrium is a good explicator of observed behavior in the experiment.

Benoît and Krishna [4] also work under the assumption of complete information. The valuations are common across bidders and objects may be complements or substitutes. They prove that selling the more valuable object first maximizes seller's revenues, which agrees with the results of Beggs and Graddy [3] in the case of capacity constraints. More importantly, Benoît and Krishna [4] show that budget constraints may arise endogenously - if bidders before auction decide on their budgets, they will choose to be budget constrained even if rising money is costless.

Pitchik [9] extends the analysis of sequential auctions with budget constraints to incomplete information, assuming that valuations of both items and income are functions of bidder's type. She demonstrates that the order of sales affects prices and revenues, the results depending on the magnitudes of valuations and incomes of bidders.

To my knowledge there is no work undertaken in the empirical auction literature to test the relevance of the order of sales. The empirical literature on sequential auctions has mainly focused on testing the 'declining price anomaly' when later objects systematically fetch lower price after controlling for valuations of objects. Starting by the work of Ashenfelter [1] there has been growing body of literature on the declining price anomaly (references can be found in the survey by Ashenfelter and Graddy [2]). One of the explanations for the declining prices comes from the heterogeneity of auctioned objects in the presence of capacity constraints, for example, Gale and Hausch [7] and Beggs and Graddy [3], mentioned above. Although there exists a link between the declining price anomaly and the optimal ordering of objects, research has focused on testing the former phenomenon, while ignoring the latter.

The remaining of the chapter is organized as follows. The following section describes the company "Latvia's State Forests" and its practices to sell state owned forests. Section 4.3 provides economic motivation, based on which I derive econometric model in section 4.4. The results of the estimations are in section 4.5 and their discussion in section 4.6. The description of variables is relegated to the Appendix.

4.2 Timber Sales by "Latvia's State Forests"

State company "Latvia's State Forests" (in sequel I use its Latvian abbreviation - LVM) was created in 1999 to administer forests owned by the state. It is organized along 8 regions. In 2003 it administered 1.43 million hectares of forests, which account for 47% of all forests in Latvia, with estimated value of 2.7 billion lats.[1,2] Two thirds of forests are occupied by coniferous trees (pine forms 47% of forests, and fir-tree 21%). Among foliage trees birch is the most common, accounting 24% of forests. Total estimated stock of timber is 272.6 million cubic meters, while annual increase is 7 million cubic meters, including the increase in the stock of pine by 2.85 mill. m^3, fir-tree by 2.03 mill. m^3, and birch by 1.56 mill. m^3. On average, 4 million m^3 of timber is cut annually in the state forests.

Felling both in private and public forests is regulated by the Law of Forest. The law distinguishes several types of felling. When the trees have reached a certain age or diameter, appropriate for that specie of trees, they are harvested in the so called 'principal' felling. The most common type of principal felling is clear-cut, after which replanting of forest is required. The other, secondary, types of felling include the stock care-cut (to improve growth conditions and quality of remaining trees), the sanitary-cut (to eliminate infected or damaged trees), the reconstructive-cut (to fell unproductive forests) and the other-cut (to create and maintain forest infrastructure).

LVM sells trees using three mechanisms: long-term felling contracts, auctions and roundwood supplies. The long-term felling contracts were introduced in the beginning of the nineties to guarantee input supplies to privatized logging companies. In 2002 around 2.5 mill. m^3 of growing trees where sold through these contracts. Since 2000, when auctions of growing trees were introduced, the

[1] The information was taken from the web site of the company: *www.lvm.lv*.

[2] During the sample period the national currency of Latvia, lat (LVL) was fixed to the IMF currency basket: 1 SDR = 0.7997 LVL. The exchange rates with respect to 1 USD and 1 EUR were in the range of 0.55-0.65 LVL.

average price realized during auctions has exceeded the average sales prices set in the contracts with the tendency for the difference to increase: for example, in 2002 the average price obtained in auctions was 9.99 LVL/m^3 against 5.24 LVL/m^3 set in contracts. Therefore, starting 2003 LVM does not sign anymore new contracts.

Starting 2003 LVM has introduced a new selling mechanism when, instead of selling growing trees, it supplies already prepared roundwood to sawmills and other wood-processing companies. The process is organized along three separate auctions: one to sell timber to its users, another to contract companies to cut trees and the last to transport prepared timber to its buyers.

I address the auctions of growing trees conducted from the start of 2001 to the middle of 2003. Each of the 8 regions runs its auctions at the frequency it finds appropriate, usually once or twice a month. At least two weeks before an auction a list of felling areas is published. An example of announcement is presented in the Table 4.3. There is available more detailed description of each lot in forestry and buyers have the right to visit and inspect the lot in nature.

Lots can be sold in oral auctions with ascending or descending price. Lots that failed to be sold in at least two oral auctions or that must be sold urgently are sold in sealed bid auctions. More than 95% of sales are done using sequential ascending price auctions. Buyers usually bid not the price per m^3 but the total price that they must pay, that is, the price does not depend on wood collected. For some lots buyers bid price per m^3 and they pay according to actual amount of wood collected; for these later lots there is an estimate of the total volume of timber, but not of the useful volume or its composition. Before auction buyers must pay a participation fee depending on the type of felling: 5-20 LVL. They must pay a deposit that is 10% of the sum of the initial prices for lots they want to buy. That is, buyer cannot participate in bidding for a given lot if remaining (unused) deposit is below 10% of the initial price of the lot. Unused deposits are returned or can be included in payment for bought lots. The initial price of a lot is based on an 'estimated value' of felling area. Bidding step depends on the initial price of lot: 50, 100, 200 LVL. Buyers raise cards with their identification number and price increases by a specified step. They can cry out higher increase in price. If during 15 workdays the winner does not pay, he loses the deposit. Firms must harvest the timber by the end of year if not specified otherwise.

Since I address the effect of lot order on the revenues of the seller, I describe what is the current practise to organize the lots in an auction. First, lots of principal cut are sold, and then the lots of secondary cut. It could be that the forestry decides to auction urgently some lots after the auction was announced. In that case these lots are attached at the end of the list, even if they belong to the principal cut. Within each group the lots are grouped according to location - forest district and forestry. Except that the principal-cut lots are, in general, more valuable than the secondary-cut lots, there is no pattern according to the initial price, taking it as a proxy for the value of the lot: more valuable lots can appear before or after less valuable lots. There is, of course, a rationale to group lots according to the location, but within each forest district or forestry there exists the scope to rearrange lots according to their valuations.

4.3 Economic Motivation

We assume a complete information and common values framework; that is, the valuations of objects are common and known to all bidders, but not to the seller. Another assumption we make is that the entry in the auction is endogenous, where bidders upon entry decide how much capacity and/or budget they commit for the given auction. These simplifying assumptions suggest certain equilibrium outcomes that allow us to derive later an econometric model to test whether order affects seller's revenues or not. The assumption of common valuations allows us to abstract from the identity of the winner of a given lot, and to ignore how many potential bidders there are. Complete information ensures that bidders will know the outcome of the auction before it starts if they follow (pure) equilibrium strategy. This implies, combined with the assumption of endogenous entry, that in equilibrium a lot will be sold as long as its value exceeds the reserve price, since otherwise somebody will find it profitable to enter in order to buy the lot.

I now present two examples that illustrate how order affects seller's revenues in the presence of capacity and budget constraints. In both examples the equilibria are derived in undominated strategies, that is, no bidder uses weakly dominated strategies. To illustrate that order might matter when there are capacity constraints consider the following simple example. There are two lots available and two bidders each willing to acquire at most one unit. Suppose that one lot is worth 10 while another 6. Consider the order where the most valuable lot is sold first. Once it is sold, the bidder who bought it will not participate in the bidding for the second lot, therefore another bidder will obtain it at zero price, and make profit equal to 6. It implies that he will be ready to pay for the first lot up to 4. The first lot will be sold at this price and the revenue of the seller will be 4. Now consider the opposite order. The only remaining bidder, once the first lot is sold, will again obtain the second lot at zero price and make profit 10. He will not bid for the first lot, therefore it will also sell at zero price, and the revenues of the seller will be zero.

The example illustrates that the revenues are a linear combination of valuations but the coefficients change depending on the order of sales: under the order where the first lot is more valuable, the coefficients are 1 and -1, while under the second order where the first lot is less valuable, the coefficients are both equal to zero. Therefore, one should estimate the model allowing for the regime switch which depends on the valuations of the objects:

$$y_t = \beta_{11} D_{t1} v_{t1} + \beta_{12} D_{t1} v_{t2} + \beta_{21} D_{t2} v_{t1} + \beta_{22} D_{t2} v_{t2} + \varepsilon_t, \tag{4.1}$$

where y_t is seller's revenue from auction t (where the auction refers to both lots sold on a given auction day), v_{ti}, $i = 1, 2$ are the valuations of the objects sold in auction t where the index i indicates whether the object was sold as the first or second, and $D_{t1} = 1$ if $v_{t1} \geq v_{t2}$ and $D_{t1} = 0$ otherwise, and $D_{t2} = 1 - D_{t1}$. We can think of the disturbance term ε_t as deviations from equilibrium strategies. The only restriction, the specification (4.1) imposes, is that equilibrium bids and hence revenues are linear in valuations. Otherwise the specification (4.1) is quite general in that it accommodates other (linear) bidding strategies and allows the coefficients to change depending on whether the most valuable lot was sold first or second.

The equilibrium strategies were derived only for two bidders. If the number of bidders is different, the equilibrium strategies will accordingly change. For example, if in the above example there are two objects but three (or more) bidders each willing to acquire one lot, it is easy to verify that prices in equilibrium will be equal to the valuations of lots, that is, all coefficients will be equal to 1 (and consequently order will not matter). It suggests that we must also assume that bidders play the same equilibrium strategies from one auction to another if there exist several equilibria. For example, from the two equilibrium strategies discussed, the participants may prefer the first equilibrium where two bidders enter, each willing to acquire one unit, and the remaining participants abstain from entering than the second equilibrium where three or more participants enter, since the first equilibrium is revenue superior from bidders point of view. Of course, we could also rule out the second equilibrium by assuming that bidders, in order to participate in the auction, must incur some small costs proportional to the capacity choice. That and the complete information framework, will ensure that in equilibrium at most two bidders will enter and compete for these two lots. However, to reiterate, my objective is not to rule out any particular equilibrium but rather to assume that agents will consistently play the same equilibrium strategies in all auctions. Optimally, we want to use the estimation results of equation (4.1) to infer what strategies bidders adopt, and not to impose any equilibrium outcome from the outset.

Suppose we have two lots A and B that have valuations v^A and v^B, and $v^A \geq v^B$. The model (4.1) implies that the order will not affect revenues if $\beta_{11}v^A + \beta_{12}v^B = \beta_{21}v^B + \beta_{22}v^A$. Since it must hold for any valuations v^A and v^B such that $v^A \geq v^B$, it follows that the hypothesis that the order does not matter is

$$\begin{aligned} H_0 &: \beta_{11} = \beta_{22}, \beta_{12} = \beta_{21} \\ H_1 &: \beta_{11} \neq \beta_{22} \text{ and/or } \beta_{12} \neq \beta_{21}. \end{aligned}$$

A remark is needed on what are the relevant valuations v_{ti} in equation (4.1) in the presence of the reserve prices, since in the timber auctions I analyze the seller sets them. In the example above we can think of the valuations of 10 and 6 as net valuations after deducting reserve prices. For example, it could be that one lot is worth 14 and its reserve price is 4 while another lot is worth 16 with reserve price equal 10. If we define the (gross) valuation of a lot as w_{ti} and reserve price by r_{ti}, the relevant variable in the model (4.1) is $v_{ti} = w_{ti} - r_{ti}$ and not w_{ti}, and the revenues of the seller we define as $y_t \equiv \sum_{i=1}^{2}(p_{ti} - r_{ti})$ where p_{ti} is the price at which lot i was sold.

Benoît and Krishna [4] have analyzed the relevance of order when bidders are budget, rather than capacity constrained, that is, they will not be able to bid more than their budgets allow. Additionally, they considered the case when bidders, before auction starts, decide about their budgets, and budget decisions are observable. Even if rising money is costless, Benoît and Krishna [4] have shown that in the equilibrium bidders choose to be budget constrained. In particular, when there are two bidders, equilibrium budget decisions and equilibrium bidding prices are:

$$\begin{array}{lll} b_m \geq v_1 - v_2 & p_1 = v_1 - v_2 & \multirow{2}{*}{if $v_1 \geq 2v_2$} \\ b_n = v_1 - v_2 & p_2 = 0 & \\ b_m \geq \frac{1}{2}v_1 + v_2 & p_1 = \frac{1}{2}v_1 & \multirow{2}{*}{if $v_1 < 2v_2$} \\ b_n = \frac{1}{2}v_1 & p_2 = 0 & \end{array}$$

where v_1 (v_2) is the valuation of the object sold first (second) and p_1 (p_2) is its price, and b_m and b_n are budgets of bidders m and n. In equilibrium bidder n will win the first object while bidder m will obtain the second. As one can see, the revenues of the seller are again a linear function of object valuations although the vector of coefficients now is $(1, -1, 0.5, 0)$. If we want to estimate revenues as a function of valuations, we can again specify the econometric model (4.1), except now $D_{t1} = 1$ if $v_1 \geq 2v_2$ and $D_{t1} = 0$ otherwise. Again we can think of v_i, $i = 1, 2$ as net valuations and p_i, $i = 1, 2$ as winning prices after subtracting reserve prices, while the equilibrium budgets of both bidders must be increased by $r_1 + r_2$.

To determine what are the relevant restrictions on the coefficients in the presence of budget constraints, consider first the case when the valuations of lots A and B are v^A and v^B, and $v^B < v^A < 2v^B$. Under both orderings of objects, condition $v_1 < 2v_2$ holds, thus $\beta_{11} = \beta_{12}$ for order not to matter. Now if the ranking of valuations is $v^A \geq 2v^B$ then under the ordering, where unit B is sold first, the condition $v_1 < 2v_2$ holds and under another ordering, where unit A is sold first, the condition is $v_1 \geq 2v_2$. It follows that for the order not to matter it must be that $\beta_{12} = \beta_{21}$ and $\beta_{11} = \beta_{22}$. Summarizing, the test that the order of sales does not matter is equivalent to testing hypothesis

$$\begin{aligned} H_0 &: \beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} \\ H_1 &: \beta_{ij} \neq \beta_{kl} \text{ for some } i \neq k \text{ or } j \neq l. \end{aligned}$$

In both examples, I have restricted attention to only two units. As the examples suggest, the number of different orders is factorial of the number of lots sold, and likewise the number of coefficients to be estimated will grow in factorial. Hence, we must restrict attention in equation (4.1) to a limited number of units even if more lots were sold on a given auction. In any moment during an auction bidding strategies will depend on the valuations of lots that are still not sold, as above examples illustrate, but we can reasonably assume that bidding strategies will not depend on the valuations of already sold units. It would not be true when the past bidding conveys information relevant for later bidding. For example, under incomplete information the prices of sold units would allow to update information about the valuations of the remaining units, or about the capacities or budgets of bidders. Even under complete information previous bids, and hence valuations of sold units, may affect bids for remaining units, if budgets of bidders are given exogenously, as shown in Benoît and Krishna [4].

Above argument implies that in equation (4.1) we must use the last units from an auction, and in the econometric analysis I will restrict attention to the last two units. If we were using, for example, the last three units, we would have 6 possible orders of lots, and we would need to estimate 3 coefficients under each order. Besides, we expect the number of equilibrium strategies to depend on whether lots are won by different bidders or the same. In the case of three lots, the coefficients would differ depending on whether these lots were bought by 1, 2 or 3 different bidders. Thus, if we limited attention to the last three units, we would need to estimate 54 different coefficients, therefore the decision to use only 2 lots.

4.4 Econometric model

Before proceeding to build an econometric model, I want to discuss the similarity between testing whether the order matters with the estimation of average treatment effects (ATE), especially widely used in labour econometrics, for example, to evaluate how much increases the probability of an unemployed person to find a job if he undergoes or not a training course (see Blundell and Costa Dias [5], Imbens [8], Wooldridge [11], ch. 18). We define expected revenues of the seller when the most valuable unit is sold first and when sold last, after controlling for the covariates, by

$$\mu_1(v_1, v_2) \equiv E(y|v_1, v_2, D_1 = 1) \tag{4.2}$$

$$\mu_2(v_1, v_2) \equiv E(y|v_1, v_2, D_1 = 0). \tag{4.3}$$

The average treatment effect is

$$\tau \equiv E[\mu_1(v_1, v_2) - \mu_2(v_1, v_2)],$$

where the expectation is over the distribution of covariates v_1 and v_2. In order to test whether the order matters, we would first obtain estimates of expected revenues of the seller $\widehat{\mu}_1(v_1, v_2)$ and $\widehat{\mu}_2(v_1, v_2)$ and, second, calculate the estimator of ATE

$$\widehat{\tau} = \frac{1}{T}\sum\nolimits_{t=1}^{T}[\widehat{\mu}_1(v_{t1}, v_{t2}) - \widehat{\mu}_2(v_{t1}, v_{t2})] \tag{4.4}$$

and test whether it is different from zero.

Note that we have already combined $\mu_1(v_1, v_2)$ and $\mu_2(v_1, v_2)$ with the help of dummy variables into single equation (4.1), under the assumption that the revenues are a linear function of valuations. The linearity was suggested by the examples in the previous section. However, instead of testing whether $\tau = 0$, we propose testing the hypothesis directly about the coefficients of (4.1), as discussed in the previous section. Testing restrictions about coefficients is more general in the sense that it could be that for some valuations of lots (v_1, v_2) one order of sales is better for the seller, while for other valuations of lots another order can be revenue superior. On the other hand, ATE would test whether on average one order of sales is better than another.[3]

If we observed the valuations of lots, the estimation of (4.1) would be trivial. Since they are not observed, we need to infer them from the available data. For example, equilibrium prices will be a function of valuations. If we assumed

[3]Of course, we can generalize the ATE. If we expect that $\tau > 0$ for a subdomain of (v_1, v_2), and $\tau < 0$ for another subdomain, then we define two separate ATEs for each subdomain and test them.

that the observed data correspond to a particular equilibrium, we could apply it to recover valuations from the observed prices. However, this approach already supposes that the particular equilibrium strategies describe the true bidding process, and it already imposes that the order does or does not affect seller's revenues depending on what the theoretical model predicts. Therefore, we must find an alternative way to recover the valuations, and once obtained we estimate the model (4.1) to test whether the order matters.

The valuation of a lot will depend on its characteristics like the size of the lot, its composition, the distance to a road and other factors captured by the error term:

$$w_{ti} = X_{ti}\gamma + \theta_{ti}, \tag{4.5}$$

where X_{ti} is a vector of explanatory variables for lot i in auction t, and θ_{ti} is an error term. I assume that the error term is independent from explanatory variables, and that $\theta_{ti} \sim i.i.d.N(0, \sigma_\theta^2)$.

The seller sets a reserve price which is a function of the characteristics of the lot:

$$r_{ti} = h(x_{ti1}, x_{ti2}, ..., x_{tim}) \tag{4.6}$$

where x_{tij} is a j-th characteristic of lot ti. In general, we do not know how the seller chooses the reserve price, although the rules of the auction state that the reserve price is set based on an estimate of the value of lot. We have argued in the previous section that if we assume that the participation in the auction is endogenous, and before the auction starts bidders choose their capacity or budgets, it is reasonable to conclude that a lot will be sold as long as its (gross) valuation exceeds the reserve price: $w_{ti} \geq r_{ti}$. That is, we have a model of discrete choice and since we assumed that θ_{ti} for all t and i comes from normal distribution, we estimate the Probit model. Thus, the probability of selling the lot is

$$\Pr(\theta_{ti} \geq r_{ti} - X_{ti}\gamma) = 1 - \Pr(\theta_{ti} \leq r_{ti} - X_{ti}\gamma)$$

and the probability of not selling the lot is

$$\Pr(\theta_{ti} \leq r_{ti} - X_{ti}\gamma).$$

Since $\theta_{ti} \sim N(0, \sigma_\theta^2)$, we can write the probabilities of selling and not selling as

$$\Phi\left(\frac{X_{ti}\gamma - r_{ti}}{\sigma_\theta}\right)$$

and

$$1 - \Phi\left(\frac{X_{ti}\gamma - r_{ti}}{\sigma_\theta}\right),$$

respectively, where $\Phi(\cdot)$ denotes standard normal distribution, and I have used the fact that the normal distribution is symmetric around the mean.

The likelihood function of the Probit model, to be maximized with respect to γ and σ_θ, is:

$$L = \prod_{t=1}^{T}\prod_{i=1}^{n_t} \Phi\left(\frac{X_{ti}\gamma - r_{ti}}{\sigma_\theta}\right)^{F_{ti}} \left[1 - \Phi\left(\frac{X_{ti}\gamma - r_{ti}}{\sigma_\theta}\right)\right]^{1-F_{ti}} \tag{4.7}$$

where $F_{ti} = 1$ if the unit is sold, and $F_{ti} = 0$ otherwise. Note that since the coefficient in front of reserve price r_{ti} is fixed equal to -1, we are able to identify estimated values of γ and σ_θ^2. Instead of (4.7), I estimate slightly modified likelihood function:

$$L = \prod_{t=1}^{T}\prod_{i=1}^{n_t} \Phi\left(X_{ti}\delta_1 + r_{ti}\delta_2\right)^{F_{ti}} \left[1 - \Phi\left(X_{ti}\delta_1 + r_{ti}\delta_2\right)\right]^{1-F_{ti}} \tag{4.8}$$

where $\delta_1 = \gamma/\sigma_\theta$ and $\delta_2 = -1/\sigma_\theta$. Once δ-s are estimated we can recover the original parameters.

So far we have assumed that the valuations are independent across and within auctions. However, we can reason that the valuations of lots sold on a particular auction are correlated because of some common factors. For example, if oil is important input in harvesting and transporting timber, the increase in oil prices will lower the values of all forests. Further, even within each auction we could group data according to regional location and assume that a common term affects the valuations of all lots within a given group. Now we generalize the data generating process for the valuations of lots and decompose the error term into two components. That is, we assume that besides the error term that is specific for each lot, there is another random variable that has common effect on the valuations of all the lots on that day. Now we write the valuation of a lot i that belongs to auction t as

$$w_{ti} = X_{ti}\gamma + \eta_t + \theta_{ti}, \tag{4.9}$$

where η_t is a random variable that affects the valuations of all lots within group t equally. Again I assume that explanatory variables are independent from random terms, and that $\eta_t \sim i.i.d.N(0, \sigma_\eta^2)$ and $\theta_{ti} \sim i.i.d.N(0, \sigma_\theta^2)$. Note that the assumption that the explanatory variables and group-specific random variable are independent is unnecessary, in the same way as it is unnecessary to assume that different components of vector X_{ti} are independent. However, while X_{ti} will mainly consist of characteristics of a particular forest, I assume that η_t captures macro-level factors, therefore we treat them as independent. One situation when X_{ti} and η_t are correlated can occur if η_t depends on the total supply of timber for sale $\sum_{i=1}^{n_t} X_{ti}$, since excess supply will depress market prices for timber and, as a result, the value of the forest. I will rule this case out, based on the observations that most of the timber is exported and Latvia supplies small amount of world timber.

Now the probability of selling the lot is

$$\Phi\left(\frac{X_{ti}\gamma + \eta_t - r_{ti}}{\sigma_\theta}\right)$$

and the probability of not selling the lot is

$$1 - \Phi\left(\frac{X_{ti}\gamma + \eta_t - r_{ti}}{\sigma_\theta}\right).$$

If the variable η_t was observable, we could write likelihood function like in (4.7). However, since η_t is unobservable, we cannot condition on it. Instead, we need to integrate it out, where now the likelihood function is

$$L = \prod_{t=1}^{T} \int_{-\infty}^{+\infty} \left(\prod_{i=1}^{n_t} \Phi \left(\frac{X_{ti}\gamma + \eta_t - r_{ti}}{\sigma_\theta} \right)^{F_{ti}} \left[1 - \Phi \left(\frac{X_{ti}\gamma + \eta_t - r_{ti}}{\sigma_\theta} \right) \right]^{1-F_{ti}} \right) \varphi \left(\frac{\eta}{\sigma_\eta} \right) d \left(\frac{\eta}{\sigma_\eta} \right) \tag{4.10}$$

where $\varphi(\cdot)$ is the density function of standard normal variable. Again, I estimate slightly modified likelihood function:

$$L = \prod_{t=1}^{T} \int_{-\infty}^{+\infty} \left(\prod_{i=1}^{n_t} \Phi \left(X_{ti}\delta_1 + r_{ti}\delta_2 + \eta_t^*\delta_3 \right)^{F_{ti}} \left[1 - \Phi \left(X_{ti}\delta_1 + r_{ti}\delta_2 + \eta_t^*\delta_3 \right) \right]^{1-F_{ti}} \right) \varphi(\eta^*) \, d\eta^* \tag{4.11}$$

where $\eta^* = \eta/\sigma_\eta$ is standardized normal variable, $\delta_1 = \gamma/\sigma_\theta$, $\delta_2 = -1/\sigma_\theta$ and $\delta_3 = \sigma_\eta/\sigma_\theta$.

Thus, we can estimate consistently the deterministic part of the valuation of each lot. However, the valuation net of the reserve price, which is defined as[4]

$$v_{ti} = X_{ti}\gamma - r_{ti} + \eta_t + \theta_{ti}, \tag{4.12}$$

depends on random terms therefore valuations v_{ti} are unobservable variables, even after estimating γ. Given distributional assumptions about θ_{ti} and η_t, net valuations also have normal distribution, $v_{ti} \sim N(X_{ti}\gamma - r_{ti}, \sigma_\theta^2 + \sigma_\eta^2)$, and if lots belong to the same auction their valuations are correlated with covariance $Cov(v_{ti}, v_{tj}) = \sigma_\eta^2$, $i \neq j$ while $Cov(v_{ti}, v_{sj}) = 0$, if $t \neq s$. As long as θ_{ti} and η_t are independent of $X_{ti}\gamma - r_{ti}$, which we have assumed, the use of $X_{ti}\gamma - r_{ti}$ instead of v_{ti} as explanatory variables in (4.1) would still provide consistent estimates β-s when estimated by OLS (since θ_{ti} and η_t would be suppressed into the error term ε_t, but the new error term would still be independent from explanatory variables).

[4]From here on I only consider the case when the valuations are given by equation (4.9), since equation (4.5) is a special case.

However, there are two issues which we must account for. First, by the construction of model, the expected value of error term $\theta_{ti} + \eta_t$ for the sold units is different from zero since in the second stage (that is, when estimating model (4.1)) we use only observations whose gross valuations exceed the reserve price: $E(\theta_{ti} + \eta_t | w_{ti} > r_{ti}) \neq E(\theta_{ti} + \eta_t) = 0$. Since $\theta_{ti} + \eta_t \sim N(0, \sigma_\theta^2 + \sigma_\eta^2)$,

$$E(\theta_{ti} + \eta_t | w_{ti} > r_{ti}) = \sqrt{\sigma_\theta^2 + \sigma_\eta^2}\lambda(z_{ti}),$$

where $z_{ti} = (r_{ti} - X_{ti}\gamma)/\sqrt{\sigma_\theta^2 + \sigma_\eta^2}$ and $\lambda(z_{ti}) = \frac{\phi(z_{ti})}{1-\Phi(z_{ti})}$.

Second, even after correcting for sample selection bias, we still do not know whether the first object is more valuable than the second or not.[5] Each variable $D_{tj}v_{ti}$, $i = 1, 2$, $j = 1, 2$ we write as sum of its mean and random part: $D_{tj}v_{ti} = x_{tij} + \varepsilon_{tij}$, where $x_{tij} \equiv E(D_{tj}v_{ti})$. For example, x_{t11} we calculate as

$$x_{t11t} = E(D_{t1}v_{t1}) = \pi_t E(v_{t1} | v_{t1} > v_{t2} > 0) + (1 - \pi_t)0$$

where

$$\pi_t \equiv \Pr(v_{t1} > v_{t2} | v_{t1} > 0, v_{t2} > 0).$$

Conditional on η_t, valuations are independent and each comes from normal distribution: $v_{ti}|\eta_t \sim N(X_{ti}\gamma - r_{ti} + \eta_t, \sigma_\theta^2)$ truncated from below at 0. Therefore, $\pi_t E(v_{t1}|v_{t1} > v_{t2} > 0)$ we obtain from

$$\begin{aligned} \Pr(v_{t1} &> 0, v_{t2} > 0)\pi_t E(v_{t1}|v_{t1} > v_{t2} > 0) \\ &= \int_{-\infty}^{+\infty} \left(\int_0^\infty \int_0^{v_{t1}} v_{t1} f(v_{t1}|\eta_t) f(v_{t2}|\eta_t) dv_{t2} dv_{t1} \right) f(\eta_t) d\eta_t \\ &= \int_{-\infty}^{+\infty} \left(\int_0^\infty v_{t1} f(v_{t1}|\eta_t)[F(v_{t1}|\eta_t) - F(0|\eta_t)] dv_{t2} dv_{t1} \right) f(\eta_t) d\eta_t. \end{aligned}$$

In similar way we calculate the remaining variables. Observe that when calculating $\pi_t E(v_{t1}|v_{t1} > v_{t2} > 0)$ we already take into account the first problem that $E(\theta_{ti} + \eta_t | w_{ti} > r_{ti}) \neq 0$ or equivalently that $E(v_{ti}|v_{ti} > 0) \neq X_{ti}\gamma - r_{ti}$.

[5] In what follows I discuss the case when the regime change occurs depending on whether $v_1 > v_2$ or $v_1 \leq v_2$, which is relevant in the presence of capacity constraints. Similar analysis can be performed if the regime change occurs depending on whether $v_1 > 2v_2$ or $v_1 \leq 2v_2$, relevant when we consider budget constraints.

Thus, instead of unobserved variables $(D_{t1}v_{t1}, D_{t1}v_{t2}, D_{t2}v_{t1}, D_{t2}v_{t2})$ we use their expected values in the equation (4.1), and we estimate the following model

$$y_t = \beta_{11}x_{t11} + \beta_{12}x_{t12} + \beta_{21}x_{t21} + \beta_{22}x_{t22} + u_t, \tag{4.13}$$

where $u_t = \varepsilon_t + \beta_{11}\varepsilon_{t11} + \beta_{12}\varepsilon_{t12} + \beta_{21}\varepsilon_{t21} + \beta_{22}\varepsilon_{t22}$. Note that the variance of u_t is a function of the characteristics of both lots $X_{t1}\gamma - r_{t1}$ and $X_{t2}\gamma - r_{t2}$. It follows that the variance of u_t will be different across auctions as long as lots will differ in their characteristics, that is, we must take into account the heteroscedasticity when estimating equation (4.13). Knowing the distributions of v_{ti}, $i = 1, 2$, the variances and covariances of ε_{tij}, $i = 1, 2$, $j = 1, 2$ can be calculated. Then, in order to estimate equation (4.13), we proceed as follows. First, we estimate equation (4.13) by OLS to obtain consistent estimates of β_{ij}, $i = 1, 2$, $j = 1, 2$. Next, we calculate the weights $\widehat{\sigma}_{u_t}$ and we estimate now equation (4.13) by Weighted Least Squares. We can iterate the process several times.

4.5 Econometric Results

4.5.1 Estimation of Probit model

In the first stage we estimate the Probit models (4.10) and (4.11) in order to obtain consistent estimates of the valuations of lots. In Section 2 when describing the rules of the auction I mentioned that forests that will be clear-cut are auctioned first, and afterwards secondary-cut forests are auctioned. Since I will select the last two units from the auction to test the hypothesis whether the order affects revenues, these lots would necessarily be from the secondary cut. If we treat the reserve price as an indicator of the forest value, then the average reserve price for principal-cut lots is 4000 LVL, while the average reserve price for secondary-cut lots is only 500 LVL. I assume that bidders regard principal-cut and secondary-cut lots as belonging to two different auctions, possibly because the timber obtained from each type is used for different purposes: the timber from principal-cut is bought by big wood processing companies or exported while the timber from the secondary-cut is used by small local companies. Therefore, I have decided to restrict the sample only to clear-cut type of lots. After omitting lots of secondary-cut and lots with missing observations[6], I was left with 4961 lots sold in 292 auctions.

First, we need to specify what are the relevant explanatory variables that affect the value of the forest w_{ti}. The particular specification I adopt for (4.9) is the following:

$$\begin{aligned} w_{ti} &= \widetilde{\gamma}_1 + \widetilde{\gamma}_2 Pine_{ti} + \widetilde{\gamma}_3 Fir_{ti} + \widetilde{\gamma}_4 Birch_{ti} + \widetilde{\gamma}_5 Aspen_{ti} \\ &+ \widetilde{\gamma}_6 Other_{ti} + \widetilde{\gamma}_7 Road_{ti} + \widetilde{\gamma}_8 Area_{ti} + \eta_t + \theta_{ti}, \end{aligned} \tag{4.14}$$

where the definitions of the variables are given in the Appendix 4.7. Thus, I assume that the value of the forest is proportional to the cubic-meters of timber available there, and different types of timber are valued differently. Variables *Road* and *Area* are meant to capture the harvesting costs. Note that coefficients $\widetilde{\gamma}_2, ..., \widetilde{\gamma}_6$ give the value of one cubic-meter of growing tree of corresponding type, and not the price of cut trees sold on the market. To obtain the latter we would need to add to the coefficients $\widetilde{\gamma}_2, ..., \widetilde{\gamma}_6$ the cutting costs. The constant term $\widetilde{\gamma}_1$ can be interpreted as fixed costs.

Since the data I have from auctions cover two and a half year period from the beginning of 2001 till the middle of 2003, it is probable that the coefficients $\widetilde{\gamma}$ did not remain fixed. When regressing the reserve price r on the explanatory variables X and the date of auction, the coefficient for the date was positive and significant. On the other hand, the proportion of sold to unsold units in each auction had the tendency to increase, implying that the value of lots has increased even faster. To account for this potential evolution of coefficients and at

[6]In particular, I needed to drop all observations from Eastern Vidzeme Region since they did not provide percentage composition in their announcements of auctions.

the same time to give it some economic justification, I assume that $\widetilde{\gamma}_j = \gamma_j GDP$ for $j = 1, ..., 8$, that is, I assume that the coefficients are proportional to the evolution of nominal GDP since it captures both inflation and economic activity. Equation (4.14) now becomes

$$\begin{aligned} w_{ti} &= \gamma_0 + \gamma_1 GDP_t + \gamma_2 GDP_t Pine_{ti} + \gamma_3 GDP_t Fir_{ti} \\ &\quad + \gamma_4 GDP_t Birch_{ti} + \gamma_5 GDP_t Aspen_{ti} + \gamma_6 GDP_t Other_{ti} \\ &\quad + \gamma_7 GDP_t Road_{ti} + \gamma_8 GDP_t Area_{ti} + \eta_t + \theta_{ti}, \end{aligned} \tag{4.15}$$

where I have introduced new intercept term γ_0.

Summary statistics of variables for all units and only for sold units is provided in the Table 4.1. I also report the information about seller's revenues.

Table 4.1: Sample summary statistics

	all units		sold units				
	average	std.	average	std.	min	max	total
pine	125.2	209.0	146.1	225.8		1,770	518,597
fir	158.2	157.8	177.2	169.3		1,551	629,022
birch	92.9	112.0	94.0	116.4		1,398	333,800
aspen	60.4	129.2	57.4	125.7		1,504	203,810
other	27.3	63.1	26.8	60.5		655	94,963
road	481.1	389.0	450.3	366.6	10	3000	
area	1.96	1.38	2.04	1.35	0.1	23.0	
r	3,945.60	3,766.57	4,409.82	3,981.76	10	30,000	
GDP			1.16	0.11	1.00	1.33	
revenue			6,211.66	5,665.69	10	52,100	22,051,385

Having specified the explanatory variables, we estimate δ-s in (4.8) and (4.11). The estimation results are given in the Table 4.2 where I report directly original parameters γ-s, σ_θ and σ_η, and their standard deviations obtained by the delta method. First, observe that the estimates of coefficients from (4.11) take larger absolute values than estimates from (4.8). The reason is that when we neglect group-specific random effects, we can still estimate the Probit model of the form (4.7), but instead of standard deviation σ_θ, now the standard devia-

Table 4.2: Results from estimation of Probit model

	equation (4.8)		equation (4.11)	
	coefficients	stand. dev.	coefficients	stand. dev.
intercept	-31303.17	4192.16	-39351.64	6704.36
GDP_t	30802.01	4073.18	40364.50	6657.37
$GDP_t Pine_{ti}$	16.85	1.08	17.39	1.27
$GDP_t Fir_{ti}$	24.84	1.86	27.00	2.28
$GDP_t Birch_{ti}$	6.33	1.37	8.60	1.89
$GDP_t Aspen_{ti}$	-0.03	1.00	-0.28	0.99
$GDP_t Other_{ti}$	5.57	2.03	-2.72	2.30
$GDP_t Road_{ti}$	-2.36	0.38	-3.50	0.55
$GDP_t Area_{ti}$	-455.64	143.54	-570.85	134.41
σ_θ	6060.61	724.90	6581.22	937.21
σ_η			4930.19	761.16
observations	4961		4961	
sold units	3550		3550	
unsold units	1411		1411	
$\log L$	-2491.60		-2206.10	
pseudo R^2	0.159		0.255	
$\Pr(y=1) \leq .5$	27.57		15.95	
$\Pr(y=1) > .5$	92.93		96.23	

tion is $\sqrt{\sigma_\theta^2 + \sigma_\eta^2}$.[7] It means that the coefficients obtained from (4.8) will be $\sigma_\theta / \sqrt{\sigma_\theta^2 + \sigma_\eta^2}$ times lower than obtained from (4.11). Using estimates of σ_θ and σ_η from (4.11), the ratio is 0.8. It means that by omitting the variable η_t from the list of explanatory variables, we are systematically underestimating the coefficients, and thus incorrectly predicting the value of a forest. The results from Table 4.2 imply that we reject $H_0 : \sigma_\eta = 0$, therefore we reject model (4.7) as misspecified.

Coefficients have signs as expected, except estimates of γ_5 and γ_6 in (4.11). However, neither of these coefficients is significantly different from zero. I had normalized GDP equal to 1 in the first quarter of 2001, while the highest value it reached during the sample period is in the last quarter of 2002, when it was 1.3312. Thus, for example, the price of cubic-meter of growing fir-tree was initially

[7]However, when estimating (4.7) we cannot obtain separate estimates for σ_θ and σ_η.

27.00 LVL/m^3 and its peak price was 35.94 LVL/m^3. Since recently LVM has started to publish on its web site the prices it uses to obtain the initial price of a lot. The prices are determined for each type of tree depending on the diameter and whether distance to road is below or above 800 m. For example, the prices of fir-tree during the month of August of 2005 with distance below 800 m were 24.90, 18.00 and 8.00 lats per m^3 depending on the diameter. The corresponding prices for pine-tree were 30.40, 23.50 and 12.00 and for birch 15.20, 9.00 and 6.80 lats per m^3. The published prices are comparable with the coefficients obtained. The most important difference is that according to my estimates cubic meter of fir-tree was more valuable than the cubic-meter of pine-tree, while according to LVM the price relationship (in August of 2005) is reversed.

Each 100 m that a forest is away from a road lowers its value by $350 \cdot GDP_t$ lats. Average amount of timber in the sample is 464 m^3 while average distance to road is 481 m, implying that transporting 1 m^3 to the road costs around $3.50 \cdot 481/464 \cdot GDP_t = 3.63 \cdot GDP_t$ LVL/m^3. As mentioned in the description of the company, it has introduced a new type of auctions where companies are just contracted to cut growing trees. The companies that participate in these auctions must submit in their bids separately the cost of harvesting 1 m^3 of timber and the cost of transporting it to the road. While one does not expect the companies bidding truthfully their costs, we can expect that the true costs are below the bid cost, the difference being the profit of company. In the auction conducted in 2003 transportation costs to the road were in the range 2.00-2.50 LVL/m^3. The results of estimation indicate that we possibly overestimate the transportation costs. However, the auctions are not directly comparable since in roundwood auctions the lot size is much bigger: 50,000 m^3 compared with approximately 500 m^3 in the auctions we analyze. It could explain the differences in costs if there are economies of scale.

The coefficient for area tells the decrease in the value of forest per hectare of forest. I assume that the variable $Area$ captures the costs of harvesting since it is reasonable that the costs are proportional to the size of area. Another way to interpret the coefficient for area is as follows: given the volume of timber, the smaller the area, the taller and bigger in the diameter the trees are, and therefore the more valuable the timber is. The variable GDP, in turn, may capture the cost savings as long as the growth in GDP indicates technological improvements.

The explanatory power of the model is not high since pseudo R^2 is 0.255, where pseudo R^2 is calculated as $1 - \log L / \log L_R$ and $\log L_R$ is the log-likelihood from the restricted model containing only the intercept term. I also report the so-called percent correctly predicted. Given the estimates of coefficients $\widehat{\delta}$ we predict the lot to be sold, that is, $y_{ti} = 1$ if $\Phi\left(X_{ti}\widehat{\delta}_1 + r_{ti}\widehat{\delta}_2\right) > 0.5$ and unsold in the opposite case. The percentage of times the predicted y_{ti} matches the actual outcome is the percent correctly predicted. From Table 4.2 we see that the model poorly predicts the cases when the unit is not sold, that is, for the most of unsold units the model says that these lots should have been sold. This result can be interpreted as the evidence against the assumption that any lot whose gross value exceeds the reserve price will be sold.

4.5.2 Testing the order effect

From 292 auctions used to estimate the Probit model, I selected 204 auctions where the last two lots of principal-cut were bought by two different bidders since the order of sales will only matter if the lots are bought by two different bidders. If both lots are bought by single bidder, his opponents' undominated strategy under the complete information assumption is to bid for each lot its valuation or up to the budget constraint which ever is smaller.

The way how we arrive at the specification in equation (4.13) has a drawback. First, the variability in the valuations can be decomposed into the variability of the observable lot characteristics and into the variability coming from random effects. By taking expectations we remove the second source of the variability. Second, even if valuations were independent (they are not since contain common term η_t), the expected values we calculate depend on the characteristics of both lots. As a result, the obtained variables are highly correlated and we face multicollinearity problem. The coefficients of correlation between all x_{ij} $i = 1, 2$, $j = 1, 2$ are above 0.9. Although the estimated coefficients are unbiased in the presence of multicollinearity, the estimates of individual coefficients can be very imprecise.

I assume that there are no errors induced by bidding, $\varepsilon_t = 0$, in order to simplify the calculation of $E(u_t^2)$. Iterative estimation of (4.13) by WLS gave

	coefficients	stand. dev.	t-statistic
x_{11}	-0.7246	2.0585	-0.3520
x_{12}	0.2804	3.5123	0.0798
x_{21}	1.0884	3.6904	0.2949
x_{22}	0.2557	1.9609	0.1304
R^2	0.5078	SCR	1.65e+09

The obtained coefficients indicate that selling less valuable object first leads to higher seller's revenues since $\widehat{\beta}_{22} > \widehat{\beta}_{11}$ and $\widehat{\beta}_{21} > \widehat{\beta}_{12}$. However, as mentioned above, in the presence of multicollinearity we can not trust obtained coefficients. Standard deviations are very high compared with the magnitudes of the coefficients and we even can not reject the joint insignificance of explanatory variables $H_0 : \beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 0$. Wald test statistic is equal to 6.01 that has $\chi^2(4)$ distribution, and we do not reject the null hypothesis at 10% significance level. The test whether the order of sales does not affect the revenues of the seller $H_0 : \beta_{11} = \beta_{22}, \beta_{12} = \beta_{21}$ gave statistic 0.13, therefore we do not reject null hypothesis that the order is irrelevant.[8] I also tested the hypothesis whether bidders play strategies implied by the presence of capacity constraints. Test statistic when testing hypothesis $H_0 : \beta_{11} = 1, \beta_{12} = -1, \beta_{21} = \beta_{22} = 0$ was 10.37, and I could reject the null hypothesis at 5% significance level. I also rejected the strategies, the bidders would adopt, if they were unconstrained: $H_0 : \beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 1$ since Wald statistic was 751.56.

[8]It was already implied by the previous test since $H_0 : \beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 0$ is more restrictive than $H_0 : \beta_{11} = \beta_{22}, \beta_{12} = \beta_{21}$.

Estimating (4.13) by WLS when regime switch occurs depending on whether $v_1 \geq 2v_2$ or not, gave the following results

	coefficients	stand. dev.	t-statistic
x_{11}	-1.0503	2.4219	-0.4337
x_{12}	0.6807	8.9690	0.0759
x_{21}	1.0394	4.5742	0.2272
x_{22}	0.4987	2.1230	0.2349
R^2	0.5225	SCR	1.60e+09

The obtained coefficients again indicate that selling less valuable object first leads to higher seller's revenues since $\widehat{\beta}_{22} > \widehat{\beta}_{11}$ and $\widehat{\beta}_{21} > \widehat{\beta}_{12}$, but we can not reject the joint hypothesis that all coefficients are equal to zero, $H_0 : \beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 0$ since Wald test statistic is equal to 3.14. An implication again is that the order of sales does not affect the revenues of the seller. On the other hand, I rejected the hypothesis that bidders play strategies implied by the presence of budget constraints. Test statistic when testing hypothesis $H_0 : \beta_{11} = 1, \beta_{12} = -1, \beta_{21} = 0.5, \beta_{22} = 0$ was 34.82, which allows to reject the null hypothesis. I also reject that bidders behave as if they were budget unconstrained: $H_0 : \beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 1$ since Wald statistic was 337.54.

4.6 Discussion

We do not find the evidence that the seller can increase the revenues from sequential auction by changing the order in which lots of forest are auctioned. Although we reject the particular strategies derived in Section 4.3 to illustrate why the order becomes relevant when there are capacity or budget constraints, the power of test, when testing the hypothesis that the order does not matter, is very low. That is, even if the order of sales affected the revenues we would not be able to detect it due to multicollinearity problem and resulting high variances. The reason of multicollinearity is the high variances obtained in the Probit model, $\widehat{\sigma}_\theta^2$ and $\widehat{\sigma}_\eta^2$ since they do not allow to classify whether the first object is more valuable or not than the second with certainty. Most of the probabilities

$\Pr(v_{t1} > v_{t2} | v_{t1} > 0, v_{t2} > 0)$ take values in the range 0.3-0.7, and just few are close to 1 or 0.

Additionally, the results hinge on many modelling assumptions. The violations of any of them can lead to erroneous conclusions. First, the valuations of lots are not known and are estimated. The estimation was based on the assumption that the lot will be bought as long as its valuation exceeds the reserve price. It implies that joint budget and/or capacity of bidders must be sufficient to exercise any profitable purchase. In reality firms may not be able to increase their capacities before the auction, and consequently, not all lots, that have positive net value, will be bought. It is suggested by the fact that the Probit model predicts that most of the unsold lots should have been sold. On the other hand, the obtained coefficients are confirmed by indirect evidence like the prices the seller uses to construct initial price and bids from related roundwood auctions.

The second stage analysis assumes that bidders regard lots of principal-cut and secondary-cut as two distinct goods. Although there may be factors that differentiate them like size and composition, the substitutability is a matter of relative prices. A company that specialized in harvesting big lots of principal cut may now find it more profitable to purchase growing trees from the secondary-cut forests. It implies that the bidding strategies for the last two units of principal cut will not be independent of the valuations from the lots of secondary cut that are sold after. The same argument applies to the independence of individual auctions. The firms when planning their bidding strategies for a particular auction will take into account the fact that there will be other auctions afterwards. Since the forest must usually be harvested by the end of the year, firms can plan over longer period how to utilize their resources. Additionally, auctions are announced two weeks in advance, and also the lots that are not sold, are offered for sale in the subsequent auctions. Thus, firms often will know what will be offered in the following auctions while bidding in the present auction.

The hypothesis that the order of sales can affect seller's revenue depends on heterogeneity of sold lots. If the number of lots is large relative to the number of firms, they can implicitly bundle lots in order to homogenize their purchases, since it eliminates the competition induced by heterogeneity. In general, the repeated interaction of firms across auctions and even within auctions allow them to mitigate the competition.

Even if the data supported the relevance of the order of sales, the use of optimal ordering to raise revenues could be limited in practice. We assume that the seller does not know how much firms value each forest, therefore it would need to order lots based on its estimates. It means that, unless estimates are very precise, there will be cases when the lots will be ordered suboptimally, limiting the gains from this policy. If LVM knew the valuations or had very precise estimates, the best it could do, would be to set the reserve price equal to the value or estimate of the forest. It suggests that the reserve price may be much more important tool to raise revenues then the optimal ordering of lots.

4.7 Appendix: The description of variables

The data sample contain most of the auctions of growing trees held from January of 2001 till July of 2003. The data combine the descriptions of lots available from the web site of the company with the information on sold units (the identity of the winner and the price he pays), obtained from sales reports, submitted by the regional forestries to the sales department of LVM. An example of announcement is provided in Table 4.3. The description of each lot contains: 1) lot number; 2) location of the lot; 3) type of felling; 4) size of area; 5) volume - total and useful; 6) composition by type of trees; 6) distance to road; 7) initial price; 8) required deposit; 9) notes if special terms apply.

The composition of forest indicates the percentage split among several types of trees. The most common types of trees are pine-tree, fir-tree and birch, other types include aspen, black and white alder, oak, ash-tree and lime-tree. To obtain the amount of timber for each type I multiplied the 'useful' timber amount, measured in cubic-meters, with share of that type of tree in the forest. I defined corresponding variables: $pine$, fir, $birch$, $aspen$, and $other$ where the last variable contains all the rest of timber types since their amount was relatively small. All variables are measured in cubic-meters. I also defined the variables $road$ - the distance to road measured in meters, $area$ - the size of forest measured in hectares, and r - the initial price measured in lats. The variable GDP is nominal quarterly GDP of Latvia obtained from the web site of Statistical Office of Latvia. The revenue y in (4.1) is defined as the difference between sales price p and the initial price r, summed over both lots, and all measured in lats.

Table 4.3: Auction held on January 24, 2003, region of Southern Kurzeme

Lot number	Forest district	Head forestry	Forestry	Type of felling	Identificators of felling area		Area, ha	Volume, m3		Composition	Initial price, LVL	Deposit, LVL	Distance to road, m
								total	useful				
1	Alsungas	Kuldīgas	Alsungas	clear-cut	143	1;2;3	4.7	1481	1420	7P3E	23000	2300	300
2	Alsungas	Kuldīgas	Alsungas	clear-cut	155	17;24; 25;27	5	1362	1340	5P3E2B	20000	2000	200
3	Apriķu	Liepājas	Aizputes	clear-cut	359	14	2.1	501	455	6E3B1Ba	5000	500	200
4	Apriķu	Liepājas	Aizputes	clear-cut	359	15	2.2	651	561	4E4B2A	4500	450	350
5	Apriķu	Liepājas	Aizputes	clear-cut	359	30	1.8	569	473	2E3B2A2M 1Ba	3000	300	300
6	Grobiņas	Liepājas	Grobiņas	clear-cut	14	3	1.3	356	242	5B2E2A1Os	2000	200	400
7	Grobiņas	Liepājas	Grobiņas	clear-cut	14	12	3.2	643	456	5E1P2Os1B 1A	5000	500	450
8	Krīvukalna	Liepājas	Priekules	clear-cut	20	14	2.8	884	747	4E4B2A	7000	700	600
9	Akmensraga	Liepājas	Apriķu	stock care-cut	22	5;6;11; 12;13	16.8	741	666	7P1E2B	4000	400	550
10	Akmensraga	Liepājas	Apriķu	stock care-cut	27	4;5;8-12	11.8	587	495	5P1E4B	3500	350	350
11	Apriķu	Liepājas	Aizputes	stock care-cut	202	1;2;3	3.5	185	104	6E2Oz1B 1Ba	1000	100	400
12	Apriķu	Liepājas	Aizputes	stock care-cut	358	10;11	7.8	508	351	6E2B1M 1Ba	3000	300	90
13	Krīvukalna	Liepājas	Priekules	stock care-cut	370	14	2.8	243	196	5E5B	1500	150	600

Composition 7P3E, for example, means 70% pine-tree, 30% fir-tree.

Bibliography

[1] O. Ashenfelter. How auctions work for wine and art. *Journal of Economic Perspectives*, 3(3):23–36, Summer 1989.

[2] O. Ashenfelter and K. Graddy. Art auctions: A survey of empirical studies. *NBER WP Series*, (8997), 2002.

[3] A. Beggs and K. Graddy. Declining values and the afternoon effect: evidence from art auctions. *RAND Journal of Economics*, 28(3):544–565, Autumn 1997.

[4] J.P. Benoît and V. Krishna. Multiple-object auctions with budget constrained bidders. *Review of Economic Studies*, 68:155–179, 2001.

[5] R. Blundell and M. Costa Dias. Alternative approaches to evaluation in empirical microeconomics. *Portuguese Economic Journal*, 1:91–115, 2002.

[6] W. Elmaghraby. The importance of ordering in sequential auctions. *Management Science*, 49(5):673–682, May 2003.

[7] I.L. Gale and D.B. Hausch. Bottom-fishing and declining prices in sequential auctions. *Games and Economic Behavior*, 7:318–331, 1994.

[8] G.W. Imbens. Nonparametric estimation of average treatment effects under exogeneity: A review. *The Review of Economics and Statistics*, 86(1):4–29, February 2004.

[9] C. Pitchik. Budget-constrained sequential auctions with incomplete information. April 2004.

[10] C. Pitchik and A. Schotter. Perfect equilibria in budget-constrained sequential auctions: an experimental study. *RAND Journal of Economics*, 19(3):363–388, Autumn 1988.

[11] J.M. Wooldridge. *Econometric Analysis of Cross Section and Panel Data*. The MIT Press.